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On Symmetries of Geometric Algebra $Cl(3,1)$ for Space-Time

Eckhard Hitzer

Abstract. From viewpoints of crystallography and of elementary particles, we explore symmetries of multivectors in the geometric algebra $Cl(3,1)$ that can be used to describe space-time.

Keywords. Geometric algebra, Clifford algebra, symmetry, multivector symmetry, crystallography, elementary particles.

1. Introduction

Recently, [9] and [8] classified multivectors based on their symmetries[1] under space inversion (main grade involution in geometric algebra) $\hat{1}$, time reversal $1'$ and reversion (called wedge reversion by them) $\tilde{1}$. [9] notes in the conclusions that *One could perhaps explore charge reversal ($\hat{C}$), parity reversal ($\hat{P}$) and time reversal ($\hat{T}$) in the relativistic context [11].* In the standard model of elementary particle physics many experiments have confirmed violation of parity symmetry by the weak interaction, and of $\hat{C}\hat{P}$ symmetry. However, strong interactions by themselves do preserve $\hat{C}\hat{P}$ symmetry [5].

Therefore one aim of this paper is to work in this direction by looking at the effect of these three symmetries on multivectors of $Cl(3,1)$, a (geometric) algebra that can be used to express space-time physics, with $\hat{C}$, $\hat{P}$ and $\hat{T}$ symmetries, e.g., defined by [6], but here we only focus on the effect of these transformations on the 16 basis blades that constitute the multivector basis of $Cl(3,1)$, and ignore any functional dependence of coefficients in linear combinations that might express spinors or other physical quantities. Furthermore, [11] and [6] have a clear preference for the use of $Cl(1,3)$, while

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Soli Deo Gloria. This work is dedicated to peace for Israel: *Pray for the peace of Jerusalem: May those who love you be secure. May there be peace within your walls and security within your citadels.* (Ps. 122:6+7, NIV). Please note that this research is subject to the Creative Peace License [15].

[1] Note that [9] and [8] use for space inversion $\bar{1}$, and for (wedge) reversion $1^{\dagger}$.

in this work we prefer[2] to use $Cl(3,1)$ because its volume-time subalgebra $\{1, e_0, e_{123}, e_{0123}\}$ isomorphic to quaternions, where e_0 expresses the time direction, at the foundation of the theory of space-time Fourier transforms [17].

[9] and [8] work signature independent for all Clifford algebras of quadratic spaces. We work in an algebra of specific signature and want to take advantage of the principal reverse[3] (see e.g. [17] (2.1.12)), which applied to $Cl(3,1)$ acts like the conventional reverse, but in addition changes the sign of the time vector $e_0 \to -e_0$. So we focus on the group of eight symmetries generated by grade involution $\hat{1}$, reversion $\tilde{1}$ and principal reverse[4] $1'$.

An introduction to geometric algebra can be found in [16]. A mathematically very thorough introductory textbook is [23]. The use of geometric algebra in physics can be found in [11] and more recently in [6]. Computations, like the ones performed in this paper can be checked with computer algebra software, e.g. with the MATLAB package [24]. In the field of computer science, the textbook [7] is a standard reference, and [12] shows how to optimize geometric algebra computations. Extensive surveys of applications can be found in [2, 14, 20]. Applications of geometric algebra to crystallography can be found in [10, 13, 19].

This paper is structured as follows. Section 2 studies the symmetries of Cl(3,1) multivectors under space inversion, reversion and principal reverse. Section 3 is devoted to aspects of charge conjugation, parity reversal and time reversal, when $Cl(3,1)$ is applied in the description of elementary particle physics. For ease of reference, four tables are included that show the application space inversion, reversion and principle reverse to the multivector basis of $Cl(3,1)$ in Table 1, the composition of the symmetries $\hat{C}, \hat{P}$ and $\hat{T}$ in Table 2, the application of the symmetries $\hat{C}, \hat{P}$ and $\hat{T}$ to the multivector basis of $Cl(3,1)$ in Table 3, and a reordered version of that in Table 4 in Appendix A.

2. Symmetries of $Cl(3,1)$ multivectors generated by space inversion, reversion and principal reverse

[9] and [8] correctly turn to Clifford algebra in order to generalize the notion of cross product that only exists in three dimensions to arbitrary dimensions. In this context [18], for crystallographers it may be of interest to know that J. G. Grassmann (Justus G. was the father of Hermann G. Grassmann) originally introduced the characterization of crystal planes by orthogonal vectors, now commonly denoted with Miller indices (see Erhard

[2] Another notable work using $Cl(3,1)$ in elementary particle physics is, e.g. [25].
[3] The principal reverse is in geometric algebra the equivalent of matrix transposition, see [1].
[4] The reader should be aware that therefore in this work we do not use a priori the notion of time reversal of [9] and [8], which there also has the symbol $1'$. Although we do obtain it for multivectors that have e_0 as a factor, by the product of reversion $\tilde{1}$ and principal reverse $1'$ (our notation).

Scholz [26], pp. 37–46). J. G. Grassmann's work, including his mathematical school textbooks, provided H. G. Grassmann with fertile ideas for his new concepts of algebra (including exterior algebra), solely defined by the relations of its elements, from which G. Peano distilled the modern concept of vectors. Grassmann's pioneering approach was so far ahead of its time that only a few bright minds (like R. W. Hamilton, F. Klein and S. Lie) recognized its genius during his lifetime, late in Grassmann's life. But the young Cambridge-educated genius W. K. Clifford was truly exceptional, and published in 1878 (one year after Grassmann's death) his seminal paper 'Applications of Grassmann's Extensive Algebra' in Am. J. Math. [4]. It elegantly unified the earlier works of Hamilton on quaternions and Grassmann's metric-free algebra of extension to geometric algebras (now known as Clifford algebras), by simply adding in the Clifford (or geometric) product the inner product of vectors (necessary for measurements) and the outer product of Grassmann. Unknowingly perhaps, [9] and [8] thus return to the origins of both crystallography and modern algebra in their search for a dimension independent mathematical framework.

The unit blade basis of the geometric algebra $Cl(3,1)$ is given by one scalar, four vectors, six bivectors, four trivectors and one pseudoscalar quadvector I,

$$\{1, e_0, e_1, e_2, e_3, e_{01}, e_{02}, e_{03}, e_{23}, e_{31}, e_{12}, e_{023}, e_{031}, e_{012}, e_{123}, I = e_{0123}\}, \quad (1)$$

with

$$e_0^2 = -1, \;\; e_1^2 = e_2^2 = e_3^2 = 1, \;\; e_j \cdot e_k = 0 \;\; \forall \; j \neq k. \quad (2)$$

We then have for $j, k \in \{1,2,3\}$, $j \neq k$,

$$e_{0j}^2 = 1, \;\; e_{jk}^2 = -1, \;\; e_{0jk}^2 = 1, \;\; e_{123}^2 = -1, \;\; e_{0123}^2 = -1. \quad (3)$$

We define the main grade involution (space inversion) for $M \in Cl(3,1)$ by

$$\hat{1}M = \widehat{M} = \sum_{k=0}^{4} (-1)^k \langle M \rangle_k, \quad (4)$$

where $\langle M \rangle_k$ is the grade k part of M. This is equivalent to reversing the direction of every vector $e_i \to -e_i$, $i \in \{0,1,2,3\}$ in the expression for M.

The reversion (reversing the geometric product order of all geometric products of vectors) of M is defined as

$$\tilde{1}M = \widetilde{M} = \sum_{k=0}^{4} (-1)^{\frac{1}{2}k(k-1)} \langle M \rangle_k. \quad (5)$$

The product of grade involution and reversion leads to Clifford conjugation

$$\bar{1}M = \overline{M} = \hat{1}\tilde{1}M = \tilde{1}\hat{1}M = \sum_{k=0}^{4} (-1)^{\frac{1}{2}k(k+1)} \langle M \rangle_k. \quad (6)$$

Finally, the principal reverse $1'M = M'$ of $M \in Cl(3,1)$ is defined identical to the reversion with additionally changing each occurrence of the

Table 1. Action of involutions of group (7) on all 16 basis elements (1) of $Cl(3,1)$. Tp. = type with scalar S, time vector V_0 multiple of e_0, space vector V, bivector B_0 with e_0 factor, space bivector B, trivector T_0 with e_0 factor, space trivector T and pseudoscalar quadvector Q. Bas. = basis element.

Tp.	Bas.	$\hat{1}$	$\tilde{1}$	$\bar{1}$	$1'$	$\hat{1}'$	$\tilde{1}'$	$\bar{1}'$
S	1	e	e	e	e	e	e	e
V_0	e_0	o	e	o	o	e	o	e
V	e_1	o	e	o	e	o	e	o
	e_2	o	e	o	e	o	e	o
	e_3	o	e	o	e	o	e	o
B_0	e_{01}	e	o	o	e	e	o	o
	e_{02}	e	o	o	e	e	o	o
	e_{03}	e	o	o	e	e	o	o
B	e_{23}	e	o	o	o	o	e	e
	e_{31}	e	o	o	o	o	e	e
	e_{12}	e	o	o	o	o	e	e
T_0	e_{023}	o	o	e	e	o	o	e
	e_{031}	o	o	e	e	o	o	e
	e_{012}	o	o	e	e	o	o	e
T	e_{123}	o	o	e	o	e	e	o
Q	e_{0123}	e	e	e	o	o	o	o

time basis vector e_0 to $-e_0$. In a general Clifford algebra simply every unit basis vector is multiplied by its own square.

These involutions generate by composition the following Abelian group of involutions

$$G = \{1, \hat{1}, \tilde{1}, \bar{1}, 1', \hat{1}', \tilde{1}', \bar{1}'\}. \tag{7}$$

In Table 1 all seven involutions (apart from the identity) of the group (7) are applied to the blade basis (1) of $Cl(3,1)$. Since the involutions only involve sign changes, the letter e for even indicates no sign change and the letter o for odd indicates a sign change.

The eight principal types $S, V_0, V, B_0, B, T_0, T$ and Q denoted in Table 1 are all uniquely characterized by the action of the elements of the group of involutions (7). It is now of course possible to follow the pattern established by [9] and [8] and regard linear combinations of principal types as new types, for which the action of the the group of involutions (7) would then be called *mixed* m. For example if we add scalars and quadvectors we would get the type $SQ = S + Q$ with the group action given by a new line in Table 1 that has the seven entries (in the same order as in the table)

$$e \quad e \quad e \quad m \quad m \quad m \quad m. \tag{8}$$

Another example is that a combination of SB_0B or of SB_0BQ has the seven group action entries

$$e \quad m \quad m \quad m \quad m \quad m \quad m. \tag{9}$$

This method leads, similar to Table 3 of [8], again to distinguish exactly 51 types of multivectors characterized by the action of the group of involutions (7).

The results on the eight principal multivector types and 43 further multivector types of [8] can be fully transferred to $Cl(3,1)$ with the following map, where the index 31 stands for $Cl(3,1)$, and the index GF for the authors Gopalan and Fabrykiewicz of [9] and [8], respectively. First we state the map of the seven involutions

$$\begin{aligned} &\bar{1}_{GF} \to \hat{1}_{31}, \qquad 1'_{GF} \to \tilde{1}'_{31}, \qquad 1^{\dagger}_{GF} \to \tilde{1}_{31}, \qquad 1'^{\dagger}_{GF} \to 1'_{31}, \\ &\bar{1}'_{GF} \to \bar{1}'_{31}, \qquad \bar{1}^{\dagger}_{GF} \to \bar{1}_{31}, \qquad \bar{1}'^{\dagger}_{GF} \to \hat{1}'_{31}. \end{aligned} \tag{10}$$

Next the map for the multivector type labels

$$\begin{aligned} &S'_{GF} \to S_{31}, \qquad V_{GF} \to V_{0\,31}, \qquad V'_{GF} \to V_{31}, \qquad B_{GF} \to B_{0\,31}, \\ &B'_{GF} \to B_{31}, \qquad T_{GF} \to T_{0\,31}, \qquad T'_{GF} \to T_{31}, \qquad S_{GF} \to Q_{31}. \end{aligned} \tag{11}$$

With the two maps (10) and (11) all results of Table 3 of page 383 of [8] can now be transferred to a classification of $Cl(3,1)$ multivectors into a total of 51 types, including the eight principal types (which appear in the first column of Table 1). The grades in Table 3 of page 383 of [8] are then of course restricted to $\{0,1,2,3,4\}$, S having grade zero and Q having grade four. For example, the label $S'VBT'$ of No. 43 in Table 3 of page 383 of [8] is mapped to SV_0B_0T, etc.

3. On symmetries of $Cl(3,1)$ related to elementary particles: charge conjugation, parity reversal and time reversal

In this section we apply the symmetry operations of charge conjugation $\hat{C}$, parity reversal $\hat{P}$ and time reversal $\hat{T}$ expressed in the geometric algebra $Cl(3,1)$ for the description of space-time as they can, e.g., be found in [6], page 283. There the application is to spinors (even grade valued multivector functions $\mathbb{R}^{1,3} \to Cl_+(1,3)$) including reflection at the time axis e_0 of the argument of the spinor. Here we prefer to work instead with $Cl(3,1)$ as explained in the introduction. And we restrict ourselves to only study the action of the three symmetry operations on the constant basis elements (1) of $Cl(3,1)$. Following [6], page 283, we therefore define for multivectors $M \in Cl(3,1)$

$$\hat{C}M = Me_1e_0, \qquad \hat{P}M = e_0Me_0, \qquad \hat{T}M = Ie_0Me_1, \tag{12}$$

where M is not restricted to the even grade subalgebra. The associativity of the geometric product has as immediate consequence that the composition of these three symmetry operations is also associative, e.g.,

$$\hat{C}(\hat{P}(\hat{T}M)) = (\hat{C}\hat{P})\hat{T}M = \hat{C}(\hat{P}\hat{T}M) = \hat{C}\hat{P}\hat{T}M, \tag{13}$$

so it is not necessary to write brackets to indicate the order of composition, and we generally drop brackets when composing the symmetry operations as in the last expression above. We further find that

$$\begin{aligned} &\hat{C}\hat{C}M = Me_{10}e_{10} = M, \qquad \hat{P}\hat{P}M = e_0^2 M e_0^2 = M, \\ &\hat{T}\hat{T}M = Ie_0 Ie_0 M e_1^2 = e_{123}^2 M = -M, \end{aligned} \tag{14}$$

which shows that

$$\hat{C}^2 = 1, \qquad \hat{P}^2 = 1, \qquad \hat{T}^2 = -1. \tag{15}$$

Computation further shows the following commutation relations

$$\begin{aligned} &\hat{P}\hat{T}M = \hat{T}\hat{P}M = I\hat{C}M, \qquad \hat{T}\hat{C}M = -\hat{C}\hat{T}M = -I\hat{P}M, \\ &\hat{C}\hat{P}M = -\hat{P}\hat{C}M = -I\hat{T}M = e_0 M e_1 \quad \Rightarrow \quad \hat{T}M = I\hat{C}\hat{P}M. \end{aligned} \tag{16}$$

Moreover,

$$\hat{C}\hat{P}\hat{T}M = IM, \qquad \hat{C}\hat{P}\hat{T}\hat{C}\hat{P}\hat{T}M = -M. \tag{17}$$

and applying the above commutation relations leads to

$$\hat{C}\hat{P}\hat{T} = \hat{P}\hat{T}\hat{C} = -\hat{P}\hat{C}\hat{T} = \hat{C}\hat{T}\hat{P} = -\hat{T}\hat{C}\hat{P} = \hat{T}\hat{P}\hat{C}, \tag{18}$$

and

$$\hat{C}\hat{C}\hat{P}\hat{T}M = \hat{P}\hat{T}M, \quad \hat{P}\hat{C}\hat{P}\hat{T}M = -\hat{C}\hat{T}M, \quad \hat{T}\hat{C}\hat{P}\hat{T}M = \hat{C}\hat{P}M. \tag{19}$$

Putting all this information together we can represent all possible compositions of the symmetry operators $\hat{C}, \hat{P}$ and $\hat{T}$ in Table 2, where the symmetry operations in the top row are applied first to $M \in Cl(3,1)$, followed by the symmetry operations in the first column.

Inspection of the table shows that under the following map from the three symmetry operations and their compositions to the elements of the geometric algebra of space $Cl(3,0)$, which itself is isomorphic to $Cl_+(3,1)$, the even subalgebra of $Cl(3,1)$, the $\hat{C}, \hat{P}, \hat{T}$ composition table Table 2 is seen to be isomorphic to the multiplication table of the basis elements $\{1, e_1, e_2, e_3, e_{12}, e_{31}, e_{23}, e_{123}\}$ of $Cl(3,0)$ itself.

$$\begin{aligned} &\hat{C} \to e_1, \quad \hat{P} \to e_2, \quad \hat{T}\hat{C} \to e_3, \\ &\hat{C}\hat{P} \to e_{12}, \quad \hat{T} \to e_{31}, \quad \hat{C}\hat{P}\hat{T} \to e_{23}, \quad \hat{P}\hat{T} \to e_{123}. \end{aligned} \tag{20}$$

The consequence is that the composition of charge conjugation $\hat{C}$, parity reversal $\hat{P}$ and time reversal $\hat{T}$ forms an algebra that is isomorphic to $Cl(3,0)$ and $Cl_+(3,1)$.

Finally, we add a table Table 3 showing the application of charge conjugation $\hat{C}$, parity reversal $\hat{P}$ and time reversal $\hat{T}$ to all elements of the basis of $Cl(3,1)$ given in (1).

We state a handful of immediate observations about Table 3.

TABLE 2. Table of all compositions of symmetry operators $\hat{C}, \hat{P}$ and $\hat{T}$, where operations in the top row are applied first to M followed by an operation from the first column. For example: combining $\hat{T}\hat{C}$ from the top row with $\hat{C}\hat{P}$ from the first column (6th row) shows that $\hat{C}\hat{P}\hat{T}\hat{C}M = \hat{P}\hat{T}M$.

1st: / 2nd:	1	$\hat{C}$	$\hat{P}$	$\hat{T}\hat{C}$	$\hat{C}\hat{P}$	$\hat{T}$	$\hat{C}\hat{P}\hat{T}$	$\hat{P}\hat{T}$
1	1	$\hat{C}$	$\hat{P}$	$\hat{T}\hat{C}$	$\hat{C}\hat{P}$	$\hat{T}$	$\hat{C}\hat{P}\hat{T}$	$\hat{P}\hat{T}$
$\hat{C}$	$\hat{C}$	1	$\hat{C}\hat{P}$	$-\hat{T}$	$\hat{P}$	$-\hat{T}\hat{C}$	$\hat{P}\hat{T}$	$\hat{C}\hat{P}\hat{T}$
$\hat{P}$	$\hat{P}$	$-\hat{C}\hat{P}$	1	$\hat{C}\hat{P}\hat{T}$	$-\hat{C}$	$\hat{P}\hat{T}$	$\hat{T}\hat{C}$	$\hat{T}$
$\hat{T}\hat{C}$	$\hat{T}\hat{C}$	$\hat{T}$	$-\hat{C}\hat{P}\hat{T}$	1	$\hat{P}\hat{T}$	$\hat{C}$	$-\hat{P}$	$\hat{C}\hat{P}$
$\hat{C}\hat{P}$	$\hat{C}\hat{P}$	$-\hat{P}$	$\hat{C}$	$\hat{P}\hat{T}$	-1	$\hat{C}\hat{P}\hat{T}$	$-\hat{T}$	$-\hat{T}\hat{C}$
$\hat{T}$	$\hat{T}$	$\hat{T}\hat{C}$	$\hat{P}\hat{T}$	$-\hat{C}$	$-\hat{C}\hat{P}\hat{T}$	-1	$\hat{C}\hat{P}$	$-\hat{P}$
$\hat{C}\hat{P}\hat{T}$	$\hat{C}\hat{P}\hat{T}$	$\hat{P}\hat{T}$	$-\hat{T}\hat{C}$	$\hat{P}$	$\hat{T}$	$-\hat{C}\hat{P}$	-1	$-\hat{C}$
$\hat{P}\hat{T}$	$\hat{P}\hat{T}$	$\hat{C}\hat{P}\hat{T}$	$\hat{T}$	$\hat{C}\hat{P}$	$-\hat{T}\hat{C}$	$-\hat{P}$	$-\hat{C}$	-1

TABLE 3. Application of charge conjugation $\hat{C}$, parity reversal $\hat{P}$ and time reversal $\hat{T}$ (top row) defined in (12), to all elements of the basis (first column) of $Cl(3,1)$ given in (1).

Basis	1	$\hat{C}$	$\hat{P}$	$\hat{T}\hat{C}$	$\hat{C}\hat{P}$	$\hat{T}$	$\hat{C}\hat{P}\hat{T}$	$\hat{P}\hat{T}$
1	1	$-e_{01}$	-1	e_{0123}	e_{01}	e_{23}	e_{0123}	$-e_{23}$
e_0	e_0	e_1	$-e_0$	e_{123}	$-e_1$	$-e_{023}$	e_{123}	e_{023}
e_1	e_1	e_0	e_1	$-e_{023}$	e_0	e_{123}	e_{023}	e_{123}
e_2	e_2	$-e_{012}$	e_2	$-e_{031}$	$-e_{012}$	e_3	e_{031}	e_3
e_3	e_3	e_{031}	e_3	$-e_{012}$	e_{031}	$-e_2$	e_{012}	$-e_2$
e_{01}	e_{01}	-1	e_{01}	$-e_{23}$	-1	$-e_{0123}$	e_{23}	$-e_{0123}$
e_{02}	e_{02}	e_{12}	e_{02}	$-e_{31}$	e_{12}	$-e_{03}$	e_{31}	e_{03}
e_{03}	e_{03}	$-e_{31}$	e_{03}	$-e_{12}$	$-e_{31}$	e_{02}	e_{12}	$-e_{02}$
e_{23}	e_{23}	$-e_{0123}$	$-e_{23}$	$-e_{01}$	e_{0123}	-1	$-e_{01}$	1
e_{31}	e_{31}	$-e_{03}$	$-e_{31}$	$-e_{02}$	e_{03}	e_{12}	$-e_{02}$	$-e_{12}$
e_{12}	e_{12}	e_{02}	$-e_{12}$	$-e_{03}$	$-e_{02}$	$-e_{31}$	$-e_{03}$	e_{31}
e_{023}	e_{023}	e_{123}	$-e_{023}$	$-e_1$	$-e_{123}$	e_0	$-e_1$	$-e_0$
e_{031}	e_{031}	e_3	$-e_{031}$	$-e_2$	$-e_3$	$-e_{012}$	$-e_2$	e_{012}
e_{012}	e_{012}	$-e_2$	$-e_{012}$	$-e_3$	e_2	e_{031}	$-e_3$	$-e_{031}$
e_{123}	e_{123}	e_{023}	e_{123}	e_0	e_{023}	$-e_1$	$-e_0$	$-e_1$
e_{0123}	e_{0123}	$-e_{23}$	e_{0123}	1	$-e_{23}$	e_{01}	-1	e_{01}

- The maps $\hat{C}, \hat{P}$ and $\hat{T}$ map even grade elements to even grade elements and odd grade elements to odd grade elements, i.e., they preserve even and odd grade multivector subspaces of $Cl(3,1)$.
- The map $\hat{P}$ only leads to sign changes.

- The rows for the even basis elements $\{1, e_{01},\ e_{23},\ e_{0123}\}$ all contain these four elements twice, they form together a commutative subalgebra generated by $\{e_{01}, e_{23}\}$. The operators $\hat{C}$, $\hat{T}$ and their composition $\hat{T}\hat{C}$, applied to any of the four elements $\{1, e_{01}, e_{23}, e_{0123}\}$, generate the other three.
- The rows for the other even basis blades, i.e., the four bivectors $\{e_{02}, e_{03}, e_{31}, e_{12}\}$ all contain precisely these bivectors twice, i.e., they exclude the bivectors $\{e_{01}, e_{23}\}$, and they obviously do not form a subalgebra. The operators $\hat{C}$, $\hat{T}$ and their composition $\hat{T}\hat{C}$, applied to any of the four bivectors, generate the other three.
- The rows for the four odd basis blades $\{e_0, e_1, e_{023}, e_{123}\}$ all contain these four elements twice. The operators $\hat{C}$, $\hat{T}$ and their composition $\hat{T}\hat{C}$, applied to any of the four elements $\{e_0, e_1, e_{023}, e_{123}\}$, generate the other three.
- The rows for the other four odd basis blades $\{e_2, e_3, e_{031}, e_{012}\}$ all contain these four elements twice. The operators $\hat{C}$, $\hat{T}$ and their composition $\hat{T}\hat{C}$, applied to any of the four elements $\{e_2, e_3, e_{031}, e_{012}\}$, generate the other three.
- Thus Table 3 has four groups (two with even blades and two with odd blades, respectively) of four rows, and inside each group each of the four rows contains the same set of elements twice in different positions. Within each group of four, the operators $\hat{C}$, $\hat{T}$ and their composition $\hat{T}\hat{C}$, applied to any of the four elements present in that group, generate the other three elements.
- The four groups can be clustered together by reordering Table 3, see Table 4 in the appendix. This reveals that each group of four contains two pairs of dual elements (dual with respect to multiplication with $\pm I$), where the duality is element by element from left to right in each pair of rows.
- The reordered table Table 4 also reveals that (up to a sign ± 1) every row can be obtained from the first row (starting with 1) by multiplication with the first element of each row. The same applies to the relation of the first column with every other column (using multiplication of the first column with the elements in the top row of each column).

4. Conclusion

In this work we have pursued the application of elementary symmetries of the geometric algebra $Cl(3,1)$ that can describe space-time. Inspired by [9] and [8], we chose three involutions of space inversion, reverse and *principal reverse* and studied the Abelian group thus generated and its action on the multivectors of $Cl(3,1)$. We found that similar to [8], a classification in eight principal and further 43 types of multivectors is thus possible, leading to a total of 51 types. Then we looked at algebraic aspects of applying charge conjugation, parity reversal and time reversal to the multivector basis of

$Cl(3,1)$. We found that the composition of the symmetry operations $\hat{C}, \hat{P}$ and $\hat{T}$ forms an algebra isomorphic to $Cl(3,0)$ and $Cl_+(3,1)$, and we commented on the structures found when $\hat{C}, \hat{P}$ and $\hat{T}$ are applied to the complete set of basis blades of $Cl(3,1)$. It may be interesting to apply both approaches in Clifford space gravity [3], and the study of elementary particles using a new embedding of octonions in geometric algebra [21, 22].

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TABLE 4. Reordered table (compare Table 3) of application of charge conjugation $\hat{C}$, parity reversal $\hat{P}$ and time reversal $\hat{T}$ (top row) defined in (12), to all elements of the basis (first column) of $Cl(3,1)$ given in (1). Double rows contain dual elements (one above the other). The top half contains only even elements, the bottom half only odd elements.

Basis	1	$\hat{C}$	$\hat{P}$	$\hat{T}\hat{C}$	$\hat{C}\hat{P}$	$\hat{T}$	$\hat{C}\hat{P}\hat{T}$	$\hat{P}\hat{T}$
1	1	$-e_{01}$	-1	e_{0123}	e_{01}	e_{23}	e_{0123}	$-e_{23}$
e_{0123}	e_{0123}	$-e_{23}$	e_{0123}	1	$-e_{23}$	e_{01}	-1	e_{01}
e_{01}	e_{01}	-1	e_{01}	$-e_{23}$	-1	$-e_{0123}$	e_{23}	$-e_{0123}$
e_{23}	e_{23}	$-e_{0123}$	$-e_{23}$	$-e_{01}$	e_{0123}	-1	$-e_{01}$	1
e_{02}	e_{02}	e_{12}	e_{02}	$-e_{31}$	e_{12}	$-e_{03}$	e_{31}	e_{03}
e_{31}	e_{31}	$-e_{03}$	$-e_{31}$	$-e_{02}$	e_{03}	e_{12}	$-e_{02}$	$-e_{12}$
e_{03}	e_{03}	$-e_{31}$	e_{03}	$-e_{12}$	$-e_{31}$	e_{02}	e_{12}	$-e_{02}$
e_{12}	e_{12}	e_{02}	$-e_{12}$	$-e_{03}$	$-e_{02}$	$-e_{31}$	$-e_{03}$	e_{31}
e_0	e_0	e_1	$-e_0$	e_{123}	$-e_1$	$-e_{023}$	e_{123}	e_{023}
e_{123}	e_{123}	e_{023}	e_{123}	e_0	e_{023}	$-e_1$	$-e_0$	$-e_1$
e_1	e_1	e_0	e_1	$-e_{023}$	e_0	e_{123}	e_{023}	e_{123}
e_{023}	e_{023}	e_{123}	$-e_{023}$	$-e_1$	$-e_{123}$	e_0	$-e_1$	$-e_0$
e_2	e_2	$-e_{012}$	e_2	$-e_{031}$	$-e_{012}$	e_3	e_{031}	e_3
e_{031}	e_{031}	e_3	$-e_{031}$	$-e_2$	$-e_3$	$-e_{012}$	$-e_2$	e_{012}
e_3	e_3	e_{031}	e_3	$-e_{012}$	e_{031}	$-e_2$	e_{012}	$-e_2$
e_{012}	e_{012}	$-e_2$	$-e_{012}$	$-e_3$	e_2	e_{031}	$-e_3$	$-e_{031}$

Appendix A. Reordered table of $\hat{C}$, $\hat{P}$ and $\hat{T}$ application to multivector basis of $Cl(3,1)$

Eckhard Hitzer
International Christian University
181-8585 Mitaka
Tokyo
Japan
e-mail: hitzer@icu.ac.jp

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