The Continuum Theorem

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Abstract

An injection from ω_1 to the paths in the binary tree leads to a bijection between the ω-length binary strings and an $\aleph_1$-sized set. Since there are $2^{\aleph_0}$ many ω-length binary strings, this proves the Continuum Hypothesis, $2^{\aleph_0} = \aleph_1$. The technique may be extended to show the Generalized Continuum Hypothesis, $\aleph_\alpha = \beth_\alpha$, which however will be reported separately.

1 Introduction

Since Cantor and Hilbert, the Continuum Hypothesis is the most prominent problem in set theory. We will show that it can be resolved with ordinary mathematical reasoning, without jeopardizing the results with problems like Russell's Paradox. Frege's Basic Law V, which gives rise to that paradox, is not used.

2 Proof

There is an injection from ω_1 to the paths in the binary tree because $\aleph_1$ is less than or equal to the continuum.

We will call any paths, regardless of whether they map to any ordinal in the injection, "tree paths." Paths that have an ordinal mapped to them by the injection we call "ordinal paths." By $\aleph_n$-node we mean a node with $\aleph_n$ ordinal paths passing through it. An $\aleph_n$-node-path is a path whose every node is an $\aleph_n$-node.

Each $\aleph_1$-node has two $\aleph_1$-node children or has a descendant that has two $\aleph_1$-node children. For suppose not. Then some $\aleph_1$-node has only one $\aleph_1$-node descendant at every lower level of the tree. In that subtree, there are $\aleph_1$ many ordinal paths corresponding to only one tree path, as shown in Figure 1.

The lack of distinctness in the mapping of ordinals to tree paths contradicts the definition of an injection.

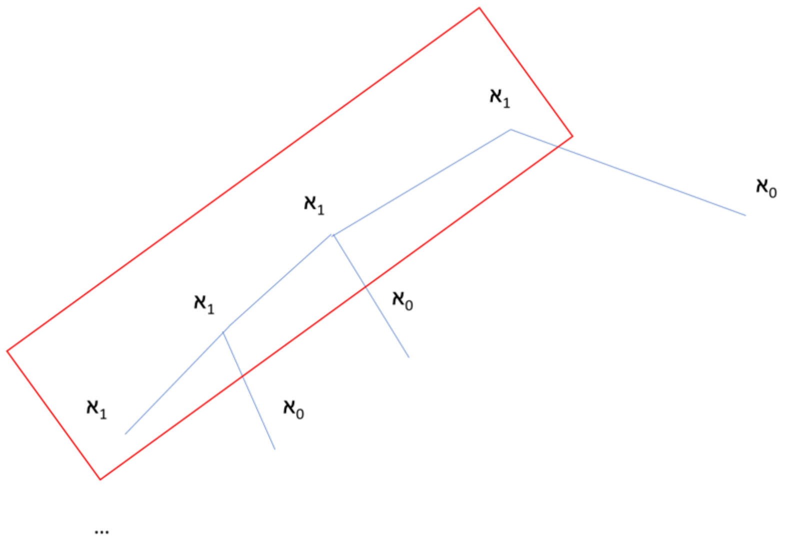


Figure 1: This (sub) tree is impossible because removing $\aleph_0$ paths, $\aleph_0$ times, does not produce distinct paths as the injection requires.

Label each pair of $\aleph_1$-nodes that share a parent with finite binary strings of increasing length as follows: the first such left child is labelled "0" and the first such right child is labelled "1." The first such left child that descends from the "0" node is labelled "00." The corresponding right child is labelled "01." The similar descendants of "1" are labelled "10" and "11." Some of these labels are shown in Figure 2. The infinite binary strings built up from these finite binary strings label the paths. There is a bijection from the strings (the continuum) to $\aleph_1$ (the $\aleph_1$-node-paths). Figure 3 shows an alternate possible tree. All possible trees encode a bijection in which the relation "is a label of" relates the infinite binary strings to the $\aleph_1$-node-paths. All the infinite binary strings are accounted for, because in each $\aleph_1$-node-path there is no last $\aleph_1$-node having two $\aleph_1$-node children.

Figure 4 shows the assignment of digits to nodes while Figure 5 shows an explicit bijection which exists for any possible tree based on the zeros and ones of the strings mapping to the nodes in the tree.

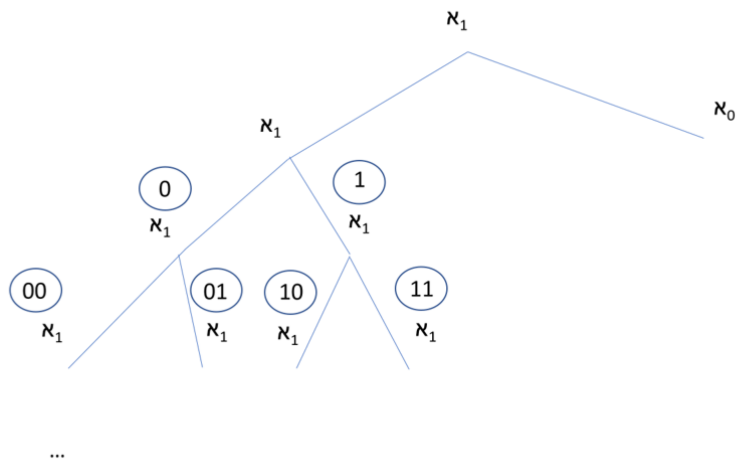


Figure 2: A tree like this remains to satisfy the requirement of an injection from ω_1 to the tree paths. Each $\aleph_1$-node-path has a unique infinite binary string and all such strings have some such path.

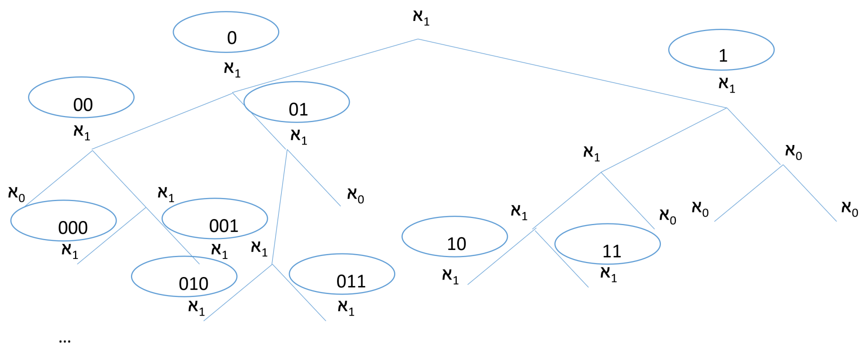


Figure 3: Alternate possible tree emphasizing that children of $\aleph_1$-nodes need not be $\aleph_1$-nodes, however each $\aleph_1$-node-path has an infinite binary string and each string has an $\aleph_1$-node-path.

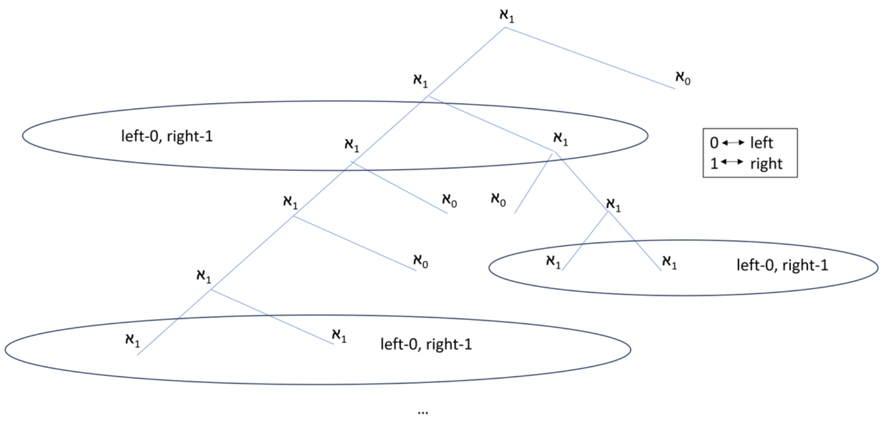
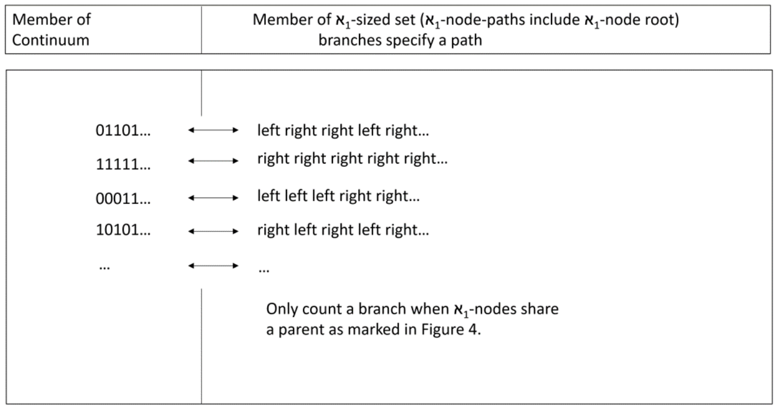


Figure 4: Assignment of digits in the binary strings to nodes in the tree.



Member of Continuum		Member of $\aleph_1$-sized set ($\aleph_1$-node-paths include $\aleph_1$-node root) branches specify a path
01101...	$\longleftrightarrow$	left right right left right...
11111...	$\longleftrightarrow$	right right right right right...
00011...	$\longleftrightarrow$	left left left right right...
10101...	$\longleftrightarrow$	right left right left right...
...	$\longleftrightarrow$	...

Only count a branch when $\aleph_1$-nodes share a parent as marked in Figure 4.

Figure 5: Bijection between continuum and $\aleph_1$ in which "0" maps to "left" and "1" maps to "right" in the strings and paths.