On the Origin and Unification of Electromagnetism, Gravitation, and Quantum Mechanics

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Abstract

Physical action is a spacetime bivector. Based on this single assumption, this paper argues that spacetime itself is responsible for much of physics. The properties of its elements and their derivatives, especially the transformations of special relativity, allow us to successively derive and confirm major empirically found laws and equations. We exemplary derive equations for electromagnetism including: Biot-Savart Law, Electrostatic Force, Lorentz Force, Maxwell's Equations including charges, and also constants like the fine-structure constant, electric permittivity, and even additional terms that are related to spin. In strict analogy, we gain very similar equations for gravitation which, e.g., include Poisson's equation for gravity. A further derivation of these equations of gravitation leads to field equations which are comparable to the field equations of general relativity. Electromagnetism and gravitation unite into one set of equations. The fact that bivectors can be written as wave equations also unify classical physics with wave mechanics.

1 Introduction

In physics, different theoretical frameworks exist to describe nature. Besides the classical theories, some of the best known are probably quantum mechanics, the standard model of particle physics" [2], or even string theory [14]. However, these theories mostly concentrate on describing particles and the short range interactions between them but fail to include gravitation.

Both electromagnetism and gravitation can describe long scale interactions. To develop a new model, we approach the problem by not starting from particles and their properties but by looking at the *long scale* aspect of their interactions and therefore looking more in depth at spacetime and its properties itself.

Spacetime already is the basis of the theories of special relativity [3] and of general relativity [4]. We will show the unifying and explanatory effect that spacetime, the transformations of special relativity, and the use of the spacetime derivative $\mathring{\nabla}$, see (2), have by deriving the equations of electromagnetism and gravitation from one of the most basic physical quantities: *Action* $\mathcal{S}$. After stating naming conventions, we begin with a very general overview before this paper then dives right into spacetime, electromagnetism, and gravitation. Much of the necessary math resides in the Appendix, not because it is less important, but because working step by step through a lot of equations before arriving at the main topic seriously would obstruct the flow of this paper.

To simplify the equations and make connections and analogies more obvious, we use the notations described below:

General:

$\mathcal{S}$	Action.
$\mathcal{E}$	Energy
$\mathcal{P}$	Momentum
$\mathcal{A}$	Angular Momentum
$\mathcal{F}$	Force

Charges / charge operators:

q_u^{0j}, q_u^{ij}	Single unit charges, affected by the respective action or force fields
m_u^{0j}, m_u^{ij}	Single unit masses, affected by the respective action or force field
q^{0j}, q^{ij}, Q^{0j}, Q^{ij}	Multiple charges, affected by the respective action or force fields
m^{0j}, m^{ij}, M^{0j}, M^{ij}	Multiple masses, affected by the respective action or force fields
q, m	Simplified names without indices

Action fields:

B_S	Magnetic action field
E_S	Electric action field
G_S	Gravitational action field
P_S	Momentum action field

Force fields:

E_F or just E	Electric force field
B_F or just B	Magnetic force field
G_F or just G	Gravitational force field
P_F or just P	Momentum force field

Action:

$\mathcal{S}_B = Q^{0j} B_S$	Magnetic action
$\mathcal{S}_E = Q^{ij} E_S$	Electric action
$\mathcal{S}_G = M^{0j} G_S$	Gravitational action
$\mathcal{S}_P = M^{ij} P_S$	Momentum action

Force:

$\mathcal{F}_B = Q^{0j} B_F = Q^{0j} B$	Magnetic force
$\mathcal{F}_E = Q_{ij} E_F = Q_{ij} E$	Electric force
$\mathcal{F}_G = M_{0j} G_F = M_{0j} G$	Gravitational force
$\mathcal{F}_P = M^{ij} P_F = M^{ij} P$	Momentum force

Momentum action field and force are new concepts that will be developed in this paper. They are the gravitational equivalent to magnetic action and force. A static magnetic force as written here $\mathcal{F}_B = qB$ has not been observed yet. It is still useful for the math in this paper as seen below in e.g. section 3.4 “Maxwell’s Equations of Static Charges” and we must include it in further derivatives.

In order to limit the complexity and allow for a mathematically manageable solution, some simplifying assumptions are made. First, all equations describe either static scenarios in rest or scenarios that can be described by the transformations of special relativity, e.g. two observer/rest frames moving with a constant relative speed with regard to each other. Second, accelerations of observer frames are not handled in this paper. Third, all spacetime geometry in a frame is described in a fixed, extrinsic/global coordinate system like in special relativity. The intrinsic geometry of the curvature of spacetime in general relativity (G.R.) will only be pointed out when comparing these two approaches in section 4.9. However, this is typical for such analysis and therefore does not limit the relevance of the results. Most of relativistic quantum mechanics follow the same approach.

Before presenting the concrete derivations of electromagnetism and gravitation, the following overview shows how applying the spacetime derivative to bivectors of *action* leads to *energy, momentum, and angular momentum*. Successively applying the spacetime derivative $\mathring{\nabla}$, see (2), to each result of the previous derivative then leads to *forces and fields*, which then leads to *Maxwell's equations* for electromagnetism and an equivalent for gravitation. The final derivative leads to equations of energy-, stress- and momentum-density comparable to *the field equations of general relativity*:

$\mathcal{S}$	Action
$\mathring{\nabla}(\mathcal{S})$	Derivative level of energy, momentum, and angular momentum
$\mathring{\nabla}\left(\mathring{\nabla}(\mathcal{S})\right)$	Derivative level of forces
$\mathring{\nabla}\left(\mathring{\nabla}\left(\mathring{\nabla}(\mathcal{S})\right)\right)$	Derivative level of Maxwell's equations, Poisson's equation for gravity
$\mathring{\nabla}\left(\mathring{\nabla}\left(\mathring{\nabla}\left(\mathring{\nabla}(\mathcal{S})\right)\right)\right)$	Derivative level of energy-stress-momentum densities, G.R.

While we only look in detail at some of the steps described above, a summary of all the equations gained in this paper – either derived directly or in analogy to other equations – can be found in mostly bullet point style in section 5.

2 Spacetime

Since around the time that special relativity was first published by Einstein [3] and Minkowski held his lecture about "Space and Time" [16], it is assumed that we live in a four-dimensional spacetime. Even before that time, four dimensional algebras were investigated by Hamilton [10], Gibbs [9], Grassmann [7], Clifford [8] and others and later popularized e.g. as Geometric Algebra by Hestenes [13] [12].

In general, a four-dimensional spacetime is described by scalars, vectors e_α, bivectors $e_{\alpha\beta}$, trivectors $e_{\alpha\beta\gamma}$, quadvectors $e_{\alpha\beta\gamma\delta}$, pseudovectors (= trivectors in 4D), and pseudoscalars (= quadvectors in 4D). As usual, Greek letters α, β, γ, δ denote dimensions 0, 1, 2, 3 (in time and space context), while Latin letters i, j, k denote only spatial dimensions 1, 2, 3. In this text, to shorten the notation, we write the unit vectors and multivectors as follows:

e_α	unit vector
$e_\alpha \wedge e_\beta \to e_{\alpha\beta}$	unit bivector
$e_\alpha \wedge e_\beta \wedge e_\gamma \to e_{\alpha\beta\gamma}$	unit trivector/pseudovector
$e_\alpha \wedge e_\beta \wedge e_\gamma \wedge e_\delta \to e_{\alpha\beta\gamma\delta}$	unit quadvector/pseudoscalar

The basic mathematics are summarized in Appendix A.1.

2.1 Derivatives of Spacetime

For the analysis we must understand the concepts of contravariant and covariant vectors which will be generalized in the following. An example for a covariant vector is the derivative $d_x e^x$ and for a contravariant vector the velocity $v^x e_x$. These covariant and contravariant types can be turned into each other by raising or lowering the indices. This is done by multiplying the vector with the metric tensor, $d^\alpha e_\alpha = g_{\mu\nu} d_\alpha e_\alpha$. The covariant derivative accounts for changes in field strength but also accounts for the changes of the local unit vectors. In general relativity, the changes of the local unit vectors are described by Christoffel symbols.

As we will see later in this paper, equations for fields (E, B, G, P) describe how spacetime (= fields) changes in a stationary frame of reference, flat spacetime. This frame of reference might be called a stationary meta spacetime. As we mostly deal with a universal, flat meta spacetime, the Minkowski metric can always be used in lieu of the metric tensor.

We can define the covariant meta-spacetime partial derivative as $d_0e^0 + d_1e^1 + d_2e^2 + d_3e^3$ with $d_0 = \frac{\partial}{c\partial t}$, $d_1 = \frac{\partial}{\partial x}$, $d_2 = \frac{\partial}{\partial y}$, $d_3 = \frac{\partial}{\partial z}$ or $d_0 = \frac{\partial}{\partial x_0}$, $d_1 = \frac{\partial}{\partial x_1}$, $d_2 = \frac{\partial}{\partial x_2}$, $d_3 = \frac{\partial}{\partial x_3}$.

By lowering the indices of the unit vectors, we get the partial contravariant spacetime derivative $d_0e_0 - d_1e_1 - d_2e_2 - d_3e_3$.

In physics, the three-dimensional partial derivative / gradient is often written with the "nabla" symbol, i.e., $\nabla = \frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}$. Furthermore, the four-dimensional partial derivative is sometimes written as $\partial_\mu = \frac{\partial}{c\partial t} + \nabla$. To distinguish this four-dimensional partial derivative from our partial contravariant spacetime derivative, and to limit the use of indices, we define the symbol $\mathring{\nabla}$ with

$$\mathring{\nabla} \overset{\text{def}}{=} d_0e_0 - d_1e_1 - d_2e_2 - d_3e_3. \tag{1}$$

To shorten the name "partial contravariant spacetime derivative", we will simply refer to it as "spacetime derivative".

Next, we determine the second spacetime derivative:

$$\begin{aligned}\mathring{\nabla}\mathring{\nabla} &= (d_0e_0 - d_1e_1 - d_2e_2 - d_3e_3)(d_0e_0 - d_1e_1 - d_2e_2 - d_3e_3)\\ &= d_0d_0 - d_1d_1 - d_2d_2 - d_3d_3\end{aligned}$$

Thus, the result is

$$\mathring{\nabla}^2 = \mathring{\nabla} \cdot \mathring{\nabla} = d_0^2 - d_1^2 - d_2^2 - d_3^2 = \frac{d^2}{c^2dt^2} - \frac{d^2}{dx^2} - \frac{d^2}{dy^2} - \frac{d^2}{dz^2} = \frac{d^2}{c^2dt^2} - \nabla^2 = \Box, \tag{2}$$

where the $\Box$-symbol is the d'Alembert operator (d'Alembertian). Besides other names, it is also called the wave operator.

It should be noted that it is a property of spacetime itself or its second partial derivative that, no matter what the argument is (scalar, vector, bivector, pseudovector or pseudoscalar), the second spacetime derivative is always a wave equation. This does, however, not imply that every argument can be quantized.

A complete step by step walk through of the first and second spacetime derivative of a bivector field is gives as an example in appendix A.2.

2.2 Movements in Spacetime

Besides spacetime and the spacetime derivative, the most important building block of the model we develop in this paper is the transformation of spacetime elements as described by special relativity. Especially the transformations of bivectors will prove to be of particular interest and relevance. To better understand the transformations and rotations in spacetime, we need to compare them with transformations and rotations in euclidean space, respectively. In euclidean 2D or 3D space,

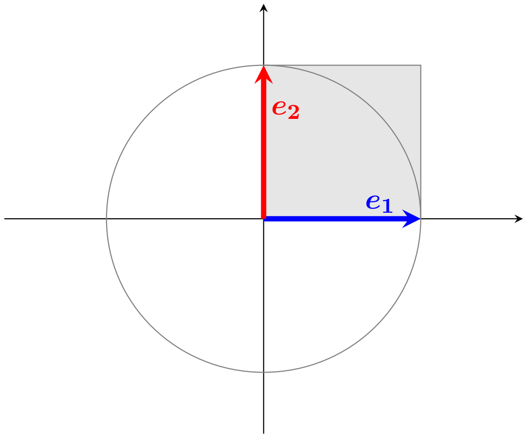

Figure 1: Invariant circle,
unit axes,
unit grid area

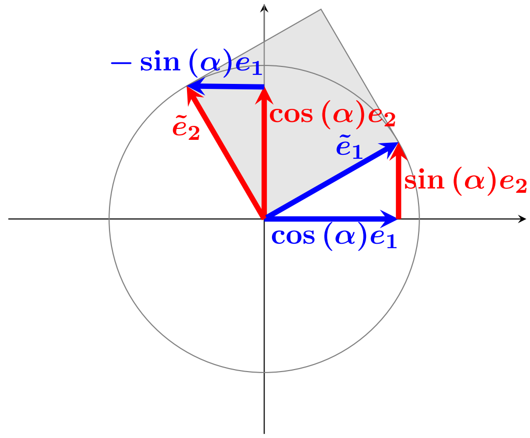

Figure 2: Invariant circle, rotated axes
$\tilde{e}_1 = +\cos(\alpha)e_1 + \sin(\alpha)e_2$
$\tilde{e}_2 = -\sin(\alpha)e_1 + \cos(\alpha)e_2$

a point $\vec{R}$ on a circle of radius $\vec{r}$ or on a sphere is an invariant. No matter the transformation of the basis vectors, the value of $\vec{r}$ stays the same. To illustrate this for a circle, compare $\vec{R}$ written with unit vectors, $\vec{R} = xe_1 + ye_2$ and written as absolute values, $R^2 = x^2(e_1)^2 + y^2(e_2)^2$ which in euclidean space simplifies to $r^2 = x^2 + y^2$. Similarly, a sphere written with unit vectors is $\vec{R} = xe_1 + ye_2 + ze_3$ and written as an absolute value, $R^2 = x^2(e_1)^2 + y^2(e_2)^2 + z^2(e_3)^2$ which again simplifies to $r^2 = x^2 + y^2 + z^2$.

Rotating the unit vectors e_i around the origin by an angle β leads to new unit vectors $\tilde{e}_i$, but the spacetime invariant $\vec{R}$ does not change its absolute value, hence the name "Invariant".

$$\begin{aligned}
\vec{S} &= xe_1 + ye_2 \\
&= \begin{bmatrix} e^1 & e^2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
&= \begin{bmatrix} e^1 & e^2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\
&= \left(\begin{bmatrix} e^1 & e^2 \end{bmatrix} \begin{bmatrix} +\cos\alpha & -\sin\alpha \\ +\sin\alpha & +\cos\alpha \end{bmatrix} \right) \left(\begin{bmatrix} +\cos\alpha & +\sin\alpha \\ -\sin\alpha & +\cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \right) \\
&= (\text{transformation of unit vectors})\ (\text{transformation of vector components}) \\
&= \begin{bmatrix} \tilde{e}^1 & \tilde{e}^2 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{y} \end{bmatrix}
\end{aligned}$$

Together, the unit vectors span a unit grid area. The size of this area is 1 and does not change under transformation of the basis vectors. The area of a rectangle is the determinant of the matrix of the transformed vectors, in this case the unit area is always $e_1 \wedge e_2 = (+\cos\alpha)(+\cos\alpha) - (+\sin\alpha)(-\sin\alpha) = \cos\alpha^2 + \sin\alpha^2 = 1$. To illustrate the rotation, figures 1 and 2 use the example of $\alpha = 30°$, with $\cos\alpha = 0.87$ and $\sin\alpha = 0.50$.

In flat 4D spacetime, an invariant similar to $\vec{R}$ is often called $\vec{S}$. The equation corresponding to a circle or sphere in unit vectors is:

$$\vec{S} = cte_0 + xe_1 + ye_2 + ze_3 \tag{3}$$

In absolute values we get $S^2 = c^2t^2(e_0)^2 + x^2(e_1)^2 + y^2(e_2)^2 + z^2(e_3)^2$ or simplified $S^2 = c^2t^2 - x^2 - y^2 - z^2$. Note that, because of the Minkowski metric of flat 4D spacetime, the spatial parts now show a negative sign.

Dynamic spacetime and special relativity are governed by the transformation of time and space components and their unit vectors. The transformation of an invariant $\vec{S}$ in spacetime is called Lorentz transformation if one of the involved unit vectors is e_0 (ce_t).

$$\begin{aligned}
\vec{S} &= cte_{ct} + xe_x \\
&= \begin{bmatrix} e^{ct} & e^x \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix} \\
&= \begin{bmatrix} e^{ct} & e^x \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix} \\
&= \left(\begin{bmatrix} e^{ct} & e^x \end{bmatrix} \gamma \begin{bmatrix} 1 & \beta \\ \beta & 1 \end{bmatrix} \right) \left(\gamma \begin{bmatrix} 1 & -\beta \\ -\beta & 1 \end{bmatrix} \begin{bmatrix} ct \\ x \end{bmatrix} \right) \\
&= (\text{transformation of unit vectors})\ (\text{transformation of vector components}) \\
&= \begin{bmatrix} \tilde{e}^{ct} & \tilde{e}^x \end{bmatrix} \begin{bmatrix} \tilde{ct} \\ \tilde{x} \end{bmatrix} \\
&= \tilde{ct}\tilde{e}_{ct} + \tilde{x}\tilde{e}_x
\end{aligned}$$

To simplify many equations of special relativity, one often uses the parameters

$$\beta = \frac{v}{c} \tag{4}$$

and

$$\gamma = \frac{1}{\sqrt{1 - \beta^2}}. \tag{5}$$

The transformation of unit vectors and the transformation of vector components can be viewed as two sides of a coin which complement each other. It is essential to point out that the "unit grid area" covered by time and space unit vectors, both in rest and moving, remains the same and makes vectors and components comparable in both cases.

Figures 3 and 4 are an example of the connections between the time- and space-axes in rest and when moving, shown for values of $\beta = 0.33$ $(\widehat{=}\, 0.33 \cdot 45°)$, therefore $\gamma = 1.06$ and $\gamma\beta = 0.35$.

From the transformations of the unit vectors as described in A.3, we can directly conclude the transformations of unit bivectors and their bivector components under movement. As unit bivectors are formed by taking the wedge product of unit vectors, their transformations are obtained by taking the wedge product of a transformed unit vector with another unit vector.

An example of a bivector transformation for a velocity v_1 (in direction 1) is

$$\begin{aligned}
\tilde{e}^{02} &= \tilde{e}^0 \wedge \tilde{e}^2 \\
&= \gamma_1(e_0 + \beta_1 e_1) \wedge e_2 \\
&= \gamma_1 e_{02} + \gamma_1\beta_1 e_{12}
\end{aligned}$$

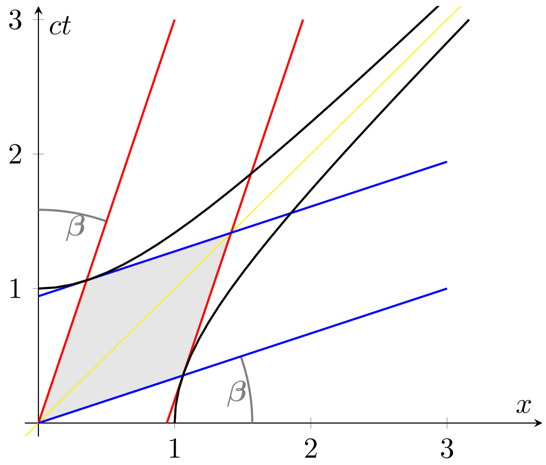

Figure 3: moving time- and space-axes, unit grid area

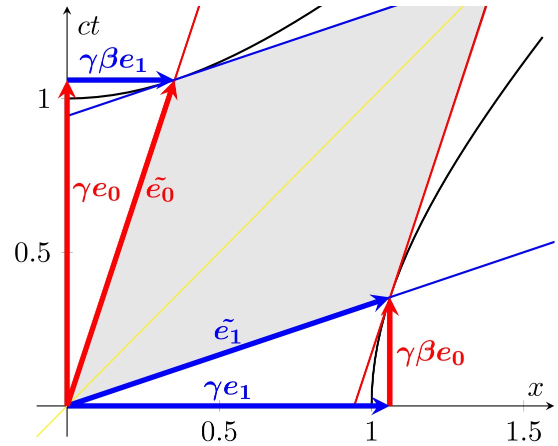

Figure 4: $\tilde{e}_0 = \gamma e_0 + \gamma\beta e_1$, $\tilde{e}_1 = \gamma e_1 + \gamma\beta e_0$

Just like a vector with the Lorentz transformation, the absolute (area) value of the bivector remains the same. This makes bivectors and their components comparable when in rest and moving systems. The area value of the transformed bivector is:

$$\begin{aligned} \text{area} &= \sqrt{\gamma_1((e_0)^2 + (\beta_1 e_1)^2)} \\ &= \sqrt{\gamma_1(1-\beta_1^2)} \\ &= \sqrt{1} \\ &= 1 \end{aligned}$$

The area value stays the same.

When dealing with movements in four dimension, it makes sense to assign indices to v, β, and γ according to the spatial direction of the movement, i.e., v_i, β_i, and γ_i when the direction of the movement is in direction $i = 1, 2, 3$.

v_1, β_1, and γ_1 when the direction of the movement is in direction 1.
v_2, β_2, and γ_2 when the direction of the movement is in direction 2.
v_3, β_3, and γ_3 when the direction of the movement is in direction 3.

Also useful are the following factors:

1_1 means this factor is 1 when the direction of the movement is in direction 1.
1_2 means this factor is 1 when the direction of the movement is in direction 2.
1_3 means this factor is 1 when the direction of the movement is in direction 3.

All parameters and factors that have other indices are 0 or ignored. Moreover, in the equations we use "$\vee$", which means "logical or". As an example, look at $(\gamma_1 \vee \gamma_2 \vee \gamma_3)$. This should be read as "if the direction of movement is 1, then use γ_1, parameters with other factors are 0 or ignored" or e.g. "if the direction of movement is 2, then use γ_2, parameters with other factors are 0 or ignored".

In appendix A.3 we generalise the transformation of all unit bivectors under movement. Also in A.3 we generalise the transformation of all bivector components under movement. To write down these transformations we rely on the indexed parameters and factors above.

2.3 Spacetime Invariants, Bivectors and Wave Equations

Gibbs [9] showed that bivectors can be written as an ellipse $xe_1 + ye_2$ or a complex number $x + iy$ with vectors xe_1 and ye_2 and $i^2 = -1$. The ellipse $xe_1 + ye_2$ can also be seen as a two dimensional subspace of a spacetime invariant (3) $\vec{S} = cte_0 + xe_1 + ye_2 + ze_3$.

Using Euler's formula [6] - that states that $e^{ix} = \cos(x) + i\sin(x)$ for every real x - we see that a bivector can be written in this exponential form. We also associate this formula with wave equations.

As we see in 5.2 the spacetime derivative of action, $\mathcal{S}$, is mainly the difference between energy, $\mathcal{E}$, and momentum, $\mathcal{P}$ (if we ignore for now angular momentum $\mathcal{A}$),

$$\mathring{\nabla}(\mathcal{S}) = 0 = \mathcal{E} - \mathcal{P}.$$

With this, we can also write for $\Delta\mathcal{S}$

$$\begin{aligned}
\Delta\mathcal{S}(t) &= \frac{\Delta\mathcal{S}(t)}{\Delta t}\Delta t = +\Delta\mathcal{E}\Delta t \\
\Delta\mathcal{S}(x) &= \frac{\Delta\mathcal{S}(x)}{\Delta x}\Delta x = -\Delta\mathcal{P}\Delta x \\
\Delta\mathcal{S}(x,t) &= \Delta\mathcal{S}(t) + \Delta\mathcal{S}(x) = \Delta\mathcal{E}\Delta t - \Delta\mathcal{P}\Delta x
\end{aligned}$$

Dropping the deltas, we can write the action bivector field as a wave equation. Because the argument of the exponential has to be unitless, one has to divide it by a unit of action, $\hbar$ in this case. We also flip the signs of momentum and energy to adjust for our usual definition of direction of travel of the wave. The wave equation of action now therefore is:

$$\mathcal{S}(x,t) = \psi(x,t) = e^{\frac{i}{\hbar}S(x,t)} = e^{\frac{i}{\hbar}(\mathcal{P}x - \mathcal{E}t)} \tag{6}$$

It is easy to see that this is a (non relativistic) wave equation as e.g. used in quantum mechanic.

On a side note: If we assumes that $\mathcal{S}$ is quantised and one quantum of $\mathcal{S}$ is h, then the minimum $\Delta\mathcal{S}$ is $\Delta\mathcal{S} = h = \Delta\mathcal{E}\Delta t$ or $\Delta\mathcal{S} = h = \Delta\mathcal{P}\Delta x$. These are the values of the original uncertainty principle of Heisenberg [11]. Because of other considerations Heisenberg's uncertainty principle has been corrected to $\Delta\mathcal{S} = \frac{\hbar}{2}$. An equivalent for the quantisation of the unexpected angular momentum - which we will derive later - should be added but is more complicated because of the curl.

As shown in a general form in appendix A.2 "First Derivative of a Bivector Field" and the more concrete section 5.2 "Summary Gravitational Energy, Momentum, and Angular Momentum" and "Summary Electromagnetic Energy, Momentum, and Angular Momentum", the actual internal structure of the spacetime derivative of $\mathcal{S}$ is more complicated than just the normally assumed $\mathcal{E}$ and $\mathcal{P}$, as it does not only contain the simple temporal and spatial derivatives of the action field but also their curls, which we associate with angular momentum $\mathcal{A}$ - possibly related to spin and magnetic moment.

To acquire an even more accurate equation, we would have to use energy, momentum and angular momentum from the dynamic relativistic equations.

2.4 Operators, Charge and Mass

As will be shown below in 3.1, B-fields are induced by the relative movement of electric charges or E-fields and have "hidden" components of the cross product between velocity- and E-field components, both only scalar values. Take for example from (10) the term

$$-B^{01}e_{01} = (\gamma_3\beta_3\tilde{E}^{31} - \gamma_2\beta_2\tilde{E}^{12})e_{01}$$

The direction or orientation is given by the unit bivector e_{0j}. We can multiply both sides with a scalar value q. Therefore, because for electric charge q no value for qB that is not zero is observed for now, electric charge cannot be a simple scalar value with units.

In current physics, electric charge is not considered to be fundamental and only one type of a number of different charges (e.g. electric charge, color charge, etc.). Generally, electric charge is considered in the context of symmetry groups (the purely spatial rotations of U(1)) and conserved quantum numbers/ conserved currents.

As we have seen in 2.3, for our at first seemingly "classical" problem here, a solution is to define a charge operator $\hat{q}^{\mu\nu}$ that acts on a bivector wave equation $e_{\mu\nu}$ and returns the charge $q^{\mu\nu}$. Note that this implies that the charge we use for action must be the same charge that we use for force, as we use them on the same bivector. Moreover this charge operator acts a lot like the energy operator on a wave equation, compare with $\hat{H}\Psi = \hat{E}\Psi$ and similar equations. Also compare with section 3.6, where an example for this "operator" is shown.

However, at least in a flat spacetime, the charge operator only appears to act on vectors or bivectors that share common indices. When acting on other vectors and bivectors so far no observations were made. After $\hat{q}^{\mu\nu}e_{\mu\nu}$ acted on the bivector, we will just write the value $q^{\mu\nu}$ instead of $\hat{q}^{\mu\nu}$.

To set up the indices for a good comparability to current physics, we could use the covariant form of the well known existing electromagnetic field strength (Faraday) tensor (Here the factor c is included in the tensor and the B-field)

$$F_{em} = \begin{pmatrix} 0 & E^1 & E^2 & E^3 \\ -E^1 & 0 & -B^3 & B^2 \\ -E^2 & B^3 & 0 & -B^1 \\ -E^3 & -B^2 & B^1 & 0 \end{pmatrix} \tag{7}$$

to guess the indices of an electric charge q^{0j} and a magnetic charge q^{ij}. However, because our matrix components should be two dimensional and because of other considerations described immediately below, we use the Hodge dual of this tensor which exchanges the indices / positions of the electric and the magnetic field, so that this tensor looks like

$$F_{em} = \begin{pmatrix} 0 & -B^{01} & -B^{02} & -B^{03} \\ B^{01} & 0 & E^{12} & -E^{31} \\ B^{02} & -E12 & 0 & E^{23} \\ B^{03} & E^{31} & -E^{23} & 0 \end{pmatrix} \tag{8}$$

Later, in the chapter on gravitation 4, we will use a similar approach and use a hodge dual gravitational tensor

$$F_{em} = \begin{pmatrix} 0 & -G^{01} & -G^{02} & -G^{03} \\ G^{01} & 0 & P^{12} & -P^{31} \\ G^{02} & -P12 & 0 & P^{23} \\ G^{03} & P^{31} & -P^{23} & 0 \end{pmatrix} \tag{9}$$

The above ordering of the indices is useful if one wanted to make a connection between the ”circles” of bivectors and spacetime invariants of the form $S = (ct)^2 - x^2 - y^2 - z^2$. In this case, every bivector e_{0j} which includes an index 0 would represent a hyperbola, while every purely spatial bivector e_{ij} represents a circle or ellipse. Derivatives of action hyperbolas and circles represent energies etc., second derivatives represent forces. The different resulting forces would therefore be attractive or repulsive. Moreover circles represent a “periodic space” which is quantizable while hyperbolas are not. It is beyond the scope of this paper to go into these details but the choice of the connection of electric charge with purely spatial bivectors and mass with temporal-spatial bivectors stems from here.

Taking this opportunity for a small detour into particle physics, this splitting might also suggests that, assuming each bivector e_{ij} is linked to a electric charge $\pm\frac{1}{3}$, then each bivector e_{0j} might be linked to a fixed amount of different kind of charge (Compare with 3.5 for some thoughts about this). The different rules of what makes up a whole, observable particle (quark, lepton, etc.) should be easily amendable and now possibly also explainable. Because of these ”observable particle” rules, all charge bivectors should always be contained in a quark / electron etc. and not be observable as an individual particle.

In an earlier, abandoned draft of this paper, it was assumed that mixed temporal-spatial bivectors and charges were related to gravitation and purely spatial bivectors and charges were related to electromagnetism. While this approach resulted in the same equations that are presented below in this paper, ”magnetic charges” and ”momentum masses/charges” were not treated as virtual (just a placeholder for the math) but treated more as an addition or something fundamental. Also, there was the possibility of negative mass, which hasn't been observed yet. Therefore, the version presented in this paper assumes that electric charges directly work on / are linked to their bivector. Because of this, a new idea for ”mass” had to be found. We now assume that mass stems from the energy of the vibration of two bivectors against each other. The vibration of a bivector $e_{\alpha\beta}$ against a bivector $e_{\beta\gamma}$ around the β-axis could now be described as a wave function / bivector in the plane $e_{\alpha\gamma}$. Therefore we can also describe mass and gravitation with a bivector field. Mass however now isn't an intrinsic fixed property of a bivector itself, but depends on the energy levels of the vibrations of the various bivectors of the particles themselves (and then additionally their interactions with other particles). With this assumption, we can still handle gravitation in complete analogy to electromagnetism and additionally make the prediction that the energy/mass of a particle is computable. Also, the ground states of these energies/wave functions are spherical (giving particles a spherical appearance) but possible higher order energies/wave functions are also of different shapes (compare with the solutions for electron shells in atoms). It is beyond the scope of this paper to go into these details.

3 Electromagnetism

Most of the equations that describe electromagnetism start at the derivative level of forces and (electric and magnetic) fields. It is therefore best to start exploring electromagnetism right there. Electromagnetic Action, Energy and Momentum and Energy Densities can be found in the Summary 5.

3.1 Electric and Magnetic Field, Biot-Savart Law

When looking at the first spacetime derivative of a bivector field, as shown in appendix A.2, we see that, except for the naming of the field components, the unusual unit vectors and pseudovectors and some signs, this resembles Maxwell's equations in vacuum. However, because we assume that the purely spatial components C^{ij} represent the electric field, the time and space related components C^{0j} cannot be original components of the magnetic field, as we will immediately below show that the magnetic field is just a relativistic effect of a moving electric field. Also from this, we assume that there are no magnetic charges.

To obtain the magnetic field, consider the following situation: An observer $\tilde{O}$ is at rest in a system $\tilde{S}$. Observer $\tilde{O}$ and system $\tilde{S}$ are moving with velocity v relative to an observer 0, who is at rest in system S. Observer $\tilde{O}$ measures an $\tilde{E}$-field but no $\tilde{B}$-field. The magnetic field is understood to be a relativistic effect of moving electric charges or fields. Utilising the reverse transformation of components of bivectors under movement from appendix A.3

$$\begin{aligned}
C^{01}e_{01} &= ((1_1 \vee \gamma_2 \vee \gamma_3)\tilde{C}^{01} + \gamma_3\beta_3\tilde{C}^{31} - \gamma_2\beta_2\tilde{C}^{12})e_{01} \\
C^{02}e_{02} &= ((\gamma_1 \vee 1_2 \vee \gamma_3)\tilde{C}^{02} + \gamma_1\beta_1\tilde{C}^{12} - \gamma_3\beta_3\tilde{C}^{23})e_{02} \\
C^{03}e_{03} &= ((\gamma_1 \vee \gamma_2 \vee 1_3)\tilde{C}^{03} + \gamma_2\beta_2\tilde{C}^{23} - \gamma_1\beta_1\tilde{C}^{31})e_{03} \\
C^{12}e_{12} &= ((\gamma_1 \vee \gamma_2 \vee 1_3)\tilde{C}^{12} + \gamma_1\beta_1\tilde{C}^{02} - \gamma_2\beta_2\tilde{C}^{01})e_{12} \\
C^{31}e_{31} &= ((\gamma_1 \vee 1_2 \vee \gamma_3)\tilde{C}^{31} + \gamma_3\beta_3\tilde{C}^{01} - \gamma_1\beta_1\tilde{C}^{03})e_{31} \\
C^{23}e_{23} &= ((1_1 \vee \gamma_2 \vee \gamma_3)\tilde{C}^{23} + \gamma_2\beta_2\tilde{C}^{03} - \gamma_3\beta_3\tilde{C}^{02})e_{23}
\end{aligned}$$

and assuming no stationary $\tilde{B}$-field exists ($\tilde{C}^{01} = \tilde{C}^{02} = \tilde{C}^{03} = 0$), we get

$$\begin{aligned}
C^{01}e_{01} &= (\gamma_3\beta_3\tilde{C}^{31} - \gamma_2\beta_2\tilde{C}^{12})e_{01} \\
C^{02}e_{02} &= (\gamma_1\beta_1\tilde{C}^{12} - \gamma_3\beta_3\tilde{C}^{23})e_{02} \\
C^{03}e_{03} &= (\gamma_2\beta_2\tilde{C}^{23} - \gamma_1\beta_1\tilde{C}^{31})e_{03} \\
C^{12}e_{12} &= (\gamma_1 \vee \gamma_2 \vee 1_3)\tilde{C}^{12})e_{12} \\
C^{31}e_{31} &= (\gamma_1 \vee 1_2 \vee \gamma_3)\tilde{C}^{31}e_{31} \\
C^{23}e_{23} &= (1_1 \vee \gamma_2 \vee \gamma_3)\tilde{C}^{23}e_{23}
\end{aligned}$$

Setting $\tilde{C}^{12} = \tilde{E}^{12}$, $\tilde{C}^{31} = \tilde{E}^{31}$ and $\tilde{C}^{23} = \tilde{E}^{23}$, we identify the resulting space and time related components as the magnetic field

$$\begin{aligned}
-B^{01}e_{01} &= (\gamma_3\beta_3\tilde{E}^{31} - \gamma_2\beta_2\tilde{E}^{12})e_{01} \\
-B^{02}e_{02} &= (\gamma_1\beta_1\tilde{E}^{12} - \gamma_3\beta_3\tilde{E}^{23})e_{02} \\
-B^{03}e_{03} &= (\gamma_2\beta_2\tilde{E}^{23} - \gamma_1\beta_1\tilde{E}^{31})e_{03} \\
E^{12}e_{12} &= (\gamma_1 \vee \gamma_2 \vee 1_3)\tilde{E}^{12}e_{12} \\
E^{31}e_{31} &= (\gamma_1 \vee 1_2 \vee \gamma_3)\tilde{E}^{31}e_{31} \\
E^{23}e_{23} &= (1_1 \vee \gamma_2 \vee \gamma_3)\tilde{E}^{23}e_{23}
\end{aligned} \tag{10}$$

To justify the assumption – that the C^{0j} are the bivector components of a B-field – compare the equations to Biot-Savart law. Just by looking at the mixed time-space components of the above equations, we can see that the C^{0j}'s are indeed a magnetic field (with same units as the E-field – use factor c to convert to "normal" B_*-field units). Combining the time-space components of B

into a vector $\vec{B}_*$, multiplied with c to adjust for units, one can look at the amount $d\vec{B}_*$ from the contribution of $d\vec{E}(r)$:

$$d\vec{B}_* c = \gamma\vec{\beta} \times d\vec{E}(r)$$

With the substitution $\beta = \frac{v}{c}$ and some rearranging this becomes

$$d\vec{B}_* = \gamma\frac{\vec{v}}{c^2} \times d\vec{E}(r)$$

Using $\frac{1}{c^2} = \epsilon_0\mu_0$ and $d\vec{E}(r) = \frac{\Delta Q_E}{4\pi\epsilon_0 r^2}\vec{e}_r$ we get $d\vec{B}_* = \gamma\epsilon_0\mu_0\vec{v} \times \frac{\Delta Q_E}{4\pi\epsilon_0 r^2}\vec{e}_r$ and then

$$d\vec{B}_* = \gamma\frac{\mu_0}{4\pi r^2}\Delta Q_E\vec{v} \times \vec{e}_r$$

With the definition of current $I = \frac{\Delta Q_E}{\Delta t}$ and $\vec{v} = \frac{d\vec{L}}{dt}$, $\Delta Q_E\vec{v}$ can be rewritten as $Id\vec{L}$, so the whole equation becomes

$$d\vec{B}_* = \gamma\frac{\mu_0}{4\pi r^2}Id\vec{L} \times \vec{e}_r$$

Except for the additional relativistic factor of γ (which can be set to 1 in non-relativistic scenarios), this is Biot-Savart law:

$$d\vec{B} = \frac{\mu_0}{4\pi r^2}Id\vec{L} \times \vec{e}_r \tag{11}$$

3.2 Static Electromagnetic Force

The electric field E is defined as $E = \frac{\text{force}}{\text{electric unit charge}}$. Multiplied with electric charge $q_e(= q)$, the resulting force is

$$\mathcal{F}_{static} = \hat{q}^{ij}E^{ij}e_{ij} = qE$$

This is the *static electric force* (electrostatic force). As stated above in section 2.4, electric charges do not seem to interact with the magnetic field B. However, we can use the idea of virtual "magnetic charges" that interact with the B-field to fill in the math. Therefore, the electrostatic force can – for symmetry reasons – be extended to and written as the static electromagnetic force. Compare with the field strength tensor (8) for the correct plus and minus signs.

$$\mathcal{F}_{static} = -\hat{q}^{0j}B^{0j}e_{0j} + \hat{q}^{ij}E^{ij}e_{ij} \tag{12}$$

3.3 Dynamic Electromagnetic Force and Lorentz Force

An observer at rest in its system $\tilde{O}$ measures the static electromagnetic field $\tilde{F}$ in this system. An observer at rest in another system O, which moves with velocity v with respect to the system of the charge, measures a different field $\mathcal{F}$. Utilising the reverse transformation formulas for bivector components under movement from appendix A.3

$$\begin{aligned}
C^{01}e_{01} &= \left((1_1 \vee \gamma_2 \vee \gamma_3)\,\tilde{C}^{01} + \gamma_3\beta_3\tilde{C}^{31} - \gamma_2\beta_2\tilde{C}^{12}\right)e_{01}\\
C^{02}e_{02} &= \left((\gamma_1 \vee 1_2 \vee \gamma_3)\,\tilde{C}^{02} + \gamma_1\beta_1\tilde{C}^{12} - \gamma_3\beta_3\tilde{C}^{23}\right)e_{02}\\
C^{03}e_{03} &= \left((\gamma_1 \vee \gamma_2 \vee 1_3)\,\tilde{C}^{03} + \gamma_2\beta_2\tilde{C}^{23} - \gamma_1\beta_1\tilde{C}^{31}\right)e_{03}\\
C^{12}e_{12} &= \left((\gamma_1 \vee \gamma_2 \vee 1_3)\,\tilde{C}^{12} + \gamma_1\beta_1\tilde{C}^{02} - \gamma_2\beta_2\tilde{C}^{01}\right)e_{12}\\
C^{31}e_{31} &= \left((\gamma_1 \vee 1_2 \vee \gamma_3)\,\tilde{C}^{31} + \gamma_3\beta_3\tilde{C}^{01} - \gamma_1\beta_1\tilde{C}^{03}\right)e_{31}\\
C^{23}e_{23} &= \left((1_1 \vee \gamma_2 \vee \gamma_3)\,\tilde{C}^{23} + \gamma_2\beta_2\tilde{C}^{03} - \gamma_3\beta_3\tilde{C}^{02}\right)e_{23}
\end{aligned}$$

and setting $\tilde{C}^{01} = -\tilde{B}^{01}$, $\tilde{C}^{02} = -\tilde{B}^{02}$, $\tilde{C}^{03} = -\tilde{B}^{03}$, $\tilde{C}^{12} = \tilde{E}^{12}$, $\tilde{C}^{31} = \tilde{E}^{31}$, $\tilde{C}^{23} = \tilde{E}^{23}$, we get

$$\begin{aligned}
C^{01}e_{01} &= \left(-(1_1 \vee \gamma_2 \vee \gamma_3)\,\tilde{B}^{01} + (\gamma_3\beta_3\tilde{E}^{31} - \gamma_2\beta_2\tilde{E}^{12})\right)e_{01}\\
C^{02}e_{02} &= \left(-(\gamma_1 \vee 1_2 \vee \gamma_3)\,\tilde{B}^{02} + (\gamma_1\beta_1\tilde{E}^{12} - \gamma_3\beta_3\tilde{E}^{23})\right)e_{02}\\
C^{03}e_{03} &= \left(-(\gamma_1 \vee \gamma_2 \vee 1_3)\,\tilde{B}^{03} + (\gamma_2\beta_2\tilde{E}^{23} - \gamma_1\beta_1\tilde{E}^{31})\right)e_{03}\\
C^{12}e_{12} &= \left((\gamma_1 \vee \gamma_2 \vee 1_3)\,\tilde{E}^{12} + (-\gamma_1\beta_1\tilde{B}^{02} + \gamma_2\beta_2\tilde{B}^{01})\right)e_{12}\\
C^{31}e_{31} &= \left((\gamma_1 \vee 1_2 \vee \gamma_3)\,\tilde{E}^{31} + (-\gamma_3\beta_3\tilde{B}^{01} + \gamma_1\beta_1\tilde{B}^{03})\right)e_{31}\\
C^{23}e_{23} &= \left((1_1 \vee \gamma_2 \vee \gamma_3)\,\tilde{E}^{23} + (-\gamma_2\beta_2\tilde{B}^{03} + \gamma_3\beta_3\tilde{B}^{02})\right)e_{23}
\end{aligned} \tag{13}$$

These are the full equations of the *dynamic electromagnetic field*. In the equations above, comparing the second terms in parenthesis of all fields F^{0j} with section 3.1, we identify these terms as newly induced B-fields and E-fields.

The resulting observable force for an electric charge q^{ij} therefore is

$$\begin{aligned}
\mathcal{F}^{12}e_{12} &= \hat{q}^{12}\left((\gamma_1 \vee \gamma_2 \vee 1_3)\,\tilde{E}^{12} - (\gamma_1\beta_1\tilde{B}^{02} - \gamma_2\beta_2\tilde{B}^{01})\right)e_{12}\\
\mathcal{F}^{31}e_{31} &= \hat{q}^{31}\left((\gamma_1 \vee 1_2 \vee \gamma_3)\,\tilde{E}^{31} - (\gamma_3\beta_3\tilde{B}^{01} - \gamma_1\beta_1\tilde{B}^{03})\right)e_{31}\\
\mathcal{F}^{23}e_{23} &= \hat{q}^{23}\left((1_1 \vee \gamma_2 \vee \gamma_3)\,\tilde{E}^{23} - (\gamma_2\beta_2\tilde{B}^{03} - \gamma_3\beta_3\tilde{B}^{02})\right)e_{23}
\end{aligned}$$

These are the components of the *dynamic electromagnetic force*.

In the non-relativistic limit we can set $\gamma_i = 1$. We can also set $B^{01} = \tilde{B}^{01}$, $B^{02} = \tilde{B}^{02}$, $B^{03} = \tilde{B}^{03}$, $E^{12} = \tilde{E}^{12}$, $E^{31} = \tilde{E}^{31}$ and $E^{23} = \tilde{E}^{23}$. Observing that $\beta_i = \frac{v_i}{c}$ and that the B-fields in our unit system have a factor of c included compared to the “normally” used B_*-fields, we can rewrite the equations above as

$$\begin{aligned}
\mathcal{F}^{12}e_{12} &= \hat{q}^{12}\left(E^{12} - (\frac{v_1}{c}cB_*^{02} - \frac{v_2}{c}cB_*^{01})\right)e_{12}\\
\mathcal{F}^{31}e_{31} &= \hat{q}^{31}\left(E^{31} - (\frac{v_3}{c}cB_*^{01} - \frac{v_1}{c}cB_*^{03})\right)e_{31}\\
\mathcal{F}^{23}e_{23} &= \hat{q}^{23}\left(E^{23} - (\frac{v_2}{c}cB_*^{03} - \frac{v_3}{c}cB_*^{02})\right)e_{23}
\end{aligned}$$

Using (8) for the correct plus and minus signs, the dynamic electromagnetic force in the non-relativistic limit is thus identified as the *Lorentz force*

$$\mathcal{F}_{Lorentz} = q(E + v \times B_*) \tag{14}$$

3.4 Maxwell's Equations for Static Charges

Some of the most important equations in electromagnetism and physics are Maxwell's equations. Instead of just using the result from the general derivative in Appendix A.2, we will now derive the equations explicitly from the electrostatic force.

Looking at the static electromagnetic force in section 3.2 and assuming that forces are conserved, we can write

$$\mathring{\nabla}\mathcal{F} = 0 = \mathring{\nabla}(-\hat{q}^{0j}B^{0j}e_{0j} + \hat{q}^{ij}E^{ij}e_{ij})) = -\mathring{\nabla}(q^{0j})B^{0j}e_{0j} - q^{0j}\mathring{\nabla}(B)e_{0j} + \mathring{\nabla}(q^{ij})E^{ij}e_{ij} + q^{ij}\mathring{\nabla}(E^{ij}e_{ij})$$

Because the components are separable by their unit bivectors, we can look at each bivector component independently, e.g. for a magnetic field component

$$q^{03}\mathring{\nabla}(B^{03})e_{03} = -\mathring{\nabla}(q^{03})B^{03}e_{03} \text{ results in } \mathring{\nabla}B^{03}e_{03} = -\mathring{\nabla}(q^{03})\frac{B^{03}}{q^{03}}e_{03}$$

and e.g. for an electric field component

$$-q^{12}\mathring{\nabla}(E^{12})e_{12} = \mathring{\nabla}(q^{12})E^{12}e_{12} \text{ results in } -\mathring{\nabla}(E^{12})e_{12} = \mathring{\nabla}(q^{12})\frac{E^{12}}{q^{12}}e_{12}$$

All components combined, we now have the equation

$$-\mathring{\nabla}(-B^{0j}e_{0j} + E^{ij}e_{ij}) = -\mathring{\nabla}(q^{0j})\frac{B^{0j}}{q^{0j}}e_{0j} + \mathring{\nabla}(q^{ij})\frac{E^{ij}}{q^{ij}}e_{ij}$$

The left side of the equation can be directly compared with $\mathring{\nabla}(-B+E) = M.E.'s\ in\ vacuum$ from appendix A.2

$$\begin{aligned}
-\mathring{\nabla}(-B^{0j}e_{0j} + E^{ij}e_{ij}) = &- (d_1B^{01} + d_2B^{02} + d_3B^{03})e_0 \\
&- (-d_0B^{01} - (d_2E^{12} - d_3E^{31}))e_1 \\
&- (-d_0B^{02} - (d_3E^{23} - d_1E^{12}))e_2 \\
&- (-d_0B^{03} - (d_1E^{31} - d_2E^{23}))e_3 \\
&- (d_0E^{12} - (d_1B^{02} - d_2B^{01}))e_{012} \\
&- (d_0E^{31} - (d_3B^{01} - d_1B^{03}))e_{031} \\
&- (d_0E^{23} - (d_2B^{03} - d_3B^{02}))e_{023} \\
&- (-d_1E^{23} - d_2E^{31} - d_3E^{12})e_{231}
\end{aligned}$$

The other, right side of the equation depends on the strengths of the B- and E-fields. With a unit electric field strength E_u and a unit magnetic field strength B_u, one can write $B^{0j} = S^{0j}B_u$ and $E^{ij} = S^{ij}E_u$.

Therefore we can write

$$-\mathring{\nabla}(q^{0j})\frac{B^{0j}}{q^{0j}}e_{0j}+\mathring{\nabla}(q^{ij})\frac{E^{ij}}{q^{ij}}e_{ij}=-\mathring{\nabla}(q^{0j})\frac{S^{0j}B_u}{q^{0j}}e_{0j}+(\mathring{\nabla}q^{ij})\frac{S^{ij}E_u}{q^{ij}}e_{ij}$$
$$=-S^{0j}\mathring{\nabla}(q^{0j})\frac{B_u}{q^{0j}}e_{0j}+S^{ij}\mathring{\nabla}(q^{ij})\frac{B_u}{q^{ij}}e_{ij}$$

This "unit" E-field E_u and an unit electric charge q_u^{ij} can be used to define a constant ϵ_0 as

$$\epsilon_0=\frac{q_u^{ij}}{E_u} \tag{15}$$

This constant is called vacuum electric permittivity ϵ_0. For its value and units see appendix C. Note that the units of charge are $[A\cdot s]$ and here an electric field has $\left[\frac{kg\cdot m}{A\cdot s^3}\right]$, which combines to $\left[\frac{A^2\cdot s^4}{kg\cdot m}\right]$. This discrepancy to the "normal" units of $\left[\frac{A^2\cdot s^4}{kg\cdot m^3}\right]$ stems from the fact that our resulting separate equations are one dimensional, not three-dimensional densities.

With the well-known equivalence

$$\epsilon_0\mu_0=\frac{1}{c^2} \tag{16}$$

we can also get the constant μ_0, the vacuum magnetic permeability. For the values of these constants also refer to section C.

In analogy to (15) we can also define a constant ϵ_b for magnetic unit charge and field

$$\epsilon_b=\frac{q_u^{0j}}{B_u} \tag{17}$$

even though we assume that magnetic charges don't exist.

Combining it all, we now can write

$$-\mathring{\nabla}(q^{0j})\frac{B^{0j}}{q^{0j}}e_{0j}+\mathring{\nabla}(q^{ij})\frac{E^{ij}}{q^{ij}}e_{ij}=-\mathring{\nabla}(q_u^{0j})\frac{S^{0j}}{\epsilon_b}e_{0j}+\mathring{\nabla}(q_u^{ij})\frac{S^{ij}}{\epsilon_0}e_{ij}$$
$$=-\frac{1}{\epsilon_b}\mathring{\nabla}(S^{ij}q_u^{0j})e_{0j}+\frac{1}{\epsilon_0}\mathring{\nabla}(Sijq_u^{ij})e_{ij}$$

Because S^{0j} and Sij are only scalar numbers, we can define $Q^{ij}=S^{0j}q_u^{0j}$ and $Q^{ij}=S^{ij}q_u^{ij}$. Using these definitions, we now have $-\frac{1}{\epsilon_b}\mathring{\nabla}(Q^{0j})e_{0j}+\frac{1}{\epsilon_0}\mathring{\nabla}(Q^{ij})e_{ij}$. It becomes clear that we can use the same formula from Appendix A.2 that led to $\mathring{\nabla}(-B+E)=$ M.E.'s in vacuum. In summary, the derivative of the (static) electromagnetic force $\mathring{\nabla}\mathcal{F}_{EB}=0=\mathring{\nabla}(-q^{0j}B^{0j}e_{0j}+q^{ij}E^{ij}e_{ij})$ leads to

$$\mathring{\nabla}(B^{0j}e_{0j}-E^{ij}e_{ij})=-\frac{1}{\epsilon_b}\mathring{\nabla}(Q^{0j})e_{0j}+\frac{1}{\epsilon_0}\mathring{\nabla}(Qij)e_{ij}$$

This result will be used in 3.6 "Fine-Structure Constant"

Components related to $-\frac{1}{\epsilon_b}\mathring{\nabla}Q^{0j}$:

$$\begin{aligned}
&\frac{1}{\epsilon_b}(d_1Q^{01}+d_2Q^{02}+d_3Q^{03})e_0\\
&-\frac{1}{\epsilon_b}(d_0Q^{01})e_1\\
&-\frac{1}{\epsilon_b}(d_0Q^{02})e_2\\
&-\frac{1}{\epsilon_b}(d_0Q^{03})e_3\\
&-\frac{1}{\epsilon_b}(d_1Q^{02}-d_2Q^{01})e_{012}\\
&-\frac{1}{\epsilon_b}(d_3Q^{01}-d_1Q^{03})e_{031}\\
&-\frac{1}{\epsilon_b}(d_2Q^{03}-d_3Q^{02})e_{023}
\end{aligned}$$

All of these components contain virtual "magnetic charges" and are normally not included in Maxwell's equations.

Components related to $\frac{1}{\epsilon_0}\mathring{\nabla}Q^{ij}$:

$$\begin{aligned}
&-\frac{1}{\epsilon_0}(d_2Q^{12}-d_3Q^{31})e_1\\
&-\frac{1}{\epsilon_0}(d_3Q^{23}-d_1Q^{12})e_2\\
&-\frac{1}{\epsilon_0}(d_1Q^{31}-d_2Q^{23})e_3\\
&+\frac{1}{\epsilon_0}(d_0Q^{12})e_{012}\\
&+\frac{1}{\epsilon_0}(d_0Q^{31})e_{031}\\
&+\frac{1}{\epsilon_0}(d_0Q^{23})e_{023}\\
&-\frac{1}{\epsilon_0}(d_1Q^{23}+d_2Q^{31}+d_3Q^{12})e_{123}
\end{aligned}$$

There are some previously unobserved vector components:

$$-\frac{1}{\epsilon_0}(d_2Q^{12}-d_3Q^{31})e_1, -\frac{1}{\epsilon_0}(d_3Q^{23}-d_1Q^{12})e_2,\ \text{and}\ -\frac{1}{\epsilon_0}(d_1Q^{31}-d_2Q^{23})e_3$$

With the electric current

$$\frac{\Delta Q^{ij}}{\Delta x_0}=j_E^{ij} \tag{18}$$

and the electric charge density

$$\frac{\Delta Q^{ij}}{\Delta x_k}=-\rho_E^{ij} \tag{19}$$

the previously observed trivector components are

$$\frac{j_E^{12}}{\epsilon_0}e_{012},\ \frac{j_E^{31}}{\epsilon_0}e_{031},\ \frac{j_E^{23}}{\epsilon_0}e_{023},\ \text{and}\ \frac{\rho_E}{\epsilon_0}e_{123}$$

Remembering that we started from

$$\begin{aligned}\mathring{\nabla}\mathcal{F} = 0 = -\mathring{\nabla}(q^{0j})B^{0j}e_{0j} + \mathring{\nabla}(q^{ij})E^{ij}e_{ij} - q^{0j}\mathring{\nabla}(B^{0j})e_{ij} + q^{ij}\mathring{\nabla}(E)e_{ij}\\ \mathring{\nabla}(B^{0j}e_{0j} - E^{ij}e_{ij}) = -\frac{1}{\epsilon_b}\nabla(Q^{0j})e_{0j} + \frac{1}{\epsilon_0}\nabla(Q^{ij})e_{ij}\end{aligned}$$

we can recombine the equations and components from above side by side:

$$\begin{aligned}-(d_1B^{01} + d_2B^{02} + d_3B^{03})e_0 &= \frac{1}{\epsilon_b}(d_1Q^{01} + d_2Q^{02} + d_3Q^{03})e_0\\ -(-d_0B^{01} - (d_2E^{12} - d_3E^{31}))e_1 &= (-\frac{1}{\epsilon_b}d_0Q^{01} - \frac{1}{\epsilon_0}(d_2Q^{12} - d_3Q^{31}))e_1\\ -(-d_0B^{02} - (d_3E^{23} - d_1E^{12}))e_2 &= (-\frac{1}{\epsilon_b}d_0Q^{02} - \frac{1}{\epsilon_0}(d_3Q^{23} - d_1Q^{12}))e_2\\ -(-d_0B^{03} - (d_1E^{31} - d_2E^{23}))e_3 &= (-\frac{1}{\epsilon_b}d_0Q^{03} - \frac{1}{\epsilon_0}(d_1Q^{31} - d_2Q^{23}))e_3\\ -(d_0E^{12} - (d_1B^{02} - d_2B^{01}))e_{012} &= (\frac{1}{\epsilon_0}d_0Q^{12} - \frac{1}{\epsilon_b}(d_1Q^{02} - d_2Q^{01}))e_{012}\\ -(d_0E^{31} - (d_3B^{01} - d_1B^{03}))e_{031} &= (\frac{1}{\epsilon_0}d_0Q^{31} - \frac{1}{\epsilon_b}(d_3Q^{01} - d_1Q^{03}))e_{031}\\ -(d_0E^{23} - (d_2B^{03} - d_3B^{02}))e_{023} &= (\frac{1}{\epsilon_0}d_0Q^{23} - \frac{1}{\epsilon_b}(d_2Q^{03} - d_3Q^{02}))e_{023}\\ -(-d_1E^{23} - d_2E^{31} - d_3E^{12})e_{123} &= -\frac{1}{\epsilon_0}(d_1Q^{23} + d_2Q^{31} + d_3Q^{12})e_{123}\end{aligned} \tag{20}$$

While the new and unobserved terms might seem surprising at first, there are engineering applications where it is helpful and customary to introduce terms like "magnetic charge", e.g. in Antenna theory [1]. However, we will have a critical look at the validity of all parts of these equations below in 3.5.

If we substitute $B^{0j} := B^j$ (No change of sign! We included this step in the beginning), $Q^{0j} := Q_B^j$, $E^{ij} := E^k$, and $Q^{ij} := Q_E^k$ and rearrange slightly we can write in short notation (with terms that are so far not included in Maxwell's Equations marked in the color gray)

$$\begin{aligned}\nabla \cdot B &= \frac{1}{\epsilon_b}(\nabla \cdot Q_B)\\ \nabla \times E &= -d_0B - \frac{1}{\epsilon_b}(d_0Q_B) - \frac{1}{\epsilon_0}(\nabla \times Q_E)\\ \nabla \times B &= d_0E + \frac{1}{\epsilon_0}J_q - \frac{1}{\epsilon_b}(\nabla \times Q_B)\\ \nabla \cdot E &= \frac{1}{\epsilon_0}\rho_q\end{aligned} \tag{21}$$

Setting the unobserved terms to 0, we get these four lines, which represent all four of Maxwell's equations:

$$\begin{aligned}\nabla \cdot B &= 0 \\ \nabla \times E &= -d_0 B \\ \nabla \times B &= d_0 E + \frac{1}{\epsilon_0} J_q \\ \nabla \cdot E &= \frac{1}{\epsilon_0} \rho_q\end{aligned} \tag{22}$$

Gauss's Law for Magnetism
Look at (20) and at the real temporal component e_0:

$$(d_1 B^{01} + d_2 B^{02} + d_3 B^{03}) = 0$$

This can be written as *Gauss's law for magnetism*

$$\nabla \cdot B = 0 \tag{23}$$

Maxwell-Faraday Equation (Faraday's Law of Induction)
Look at e.g. the real spatial components e_1 (and at e_2 and e_3):

$$d_2 E^{12} - d_3 E^{31} = d_0 B^{01}$$

Up to here, B was measured in the same units as E. With the substitutions $B = cB_*$ and $d_0 = \frac{1}{c}\frac{\partial}{\partial t}$ and eliminating c, these can be combined into *Maxwell-Faraday equation (Faraday's law of induction)*

$$\nabla \times E = -\frac{\partial}{\partial t} B_* \tag{24}$$

Ampère's Circuital Law (with Maxwell's Addition)
Look at the imaginary component e_{012} or similar at components e_{031} and e_{023}: The familiar components here are

$$\epsilon_0 (d_1 B^{02} - d_2 B^{01}) = J_q^{12} + \epsilon_0 d_0 E^{12}$$

With $B = cB_*$ and $J_q = d_0 Q_e = \frac{1}{c}\frac{\partial}{\partial t} Q_e = \frac{1}{c} j_q$ this becomes

$$\begin{aligned}c\epsilon_0 (c d_1 B_*^{02} - c d_2 B_*^{01}) &= (j_q^{12} + \epsilon_0 \frac{\partial}{\partial t} E^{12}) \\ \frac{1}{\mu_0}(d_1 B_*^{02} - d_2 B_*^{01}) &= (j_q^{12} + \epsilon_0 \frac{\partial}{\partial t} E^{12}) \\ (d_1 B_*^{02} - d_2 B_*^{01}) &= \mu_0 (j_q^{12} + \epsilon_0 \frac{\partial}{\partial t} E^{12})\end{aligned}$$

These components can be combined into *Ampère's circuital law (with Maxwell's addition)*

$$\nabla \times B_* = \mu_0 (j_q + \epsilon_0 \frac{\partial}{\partial t} E) \tag{25}$$

Gauss's Law
Look at the imaginary temporal component e_{123}:

$$(d_1 E^{23} + d_2 E^{31} + d_3 E^{12}) e_{123} = (\frac{\rho_q}{\epsilon_0}) e_{123}$$

This can be written as *Gauss's law*

$$\nabla \cdot E = \frac{\rho_q}{\epsilon_0} \tag{26}$$

3.5 Interpretation of the Equations Leading to Maxwell's Equations

Looking at equation (20) and comparing them with the known form of Maxwell's equations (22) the most obvious thing to notice is that no terms including Q^{0j} have been observed. As the magnetic field B is only a relativistic effect of the electric field E, maybe the terms including Q^{0j} are really zero or non-existent. Perhaps they are some other kind of charge which work on original, non derived action and force fields $Field_S^{0j}$ and $Field^{0j}$.

Before we take a guess about these new kind of charges and fields, there is one more unknown term in (21) which contain the definitely existing Q^{ij}, this is $-\frac{1}{\epsilon_0}(\nabla \times Q_E)$. Looking at the units and the components, this is a spatial derivative - which makes it equivalent to a momentum (at the derivative level of energy, momentum ...). However, it also adds a rotation, which, when combined, makes this equivalent to an angular momentum. Looking a bit further back in the history of the derivation of this term, we see that it comes actually from the (internal) rotation of a unit charge. A reasonable assumption therefore seems to be that this term is connected to some kind of spin and should be included in extended Maxwell's equations:

$$\begin{aligned} \nabla \cdot B &= 0 \\ \nabla \times E &= -d_0 B - \frac{1}{\epsilon_0}(\nabla \times Q_E) \\ \nabla \times B &= d_0 E + \frac{1}{\epsilon_0} J_q \\ \nabla \cdot E &= \frac{1}{\epsilon_0}\rho_q \end{aligned} \tag{27}$$

This leaves the question of what Charge^{0j} represents and to which fields they connect. A possibly far fetched idea that would need serious research might be the following: For symmetry reasons, these charges should show similar behaviour to electric charges. There should be two or three different fractional values of them which can be summed up to a whole or zero and there should be a positive and a negative value to each (charge / anti-charge). Thinking about quarks and their properties, the idea presents itself that these charges Charge^{0j} might represent color charges which work on their own color field (which of course must have a complementary derived field in analogy to the B-field of the electric E-field. Also equations equivalent to Maxwell's equations must exist, including some kind of color spin).

All in all, from this we now could assume that quarks and leptons are made up of combinations of up to six independent bivectors. These are acted upon by color and electric charge operators.

If one wanted to continue to leave the realm of "reasonable assumptions" and go a bit further then one could come up with the idea that, when viewing these bivectors as spacetime invariants, it might also be possible to view them as strings (which are described by the equation of spacetime invariants). One step further might lead to the assumption that these are not strings of some kind in spacetime, but that these strings actually represent spacetime itself. However, when thinking about the field equations of general relativity and how they are interpreted as "mass tells spacetime how to bend" then one could also actually take the equality sign to literally mean "mass is bent spacetime". In this way, the idea that these bivectors represent/are spacetime does not seem to be too unlikely. The superposition of all the bivectors/wave functions would make up all of spacetime. Particle/wave duality could also easily be explained this way. Energy or force singularities at $r = 0$

would no longer be problematic (Derivative/slope of a circle at angle $\alpha = 0$. α stands in for radius / distance from origin)

3.6 Fine-Structure Constant

Generally speaking, a bivector field $C^{\mu\nu}e_{\mu\nu}$ can be written as $q^{\mu\nu}F^{\mu\nu}e_{\mu\nu}$.

$$C^{\mu\nu}e_{\mu\nu} = q^{\mu\nu}\frac{\mathcal{C}^{\mu\nu}}{q^{\mu\nu}}e_{\mu\nu} = q^{\mu\nu}F^{\mu\nu}e_{\mu\nu}$$

For simplification, we will not write out $e_{\mu\nu}$ or any unit vectors below. The first spacetime derivative yields

$$\mathring{\nabla}C^{\mu\nu} = \mathring{\nabla}(q^{\mu\nu}F^{\mu\nu}) = F^{\mu\nu}\mathring{\nabla}(q^{\mu\nu}) + q^{\mu\nu}\mathring{\nabla}(F^{\mu\nu})$$

With the condition $\mathring{\nabla}C^{\mu\nu} = 0$, we can write

$$\begin{aligned} 0 &= F^{\mu\nu}\mathring{\nabla}(q^{\mu\nu}) + q^{\mu\nu}\mathring{\nabla}(F^{\mu\nu}) \\ -q^{\mu\nu}\mathring{\nabla}(F^{\mu\nu}) &= F^{\mu\nu}\mathring{\nabla}(q^{\mu\nu}) \\ -\mathring{\nabla}(F^{\mu\nu}) &= \frac{F^{\mu\nu}}{q^{\mu\nu}}\mathring{\nabla}(q^{\mu\nu}) \end{aligned}$$

With $q^{\mu\nu} = S_q^{\mu\nu}q_u^{\mu\nu}$ and $F^{\mu\nu} = S_F^{\mu\nu}F_u^{\mu\nu}$ this leads to

$$-\mathring{\nabla}(F^{\mu\nu}) = \frac{F_u^{\mu\nu}}{q_u^{\mu\nu}}\mathring{\nabla}(S_q^{\mu\nu}q_u^{\mu\nu})\frac{S_F^{\mu\nu}}{S_q^{\mu\nu}}$$

and finally, using a constant proportionality factor $\frac{1}{\epsilon} = \frac{F_u^{\mu\nu}}{q_u^{\mu\nu}}$

$$-\mathring{\nabla}(F^{\mu\nu}) = \frac{1}{\epsilon}\mathring{\nabla}(S_F^{\mu\nu}q_u^{\mu\nu})$$

$$-\mathring{\nabla}(F^{\mu\nu}) = \frac{1}{\epsilon}\mathring{\nabla}(Q_F^{\mu\nu}) \tag{28}$$

An example of such a proportionality factor can be seen above in 3.4 when deriving the electric permittivity ϵ_0 from the electromagnetic force:

$$\mathcal{F}_{EM}^{\mu\nu} = q^{\mu\nu}\frac{\mathcal{F}_{EM}^{\mu\nu}}{q^{\mu\nu}} = q^{\mu\nu}E^{\mu\nu}$$

Taking the derivative leads to the combined form of Maxwell's Equations

$$-\mathring{\nabla}(E^{\mu\nu}) = \frac{1}{\epsilon_0}\mathring{\nabla}(S^{\mu\nu}q_u^{\mu\nu}) = \frac{1}{\epsilon_0}\mathring{\nabla}(Q^{\mu\nu})$$

After this – and according to equation (61) – the second derivative of electromagnetic force leads to

$$-(\mathring{\nabla}\cdot\mathring{\nabla}\cdot E^{\mu\nu}) = \frac{1}{\epsilon_0}(\mathring{\nabla}\cdot\mathring{\nabla}\cdot Q^{\mu\nu})$$

and more general, the second derivative of a bivector field $C^{\mu\nu}$ leads to

$$-(\mathring{\nabla} \cdot \mathring{\nabla} \cdot F^{\mu\nu}) = \frac{1}{\epsilon}(\mathring{\nabla} \cdot \mathring{\nabla} \cdot S^{\mu\nu} q_u^{\mu\nu}) \tag{29}$$

Remembering from 2.3 that bivectors can be written as wave functions and taking the second derivative of a complex wave function, we get

$$d_x \cdot d_x \cdot e^{ikx} = ikd_x \cdot e^{ikx} = -k^2 e^{ikx}$$

Because each side of equation (29) is such a wave equation, this can also be written as

$$-(\mathring{\nabla} \cdot \mathring{\nabla} \cdot F^{\mu\nu}) = \frac{-k^2}{\epsilon}(S^{\mu\nu} q_u^{\mu\nu}) = \frac{-2\alpha}{\epsilon}(S^{\mu\nu} q_u^{\mu\nu}) \tag{30}$$

Here, we use the constant $-\alpha$ instead of the constant $-k^2$ for reasons that will become clear below in 3.6.

In analogy, the second derivative of the general bivector field from 28 and above is

$$-(\mathring{\nabla} \cdot \mathring{\nabla} \cdot F^{\mu\nu}) = \frac{-\alpha}{\epsilon} Q^{\mu\nu} \tag{31}$$

Another concrete example of this can be seen later in 4.8 "Field Equations Of Gravitation".

After looking at this general derivation, we can now look at a concrete example. Just above, as an example, we looked at the first and second derivative of a general bivector field. Looking at the concrete bivector field of electric action $\mathcal{S}$, we know that action, with the right scaling, is

$$\mathcal{S}_u = h = 2\pi\hbar \tag{32}$$

we can also write

$$\mathcal{S}_u = q\frac{\mathcal{S}_u}{q} = q\frac{2\pi\hbar}{q} = qE_S$$

With $\mathring{\nabla}\mathcal{S} = 0 = \mathring{\nabla}(qE_S) = \mathring{\nabla}(q)E_S + q\mathring{\nabla}(E_S)$ the first derivative becomes

$$-\mathring{\nabla}(E_S) = \frac{2\pi\hbar}{q^2}\mathring{\nabla}(q) \tag{33}$$

Note that this is the full spacetime derivative that not only includes energy but also momentum and angular momentum. We chose to ignore Heisenberg's uncertainty principle here which does apply to each single term of the derivative. The second derivative, a force, becomes

$$-(\mathring{\nabla} \cdot \mathring{\nabla} \cdot E_S) = \frac{2\pi\hbar}{q^2}(\mathring{\nabla} \cdot \mathring{\nabla} \cdot q) \tag{34}$$

We contract the terms of the second derivative to gain a single term for force. This of course introduces a proportianality factor. As we have seen in 3.4 "Maxwell's Equations for Static Charges", we can assume that the static electric force is $\mathcal{F} = q^{ij}E^{ij} = 0$. We can therefore write the first derivative of this force as

$$-\mathring{\nabla}E = \frac{1}{\epsilon_0}\mathring{\nabla}q = \mathring{\nabla}(\frac{1}{\epsilon_0}q)$$

The factor $\frac{1}{\epsilon_0}$ was also derived while deriving Maxwell's equations. We can compare this with the derivative of (34), the third derivative of electromagnetic action

$$-\mathring{\nabla}(\mathring{\nabla} \cdot \mathring{\nabla} \cdot E_S) = \mathring{\nabla}\left(\frac{2\pi\hbar}{q^2}(\mathring{\nabla} \cdot \mathring{\nabla} \cdot q)\right)$$

By comparing the arguments in the parentheses of the derivatives with charge, we see that

$$\frac{2\pi\hbar}{q^2}(\mathring{\nabla} \cdot \mathring{\nabla} \cdot q) = \frac{1}{\epsilon_0}q$$

and

$$\mathring{\nabla} \cdot \mathring{\nabla} \cdot q = \frac{q^2}{2\pi\epsilon_0\hbar}q \tag{35}$$

With the fine-structure constant α

$$\alpha = \frac{q^2}{4\pi\epsilon_0\hbar c} \tag{36}$$

this leads to

$$\mathring{\nabla} \cdot \mathring{\nabla} \cdot q = \frac{2}{c}\alpha q \tag{37}$$

The somewhat unexpected factor of $\frac{2}{c}$ is explainable by only looking at energy and observing Heisenberg's uncertainty principle. Using $\frac{\hbar}{2}$ instead of just $\hbar$ in the first derivative of action equation (33) and c, which often is the conversion factor between t and x_0, lets us derive the fine-structure constant α.

The above is actually the equation of a bivector component $qe_{\mu\nu}$. We therefore might assume that q can be written as a wave equation

$$q = qe^{ikx} \tag{38}$$

and

$$\mathring{\nabla} \cdot \mathring{\nabla} \cdot q = -k^2 qe^{ikx} = -k^2 q \tag{39}$$

Comparing equations (37) and (39) and removing the factor $\frac{2}{c}$ as described above, we can see that the wave number k of electromagnetic action charge would be $k = \sqrt{\alpha} = \sqrt{\frac{q^2}{4\pi\epsilon_0\hbar c}} = \frac{q}{\sqrt{2\epsilon_0\hbar c}}$

$$k = \frac{q}{\sqrt{2\epsilon_0\hbar c}} \tag{40}$$

While the above only shows the example of electromagnetic action, the same should be true for all kinds of "charge", electromagnetic, gravitational, and others.

4 Gravitation

4.1 Action and Angular Momentum, Gravitational and Momentum Action Field

The second derivative of action is force. As shown in (2) and equation (61), the second derivative of a bivector field is also a bivector field. Therefore, in analogy to electric forces and electric force fields, e.g. $\mathcal{F}_E = qE$ and $E = \frac{\mathcal{F}_E}{q}$, we can define a gravitational action $\mathcal{S}_G = mG_S$ with a gravitational action field $G_S = \frac{\mathcal{S}_G}{m}$ with units $\left[\frac{action}{mass}\right]$. In analogy to E and the motion induced field $B \propto \beta \times E$ we introduce another field $P_S \propto \beta \times G_S$. This leads to similar equations to equation (10) and (11), "Biot-Savart law".

$$\begin{aligned}
-G_S^{01} e_{01} &= -(1_1 \vee \gamma_2 \vee \gamma_3)\tilde{G_S}^{01} e_{01} \\
-G_S^{02} e_{02} &= -(\gamma_1 \vee 1_2 \vee \gamma_3)\tilde{G_S}^{02} e_{02} \\
-G_S^{03} e_{03} &= -(\gamma_1 \vee \gamma_2 \vee 1_3)\tilde{G_S}^{03} e_{03} \\
P_S^{12} e_{12} &= -(\gamma_2\beta_2\tilde{G_S}^{23} - \gamma_1\beta_1\tilde{G_S}^{31}) e_{12} \\
P_S^{31} e_{31} &= -(\gamma_1\beta_1\tilde{G_S}^{12} - \gamma_3\beta_3\tilde{G_S}^{23}) e_{31} \\
P_S^{23} e_{23} &= -(\gamma_3\beta_3\tilde{G_S}^{31} - \gamma_2\beta_2\tilde{G_S}^{12}) e_{23}
\end{aligned} \tag{41}$$

In analogy to the static electromagnetic action and force, and with electromagnetic charge replaced be mass $m^{\mu\nu}$ we can write down a static gravitational action

$$\mathcal{S}_{static} = -m^{0j} G_S^{0j} e_{0j} + m^{ij} P_S^{ij} e_{ij} \tag{42}$$

4.2 Dynamic Gravitational Action

In analogy to 3.3 "Dynamic Electromagnetic Force" and equation (13) we can write

$$\begin{aligned}
\mathcal{S}^{01} e_{01} &= m^{01}\left(-(1_1 \vee \gamma_2 \vee \gamma_3)\,\tilde{G_S}^{01} + (\gamma_3\beta_3\tilde{P_S}^{31} - \gamma_2\beta_2\tilde{P_S}^{12})\right) e_{01} \\
\mathcal{S}^{02} e_{02} &= m^{02}\left(-(\gamma_1 \vee 1_2 \vee \gamma_3)\,\tilde{G_S}^{02} + (\gamma_1\beta_1\tilde{P_S}^{12} - \gamma_3\beta_3\tilde{P_S}^{23})\right) e_{02} \\
\mathcal{S}^{03} e_{03} &= m^{03}\left(-(\gamma_1 \vee \gamma_2 \vee 1_3)\,\tilde{G_S}^{03} + (\gamma_2\beta_2\tilde{P_S}^{23} - \gamma_1\beta_1\tilde{P_S}^{31})\right) e_{03} \\
\mathcal{S}^{12} e_{12} &= m^{12}\left((\gamma_1 \vee \gamma_2 \vee 1_3)\,\tilde{P_S}^{12} - (\gamma_1\beta_1\tilde{G_S}^{02} - \gamma_2\beta_2\tilde{G_S}^{01})\right) e_{12} \\
\mathcal{S}^{31} e_{31} &= m^{31}\left((\gamma_1 \vee 1_2 \vee \gamma_3)\,\tilde{P_S}^{31} - (\gamma_3\beta_3\tilde{G_S}^{01} - \gamma_1\beta_1\tilde{G_S}^{03})\right) e_{31} \\
\mathcal{S}^{23} e_{23} &= m^{23}\left((1_1 \vee \gamma_2 \vee \gamma_3)\,\tilde{P_S}^{23} - (\gamma_2\beta_2\tilde{G_S}^{03} - \gamma_3\beta_3\tilde{G_S}^{02})\right) e_{23}
\end{aligned} \tag{43}$$

In analogy to 3.3 "Dynamic Electomagnetic Force and Lorentz Force" and equation (14) we can write in the non-relativistic limit the dynamic gravitational action

$$\mathcal{S}_{dynamic} = m(G_S + v \times P_S) \tag{44}$$

4.3 Energy, Mass, Momentum, and Angular Momentum

Assuming action is conserved, the spacetime derivative of the static gravitational action is

$$0 = \mathring{\nabla}\mathcal{S} = \mathring{\nabla}(-m^{0j} G_S^{0j} e_{0j} + m^{ij} P_S^{ij} e_{ij}) \tag{45}$$

With $G_S^{0j} = S^{0j} G_{Su}^{0j}$ (scalar strength of G multiplied by unit action field), $P_S^{ij} = S^{ij} P_{Su}^{ij}$ (scalar strength of P multiplied by unit action field), $\frac{1}{\alpha_G} = \frac{G_{Su}^{0j}}{m_u^{0j}}$, and $\frac{1}{\alpha_P} = \frac{P_{Su}^{ij}}{m_u^{ij}}$, equation (45) can be split into

$$\begin{aligned}
-\mathring{\nabla}(-G_S^{0j} e_{0j} + P_S^{ij} e_{ij}) &= -\frac{S^{0j}}{\alpha_G}\mathring{\nabla}(m_u^{0j}) e_{0j} + \frac{S^{ij}}{\alpha_P}\mathring{\nabla}(m_u^{ij}) e_{ij} \\
-\mathring{\nabla}(-G_S^{0j} e_{0j} + P_S^{ij} e_{ij}) &= -\frac{1}{\alpha_G}\mathring{\nabla} M^{0j} + \frac{1}{\alpha_G}\mathring{\nabla} M^{ij}
\end{aligned}$$

Even though the equations above describe derivatives of action and Maxwell's equations describe derivatives of force, both sides transform like Maxwell's equations, with M_G and G_S taking the place of components related to the B-field and M_P and P_S taking the place of components related to the E-field. Comparing with the short form of Maxwell's Equations (21) gravitation has its analogy in

$$\begin{aligned}
\nabla \cdot G_S &= \frac{1}{\alpha_G}(\nabla \cdot M_G) \\
\nabla \times P_S &= -d_0 G_S - \frac{1}{\alpha_G}(d_0 M_G) - \frac{1}{\alpha_P}(\nabla \times M_P) \\
\nabla \times G_S &= d_0 P_S + \frac{1}{\alpha_P}(d_0 M_P) - \frac{1}{\alpha_G}(\nabla \times M_G) \\
\nabla \cdot P_S &= -\frac{1}{\alpha_P}(\nabla \cdot M_P)
\end{aligned}$$

When applying the same logic as we did when interpreting the equations leading to Maxwell's equations in 3.5, we assume that all terms with M_P can be dropped and that the likely equations for gravitation are

$$\begin{aligned}
\nabla \cdot G_S &= \frac{1}{\alpha_G}(\nabla \cdot M_G) \\
\nabla \times P_S &= -d_0 G_S - \frac{1}{\alpha_G}(d_0 M_G) \\
\nabla \times G_S &= d_0 P_S + \frac{1}{\alpha_G}(\nabla \times M_G) \\
\nabla \cdot P_S &= 0
\end{aligned} \tag{46}$$

All the terms that are derivatives of action with respect to time, d_0, are related to energy. All the terms that are derivatives of action with respect to spatial directions, d_i, are related to momentum. The crossproduct terms are related to angular momentum (spin?).

Mass does not act like charges (eg. it is only positive and only a measure of the energy levels of the vibrations of the bivectors), so there might be some differences in the logic. However, there should be an equivalent to the color charge and the color fields, just like mass is the equivalent to the electric charge.ON the other hand, if mass acts differently than charge and the masses connected with the momentum filed cannot be dropped, then this might play a role in the different energy levels caused by spin that should be investigated.

Effectively, the sum of these energy, momentum, and angular momentum terms represent the gravitational/mass related Lagrangian of this system. If there is an equivalent to the color field, this adds another Lagrangian. Combined with the electromagnetic Lagrangian and the color Lagrangian we then have a total of four Lagrangian describing the whole system. This might be closely related to the Standard Model of Particle Physics.

4.4 Perihelion Shift of Mercury

One of the earliest confirmations of general relativity is the successful prediction of the *Perihelion shift of Mercury.* To see if our approach to gravitation can yield the same results, we can look at the equations of energies in the different approaches to gravitation.

Orbital Energies in Newtonian Gravitation

$$\text{total energy} = \text{rest energy} + \text{kinetic energy} + \text{potential energy}$$
$$\mathcal{E}_t = mc^2 + \frac{1}{2}mv^2 + mV_G(r)$$

With (63), the velocity v in polar coordinates, this becomes

$$\mathcal{E}_t = mc^2 + \frac{1}{2}m(\dot{r}^2 + r^2\dot{\phi}^2) + mV_G(r)$$
$$\mathcal{E}_t = mc^2 + \frac{1}{2}m\dot{r}^2 + m\frac{1}{2}r^2\dot{\phi}^2 + mV_G(r)$$

With equation (64), angular velocity and angular momentum, this becomes

$$\mathcal{E}_t = mc^2 + \frac{1}{2}m\dot{r}^2 + m\frac{1}{2}r^2(\frac{L}{mr^2})^2 + mV_G(r)$$
$$\mathcal{E}_t = mc^2 + \frac{1}{2}m\dot{r}^2 + \frac{mL^2}{2m^2r^2} - \frac{G_n mM}{r}$$
$$\mathcal{E}_t - mc^2 = \frac{1}{2}m\dot{r}^2 + m\left(\frac{L^2}{2m^2r^2} - \frac{G_nM}{r}\right) \tag{47}$$

Orbital Energies in General Relativity: Looking at energy from the viewpoint of gravitation and general relativity, as described e.g. in [17], one finds the following equation. With eigentime τ and angular momentum per unit mass $\mathcal{L} = \frac{L}{m}$ we have

$$\frac{1}{2}(\frac{\mathcal{E}^2}{m} - mc^2) = \frac{1}{2}m(\frac{dr}{d\tau})^2 + m\left(\frac{\mathcal{L}}{2r^2} - \frac{G_nM}{r} - \frac{G_nM}{c^2}\frac{\mathcal{L}^2}{r^3}\right)$$
$$\frac{1}{2}(\frac{\mathcal{E}^2}{m} - mc^2) = \frac{1}{2}m(\frac{dr}{d\tau})^2 + m\left(\frac{L^2}{2m^2r^2} - \frac{G_nM}{r} - \frac{G_nM}{c^2}\frac{L^2}{m^2r^3}\right)$$
$$\frac{1}{2}(\frac{\mathcal{E}^2}{m} - mc^2) = \frac{1}{2}m(\frac{dr}{d\tau})^2 + m\left(\frac{L^2}{2m^2r^2} - \frac{G_nM}{r}\right) - \frac{G_nM}{c^2}\frac{L^2}{mr^3} \tag{48}$$

Comparing this equation with equation (47) from Newtonian gravitation, we find that the significant term responsible for the precession of the perihelion of Mercury is the last term,

$$-\frac{G_nM}{c^2}\frac{L^2}{mr^3} \tag{49}$$

Derivation of the additional energy term With our assumptions about relativistic effects, we can look again at the energies of Newtonian gravitation. We also assume the simplification that – for the instant of our observation – Mercury moves on a circular geodesic around the sun. Circular and geodesic here means that r is constant and we can still apply special relativity because no acceleration takes place. The velocity v of Mercury is parallel to the tangent of movement to the circle, v is perpendicular to r.

Static case / energies: $\mathcal{E}_{static} = mc^2 - m\frac{G_nM}{r}$

Dynamic case / total energies: Just like rest mass/energy gets an additional dynamic term – the kinetic energy – the potential energy V_G becomes $V_G + \beta \times V_P$. Remembering that V_P originated from $\beta \times V_G$, we get $V_G + \beta \times \beta \times V_G$. Because v is perpendicular to V_G, γ is only applied to the first term of the energy, the rest mass.

$$\mathcal{E}_t = \gamma mc^2 - V_G - \beta \times \beta \times V_G$$
$$\mathcal{E}_t = \gamma mc^2 - m\frac{G_n M}{r} - \beta \times \beta \times m\frac{G_n M}{r}$$

$\beta \times \beta \times V_G$ is a vector triple product, see B, but because v (and therefore β) is perpendicular to V_G, we get $\beta \times \beta \times V_G = \beta^2 V_G = V_G \frac{v^2}{c^2}$. With $\gamma mc^2 = mc^2 + \frac{1}{2}\frac{v^2}{c^2}mc^2 = mc^2 + \frac{1}{2}mv^2$ the total energy becomes

$$\mathcal{E}_t = mc^2 + \frac{1}{2}mv^2 - m\frac{G_n M}{r} - m\frac{G_n M}{rc^2}v^2$$

The last term, $-m\frac{G_n M}{rc^2}v^2$, is a new term compared with Newtonian gravitation. It is a relativistic effect of potential energy.

Transforming the new energy term Using equations (63) and (64) we can write $|v|^2 = \dot{r}^2 + \dot{\phi}^2 r^2 = |v|^2 = \dot{r}^2 + \frac{L^2}{m^2 r^2}$. Using this, we write

$$\begin{aligned} & - m\frac{G_n M}{rc^2}v^2 \\ & = -m\frac{G_n M}{rc^2}(\dot{r}^2 + \frac{L^2}{m^2 r^2}) \\ & = -m\frac{G_n M}{rc^2}(\dot{r})^2 - \frac{G_n M}{c^2}\frac{L^2}{mr^3} \end{aligned}$$

Because we assumed that Mercury – in the instant of our observation – is moving on the tangent of a circular orbit and therefore r – in that instant – is constant, the first term vanishes and we are left with the term

$$-\frac{G_n M}{c^2}\frac{L^2}{mr^3} \tag{50}$$

This matches with the last term from (48) “Orbital Energies in General Relativity” above, the significant term for the perihelion shift of Mercury.

4.5 Static Gravitational Force

In analogy to section 3.2 “Static Electromagnetic Force”, the gravitational force field G is defined as $G = \frac{force}{mass}$. Multiplied with a mass m, the resulting force is

$$\mathcal{F}_{static} = mG = -\hat{m}^{0j}G^{0j}e_{0j}$$

This is the static gravitational force.

The static gravitational force can also be extended as

$$\mathcal{F}_{static} = -\hat{m}^{0j}G^{0j}e_{0j} + \hat{m}^{ij}P^{ij}e_{ij}.$$

4.6 Dynamic Gravitational Force

In analogy to section 3.3 "Dynamic Electromagnetic Force" we can derive the dynamic gravitational force. Utilising the reverse transformation formulas for bivector components under movement from appendix A.3

$$
\begin{aligned}
C^{01}e_{01} &= \Big((1_1 \vee \gamma_2 \vee \gamma_3)\,\tilde{C}^{01} + \gamma_3\beta_3\tilde{C}^{31} - \gamma_2\beta_2\tilde{C}^{12}\Big)\,e_{01}\\
C^{02}e_{02} &= \Big((\gamma_1 \vee 1_2 \vee \gamma_3)\,\tilde{C}^{02} + \gamma_1\beta_1\tilde{C}^{12} - \gamma_3\beta_3\tilde{C}^{23}\Big)\,e_{02}\\
C^{03}e_{03} &= \Big((\gamma_1 \vee \gamma_2 \vee 1_3)\,\tilde{C}^{03} + \gamma_2\beta_2\tilde{C}^{23} - \gamma_1\beta_1\tilde{C}^{31}\Big)\,e_{03}\\
C^{12}e_{12} &= \Big((\gamma_1 \vee \gamma_2 \vee 1_3)\,\tilde{C}^{12} + \gamma_1\beta_1\tilde{C}^{02} - \gamma_2\beta_2\tilde{C}^{01}\Big)\,e_{12}\\
C^{31}e_{31} &= \Big((\gamma_1 \vee 1_2 \vee \gamma_3)\,\tilde{C}^{31} + \gamma_3\beta_3\tilde{C}^{01} - \gamma_1\beta_1\tilde{C}^{03}\Big)\,e_{31}\\
C^{23}e_{23} &= \Big((1_1 \vee \gamma_2 \vee \gamma_3)\,\tilde{C}^{23} + \gamma_2\beta_2\tilde{C}^{03} - \gamma_3\beta_3\tilde{C}^{02}\Big)\,e_{23}
\end{aligned}
$$

Setting $C^{01} = -G^{01}$, $C^{02} = -G^{02}$, $C^{03} = -G^{03}$, $C^{12} = P^{12}$, $C^{31} = P^{31}$, and $C^{23} = P^{23}$, we get

$$
\begin{aligned}
-G^{01}e_{01} &= \big(-(1_1 \vee \gamma_2 \vee \gamma_3)\,G^{01} + (\gamma_3\beta_3 P^{31} - \gamma_2\beta_2 P^{12})\big)\,e_{01}\\
-G^{02}e_{02} &= \big(-(\gamma_1 \vee 1_2 \vee \gamma_3)\,G^{02} + (\gamma_1\beta_1 P^{12} - \gamma_3\beta_3 P^{23})\big)\,e_{02}\\
-G^{03}e_{03} &= \big(-(\gamma_1 \vee \gamma_2 \vee 1_3)\,G^{03} + (\gamma_2\beta_2 P^{23} - \gamma_1\beta_1 P^{31})\big)\,e_{03}\\
P^{12}e_{12} &= \big((\gamma_1 \vee \gamma_2 \vee 1_3)\,P^{12} - (\gamma_1\beta_1 G^{02} - \gamma_2\beta_2 G^{01})\big)\,e_{12}\\
P^{31}e_{31} &= \big((\gamma_1 \vee 1_2 \vee \gamma_3)\,P^{31} - (\gamma_3\beta_3 G^{01} - \gamma_1\beta_1 G^{03})\big)\,e_{31}\\
P^{23}e_{23} &= \big((1_1 \vee \gamma_2 \vee \gamma_3)\,P^{23} - (\gamma_2\beta_2 G^{03} - \gamma_3\beta_3 G^{02})\big)\,e_{23}
\end{aligned}
$$

These are the full equations of the dynamic gravitational force field.

In the equations above, comparing the second terms in parenthesis of all fields $\mathcal{F}^{ij}$ with equation (41) we can identify these terms as the definition of a newly induced P-fields, much like the definition of the B-field in (11) "Biot-Savart Law".

$$
\begin{aligned}
P^{12}_{induced} &= -(\gamma_1\beta_1 G^{02} - \gamma_2\beta_2 G^{01})\\
P^{31}_{induced} &= -(\gamma_3\beta_3 G^{01} - \gamma_1\beta_1 G^{03})\\
P^{23}_{induced} &= -(\gamma_2\beta_2 G^{03} - \gamma_3\beta_3 G^{02})
\end{aligned}
\tag{51}
$$

Assuming, in analogy to electromagnetism, that all forces $\mathcal{F}^{ij}$ are unobserved and no masses m^{ij} exist to work on these fields/bivectors, the remaining mixed time and spatial terms $\mathcal{F}^{0j}$ of the force should be observable.

$$
\begin{aligned}
\mathcal{F}^{01}e_{01} &= \hat{m}^{01}\left(-(1_1 \vee \gamma_2 \vee \gamma_3)\,G^{01} + (\gamma_3\beta_3 P^{31} - \gamma_2\beta_2 P^{12})\right)e_{01}\\
\mathcal{F}^{02}e_{02} &= \hat{m}^{02}\left(-(\gamma_1 \vee 1_2 \vee \gamma_3)\,G^{02} + (\gamma_1\beta_1 P^{12} - \gamma_3\beta_3 P^{23})\right)e_{02}\\
\mathcal{F}^{03}e_{03} &= \hat{m}^{03}\left(-(\gamma_1 \vee \gamma_2 \vee 1_3)\,G^{03} + (\gamma_2\beta_2 P^{23} - \gamma_1\beta_1 P^{31})\right)e_{03}
\end{aligned}
$$

These are the components of the dynamic gravitational force.

Reminder: The G- and P-field each include a factor c because of e_{0j}. In the non-relativistic limit, one could also write the equations in analogy to the Lorentz force as

$$\mathcal{F}_{Lorentz-like} = m(G + v \times P)$$

A "new" force/field in nature should have been observed by now. However, just like the magnetic field is much weaker than the electric field in non-relativistic scenarios, the momentum field should also be much weaker than the gravitational field. Small variations of the gravitational field are very hard to detect with today's technologies. To be observable, one would have to study something with a lot of mass at considerable speed. This is only possible for heavy astronomical objects like stars and planets – see above in 4.4 "Perihelion Shift Of Mercury" – or even more extreme objects like black holes or galaxies. Indeed, one of the "riddles" of galaxies is that they appear to be flatter than one would expect from gravitation and angular momentum alone, an effect that could be attributed to the momentum field. Also the tendency of galaxies to develop spiral arms might be an effect of the momentum field.

4.7 Maxwell-like Equations of Gravitation

In analogy to section 3.4 "Maxwell's Equations of Static Charges" we can write

$$\mathring{\nabla}\mathcal{F} = 0 = \mathring{\nabla}(-\hat{m}^{0j}G^{0j}e_{0j} + \hat{m}^{ij}P^{ij}e_{ij}) = -\mathring{\nabla}(m_g)G + \mathring{\nabla}(m_p)P - m_g\mathring{\nabla}(G) + m_p\mathring{\nabla}(P)$$

Based on our experience with Maxwell's equations we can define some factors $\frac{1}{g_0} = \frac{G_u}{m_{gu}}$ and $\frac{1}{g_p} = \frac{P_u}{m_{pu}}$.

While for Maxwell's Equations we used both the equivalent factors $\frac{1}{\epsilon_0}$ and $\frac{1}{\epsilon_b}$ (which both have the same units), we now instead use only the factor $\frac{1}{g_0}$ and absorb the unit less scalar factor s_ϵ from $\frac{1}{g_p} = \frac{s_\epsilon}{g_0}$ into the strengths of all M_P^{ij}. $S^{0j}m_g^{0j} = M_G^{0j}$ and $s_\epsilon S^{ij}m_p^{ij} = M_P^{ij}$

Now we can split and transform the equation above immediately into a left side

$$\mathring{\nabla}(G^{01}e_{01} + G^{02}e_{02} + G^{03}e_{03} - P^{12}e_{12} - P^{31}e_{31} - P^{23}e_{23})$$

and a right side

$$\mathring{\nabla}(-S_G^{01}\frac{m_g}{g_0}e_{01} - S_G^{02}\frac{m_g}{g_0}e_{02} - S_G^{03}\frac{m_g}{g_0}e_{03} + s_\epsilon S_P^{12}\frac{m_p}{g_0}e_{12} + s_\epsilon S_P^{31}\frac{m_p}{g_0}e_{31} + s_\epsilon S_P^{23}\frac{m_p}{g_0}e_{23}) =$$
$$\mathring{\nabla}(-\frac{M_G^{01}}{g_0}e_{01} - \frac{M_G^{02}}{g_0}e_{02} - \frac{M_G^{03}}{g_0}e_{03} + \frac{M_P^{12}}{g_0}e_{12} + \frac{M_P^{31}}{g_0}e_{31} + \frac{M_P^{23}}{g_0}e_{23})$$

Here, by comparing the factor $\frac{1}{4\pi\epsilon_0}$ of the electric potential $\frac{1}{4\pi\epsilon_0}\frac{qQ}{r}e_r$ with Newton's gravitational constant G_n of the gravitational potential $G_n\frac{mM}{r}e_r$ we can replace the constant $G_n = \frac{1}{4\pi g_0}$ with

$$\frac{1}{g_0} = 4\pi G_n \tag{52}$$

In analogy to (20) from 3.4 "Maxwell's Equations for Static Charges", with some rearranging, the equivalent gravitational and momentum terms above become

$$\begin{aligned}
-(d_1G^{01}+d_2G^{02}+d_3G^{03})e_0 &= 4\pi G_n(d_1M_G^{01}+d_2M_G^{02}+d_3M_G^{03})e_0\\
-(-d_0G^{01}-(d_2P^{12}-d_3P^{31}))e_1 &= 4\pi G_n(-d_0M_G^{01}-(d_2M_P^{12}-d_3M_P^{31}))e_1\\
-(-d_0G^{02}-(d_3P^{23}-d_1P^{12}))e_2 &= 4\pi G_n(-d_0M_G^{02}-(d_3M_P^{23}-d_1M_P^{12}))e_2\\
-(-d_0G^{03}-(d_1P^{31}-d_2P^{23}))e_3 &= 4\pi G_n(-d_0M_G^{03}-(d_1M_P^{31}-d_2M_P^{23}))e_3\\
-(d_0P^{12}-(d_1G^{02}-d_2G^{01}))e_{012} &= 4\pi G_n(d_0M_P^{12}-(d_1M_G^{02}-d_2M_G^{01}))e_{012}\\
-(d_0P^{31}-(d_3G^{01}-d_1G^{03}))e_{031} &= 4\pi G_n(d_0M_P^{31}-(d_3M_G^{01}-d_1M_G^{03}))e_{031}\\
-(d_0P^{23}-(d_2G^{03}-d_3G^{02}))e_{023} &= 4\pi G_n(d_0M_P^{23}-(d_2M_G^{03}-d_3M_G^{02}))e_{023}\\
-(-d_1P^{23}-d_2P^{31}-d_3P^{12})e_{123} &= 4\pi G_n(-d_1M_P^{23}-d_2M_P^{31}-d_3M_P^{12})e_{123}
\end{aligned} \tag{53}$$

In compact notation, with $G^{0j}=G^j$, $M_G^{0j}=M_G^j$, $P^{ij}=P^k$, and $M_P^{ij}=M_P^k$:

$$\begin{aligned}
-\nabla\cdot G &= \frac{1}{g_0}(\nabla\cdot M_G)\\
d_0G+\nabla\times P &= \frac{1}{g_0}(-d_0M_G-\nabla\times M_P)\\
-d_0P+\nabla\times G &= \frac{1}{g_0}(d_0M_P-\nabla\times M_G)\\
\nabla\cdot P &= \frac{1}{g_0}(-\nabla\cdot M_P)
\end{aligned}$$

Dropping all terms with M_P^k leads to

$$\begin{aligned}
-\nabla\cdot G &= \frac{1}{g_0}(-\nabla\cdot M_G)\\
d_0G+\nabla\times P &= -\frac{1}{g_0}(d_0M_G)\\
-d_0P+\nabla\times G &= -\frac{1}{g_0}(\nabla\times M_G)\\
\nabla\cdot P &= 0
\end{aligned}$$

Something like an internal mass-spin as in $-\frac{1}{g_0}(\nabla\times M_G)$ has not been observed yet.

4.8 Field Equations of Gravitation

From section A.2 we know that the second derivative of a bivector field C is

$$\begin{aligned}
&\mathring{\nabla}(\mathring{\nabla}(C^{01}e_{01}+C^{02}e_{02}+C^{03}e_{03}+C^{12}e_{12}+C^{31}e_{31}+C^{23}e_{23}))=\\
&+d_0d_0C^{01}e_{01}-d_3d_3C^{01}e_{01}+d_1d_1C^{01}e_{10}+d_2d_2C^{01}e_{10}\\
&+d_0d_0C^{02}e_{02}-d_1d_1C^{02}e_{02}+d_2d_2C^{02}e_{20}+d_3d_3C^{02}e_{20}\\
&+d_0d_0C^{03}e_{03}-d_2d_2C^{03}e_{03}+d_1d_1C^{03}e_{30}+d_3d_3C^{03}e_{30}\\
&+d_0d_0C^{12}e_{12}-d_1d_1C^{12}e_{12}+d_2d_2C^{12}e_{21}+d_3d_1C^{23}e_{21}\\
&+d_0d_0C^{31}e_{31}-d_3d_3C^{31}e_{31}+d_1d_1C^{31}e_{13}+d_2d_2C^{31}e_{13}\\
&+d_0d_0C^{23}e_{23}-d_2d_2C^{23}e_{23}+d_1d_1C^{23}e_{32}+d_3d_3C^{23}e_{32}
\end{aligned}$$

Looking at the components of e_{01} and e_{10} we realize that this can be written as

$$
\begin{aligned}
&+d_0d_0C^{01}e_{01} - d_3d_3C^{01}e_{01} + d_1d_1C^{01}e_{10} + d_2d_2C^{01}e_{10} = \\
&+\frac{1}{2}d_0d_0C^{01}e_{01} + \frac{1}{2}d_0d_0C^{01}e_{01} - \frac{1}{2}d_3d_3C^{01}e_{01} - \frac{1}{2}d_3d_3C^{01}e_{01} \\
&+\frac{1}{2}d_1d_1C^{01}e_{10} + \frac{1}{2}d_1d_1C^{01}e_{10} + \frac{1}{2}d_2d_2C^{01}e_{10} + \frac{1}{2}d_2d_2C^{01}e_{10}
\end{aligned}
$$

Reversing the direction of half of the bivectors and resorting the components accordingly results in

$$
\begin{aligned}
&= \\
&+\frac{1}{2}d_0d_0C^{01}e_{01} - \frac{1}{2}d_1d_1C^{01}e_{01} - \frac{1}{2}d_2d_2C^{01}e_{01} - \frac{1}{2}d_3d_3C^{01}e_{01} \\
&-\frac{1}{2}d_0d_0C^{01}e_{10} + \frac{1}{2}d_1d_1C^{01}e_{10} + \frac{1}{2}d_2d_2C^{01}e_{10} + \frac{1}{2}d_3d_3C^{01}e_{10}
\end{aligned}
$$

It should be remembered that we actually have values for the off diagonal elements of $R^{\mu\nu}$. $C^{01} = -G^{01}$, $C^{02} = -G^{02}$, $C^{03} = -G^{03}$, $C^{12} = P^{12}$, $C^{31} = P^{31}$, and $C^{23} = P^{23}$. Contracting the components of these bivectors like

$$R^{\mu\nu} = +d_0d_0C^{\mu\nu} - d_1d_1C^{\mu\nu} - d_2d_2C^{\mu\nu} - d_3d_3C^{\mu\nu}$$

leaves us with $R^{\mu\nu}e_{\mu\nu}$

$$
\begin{aligned}
=&\frac{1}{2}(+d_0d_0C^{\mu\nu} - d_1d_1C^{\mu\nu} - d_2d_2C^{\mu\nu} - d_3d_3C^{\mu\nu})e_{\mu\nu} \\
+&\frac{1}{2}(+d_0d_0C^{\mu\nu} - d_1d_1C^{\mu\nu} - d_2d_2C^{\mu\nu} - d_3d_3C^{\mu\nu})e_{\mu\nu} \\
=&\frac{1}{2}(+d_0d_0C^{\mu\nu} - d_1d_1C^{\mu\nu} - d_2d_2C^{\mu\nu} - d_3d_3C^{\mu\nu})e_{\mu\nu} \\
-&\frac{1}{2}(+d_0d_0C^{\mu\nu} - d_1d_1C^{\mu\nu} - d_2d_2C^{\mu\nu} - d_3d_3C^{\mu\nu})e_{\nu\mu}
\end{aligned}
$$

so that all tensor components $R^{\mu\nu}e_{\mu\nu}$ written out are

$$
\begin{aligned}
&+\frac{1}{2}R^{01}e_{01} - \frac{1}{2}R^{01}e_{10} \\
&+\frac{1}{2}R^{02}e_{02} - \frac{1}{2}R^{02}e_{20} \\
&+\frac{1}{2}R^{03}e_{03} - \frac{1}{2}R^{03}e_{30} \\
&+\frac{1}{2}R^{12}e_{12} - \frac{1}{2}R^{23}e_{21} \\
&+\frac{1}{2}R^{31}e_{31} - \frac{1}{2}R^{31}e_{13} \\
&+\frac{1}{2}R^{23}e_{23} - \frac{1}{2}R^{23}e_{32}
\end{aligned}
$$

The above is a symmetric field tensor where all the diagonal elements are 0.

$$
= \frac{1}{2}\begin{pmatrix} 0 & R^{01} & R^{02} & R^{03} \\ -R^{01} & 0 & R^{12} & R^{31} \\ -R^{02} & -R^{12} & 0 & R^{23} \\ -R^{03} & -R^{31} & -R^{23} & 0 \end{pmatrix}
$$

Remembering that the diagonal elements are 0, we can write this, the *left side of equation* (53), as the field tensor

$$\frac{1}{2}R^{\mu\nu} \tag{54}$$

In analogy to the above, for the *right side of the equation* we can set $C^{01} = -M_G^{01}$, $C^{02} = -M_G^{02}$, $C^{03} = -M_G^{03}$, $C^{12} = M_P^{12}$, $C^{31} = M_P^{31}$, and $C^{23} = M_P^{23}$. In analogy to $\frac{1}{2}R^{\mu\nu}$ we can now contract the *right side of equation* (53) to

$$4\pi G_n \frac{1}{2} T^{\mu\nu}. \tag{55}$$

Combing both equations (54) and (55) and dropping the factor $\frac{1}{2}$ on both sides, we now have the complete field equations of gravitation

$$R^{\mu\nu} = 4\pi G_n T^{\mu\nu} \tag{56}$$

However, because some of the $T^{\mu\nu}$ terms containing mass are not observable, it is better to write

$$\begin{aligned}
R^{01} &= 4\pi G_n T^{01} \\
R^{02} &= 4\pi G_n T^{02} \\
R^{03} &= 4\pi G_n T^{03} \\
R^{12} &= 4\pi G_n T^{12} \\
R^{31} &= 4\pi G_n T^{31} \\
R^{23} &= 4\pi G_n T^{23}
\end{aligned}$$

Einstein's field equations have a different factor of $\frac{8\pi G_n}{c^4}$ on the right side. This difference is explained in the next section.

4.9 Einstein's Field Equations of General Relativity

The Maxwell-like equations of gravitation and the Maxwell equations of electromagnetism are the second spacetime derivative of their potential energies. They describe vectors. Sorting the equations as they are (in component order) we have all the fields on the left hand side and all energy/mass/charge densities and currents on the right hand side. In equation (53), the line with the e_0 components is Poisson's equation for gravitation:

$$\begin{aligned}
(d_1 G^{01} + d_2 G^{02} + d_3 G^{03}) = \nabla G = \nabla^2 \phi_g \\
\nabla^2 \phi_g = 4\pi G_n \rho_m
\end{aligned}$$

Here, ϕ_g is the gravitational scalar potential. "Normally", from this equation, one would start to derive the stress-energy tensor $T^{\mu\nu}$ and the Einstein field equations of general relativity.

$$R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} + \lambda g^{\mu\nu} = \frac{8\pi G_n}{c^4} T^{\mu\nu} \tag{57}$$

To derive the Energy-Stress-Momentum Tensor in Einstein's Field equations of general relativity, we can look these field equations and compare them with our equations above. There is a 4 x 4 Tensor both on the left and on the right hand side. We have a 4 vector on each side which, when we once again take the spacetime derivative, gives back bivector components, which can be written as a 4 x 4 Tensor.

Concentrating on the right hand side first, dropping grayed out and spin related terms, and using $\rho = \nabla \cdot Q_G$ and $J = d_0 Q_G$, with $\frac{1}{g_0} = 4\pi G_n$ we get the vector components

$$\begin{aligned} &-4\pi G_n \rho e_0 \\ &\ \ 4\pi G_n J^1 e_1 \\ &\ \ 4\pi G_n J^2 e_2 \\ &\ \ 4\pi G_n J^3 e_3 \end{aligned}$$

Taking the spacetime derivative of the right side:

$$\begin{aligned} (d_0e_0 - d_1e_1 - d_2e_2 - d_3e_3)4\pi G_n(-\rho e_0 + J^1e_1 + J^2e_2 + J^3e_3) \\ = 4\pi G_n(- d_0\rho e_{00} + d_0J^1e_{01} + d_0J^2e_{02} + d_0J^3e_{03} \\ + d_1\rho e_{10} - d_1J^1e_{11} - d_1J^2e_{12} - d_1J^3e_{13} \\ + d_2\rho e_{20} - d_2J^1e_{21} - d_2J^2e_{22} - d_2J^3e_{23} \\ + d_3\rho e_{30} - d_3J^1e_{31} - d_3J^2e_{32} - d_3J^3e_{33}) \end{aligned}$$

The tensor above is the second derivative of a force.

In Einstein's field equations, the factor out front is multiplied by $\frac{1}{c^4}$. The most important term in the tensor is the component C^{00} which, in Einstein's field equations, is an energy density but in our case is a mass density ρ_m. With $\mathcal{E} = mc^2$, we can extract a factor of $\frac{1}{c^2}$ from our mass density ρ_m and now have an energy density $\frac{1}{c^2}\rho_{\mathcal{E}}$. Furthermore, in our case the units of the bivector e_{00} are in meter squared $[m^2]$, while Einstein's units are seconds squared $[s^2]$, the result of two derivation with respect to time. To align our tensor with Einstein's, we must therefore extract another factor of $\frac{1}{c^2}$.

Combined, this results in the energy-stress-momentum tensor

$$\begin{aligned} (d_0e_0 - d_1e_1 - d_2e_2 - d_3e_3)\frac{4\pi G_n}{c^4}(-\rho_{\mathcal{E}} e_0 + J^1_{\mathcal{E}}e_1 + J^2_{\mathcal{E}}e_2 + J^3_{\mathcal{E}}e_3) \\ = \frac{4\pi G_n}{c^4}(- d_0\rho_{\mathcal{E}} e_{00} + d_0J^1_{\mathcal{E}}e_{01} + d_0J^2_{\mathcal{E}}e_{02} + d_0J^3_{\mathcal{E}}e_{03} \\ + d_1\rho_{\mathcal{E}} e_{10} - d_1J^1_{\mathcal{E}}e_{11} - d_1J^2_{\mathcal{E}}e_{12} - d_1J^3_{\mathcal{E}}e_{13} \\ + d_2\rho_{\mathcal{E}} e_{20} - d_2J^1_{\mathcal{E}}e_{21} - d_2J^2_{\mathcal{E}}e_{22} - d_2J^3_{\mathcal{E}}e_{23} \\ + d_3\rho_{\mathcal{E}} e_{30} - d_3J^1_{\mathcal{E}}e_{31} - d_3J^2_{\mathcal{E}}e_{32} - d_3J^3_{\mathcal{E}}e_{33}) \\ = \frac{4\pi G_n}{c^4}T^{\mu\nu} \end{aligned}$$

Equating this to the field tensor from (54), we get $\frac{1}{2}R^{\mu\nu} = \frac{4\pi G_n}{c^4}T^{\mu\nu}$ or

$$R^{\mu\nu} = \frac{8\pi G_n}{c^4}T^{\mu\nu} \tag{58}$$

This looks like the original version of Einstein's field equations of general relativity [5]. To arrive at equation (57), Einstein later [4] amended the left side of this equation with the term $-\frac{1}{2}Rg^{\mu\nu} + \lambda g^{\mu\nu}$.

The addition of the cosmological constant $+\lambda g^{\mu\nu}$ is just a suggestion and can be left out. The terms with the Ricci scalars $-\frac{1}{2}Rg^{\mu\nu}$ are due to the fact that Einstein's equations describe the intrinsic geometry of spacetime – with the coordinate system depending on each point of spacetime itself – while this paper only describes an extrinsic geometry of spacetime.

5 Summary

To simplyfy some of the equations in the following summary, we use "$\vee$", which stands for "logical or", and we substitute

$$(\gamma_1 \vee \gamma_2 \vee \gamma_3) = \gamma_{123}$$

Moreover 1_i stands for "this is one if $v_i \neq 0$" with $i = 1, 2, 3$.

$$\begin{aligned}
&1_1 \text{ stands for "this is one if } v_1 \neq 0\text{".}\\
&1_2 \text{ stands for "this is one if } v_2 \neq 0\text{".}\\
&1_3 \text{ stands for "this is one if } v_3 \neq 0\text{".}
\end{aligned}$$

5.1 Action

$$\mathcal{S} = \text{Constant}$$

$$\begin{aligned}
\mathcal{S} = &- \mathcal{S}^{01}e_{01} - \mathcal{S}^{02}e_{02} - \mathcal{S}^{03}e_{03} + \mathcal{S}^{12}e_{12} + \mathcal{S}^{31}e_{31} + \mathcal{S}^{23}e_{23}\\
\mathcal{S} = &- \mathcal{S}_G^{01}e_{01} - \mathcal{S}_G^{02}e_{02} - \mathcal{S}_G^{03}e_{03} + \mathcal{S}_P^{12}e_{12} + \mathcal{S}_P^{31}e_{31} + \mathcal{S}_P^{23}e_{23}\\
&- \mathcal{S}_B^{01}e_{01} - \mathcal{S}_B^{02}e_{02} - \mathcal{S}_B^{03}e_{03} + \mathcal{S}_E^{12}e_{12} + \mathcal{S}_E^{31}e_{31} + \mathcal{S}_E^{23}e_{23}
\end{aligned}$$

Gravitational Action

Starting off with only gravitational components of an action bivector field, we gain additional – momentum related – components by investigating relativistic effects of the moving bivector field (in analogy to 3.1 "Electric and Magnetic Field"). If we then assume that these additional components pre-exist or exist independently from our gravitational field, we can write

$$\mathcal{S}_{GP} = \text{Constant}$$

$$\mathcal{S}_{GP} = -\mathcal{S}_G^{01}e_{01} - \mathcal{S}_G^{02}e_{02} - \mathcal{S}_G^{03}e_{03} + \mathcal{S}_P^{12}e_{12} + \mathcal{S}_P^{31}e_{31} + \mathcal{S}_P^{23}e_{23}$$

Static:

$$\mathcal{S}_{GP} = -\hat{m}^{01}G_S^{01}e_{01} - \hat{m}^{02}G_S^{02}e_{02} - \hat{m}^{03}G_S^{03}e_{03} + \hat{m}^{12}P_S^{12}e_{12} + \hat{m}^{31}P_S^{31}e_{31} + \hat{m}^{23}P_S^{23}e_{23}$$

Dynamic: In the static equations, replace

$$\begin{aligned}
G_S &\text{ with } \gamma_{123}\tilde{G}_S + \gamma\beta \times \tilde{P}_S\\
P_S &\text{ with } \gamma_{123}\tilde{P}_S + \gamma\beta \times \tilde{G}_S
\end{aligned}$$

This leads to

$$S_{GP}^{01}e_{01} = \hat{m}^{01}(-(1_1 \vee \gamma_2 \vee \gamma_3)\tilde{G}_S^{01} + (\gamma_3\beta_3\tilde{P}_S^{31} - \gamma_2\beta_2\tilde{P}_S^{12}))e_{01}$$
$$S_{GP}^{02}e_{02} = \hat{m}^{02}(-(\gamma_1 \vee 1_2 \vee \gamma_3)\tilde{G}_S^{02} + (\gamma_1\beta_1\tilde{P}_S^{12} - \gamma_3\beta_3\tilde{P}_S^{23}))e_{02}$$
$$S_{GP}^{03}e_{03} = \hat{m}^{03}(-(\gamma_1 \vee \gamma_2 \vee 1_3)\tilde{G}_S^{03} + (\gamma_2\beta_2\tilde{P}_S^{23} - \gamma_1\beta_1\tilde{P}_S^{31}))e_{03}$$
$$S_{GP}^{12}e_{12} = \hat{m}^{12}((\gamma_1 \vee \gamma_2 \vee 1_3)\tilde{P}_S^{12} - (\gamma_1\beta_1\tilde{G}_S^{02} - \gamma_2\beta_2\tilde{G}_S^{01}))e_{12}$$
$$S_{GP}^{31}e_{31} = \hat{m}^{31}((\gamma_1 \vee 1_2 \vee \gamma_3)\tilde{P}_S^{31} - (\gamma_3\beta_3\tilde{G}_S^{01} - \gamma_1\beta_1\tilde{G}_S^{03}))e_{31}$$
$$S_{GP}^{23}e_{23} = \hat{m}^{23}((1_1 \vee \gamma_2 \vee \gamma_3)\tilde{P}_S^{23} - (\gamma_2\beta_2\tilde{G}_S^{03} - \gamma_3\beta_3\tilde{G}_S^{02}))e_{23}$$

$$S_{GP}^{0j} = \hat{m}^{0j}(-\gamma_{123}\tilde{G}_S^{0j} + \gamma\beta \times \tilde{P}_S) \text{ for all } j = 1, 2, 3$$
$$S_{Gp}^{ij} = \hat{m}^{ij}(\gamma_{123}\tilde{P}_S^{ij} - \gamma\beta \times \tilde{G}_S) \text{ for all } i, j = 1, 2, 3 \text{ with } i \neq j$$

Electromagnetic Action

Starting off with only electric components of an action bivector field, we gain additional – magnetism related – components by investigating relativistic effects of the moving bivector field (in analogy to 3.1 "Electric and Magnetic Field"). If we then assume that these additional components pre-exist or exist independently from our electric field. We can write

$$\mathcal{S}_{EB} = \text{Constant}$$

$$\mathcal{S}_{EB} = -\mathcal{S}_B^{01}e_{01} - \mathcal{S}_B^{02}e_{02} - \mathcal{S}_B^{03}e_{03} + \mathcal{S}_E^{12}e_{12} + \mathcal{S}_E^{31}e_{31} + \mathcal{S}_E^{23}e_{23}$$

Static:

$$\mathcal{S}_{EB} = \text{Constant} = -\hat{q}^{01}B_S^{01}e_{01} - \hat{q}^{02}B_S^{02}e_{02} - \hat{q}^{03}B_S^{03}e_{03} + \hat{q}^{12}E_S^{12}e_{12} + \hat{q}^{31}E_S^{31}e_{31} + \hat{q}^{23}E_S^{23}e_{23}$$

Dynamic: In the static equations, replace

$$B_S \text{ with } \gamma_{123}\tilde{B}_S + \gamma\beta \times \tilde{E}_S$$
$$E_S \text{ with } \gamma_{123}\tilde{E}_S + \gamma\beta \times \tilde{B}_S$$

This leads to:

$$S_{EB}^{01}e_{01} = \hat{q}^{01}(-(1_1 \vee \gamma_2 \vee \gamma_3)\tilde{B}_S^{01} + (\gamma_3\beta_3\tilde{E}_S^{31} - \gamma_2\beta_2\tilde{E}_S^{12}))e_{01}$$
$$S_{EB}^{02}e_{02} = \hat{q}^{02}(-(\gamma_1 \vee 1_2 \vee \gamma_3)\tilde{B}_S^{02} + (\gamma_1\beta_1\tilde{E}_S^{12} - \gamma_3\beta_3\tilde{E}_S^{23}))e_{02}$$
$$S_{EB}^{03}e_{03} = \hat{q}^{03}(-(\gamma_1 \vee \gamma_2 \vee 1_3)\tilde{B}_S^{03} + (\gamma_2\beta_2\tilde{E}_S^{23} - \gamma_1\beta_1\tilde{E}_S^{31}))e_{03}$$
$$S_{EB}^{12}e_{12} = \hat{q}^{12}((\gamma_1 \vee \gamma_2 \vee 1_3)\tilde{E}_S^{12} - (\gamma_1\beta_1\tilde{B}_S^{02} - \gamma_2\beta_2\tilde{B}_S^{01}))e_{12}$$
$$S_{EB}^{31}e_{31} = \hat{q}^{31}((\gamma_1 \vee 1_2 \vee \gamma_3)\tilde{E}_S^{31} - (\gamma_3\beta_3\tilde{B}_S^{01} - \gamma_1\beta_1\tilde{B}_S^{03}))e_{31}$$
$$S_{EB}^{23}e_{23} = \hat{q}^{23}((1_1 \vee \gamma_2 \vee \gamma_3)\tilde{E}_S^{23} - (\gamma_2\beta_2\tilde{B}_S^{03} - \gamma_3\beta_3\tilde{B}_S^{02}))e_{23}$$

$$S_{EB}^{0j} = \hat{q}^{0j}(-\gamma_{123}\tilde{B}_S^{0j} + \gamma\beta \times \tilde{E}_S) \text{ for all } j = 1, 2, 3$$
$$S_{EB}^{ij} = \hat{q}^{ij}(\gamma_{123}\tilde{E}_S^{ij} - \gamma\beta \times \tilde{B}_S) \text{ for all } i, j = 1, 2, 3 \text{ with } i \neq j$$

5.2 Derivative Level of Energy, Momentum, and Angular Momentum

With $S^{01} := -S^{01}$, $S^{02} := -S^{02}$, and $S^{03} := -S^{03}$

$$
\begin{aligned}
\mathring{\nabla}\mathcal{S} = 0 = \\
-(d_1\mathcal{S}^{01} + d_2\mathcal{S}^{02} + d_3\mathcal{S}^{03})e_0 &\text{ (-Momentum)} \\
-(-d_0\mathcal{S}^{01} - (d_2\mathcal{S}^{12} - d_3\mathcal{S}^{31}))e_1 &\text{ (Energy - Angular Momentum)} \\
-(-d_0\mathcal{S}^{02} - (d_3\mathcal{S}^{23} - d_1\mathcal{S}^{12}))e_2 &\text{ (Energy - Angular Momentum)} \\
-(-d_0\mathcal{S}^{03} - (d_1\mathcal{S}^{31} - d_2\mathcal{S}^{23}))e_3 &\text{ (Energy - Angular Momentum)} \\
-(d_0\mathcal{S}^{12} - (d_1\mathcal{S}^{02} - d_2\mathcal{S}^{01}))e_{012} &\text{ (Energy + Angular Momentum)} \\
-(d_0\mathcal{S}^{31} - (d_3\mathcal{S}^{01} - d_1\mathcal{S}^{03}))e_{031} &\text{ (Energy + Angular Momentum)} \\
-(d_0\mathcal{S}^{23} - (d_2\mathcal{S}^{03} - d_3\mathcal{S}^{02}))e_{023} &\text{ (Energy + Angular Momentum)} \\
-(-d_1\mathcal{S}^{23} - d_2\mathcal{S}^{31} - d_3\mathcal{S}^{12})e_{123} &\text{ (-Momentum)}
\end{aligned}
$$

Each $\mathcal{S}^{0j}$ can be replaced by S_G^{0j} or S_B^{0j}, $m^{0j}G_S^{0j}$ or $q^{0j}B_S^{0j}$,
each $\mathcal{S}^{ij}$ can be replaced by S_P^{ij} or S_E^{ij}, $m^{ij}P_S^{ij}$ or $q^{ij}E_S^{ij}$.

Separating the masses/charges and their action fields from each other generates a proportionality factor α (or its reciprocal value $\frac{1}{\alpha}$), which relates the different units of the charges/masses and the fields to each other. Examples for this separation are given below.

Gravitational Energy $\mathcal{E}$, Momentum $\mathcal{P}$, and Angular Momentum $\mathcal{A}$

$$\mathring{\nabla}\mathcal{S}_{GP} = 0$$

Static:

$$
\begin{aligned}
-(d_1G_S^{01} + d_2G_S^{02} + d_3G_S^{03})e_0 &= \frac{1}{\alpha_{GP}}(d_1m^{01} + d_2m^{02} + d_3m^{03})e_0 \\
-(-d_0G_S^{01} - (d_2P_S^{12} - d_3P_S^{31}))e_1 &= \frac{1}{\alpha_{GP}}(-d_0m^{01} - (d_2m^{12} - d_3m^{31}))e_1 \\
-(-d_0G_S^{02} - (d_3P_S^{23} - d_1P_S^{12}))e_2 &= \frac{1}{\alpha_{GP}}(-d_0m^{02} - (d_3m^{23} - d_1m^{12}))e_2 \\
-(-d_0G_S^{03} - (d_1P_S^{31} - d_2P_S^{23}))e_3 &= \frac{1}{\alpha_{GP}}(-d_0m^{03} - (d_1m^{31} - d_2m^{23}))e_3 \\
-(d_0P_S^{12} - (d_1G_S^{02} - d_2G_S^{01}))e_{012} &= \frac{1}{\alpha_{GP}}(d_0m^{12} - (d_1m^{02} - d_2m^{01}))e_{012} \\
-(d_0P_S^{31} - (d_3G_S^{01} - d_1G_S^{03}))e_{031} &= \frac{1}{\alpha_{GP}}(d_0m^{31} - (d_3m^{01} - d_1m^{03}))e_{031} \\
-(d_0P_S^{23} - (d_2G_S^{03} - d_3G_S^{02}))e_{023} &= \frac{1}{\alpha_{GP}}(d_0m^{23} - (d_2m^{03} - d_3m^{02}))e_{023} \\
-(-d_1P_S^{23} - d_2P_S^{31} - d_3P_S^{12})e_{123} &= -\frac{1}{\alpha_{GP}}(d_1m^{23} + d_2m^{31} + d_3m^{12})e_{123}
\end{aligned}
$$

In short notation with unobserved terms grayed out

$$
\begin{aligned}
-\nabla \cdot \epsilon_G &= \frac{1}{\alpha_{GP}}(\nabla \cdot m^{0j}) \\
d_0 G_S + \nabla \times P_S &= \frac{1}{\alpha_{GP}}(-d_0 m^{0j} - \nabla \times m^{ij}) \\
-d_0 P_S + \nabla \times G_S &= \frac{1}{\alpha_{GP}}(d_0 m^{ij} - \nabla \times m^{0j}) \\
\nabla \cdot P_S &= -\frac{1}{\alpha_{GP}}(\nabla \cdot m^{ij})
\end{aligned}
$$

Lines 1 and 4 of the short notation represent momentum $\mathcal{P}$.
Lines 2 and 3 of the short notation represent energy $\mathcal{E}$ and angular momentum/spin.

Dynamic: In the static equations, replace

$$
\begin{aligned}
G_S &\text{ with } \gamma_{123}\tilde{G}_S + \gamma\beta \times \tilde{P}_S \\
P_S &\text{ with } \gamma_{123}\tilde{P}_S + \gamma\beta \times \tilde{G}_S
\end{aligned}
$$

Electromagnetic Energy $\mathcal{E}$, Momentum $\mathcal{P}$, and Angular Momentum $\mathcal{A}$

$$\mathring{\nabla}\mathcal{S}_{EB} = 0$$

Static:

$$
\begin{aligned}
-(d_1 B_S^{01} + d_2 B_S^{02} + d_3 B_S^{03})e_0 &= \frac{1}{\alpha_{EB}}(d_1 q_{BS}^{01} + d_2 q_{BS}^{02} + d_3 q_{BS}^{03})e_0 \\
-(-d_0 B_S^{01} - (d_2 E_S^{12} - d_3 E_S^{31}))e_1 &= \frac{1}{\alpha_{EB}}(-d_0 q_{BS}^{01} - (d_2 q_{ES}^{12} - d_3 q_{ES}^{31}))e_1 \\
-(-d_0 B_S^{02} - (d_3 E_S^{23} - d_1 E_S^{12}))e_2 &= \frac{1}{\alpha_{EB}}(-d_0 q_{BS}^{02} - (d_3 q_{ES}^{23} - d_1 q_{ES}^{12}))e_2 \\
-(-d_0 B_S^{03} - (d_1 E_S^{31} - d_2 E_S^{23}))e_3 &= \frac{1}{\alpha_{EB}}(-d_0 q_{BS}^{03} - (d_1 q_{ES}^{31} - d_2 q_{ES}^{23}))e_3 \\
-(d_0 E_S^{12} - (d_1 B_S^{02} - d_2 B_S^{01}))e_{012} &= \frac{1}{\alpha_{EB}}(d_0 q_{ES}^{12} - (d_1 q_{BS}^{02} - d_2 q_{BS}^{01}))e_{012} \\
-(d_0 E_S^{31} - (d_3 B_S^{01} - d_1 B_S^{03}))e_{031} &= \frac{1}{\alpha_{EB}}(d_0 q_{ES}^{31} - (d_3 q_{BS}^{01} - d_1 q_{BS}^{03}))e_{031} \\
-(d_0 E_S^{23} - (d_2 B_S^{03} - d_3 B_S^{02}))e_{023} &= \frac{1}{\alpha_{EB}}(d_0 q_{ES}^{23} - (d_2 q_{BS}^{03} - d_3 q_{BS}^{02}))e_{023} \\
-(-d_1 E_S^{23} - d_2 E_S^{31} - d_3 E_S^{12})e_{123} &= -\frac{1}{\alpha_{EB}}(d_1 q_{ES}^{23} + d_2 q_{ES}^{31} + d_3 q_{ES}^{12})e_{123}
\end{aligned}
$$

In short notation with unobserved terms

$$
\begin{aligned}
-\nabla \cdot B_S &= \frac{1}{\alpha_{EB}}(\nabla \cdot q^{0j}) \\
d_0 B_S + \nabla \times E_S &= \frac{1}{\alpha_{EB}}(-d_0 q^{0j} - \nabla \times q^{ij}) \\
-d_0 E_S + \nabla \times B_S &= \frac{1}{\alpha_{EB}}(d_0 q^{ij} - \nabla \times q^{0j}) \\
\nabla \cdot E_S &= -\frac{1}{\alpha_{EB}}(\nabla \cdot q^{ij})
\end{aligned}
$$

Lines 1 and 4 of the short notation represent momentum $\mathcal{P}$.
Lines 2 and 3 of the short notation represent energy $\mathcal{E}$ and angular momentum/spin.

Dynamic: In the static equations, replace

$$
\begin{aligned}
&B_S \text{ with } \gamma_{123}\tilde{B}_S + \gamma\beta \times \tilde{E}_S \\
&E_S \text{ with } \gamma_{123}\tilde{E}_S + \gamma\beta \times \tilde{B}_S
\end{aligned}
$$

5.3 Derivative Level of Forces and Fields

$$\mathring{\nabla}\left(\mathring{\nabla}(\mathcal{S})\right) = \mathring{\nabla}(\mathcal{E} + \mathcal{P}) = 0$$

$$\mathring{\nabla}\left(\mathring{\nabla}(\mathcal{S})\right) = (\mathring{\nabla} \cdot \mathring{\nabla})(-\mathcal{S}^{01}e_{01} - \mathcal{S}^{02}e_{02} - \mathcal{S}^{03}e_{03} + \mathcal{S}^{12}e_{12} + \mathcal{S}^{31}e_{31} + \mathcal{S}^{23}e_{23})$$

Note that each component $(\mathring{\nabla} \cdot \mathring{\nabla})\mathcal{S}^{ij}e_{ij}$ represents a wave function

$$(d_0 d_0 \mathcal{S}^{ij} - d_1 d_1 \mathcal{S}^{ij} - d_2 d_2 \mathcal{S}^{ij} - d_3 d_3 \mathcal{S}^{ij})e_{ij} = (\frac{1}{c^2}d_t d_t \mathcal{S}^{ij} - d_x d_x S^{ij} - d_y d_y \mathcal{S}^{ij} - d_z d_z \mathcal{S}^{ij})e_{ij}$$

In analogy to the contraction of the Riemann Tensor to the Ricci Tensor, we can set

$$(d_0 d_0 \mathcal{S}^{ij} - d_1 d_1 \mathcal{S}^{ij} - d_2 d_2 \mathcal{S}^{ij} - d_3 d_3 \mathcal{S}^{ij})e_{ij} = \mathcal{F}^{ij}e_{ij}$$

$$\mathring{\nabla}\left(\mathring{\nabla}(\mathcal{S})\right) = \mathcal{F} = -\mathcal{F}^{01}e_{01} - \mathcal{F}^{02}e_{02} - \mathcal{F}i^{03}e_{03} + \mathcal{F}^{12}e_{12} + \mathcal{F}^{31}e_{31} + \mathcal{F}^{23}e_{23}$$

Gravitational Forces and Fields

$$(\mathring{\nabla} \cdot \mathring{\nabla})(-\mathcal{S}_G^{01}e_{01} - \mathcal{S}_G^{02}e_{02} - \mathcal{S}_G^{03}e_{03} + \mathcal{S}_P^{12}e_{12} + \mathcal{S}_P^{31}e_{31} + \mathcal{S}_P^{23}e_{23}) = 0$$

Static:

$$\mathcal{F}_{GP} = -\hat{m}^{01}G^{01}e_{01} - \hat{m}^{02}G^{02}e_{02} - \hat{m}^{03}G^{03}e_{03} + \hat{m}^{12}P^{12}e_{12} + \hat{m}^{31}P^{31}e_{31} + \hat{m}^{23}P^{23}e_{23}$$

Dynamic: In the static equations, replace

$$G \text{ with } \gamma_{123}\tilde{G} + \gamma\beta \times \tilde{P}$$
$$P \text{ with } \gamma_{123}\tilde{P} + \gamma\beta \times \tilde{G}$$

Electromagnetic Forces and Fields

$$(\mathring{\nabla} \cdot \mathring{\nabla})(-\mathcal{S}^{01}_{EB}e_{01} - \mathcal{S}^{02}_{EB}e_{02} - \mathcal{S}^{03}_{EM}e_{03} + \mathcal{S}^{12}_{EB}e_{12} + \mathcal{S}^{31}_{EB}e_{31} + \mathcal{S}^{23}_{EB}e_{23}) = 0$$

Static:

$$\mathcal{F}_{EB} = -\hat{q}^{01}B^{01}e_{01} - \hat{q}^{02}B^{02}e_{02} - \hat{q}^{03}B^{03}e_{03} + \hat{q}^{12}E^{12}e_{12} + \hat{q}^{31}E^{31}e_{31} + \hat{q}^{2}3E^{23}e_{23}$$

Dynamic: In the static equations, replace

$$B \text{ with } \gamma_{123}\tilde{B} + \gamma\beta \times \tilde{E}$$
$$E \text{ with } \gamma_{123}\tilde{E} + \gamma\beta \times \tilde{B}$$

5.4 Derivative Level of Maxwell's Equations and Action Densities

$$\mathring{\nabla}\left(\mathring{\nabla}\left(\mathring{\nabla}\left(\mathcal{S}\right)\right)\right) = \mathring{\nabla}\left(\mathring{\nabla}\left(\mathcal{E} + \mathcal{P}\right)\right) = \mathring{\nabla}\left(\mathcal{F}\right) = 0$$

The first derivative of a force bivector field $\mathring{\nabla}\mathcal{F} = \mathring{\nabla}(-\mathcal{F}^{01}e_{01} - \mathcal{F}^{02}e_{02} - \mathcal{F}^{03}e_{03} + \mathcal{F}^{12}e_{12} + \mathcal{F}^{31}e_{31} + \mathcal{F}^{23}e_{23})$ is

$$\begin{aligned}
&\mathring{\nabla}\mathcal{F} = \\
&\quad + (d_1\mathcal{F}^{01} + d_2\mathcal{F}^{02} + d_3\mathcal{F}^{03})e_0 \\
&\quad + (-d_0\mathcal{F}^{01} - (d_2\mathcal{F}^{12} - d_3\mathcal{F}^{31}))e_1 \\
&\quad + (-d_0\mathcal{F}^{02} - (d_3\mathcal{F}^{23} - d_1\mathcal{F}^{12}))e_2 \\
&\quad + (-d_0\mathcal{F}^{03} - (d_1\mathcal{F}^{31} - d_2\mathcal{F}^{23}))e_3 \\
&\quad + (d_0\mathcal{F}^{12} - (d_1\mathcal{F}^{02} - d_2\mathcal{F}^{01}))e_{012} \\
&\quad + (d_0\mathcal{F}^{31} - (d_3\mathcal{F}^{01} - d_1\mathcal{F}^{03}))e_{031} \\
&\quad + (d_0\mathcal{F}^{23} - (d_2\mathcal{F}^{03} - d_3\mathcal{F}^{02}))e_{023} \\
&\quad + (-d_1\mathcal{F}^{23} - d_2\mathcal{F}^{31} - d_3\mathcal{F}^{12})e_{123}
\end{aligned}$$

Gravitational Equivalent to Maxwell's Equations

$$\mathring{\nabla}\mathcal{F}_{GP} = 0$$

Static:

$$
\begin{aligned}
-(d_1G^{01} + d_2G^{02} + d_3G^{03})e_0 &= \frac{1}{\epsilon_{GP}}(d_1Q_G^{01} + d_2Q_G^{02} + d_3Q_G^{03})e_0 \\
-(-d_0G^{01} - (d_2P^{12} - d_3P^{31}))e_1 &= \frac{1}{\epsilon_{GP}}(-d_0Q_G^{01} - (d_2Q_P^{12} - d_3Q_P^{31}))e_1 \\
-(-d_0G^{02} - (d_3P^{23} - d_1P^{12}))e_2 &= \frac{1}{\epsilon_{GP}}(-d_0Q_G^{02} - (d_3Q_P^{23} - d_1Q_P^{12}))e_2 \\
-(-d_0G^{03} - (d_1P^{31} - d_2P^{23}))e_3 &= \frac{1}{\epsilon_{GP}}(-d_0Q_G^{03} - (d_1Q_P^{31} - d_2Q_P^{23}))e_3 \\
-(d_0P^{12} - (d_1G^{02} - d_2G^{01}))e_{012} &= \frac{1}{\epsilon_{GP}}(d_0Q_P^{12} - (d_1Q_G^{02} - d_2Q_G^{01}))e_{012} \\
-(d_0P^{31} - (d_3G^{01} - d_1G^{03}))e_{031} &= \frac{1}{\epsilon_{GP}}(d_0Q_P^{31} - (d_3Q_G^{01} - d_1Q_G^{03}))e_{031} \\
-(d_0P^{23} - (d_2G^{03} - d_3G^{02}))e_{023} &= \frac{1}{\epsilon_{GP}}(d_0Q_P^{23} - (d_2Q_G^{03} - d_3Q_G^{02}))e_{023} \\
-(-d_1P^{23} - d_2P^{31} - d_3P^{12})e_{123} &= -\frac{1}{\epsilon_{GP}}(d_1Q_P^{23} + d_2Q_P^{31} + d_3Q_P^{12})e_{123}
\end{aligned}
$$

In short notation with unobserved terms

$$
\begin{aligned}
-\nabla \cdot G &= \frac{1}{\epsilon_{GP}}(\nabla \cdot Q_G) \\
d_0G + \nabla \times P &= \frac{1}{\epsilon_{GP}}(-d_0Q_G - \nabla \times Q_P) \\
-d_0P + \nabla \times G &= \frac{1}{\epsilon_{GP}}(d_0Q_P - \nabla \times Q_G) \\
\nabla \cdot P &= -\frac{1}{\epsilon_{GP}}(\nabla \cdot Q_P)
\end{aligned}
$$

Lines 1 and 4 of the short notation represent momentum $\mathcal{P}$.
Lines 2 and 3 of the short notation represent energy $\mathcal{E}$.

Dynamic: In the static equations, replace

$$
\begin{aligned}
G &\text{ with } \gamma_{123}\tilde{G} + \gamma\beta \times \tilde{P} \\
P &\text{ with } \gamma_{123}\tilde{P} + \gamma\beta \times \tilde{G} \\
Q_G &\text{ with } \gamma_{123}\tilde{Q}_G + \gamma\beta \times \tilde{Q}_P \\
Q_P &\text{ with } \gamma_{123}\tilde{Q}_P + \gamma\beta \times \tilde{Q}_G
\end{aligned}
$$

Maxwell's Equations

$$\mathring{\nabla}\mathcal{F}_{EB} = 0$$

Static (with $\epsilon_{EB} = \epsilon_0$):

$$
\begin{aligned}
-(d_1B^{01} + d_2B^{02} + d_3B^{03})e_0 &= \frac{1}{\epsilon_b}(d_1Q^{01} + d_2Q^{02} + d_3Q^{03})e_0 \\
-(-d_0B^{01} - (d_2E^{12} - d_3E^{31}))e_1 &= (-\frac{1}{\epsilon_b}d_0Q^{01} - \frac{1}{\epsilon_0}(d_2Q^{12} - d_3Q^{31}))e_1 \\
-(-d_0B^{02} - (d_3E^{23} - d_1E^{12}))e_2 &= (-\frac{1}{\epsilon_b}d_0Q^{02} - \frac{1}{\epsilon_0}(d_3Q^{23} - d_1Q^{12}))e_2 \\
-(-d_0B^{03} - (d_1E^{31} - d_2E^{23}))e_3 &= (-\frac{1}{\epsilon_b}d_0Q^{03} - \frac{1}{\epsilon_0}(d_1Q^{31} - d_2Q^{23}))e_3 \\
-(d_0E^{12} - (d_1B^{02} - d_2B^{01}))e_{012} &= (\frac{1}{\epsilon_0}d_0Q^{12} - \frac{1}{\epsilon_b}(d_1Q^{02} - d_2Q^{01}))e_{012} \\
-(d_0E^{31} - (d_3B^{01} - d_1B^{03}))e_{031} &= (\frac{1}{\epsilon_0}d_0Q^{31} - \frac{1}{\epsilon_b}(d_3Q^{01} - d_1Q^{03}))e_{031} \\
-(d_0E^{23} - (d_2B^{03} - d_3B^{02}))e_{023} &= (\frac{1}{\epsilon_0}d_0Q^{23} - \frac{1}{\epsilon_b}(d_2Q^{03} - d_3Q^{02}))e_{023} \\
-(-d_1E^{23} - d_2E^{31} - d_3E^{12})e_{123} &= -\frac{1}{\epsilon_0}(d_1Q^{23} + d_2Q^{31} + d_3Q^{12})e_{123}
\end{aligned}
$$

with unobserved terms

$$
\begin{aligned}
-\nabla \cdot B &= \frac{1}{\epsilon_0}(\nabla \cdot Q_B) \\
d_0B + \nabla \times E &= \frac{1}{\epsilon_0}(-d_0Q_B - \nabla \times Q_E) \\
-d_0E + \nabla \times B &= \frac{1}{\epsilon_0}(d_0Q_E + \nabla \times Q_B) \\
\nabla \cdot E &= -\frac{1}{\epsilon_0}(\nabla \cdot Q_E)
\end{aligned}
$$

Lines 1 and 4 of the short notation represent momentum $\mathcal{P}$.
Lines 2 and 3 of the short notation represent energy $\mathcal{E}$.

Dynamic: In the static equations, replace

$$
\begin{aligned}
B &\text{ with } \gamma_{123}\tilde{B} + \gamma\beta \times \tilde{E} \\
E &\text{ with } \gamma_{123}\tilde{E} + \gamma\beta \times \tilde{B} \\
Q_B &\text{ with } \gamma_{123}\tilde{Q}_B + \gamma\beta \times \tilde{Q}_E \\
Q_E &\text{ with } \gamma_{123}\tilde{Q}_E + \gamma\beta \times \tilde{Q}_B
\end{aligned}
$$

5.5 Derivative Level of General Relativity and Energy, Momentum and Stress Density

$$
\mathring{\nabla}\left(\mathring{\nabla}\left(\mathring{\nabla}\left(\mathring{\nabla}\left(\mathcal{S}\right)\right)\right)\right) = 0
$$

$$
(\mathring{\nabla} \cdot \mathring{\nabla})(\mathcal{F}^{01}e_{01} + \mathcal{F}^{02}e_{02} + \mathcal{F}^{03}e_{03} + \mathcal{F}^{12}e_{12} + \mathcal{F}^{31}e_{31} + \mathcal{F}^{23}e_{23}) = 0
$$

Note that each component $(\mathring{\nabla} \cdot \mathring{\nabla})\mathcal{F}^{ij} e_{ij}$ represents a wave function

$$(d_0 d_0 \mathcal{F}^{ij} - d_1 d_1 \mathcal{F}^{ij} - d_2 d_2 \mathcal{F}^{ij} - d_3 d_3 \mathcal{F}^{ij}) e_{ij} = (\frac{1}{c^2} d_t d_t \mathcal{F}^{ij} - d_x d_x F^{ij} - d_y d_y \mathcal{F}^{ij} - d_z d_z \mathcal{F}^{ij}) e_{ij}$$

In analogy to the contraction of the Riemann Tensor to the Ricci Tensor, we can set

$$(d_0 d_0 \mathcal{F}^{ij} - d_1 d_1 \mathcal{F}^{ij} - d_2 d_2 \mathcal{F}^{ij} - d_3 d_3 \mathcal{F}^{ij}) e_{ij} = \mathcal{R}^{ij} e_{ij}$$

$$\mathring{\nabla} \left(\mathring{\nabla} \left(\mathcal{F} \right) \right) = \mathcal{R} = \mathcal{R}^{01} e_{01} + \mathcal{R}^{02} e_{02} + \mathcal{R} i^{03} e_{03} + \mathcal{R}^{12} e_{12} + \mathcal{R}^{31} e_{31} + \mathcal{R}^{23} e_{23}$$

Gravitational Energy, Momentum and Stress Density

$$(\mathring{\nabla} \cdot \mathring{\nabla})(\mathcal{F}_{GP}^{01} e_{01} + \mathcal{F}_{GP}^{02} e_{02} + \mathcal{F}_{GP}^{03} e_{03} + \mathcal{F}_{GP}^{12} e_{12} + \mathcal{F}_{GP}^{31} e_{31} + \mathcal{F}_{GP}^{23} e_{23}) = 0$$

$$\begin{aligned}
(d_0 d_0 G^{01} - d_1 d_1 G^{01} - d_2 d_2 G^{01} - d_3 d_3 G^{01}) e_{01} &= -\frac{1}{\epsilon_{GP}} Q_G^{01} e_{01} \\
(d_0 d_0 G^{02} - d_1 d_1 G^{02} - d_2 d_2 G^{02} - d_3 d_3 G^{02}) e_{02} &= -\frac{1}{\epsilon_{GP}} Q_G^{02} e_{02} \\
(d_0 d_0 G^{03} - d_1 d_1 G^{03} - d_2 d_2 G^{03} - d_3 d_3 G^{01}) e_{03} &= -\frac{1}{\epsilon_{GP}} Q_G^{03} e_{03} \\
(d_0 d_0 P^{12} - d_1 d_1 P^{12} - d_2 d_2 P^{12} - d_3 d_3 P^{12}) e_{12} &= -\frac{1}{\epsilon_{GP}} Q_P^{12} e_{12} \\
(d_0 d_0 P^{31} - d_1 d_1 P^{31} - d_2 d_2 P^{31} - d_3 d_3 P^{31}) e_{31} &= -\frac{1}{\epsilon_{GP}} Q_P^{31} e_{31} \\
(d_0 d_0 P^{23} - d_1 d_1 P^{23} - d_2 d_2 P^{23} - d_3 d_3 P^{23}) e_{23} &= -\frac{1}{\epsilon_{GP}} Q_P^{23} e_{23}
\end{aligned}$$

$$\mathring{\nabla} \left(\mathring{\nabla} \left(\mathring{\nabla} \left(\mathring{\nabla} \left(\mathcal{S}_{GP} \right) \right) \right) \right) = \mathcal{R}_{GP}^{01} e_{01} + \mathcal{R}_{GP}^{02} e_{02} + \mathcal{R}_{GP}^{03} e_{03} + \mathcal{R}_{GP}^{12} e_{12} + \mathcal{R}_{GP}^{31} e_{31} + \mathcal{R}_{GP}^{23} e_{23}$$

Static:

$$\mathcal{R}_{GP} = q_G R^{01} e_{01} + q_G R^{02} e_{02} + q_G R^{03} e_{03} + q_G R^{12} e_{12} + q_G R^{31} e_{31} + q_G R^{23} e_{23}$$

Dynamic: In the static equations, replace

$$G \text{ with } \gamma_{123} \tilde{G} + \gamma \beta \times \tilde{P}$$
$$P \text{ with } \gamma_{123} \tilde{P} + \gamma \beta \times \tilde{G}$$

Electromagnetic Energy, Momentum and Stress Density

$$(\mathring{\nabla} \cdot \mathring{\nabla})(\mathcal{F}_{EB}^{01} e_{01} + \mathcal{F}_{EB}^{02} e_{02} + \mathcal{F}_{EB}^{03} e_{03} + \mathcal{F}_{EB}^{12} e_{12} + \mathcal{F}_{EB}^{31} e_{31} + \mathcal{F}_{EB}^{23} e_{23}) = 0$$

$$
\begin{aligned}
(d_0d_0B^{01} - d_1d_1B^{01} - d_2d_2B^{01} - d_3d_3B^{01})e_{01} &= -\frac{1}{\epsilon_{EB}}Q_B^{01}e_{01}\\
(d_0d_0B^{02} - d_1d_1B^{02} - d_2d_2B^{02} - d_3d_3B^{02})e_{02} &= -\frac{1}{\epsilon_{EB}}Q_B^{02}e_{02}\\
(d_0d_0B^{03} - d_1d_1B^{03} - d_2d_2B^{03} - d_3d_3B^{01})e_{03} &= -\frac{1}{\epsilon_{EB}}Q_B^{03}e_{03}\\
(d_0d_0E^{12} - d_1d_1E^{12} - d_2d_2E^{12} - d_3d_3E^{12})e_{12} &= -\frac{1}{\epsilon_{EB}}Q_E^{12}e_{12}\\
(d_0d_0E^{31} - d_1d_1E^{31} - d_2d_2E^{31} - d_3d_3E^{31})e_{31} &= -\frac{1}{\epsilon_{EB}}Q_E^{31}e_{31}\\
(d_0d_0E^{23} - d_1d_1E^{23} - d_2d_2E^{23} - d_3d_3E^{23})e_{23} &= -\frac{1}{\epsilon_{EB}}Q_E^{23}e_{23}
\end{aligned}
$$

$$
\mathring{\nabla}\left(\mathring{\nabla}\left(\mathcal{S}_{EB}\right)\right) = \mathcal{F}_{EB}^{01}e_{01} + \mathcal{F}_{EB}^{02}e_{02} + \mathcal{F}_{EB}i^{03}e_{03} + \mathcal{F}_{EB}^{12}e_{12} + \mathcal{F}_{EB}^{31}e_{31} + \mathcal{F}_{EB}^{23}e_{23}
$$

Static:

$$
\mathcal{F}_{EB} = q_EB^{01}e_{01} + q_EB^{02}e_{02} + q_EB^{03}e_{03} + q_EE^{12}e_{12} + q_EE^{31}e_{31} + q_EE^{23}e_{23}
$$

Dynamic: In the static equations, replace

$$
\begin{aligned}
&B \text{ with } \gamma_{123}\tilde{B} + \gamma\beta \times \tilde{E}\\
&E \text{ with } \gamma_{123}\tilde{E} + \gamma\beta \times \tilde{B}
\end{aligned}
$$

6 Conclusion and Outlook

This paper showed that many major laws, equations and even constants of electromagnetism and gravitation can be viewed as and derived from properties of spacetime. This is achieved by consecutively applying the spacetime derivative and utilizing the effects of movement on the elements of spacetime. Through the assumption that gravitational action depends on the vibrational movement of the electromagnetic (and possibly color) bivectors, gravitation and electromagnetism can be unified into one common framework or model. All the physics in this paper can be combined into the equation

$$
\mathring{\nabla}\left(\mathring{\nabla}\left(\mathring{\nabla}\left(\mathring{\nabla}\left(\mathcal{S}\right)\right)\right)\right) = \mathring{\nabla}^4\mathcal{S} = 0 \tag{59}
$$

Here, each pair of parentheses () represents a different level of derivation and therefore different physical laws and equations. The analogies between electromagnetism and gravitation introduces new ideas like a momentum action, force, and others. Moreover, the connections to quantum mechanics is present throughout.

This invites to ask new questions that deserve further research. Some of these questions have been mentioned earlier in this paper, some of them are repeated and some of them are added in the following short sections.

Gravitation is the result of "curved spacetime". Because of this, electromagnetism can also be viewed as being the result of "curved spacetime". Because of the analogies between electromagnetism and gravitation and both their tight connection to spacetime, it might be of some interest to investigate the idea that the atomic orbitals electrons move on are – in analogy to gravitation –

geodesics and electrons would therefore move around the nucleus without experiencing any force. This might be a solution to the problem that charged particles that are accelerated give off photons because of energy conservation. Classically one would therefore expect that electrons, bound in an atom by the electric force, should emit photons while moving around the atomic nucleus. This is however not the case. One existing possible attempt of explanation is the idea that an electron – or rather its quantum mechanical wave function – forms a standing wave around the nucleus. However, if electric force, just like gravitational force, is not really a force at all but is also caused by the curvature of spacetime, then no acceleration takes place. Note that the quantum mechanical description of the electric force by the exchange of virtual photons should not be affected by this different view of the "force".

Taking the interpretation of this paper's results even further, one can start to speculate even more. Einstein's field equations (57) $R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} + \lambda g^{\mu\nu} = \frac{8\pi G_n}{c^4}T^{\mu\nu}$ are often interpreted as "Energy and mass tell spacetime how to curve". However, if one takes the equality sign"=" literally, then one could say that energy and mass *are* curved spacetime. This opens up a completely new way to think about and explore physics, which might combine some aspects of particle physics and string theory.

Several ideas worth to research have been presented throughout this paper. Especially the much stronger than previously realized connections between classical physics and quantum mechanics suggest many areas for further research.

A Appendix Spacetime

A.1 Elements

Imaginary Unit I
The pseudoscalar from above is also called the imaginary unit I

$$\begin{aligned} e_{0123} &= I \\ e_{0123}^2 &= I^2 = -1 \\ -e_{0123}e_{0123} &= 1 \end{aligned}$$

This is a generalisation of imaginary "i" of complex numbers in 2D.

Metric Tensor
The metric of spacetime is given by the metric tensor $g_{\mu\nu}$. In a curved spacetime like in general relativity all elements $g_{\mu\nu}$ of the metric tensor can be non-zero. In a flat spacetime like in special relativity only the diagonal elements are non-zero and the metric tensor is sometimes called Minkowski tensor $\eta_{\mu\nu}$.

$$g_{\mu\nu} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}$$

Figure 5: The metric tensor in general relativity

$$\eta_{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Figure 6: The metric tensor in flat spacetime

Contraction

Two equal indices can be contracted. The respective signs for this contraction are defined by the Minkowski metric (+,-,-,-).

$$\begin{aligned} e_{00} &= +1 \\ e_{11} &= -1 \\ e_{22} &= -1 \\ e_{33} &= -1 \end{aligned}$$

More generally, if two adjacent indices are equal, they can be contracted

$$\begin{aligned} e_{\alpha 00\beta} &= +e_{\alpha\beta} \\ e_{\alpha ii\beta} &= -e_{\alpha\beta} (\text{ for } i \neq 0) \end{aligned}$$

Reordering

The order of two non-equal indices can be reversed.

$$e_{\alpha\beta} = -e_{\beta\alpha} \text{ for } (\alpha \neq \beta)$$

More generally, reversing the order of two adjacent, non-equal indices reverses the sign

$$e_{\mu\alpha\beta\nu} = -e_{\mu\beta\alpha\nu} \text{ for } (\alpha \neq \beta)$$

Note: The "order" of two adjacent, equal indices cannot be reversed.

Identities of Unit-Vectors and -Pseudovectors

$$\begin{aligned} e_0 &= -e_{0123}e_{0123}e_0 = +e_{0123}e_{00123} = Ie_{123} = Ie_{231} \\ e_1 &= -e_{0123}e_{0123}e_1 = -e_{0123}e_{01123} = Ie_{023} \\ e_2 &= -e_{0123}e_{0123}e_2 = +e_{0123}e_{01223} = -Ie_{013} = Ie_{031} \\ e_3 &= -e_{0123}e_{0123}e_3 = -e_{0123}e_{01233} = Ie_{012} \end{aligned}$$

$$\begin{aligned} e_0 &= Ie_{231}, \quad e_{231} = -Ie_0 \\ e_1 &= Ie_{023}, \quad e_{023} = -Ie_1 \\ e_2 &= Ie_{031}, \quad e_{031} = -Ie_2 \\ e_3 &= Ie_{012}, \quad e_{012} = -Ie_3 \end{aligned}$$

Identities of Unit-Bivectors

$$\begin{aligned} e_{01} &= -e_{0123}e_{0123}e_{01} = +e_{0123}e_{001123} = -Ie_{23} \\ e_{02} &= -e_{0123}e_{0123}e_{02} = -e_{0123}e_{001223} = +Ie_{13} = -Ie_{31} \\ e_{03} &= -e_{0123}e_{0123}e_{03} = +e_{0123}e_{001233} = -Ie_{12} \\ e_{12} &= -e_{0123}e_{0123}e_{12} = +e_{0123}e_{011223} = +Ie_{03} \\ e_{31} &= -e_{0123}e_{0123}e_{31} = +e_{0123}e_{011233} = +Ie_{02} \\ e_{23} &= -e_{0123}e_{0123}e_{23} = +e_{0123}e_{01} = +Ie_{01} \end{aligned}$$

Together with the anti unit-bivectors:

$$\begin{aligned}
e_{01} &= -e_{10} = +Ie_{32} = -Ie_{23} \\
e_{02} &= -e_{20} = +Ie_{13} = -Ie_{31} \\
e_{03} &= -e_{30} = +Ie_{21} = -Ie_{12} \\
e_{12} &= -e_{21} = -Ie_{30} = +Ie_{03} \\
e_{31} &= -e_{13} = -Ie_{20} = +Ie_{02} \\
e_{23} &= -e_{32} = -Ie_{10} = +Ie_{01}
\end{aligned}$$

Identities of imaginary unit-bivectors:

$$\begin{array}{ll}
Ie_{23} = -e_{01}, & Ie_{32} = -e_{10} \\
Ie_{31} = -e_{02}, & Ie_{13} = -e_{20} \\
Ie_{12} = -e_{03}, & Ie_{21} = -e_{30} \\
Ie_{03} = e_{12}, & Ie_{30} = e_{21} \\
Ie_{02} = e_{31}, & Ie_{20} = e_{13} \\
Ie_{01} = e_{23}, & Ie_{10} = e_{32}
\end{array}$$

Real and Imaginary Unit-Vectors and Unit-Bivectors

In this paper we often use the following identities from the previous sections:

Convert "space only" bivectors to imaginary "time and space" bivectors (with only one spatial direction):

$$\begin{aligned}
e_{12} &= Ie_{03} \\
e_{31} &= Ie_{02} \\
e_{23} &= Ie_{01}
\end{aligned}$$

Convert trivectors into imaginary vectors (with only one spatial direction):

$$\begin{aligned}
e_{231} &= -Ie_0 \\
e_{023} &= -Ie_1 \\
e_{031} &= -Ie_2 \\
e_{012} &= -Ie_3
\end{aligned}$$

To complete the symmetry, remember

$$e_{0123} = I$$

Geometric Product

As an example for the usage of some of the above mentioned rules and identities, we can derive the geometric product. Besides addition, it is considered to be one of the two fundamental operations of Geometric Algebra (also called Clifford Algebra) [8].

Consider a vector/tensor multiplication $\vec{v}\vec{w}$ in three dimensions. Example with $\vec{v} = (v^1 e_1 + v^2 e_2 + v^3 e_3)$ and $\vec{w} = (w^1 e_1 + w^2 e_2 + w^3 e_3)$:

$$
\begin{aligned}
&\vec{v}\vec{w} \\
&= (v^1 e_1 + v^2 e_2 + v^3 e_3)(w^1 e_1 + w^2 e_2 + w^3 e_3) \\
&= +v^1 w^1 e_{11} + v^1 w^2 e_{12} + v^1 w^3 e_{13} + v^2 w^1 e_{21} + v^2 w^2 e_{22} + v^2 w^3 e_{23} + v^3 w^1 e_{31} + v^3 w^2 e_{32} + v^3 w^3 e_{33} \\
&= +v^1 w^1 e_{11} + v^2 w^2 e_{22} + v^3 w^3 e_{33} + v^1 w^2 e_{12} + v^2 w^1 e_{21} + v^3 w^1 e_{31} + v^1 w^3 e_{13} + v^2 w^3 e_{23} + v^3 w^2 e_{32} \\
&= +v^1 w^1 e_{11} + v^2 w^2 e_{22} + v^3 w^3 e_{33} + v^1 w^2 e_{12} - v^2 w^1 e_{12} + v^3 w^1 e_{31} - v^1 w^3 e_{31} + v^2 w^3 e_{23} - v^3 w^2 e_{23} \\
&= \vec{v} \cdot \vec{w} + \vec{v} \times \vec{w}
\end{aligned}
$$

The result of this multiplication is called "Geometric Product". It is described as:

$$
\begin{aligned}
&\text{Geometric Product} = \text{Inner Product} + \text{Outer Product} \\
&\text{Geometric Product} = \text{Scalar Product} + \text{Cross (Wedge) Product}
\end{aligned}
$$

$\vec{v} \cdot \vec{w}$ is equivalent to the scalar product, the wedge product $\vec{v} \wedge \vec{w}$ is sometimes considered the 4D generalisation of the cross product $\vec{v} \times \vec{w}$. For unit vectors, unit bivectors etc., this product effectively contracts or expands the indices.

A.2 First and Second Derivative of Bivector Fields

First Derivative of a Bivector Field

Using the spacetime derivative $\mathring{\nabla} = d_0 e_0 - d_1 e_1 - d_2 e_2 - d_3 e_3$ from equation (1) we can get the first derivative of a bivector field

$$
\begin{aligned}
\mathring{\nabla} C^{\alpha\beta} e_{\alpha\beta} = & \\
& + d_0 C^{01} e_0 e_{01} + d_0 C^{02} e_0 e_{02} + d_0 C^{03} e_0 e_{03} + d_0 C^{12} e_0 e_{12} + d_0 C^{31} e_0 e_{31} + d_0 C^{23} e_0 e_{23} \\
& - d_1 C^{01} e_1 e_{01} - d_1 C^{02} e_1 e_{02} - d_1 C^{03} e_1 e_{03} - d_1 C^{12} e_1 e_{12} - d_1 C^{31} e_1 e_{31} - d_1 C^{23} e_1 e_{23} \\
& - d_2 C^{01} e_2 e_{01} - d_2 C^{02} e_2 e_{02} - d_2 C^{03} e_2 e_{03} - d_2 C^{12} e_2 e_{12} - d_2 C^{31} e_2 e_{31} - d_2 C^{23} e_2 e_{23} \\
& - d_3 C^{01} e_3 e_{01} - d_3 C^{02} e_3 e_{02} - d_3 C^{03} e_3 e_{03} - d_3 C^{12} e_3 e_{12} - d_3 C^{31} e_3 e_{31} - d_3 C^{23} e_3 e_{23} \\
& \\
& = \\
& + d_0 C^{01} e_1 + d_0 C^{02} e_2 + d_0 C^{03} e_3 + d_0 C^{12} e_{012} + d_0 C^{31} e_{031} + d_0 C^{23} e_{023} \\
& - d_1 C^{01} e_0 + d_1 C^{02} e_{012} - d_1 C^{03} e_{031} + d_1 C^{12} e_2 - d_1 C^{31} e_3 - d_1 C^{23} e_{123} \\
& - d_2 C^{01} e_{012} - d_2 C^{02} e_0 + d_2 C^{03} e_{023} - d_2 C^{12} e_1 - d_2 C^{31} e_{123} + d_2 C^{23} e_3 \\
& + d_3 C^{01} e_{031} - d_3 C^{02} e_{023} - d_3 C^{03} e_0 - d_3 C^{12} e_{123} + d_3 C^{31} e_1 - d_3 C^{23} e_2 \\
& \\
& = \\
& - d_1 C^{01} e_0 - d_2 C^{02} e_0 - d_3 C^{03} e_0 \\
& + d_0 C^{01} e_1 - d_2 C^{12} e_1 + d_3 C^{31} e_1 \\
& + d_0 C^{02} e_2 - d_3 C^{23} e_2 + d_1 C^{12} e_2 \\
& + d_0 C^{03} e_3 - d_1 C^{31} e_3 + d_2 C^{23} e_3 \\
& + d_0 C^{12} e_{012} - d_2 C^{01} e_{012} + d_1 C^{02} e_{012} \\
& + d_0 C^{31} e_{031} - d_1 C^{03} e_{031} + d_3 C^{01} e_{031} \\
& + d_0 C^{23} e_{023} - d_3 C^{02} e_{023} + d_2 C^{03} e_{023} \\
& - d_1 C^{23} e_{123} - d_2 C^{31} e_{123} - d_3 C^{12} e_{123}
\end{aligned}
$$

In general, the first derivative of a bivector field therefore is

$$
\begin{aligned}
\mathring{\nabla} C^{\alpha\beta} e_{\alpha\beta} = & \\
& + (-d_1 C^{01} - d_2 C^{02} - d_3 C^{03}) e_0 \\
& + (d_0 C^{01} - (d_2 C^{12} - d_3 C^{31})) e_1 \\
& + (d_0 C^{02} - (d_3 C^{23} - d_1 C^{12})) e_2 \\
& + (d_0 C^{03} - (d_1 C^{31} - d_2 C^{23})) e_3 \\
& + (d_0 C^{12} + (d_1 C^{02} - d_2 C^{01})) e_{012} \\
& + (d_0 C^{31} + (d_3 C^{01} - d_1 C^{03})) e_{031} \\
& + (d_0 C^{23} + (d_2 C^{03} - d_3 C^{02})) e_{023} \\
& + (-d_1 C^{23} - d_2 C^{31} - d_3 C^{12}) e_{123}
\end{aligned}
\tag{60}
$$

When adhering to the $\pm$ signs suggested in 2.4 we exchange C^{0j} with $-C_*^{0j}$. This makes it easier to compare this equation with other equations throughout the paper.

$$
\begin{aligned}
&\mathring{\nabla} C^{\alpha\beta} e_{\alpha\beta} = \\
&\quad + (+d_1 C_*^{01} + d_2 C_*^{02} + d_3 C_*^{03}) e_0 \\
&\quad + (-d_0 C_*^{01} - (d_2 C^{12} - d_3 C^{31})) e_1 \\
&\quad + (-d_0 C_*^{02} - (d_3 C^{23} - d_1 C^{12})) e_2 \\
&\quad + (-d_0 C_*^{03} - (d_1 C^{31} - d_2 C^{23})) e_3 \\
&\quad + (d_0 C^{12} - (d_1 C_*^{02} - d_2 C_*^{01})) e_{012} \\
&\quad + (d_0 C^{31} - (d_3 C_*^{01} - d_1 C_*^{03})) e_{031} \\
&\quad + (d_0 C^{23} - (d_2 C_*^{03} - d_3 C_*^{02})) e_{023} \\
&\quad + (-d_1 C^{23} - d_2 C^{31} - d_3 C^{12}) e_{123}
\end{aligned}
$$

Second Derivative of a Bivector Field

Once again applying the spacetime derivative to the previous result (60) gives us the second derivative of a bivector field

$$
\begin{aligned}
&\mathring{\nabla}\mathring{\nabla} C^{\alpha\beta} e_{\alpha\beta} = \\
&\quad \mathring{\nabla}(\\
&\qquad - (d_1 C^{01} + d_2 C^{02} + d_3 C^{03}) e_0 \\
&\qquad + (d_0 C^{01} - (d_2 C^{12} - d_3 C^{31})) e_1 \\
&\qquad + (d_0 C^{02} - (d_3 C^{23} - d_1 C^{12})) e_2 \\
&\qquad + (d_0 C^{03} - (d_1 C^{31} - d_2 C^{23})) e_3 \\
&\qquad + (d_0 C^{12} + (d_1 C^{02} - d_2 C^{01})) e_{012} \\
&\qquad + (d_0 C^{31} + (d_3 C^{01} - d_1 C^{03})) e_{031} \\
&\qquad + (d_0 C^{23} + (d_2 C^{03} - d_3 C^{02})) e_{023} \\
&\qquad - (d_1 C^{23} + d_2 C^{31} + d_3 C^{12}) e_{123} \\
&\qquad)
\end{aligned}
$$

$$=$$
$$+ (-d_0d_1C^{01} - d_0d_2C^{02} - d_0d_3C^{03})e_{00}$$
$$+ (d_0d_0C^{01} - (d_0d_2C^{12} - d_0d_3C^{31}))e_{01}$$
$$+ (d_0d_0C^{02} - (d_0d_3C^{23} - d_0d_1C^{12}))e_{02}$$
$$+ (d_0d_0C^{03} - (d_0d_1C^{31} - d_0d_2C^{23}))e_{03}$$
$$+ (d_0d_0C^{12} + (d_0d_1C^{02} - d_0d_2C^{01}))e_{0012}$$
$$+ (d_0d_0C^{31} + (d_0d_3C^{01} - d_0d_1C^{03}))e_{0031}$$
$$+ (d_0d_0C^{23} + (d_0d_2C^{03} - d_0d_3C^{02}))e_{0023}$$
$$+ (-d_0d_1C^{23} - d_0d_2C^{31} - d_0d_3C^{12})e_{0123}$$

$$+ (+d_1d_1C^{01} + d_1d_2C^{02} + d_1d_3C^{03})e_{10}$$
$$+ (-d_1d_0C^{01} + (d_1d_2C^{12} - d_1d_3C^{31}))e_{11}$$
$$+ (-d_1d_0C^{02} + (d_1d_3C^{23} - d_1d_1C^{12}))e_{12}$$
$$+ (-d_1d_0C^{03} + (d_1d_1C^{31} - d_1d_2C^{23}))e_{13}$$
$$+ (-d_1d_0C^{12} - (d_1d_1C^{02} - d_1d_2C^{01}))e_{1012}$$
$$+ (-d_1d_0C^{31} - (d_1d_3C^{01} - d_1d_1C^{03}))e_{1031}$$
$$+ (-d_1d_0C^{23} - (d_1d_2C^{03} - d_1d_3C^{02}))e_{1023}$$
$$+ (+d_1C^{23} + d_1d_2C^{31} + d_1d_3C^{12})e_{1123}$$

$$+ (+d_2d_1C^{01} + d_2d_2C^{02} + d_2d_3C^{03})e_{20}$$
$$+ (-d_2d_0C^{01} + (d_2d_2C^{12} - d_2d_3C^{31}))e_{21}$$
$$+ (-d_2d_0C^{02} + (d_2d_3C^{23} - d_2d_1C^{12}))e_{22}$$
$$+ (-d_2d_0C^{03} + (d_2d_1C^{31} - d_2d_2C^{23}))e_{23}$$
$$+ (-d_2d_0C^{12} - (d_2d_1C^{02} - d_2d_2C^{01}))e_{2012}$$
$$+ (-d_2d_0C^{31} - (d_2d_3C^{01} - d_2d_1C^{03}))e_{2031}$$
$$+ (-d_2d_0C^{23} - (d_2d_2C^{03} - d_2d_3C^{02}))e_{2023}$$
$$+ (+d_2d_1C^{23} + d_2d_2C^{31} + d_2d_3C^{12})e_{2123}$$

$$+ (+d_3d_1C^{01} + d_3d_2C^{02} + d_3d_3C^{03})e_{30}$$
$$+ (-d_3d_0C^{01} + (d_3d_2C^{12} - d_3d_3C^{31}))e_{31}$$
$$+ (-d_3d_0C^{02} + (d_3d_3C^{23} - d_3d_1C^{12}))e_{32}$$
$$+ (-d_3d_0C^{03} + (d_3d_1C^{31} - d_3d_2C^{23}))e_{33}$$
$$+ (-d_3d_0C^{12} - (d_3d_1C^{02} - d_3d_2C^{01}))e_{3012}$$
$$+ (-d_3d_0C^{31} - (d_3d_3C^{01} - d_3d_1C^{03}))e_{3031}$$
$$+ (-d_3d_0C^{23} - (d_3d_2C^{03} - d_3d_3C^{02}))e_{3023}$$
$$+ (+d_3d_1C^{23} + d_3d_2C^{31} + d_3d_3C^{12})e_{3123}$$

$$
\begin{aligned}
&= \\
&+ (-d_0 d_1 C^{01} - d_0 d_2 C^{02} - d_0 d_3 C^{03}) e_{00} \\
&+ (d_0 d_0 C^{01} - (d_0 d_2 C^{12} - d_0 d_3 C^{31})) e_{01} \\
&+ (d_0 d_0 C^{02} - (d_0 d_3 C^{23} - d_0 d_1 C^{12})) e_{02} \\
&+ (d_0 d_0 C^{03} - (d_0 d_1 C^{31} - d_0 d_2 C^{23})) e_{03} \\
&+ (d_0 d_0 C^{12} + (d_0 d_1 C^{02} - d_0 d_2 C^{01})) e_{12} \\
&+ (d_0 d_0 C^{31} + (d_0 d_3 C^{01} - d_0 d_1 C^{03})) e_{31} \\
&+ (d_0 d_0 C^{23} + (d_0 d_2 C^{03} - d_0 d_3 C^{02})) e_{23} \\
&+ (-d_0 d_1 C^{23} - d_0 d_2 C^{31} - d_0 d_3 C^{12}) e_{0123}
\end{aligned}
$$

$$
\begin{aligned}
&+ (+d_1 d_1 C^{01} + d_1 d_2 C^{02} + d_1 d_3 C^{03}) e_{10} \\
&+ (-d_1 d_0 C^{01} + (d_1 d_2 C^{12} - d_1 d_3 C^{31})) e_{11} \\
&+ (-d_1 d_0 C^{02} + (d_1 d_3 C^{23} - d_1 d_1 C^{12})) e_{12} \\
&+ (-d_1 d_0 C^{03} + (d_1 d_1 C^{31} - d_1 d_2 C^{23})) e_{13} \\
&+ (-d_1 d_0 C^{12} - (d_1 d_1 C^{02} - d_1 d_2 C^{01})) e_{02} \\
&+ (-d_1 d_0 C^{31} - (d_1 d_3 C^{01} - d_1 d_1 C^{03}))(-e_{03}) \\
&+ (-d_1 d_0 C^{23} - (d_1 d_2 C^{03} - d_1 d_3 C^{02}))(-e_{0123}) \\
&+ (+d_1 C^{23} + d_1 d_2 C^{31} + d_1 d_3 C^{12})(-e_{23})
\end{aligned}
$$

$$
\begin{aligned}
&+ (+d_2 d_1 C^{01} + d_2 d_2 C^{02} + d_2 d_3 C^{03}) e_{20} \\
&+ (-d_2 d_0 C^{01} + (d_2 d_2 C^{12} - d_2 d_3 C^{31})) e_{21} \\
&+ (-d_2 d_0 C^{02} + (d_2 d_3 C^{23} - d_2 d_1 C^{12})) e_{22} \\
&+ (-d_2 d_0 C^{03} + (d_2 d_1 C^{31} - d_2 d_2 C^{23})) e_{23} \\
&+ (-d_2 d_0 C^{12} - (d_2 d_1 C^{02} - d_2 d_2 C^{01}))(-e_{01}) \\
&+ (-d_2 d_0 C^{31} - (d_2 d_3 C^{01} - d_2 d_1 C^{03}))(-e_{0123}) \\
&+ (-d_2 d_0 C^{23} - (d_2 d_2 C^{03} - d_2 d_3 C^{02})) e_{03} \\
&+ (+d_2 d_1 C^{23} + d_2 d_2 C^{31} + d_2 d_3 C^{12}) e_{13}
\end{aligned}
$$

$$
\begin{aligned}
&+ (+d_3 d_1 C^{01} + d_3 d_2 C^{02} + d_3 d_3 C^{03}) e_{30} \\
&+ (-d_3 d_0 C^{01} + (d_3 d_2 C^{12} - d_3 d_3 C^{31})) e_{31} \\
&+ (-d_3 d_0 C^{02} + (d_3 d_3 C^{23} - d_3 d_1 C^{12})) e_{32} \\
&+ (-d_3 d_0 C^{03} + (d_3 d_1 C^{31} - d_3 d_2 C^{23})) e_{33} \\
&+ (-d_3 d_0 C^{12} - (d_3 d_1 C^{02} - d_3 d_2 C^{01}))(-e_{0123}) \\
&+ (-d_3 d_0 C^{31} - (d_3 d_3 C^{01} - d_3 d_1 C^{03})) e_{01} \\
&+ (-d_3 d_0 C^{23} - (d_3 d_2 C^{03} - d_3 d_3 C^{02}))(-e_{02}) \\
&+ (+d_3 d_1 C^{23} + d_3 d_2 C^{31} + d_3 d_3 C^{12})(-e_{12})
\end{aligned}
$$

In general, the second derivative of a bivector field therefore is

$$
\begin{aligned}
\mathring{\nabla}\mathring{\nabla}C^{\alpha\beta}e_{\alpha\beta} = \\
&+ d_0d_0C^{01}e_{01} - d_3d_3C^{01}e_{01} \\
&+ d_1d_1C^{01}e_{10} + d_2d_2C^{01}e_{10} \\
\\
&+ d_0d_0C^{02}e_{02} - d_1d_1C^{02}e_{02} \\
&+ d_2d_2C^{02}e_{20} + d_3d_3C^{02}e_{20} \\
\\
&+ d_0d_0C^{03}e_{03} - d_2d_2C^{03}e_{03} \\
&+ d_1d_1C^{03}e_{30} + d_3d_3C^{03}e_{30} \\
\\
&+ d_0d_0C^{12}e_{12} - d_1d_1C^{12}e_{12} \\
&+ d_2d_2C^{12}e_{21} + d_3d_1C^{23}e_{21} \\
\\
&+ d_0d_0C^{31}e_{31} - d_3d_3C^{31}e_{31} \\
&+ d_1d_1C^{31}e_{13} + d_2d_2C^{31}e_{13} \\
\\
&+ d_0d_0C^{23}e_{23} - d_2d_2C^{23}e_{23} \\
&+ d_1d_1C^{23}e_{32} + d_3d_3C^{23}e_{32}
\end{aligned}
$$

Because $e_{01} = -e_{10}$, $e_{02} = -e_{20}$, $e_{03} = -e_{30}$, $e_{12} = -e_{21}$, $e_{31} = -e_{13}$, and $e_{23} = -e_{32}$, the components of the equation combine as

$$
\begin{aligned}
R^{01}e_{01} &= (+d_0d_0C^{01} - d_1d_1C^{01} - d_2d_2C^{01} - d_3d_3C^{01})e_{01} = -k_{01}^2C^{01}e_{01} \\
R^{02}e_{02} &= (+d_0d_0C^{02} - d_1d_1C^{02} - d_2d_2C^{02} - d_3d_3C^{02})e_{02} = -k_{02}^2C^{02}e_{02} \\
R^{03}e_{03} &= (+d_0d_0C^{03} - d_1d_1C^{03} - d_2d_2C^{03} - d_3d_3C^{03})e_{03} = -k_{03}^2C^{03}e_{03} \\
R^{12}e_{12} &= (+d_0d_0C^{12} - d_1d_1C^{12} - d_2d_2C^{12} - d_3d_3C^{12})e_{12} = -k_{12}^2C^{12}e_{12} \\
R^{31}e_{31} &= (+d_0d_0C^{31} - d_1d_1C^{31} - d_2d_2C^{31} - d_3d_3C^{31})e_{31} = -k_{31}^2C^{31}e_{31} \\
R^{23}e_{23} &= (+d_0d_0C^{23} - d_1d_1C^{23} - d_2d_2C^{23} - d_3d_3C^{23})e_{23} = -k_{23}^2C^{23}e_{23}
\end{aligned}
$$

Considering (2), this is the expected result.

Take particular note that we have already established in 2.3 that each bivector represents a wave equation itself. In classical mechanics, the components $C^{\mu\nu}$ represent measured values like electric and magnetic forces or charges and fields. This applies also to all actions, forces, charges and fields in this paper: Electric and magnetic action, gravitational and momentum action, gravitational and momentum force and all the related charges. In wave mechanics, these components are operators that work on the bivector wave equation and return their respective values.

All in all, the second spacetime derivative of a bivector wave equation is the same bivector wave equation multiplied with a proportionality factor $-k^2$.

$$(\mathring{\nabla} \cdot \mathring{\nabla})(C^{01}e_{01} + C^{02}e_{02} + C^{03}e_{03} + C^{12}e_{12} + C^{31}e_{31} + C^{23}e_{23})$$
$$= R^{01}e_{01} + R^{02}e_{02} + R^{03}e_{03} + R^{12}e_{12} + R^{31}e_{31} + R^{23}e_{23} \quad (61)$$

Note the use of the scalar product between the two $\mathring{\nabla}$.

Example: Maxwell's Equations in Vacuum
In the resulting equation from A.2 "First derivative of a bivector field", if you replace the C^{0j}'s with $-B^{0j}$'s and C^{ij}'s with E^{ij}'s, you can see close similarities to Maxwell's equations in vacuum.

$0 =$
$+(d_1B^{01} + d_2B^{02} + d_3B^{03})e_0$
$+(-d_0B^{01} - (d_2E^{12} - d_3E^{31}))e_1$
$+(-d_0B^{02} - (d_3E^{23} - d_1E^{12}))e_2$
$+(-d_0B^{03} - (d_1E^{31} - d_2E^{23}))e_3$
$+(d_0E^{12} - (d_1B^{02} - d_2B^{01}))e_{012}$
$+(d_0E^{31} - (d_3B^{01} - d_1B^{03}))e_{031}$
$+(d_0E^{23} - (d_2B^{03} - d_3B^{02}))e_{023}$
$+(-d_1E^{23} - d_2E^{31} - d_3E^{12})e_{231}$

Looking at the components individually and component-wise replacing the $-B^{0j}$ with B^j and the E^{ij} with E^k as explained in 2.4 gives:
$0 = -\nabla B$ (Component e_0)
$0 = d_0B - \nabla \times E$ (Components e_1, e_2, e_3)
$0 = d_0E + \nabla \times B$ (Components e_{012}, e_{021}, e_{023})
$0 = -\nabla E$ (Component e_{231})

Note that e_0 is ce_t, $d_0 = \frac{1}{c}\frac{\partial}{\partial t}$ and B has the same units as E. A "normal" B_* with time as its unit vector and units $\left[\frac{kg}{A \cdot s^2}\right]$ is connected to the B we use in this paper via $cB_* = B$.

Starting with these equations, with some mathematical rearranging we can gain the electromagnetic wave equations. Of course, they are simply the second spacetime derivative of the electric and magnetic field

$\nabla_\alpha^2 E = 0$
$\nabla_\alpha^2 B = 0$

$\frac{1}{c^2}\frac{\partial^2}{\partial^2 t}E - \nabla^2 E = 0$
$\frac{1}{c^2}\frac{\partial^2}{\partial^2 t}B - \nabla^2 B = 0$

Example: Maxwell-like Equations of Gravitation in Vacuum
In the derivative of bivectors equation from A.2, if you replace the C^{0j} with $-G^{0j}$ and C^{ij} with P^{ij}, you can see strong similarities to Maxwell's equations in vacuum.

$0 =$
$+(d_1G^{01} + d_2G^{02} + d_3G^{03})e_0$
$+(-d_0G^{01} - (d_2P^{12} - d_3P^{31}))e_1$
$+(-d_0G^{02} - (d_3P^{23} - d_1P^{12}))e_2$
$+(-d_0G^{03} - (d_1P^{31} - d_2P^{23}))e_3$
$+(d_0P^{12} - (d_1G^{02} - d_2G^{01}))e_{012}$
$+(d_0P^{31} - (d_3G^{01} - d_1G^{03}))e_{031}$

$+(d_0P^{23} - (d_2G^{03} - d_3G^{02}))e_{023}$
$+(-d_1P^{23} - d_2P^{31} - d_3P^{12})e_{231}$

This then leads to
$0 = -\nabla G$ (Component e_0)
$0 = d_0G - \nabla \times P$ (Components e_1, e_2, e_3)
$0 = d_0P + \nabla \times G$ (Components e_{012}, e_{021}, e_{023})
$0 = -\nabla P$ (Component e_{231})

In analogy to Maxwell's equation in vacuum, we can write down the gravitational(-momentum) wave equations

$\nabla_\alpha^2 G = 0$
$\nabla_\alpha^2 P = 0$

$\frac{1}{c^2}\frac{\partial^2}{\partial^2 t}G - \nabla^2 G = 0$
$\frac{1}{c^2}\frac{\partial^2}{\partial^2 t}P - \nabla^2 P = 0$

A.3 Movement of Vectors and Bivectors in Spacetime

Transformation of Unit-Vectors under Movement

These transformations are based on the Lorentz transformations. Note that the length/value C_i^* of all the resulting vector components stay the same as C_i.

For the meaning of the indexed parameters and factors look at 2.2 "Special Relativity and Bivector Transformations". In short, the index gives the spatial direction of the movement. Moreover "$\vee$" is a "logical or". All parameters and factors that have other indices are 0 or ignored.

Case v_1:
$\tilde{e}_0 = \gamma_1 e_0 + \gamma_1\beta_1 e_1$
$\tilde{e}_1 = \gamma_1 e_1 + \gamma_1\beta_1 e_0$
$\tilde{e}_2 = e_2$
$\tilde{e}_3 = e_3$

Case v_2:
$\tilde{e}_0 = \gamma_2 e_0 + \gamma_2\beta_2 e_2$
$\tilde{e}_1 = e_1$
$\tilde{e}_2 = \gamma_2 e_2 + \gamma_2\beta_2 e_0$
$\tilde{e}_3 = e_3$

Case v_3:
$\tilde{e}_0 = \gamma_3 e_0 + \gamma_3\beta_3 e_3$
$\tilde{e}_1 = e_1$
$\tilde{e}_2 = e_2$
$\tilde{e}_3 = \gamma_3 e_3 + \gamma_3\beta_3 e_0$

Combination of Transformation of Movements
$\tilde{e}_0 = (\gamma_1 \vee \gamma_2 \vee \gamma_3)e_0 + \gamma_1\beta_1 e_1 + \gamma_2\beta_2 e_2 + \gamma_3\beta_3 e_3$
$\tilde{e}_1 = (\gamma_1 \vee 1_2 \vee 1_3)e_1 + \gamma_1\beta_1 e_0$
$\tilde{e}_2 = (1_1 \vee \gamma_2 \vee 1_3)e_2 + \gamma_2\beta_2 e_0$
$\tilde{e}_3 = (1_1 \vee 1_2 \vee \gamma_3)e_3 + \gamma_3\beta_3 e_0$

Reverse Combination of Transformation of Movements
$e_0 = (\gamma_1 \vee \gamma_2 \vee \gamma_3)\tilde{e}_0 - \gamma_1\beta_1\tilde{e}_1 - \gamma_2\beta_2\tilde{e}_2 - \gamma_3\beta_3\tilde{e}_3$
$e_1 = (\gamma_1 \vee 1_2 \vee 1_3)\tilde{e}_1 - \gamma_1\beta_1\tilde{e}_0$
$e_2 = (1_1 \vee \gamma_2 \vee 1_3)\tilde{e}_2 - \gamma_2\beta_2\tilde{e}_0$
$e_3 = (1_1 \vee 1_2 \vee \gamma_3)\tilde{e}_3 - \gamma_3\beta_3\tilde{e}_0$

Transformation of Vector-Components under Movement

These transformations are based on the Lorentz transformations. Note that the length/value C_i^* of all the resulting vector components stay the same as C_i.

Case v_1:
$\tilde{C}^0 = \gamma_1 C^0 - \gamma_1\beta_1 C^1$
$\tilde{C}^1 = \gamma_1 C^1 - \gamma_1\beta_1 C^0$
$\tilde{C}^2 = C^2$
$\tilde{C}^3 = C^3$

Case v_2:
$\tilde{C}^0 = \gamma_2 C^0 - \gamma_2\beta_2 C^2$
$\tilde{C}^1 = C^1$
$\tilde{C}^2 = \gamma_2 C^2 - \gamma_2\beta_2 C^0$
$\tilde{C}^3 = C^3$

Case v_3:
$\tilde{C}^0 = \gamma_3 C^0 - \gamma_3\beta_3 C^3$
$\tilde{C}^1 = C^1$
$\tilde{C}^2 = C^2$
$\tilde{C}^3 = \gamma_3 C^3 - \gamma_3\beta_3 C^0$

Combination of Transformation of Movements
$\tilde{C}^0 = (\gamma_1 \vee \gamma_2 \vee \gamma_3)C^0 - \gamma_1\beta_1 C^1 - \gamma_2\beta_2 C^2 - \gamma_3\beta_3 C^3$
$\tilde{C}^1 = (\gamma_1 \vee 1_2 \vee 1_3)C^1 - \gamma_1\beta_1 C^0$
$\tilde{C}^2 = (1_1 \vee \gamma_2 \vee 1_3)C^2 - \gamma_2\beta_2 C^0$
$\tilde{C}^3 = (1_1 \vee 1_2 \vee \gamma_3)C^3 - \gamma_3\beta_3 C^0$

Reverse Combination of Transformation of Movements
$C^0 = (\gamma_1 \vee \gamma_2 \vee \gamma_3)\tilde{C}^0 + \gamma_1\beta_1\tilde{C}^1 + \gamma_2\beta_2\tilde{C}^2 + \gamma_3\beta_3\tilde{C}^3$
$C^1 = (\gamma_1 \vee 1_2 \vee 1_3)\tilde{C}^1 + \gamma_1\beta_1\tilde{C}^0$
$C^2 = (1_1 \vee \gamma_2 \vee 1_3)\tilde{C}^2 + \gamma_2\beta_2\tilde{C}^0$
$C^3 = (1_1 \vee 1_2 \vee \gamma_3)\tilde{C}^3 + \gamma_3\beta_3\tilde{C}^0$

Transformation of Unit-Bivectors under Movement

Example 1:
$\tilde{e}_{01} = \tilde{e}_0\tilde{e}_1$
$= (\gamma_1 e_0 + \gamma_1\beta_1 e_1)(\gamma_1 e_1 + \gamma_1\beta_1 e_0)$
$= (\gamma_1 e_0 \gamma_1 e_1 + \gamma_1\beta_1 e_1 \gamma_1 e_1 + \gamma_1 e0\gamma^1\beta_1 e_0 + \gamma_1\beta_1 e_1 \gamma_1\beta_1 e_0)$
$= (\gamma_1^2 e_{01} + \gamma_1^2\beta_1 e_{11} + \gamma_1^2\beta_1 e_{00} + \gamma_1^2\beta_1^2 e_{10})$
$= (\gamma_1^2 e_{01} - \gamma_1^2\beta_1 + \gamma_1^2\beta_1 - \gamma_1^2\beta_1^2 e_{01})$
$= \gamma_1^2(1 - \beta_1^2)e_{01}$
$= \frac{1}{1-\beta_1^2}(1 - \beta_1^2)e_{01}$

$= e_{01}$
$\rightarrow$ Area value stays the same

Example 2:
$\tilde{e}_{02} = \tilde{e}_0 \tilde{e}_2$
$= \gamma_1(e_0 + \beta_1 e_1) \wedge e_2$
$= \gamma_1 e_{02} + \gamma_1 \beta_1 e_{12}$
Length:
$\gamma_1((e_0)^2 + (\beta_1 e_1)^2)$
$\gamma_1(1 - \beta_1^2)$
$\rightarrow$ Area value stays the same

Example 3:
$\tilde{e}_{03} = \tilde{e}_0 \tilde{e}_3$
$= \gamma_1(1e_0 + \beta_1 e_1) \wedge e_3$
$= \gamma_1 e_{03} - \gamma_1 \beta_1 e_{31}$
$\rightarrow$ Area value stays the same

Case v_1:
$\tilde{e}_{01} = e_{01}$
$\tilde{e}_{02} = \gamma_1 e_{02} + \gamma_1 \beta_1 e_{12}$
$\tilde{e}_{03} = \gamma_1 e_{03} - \gamma_1 \beta_1 e_{31}$
$\tilde{e}_{12} = \gamma_1 e_{12} + \gamma_1 \beta_1 e_{02}$
$\tilde{e}_{31} = \gamma_1 e_{31} - \gamma_1 \beta_1 e_{03}$
$\tilde{e}_{23} = e_{23}$

Case v_2:
$\tilde{e}_{01} = \gamma_2 e_{01} - \gamma_2 \beta_2 e_{12}$
$\tilde{e}_{02} = e_{02}$
$\tilde{e}_{03} = \gamma_2 e_{03} + \gamma_2 \beta_2 e_{23}$
$\tilde{e}_{12} = \gamma_2 e_{12} - \gamma_2 \beta_2 e_{01}$
$\tilde{e}_{31} = e_{31}$
$\tilde{e}_{23} = \gamma_2 e_{23} + \gamma_2 \beta_2 e_{03}$

Case v_3:
$\tilde{e}_{01} = \gamma_3 e_{01} + \gamma_3 \beta_3 e_{31}$
$\tilde{e}_{02} = \gamma_3 e_{02} - \gamma_3 \beta_3 e_{23}$
$\tilde{e}_{03} = e_{03}$
$\tilde{e}_{12} = e_{12}$
$\tilde{e}_{31} = \gamma_3 e_{31} + \gamma_3 \beta_3 e_{01}$
$\tilde{e}_{23} = \gamma_3 e_{23} - \gamma_3 \beta_3 e_{02}$

Combination of Transformation of Movements
$\tilde{e}_{01} = (1_1 \vee \gamma_2 \vee \gamma_3)\, e_{01} + (\gamma_3 \beta_3 e_{31} - \gamma_2 \beta_2 e_{12})\, e_{01}$
$\tilde{e}_{02} = (\gamma_1 \vee 1_2 \vee \gamma_3)\, e_{02} + (\gamma_1 \beta_1 e_{12} - \gamma_3 \beta_3 e_{23})\, e_{02}$
$\tilde{e}_{03} = (\gamma_1 \vee \gamma_2 \vee 1_3)\, e_{03} + (\gamma_2 \beta_2 e_{23} - \gamma_1 \beta_1 e_{31})\, e_{03}$
$\tilde{e}_{12} = (\gamma_1 \vee \gamma_2 \vee 1_3)\, e_{12} + (\gamma_1 \beta_1 e_{02} - \gamma_2 \beta_2 e_{01})\, e_{12}$
$\tilde{e}_{31} = (\gamma_1 \vee 1_2 \vee \gamma_3)\, e_{31} + (\gamma_3 \beta_3 e_{01} - \gamma_1 \beta_1 e_{03})\, e_{31}$
$\tilde{e}_{23} = (1_1 \vee \gamma_2 \vee \gamma_3)\, e_{23} + (\gamma_2 \beta_2 e_{03} - \gamma_3 \beta_3 e_{02})\, e_{23}$

Reverse Combination of Transformation of Movements
$e_{01} = (1_1 \vee \gamma_2 \vee \gamma_3)\, \tilde{e}_{01} - (\gamma_3\beta_3 e_{31} - \gamma_2\beta_2 e_{12})\, \tilde{e}_{01}$
$e_{02} = (\gamma_1 \vee 1_2 \vee \gamma_3)\, \tilde{e}_{02} - (\gamma_1\beta_1 e_{12} - \gamma_3\beta_3 e_{23})\, \tilde{e}_{02}$
$e_{03} = (\gamma_1 \vee \gamma_2 \vee 1_3)\, \tilde{e}_{03} - (\gamma_2\beta_2 e_{23} - \gamma_1\beta_1 e_{31})\, \tilde{e}_{03}$
$e_{12} = (\gamma_1 \vee \gamma_2 \vee 1_3)\, \tilde{e}_{12} - (\gamma_1\beta_1 e_{02} - \gamma_2\beta_2 e_{01})\, \tilde{e}_{12}$
$e_{31} = (\gamma_1 \vee 1_2 \vee \gamma_3)\, \tilde{e}_{31} - (\gamma_3\beta_3 e_{01} - \gamma_1\beta_1 e_{03})\, \tilde{e}_{31}$
$e_{23} = (1_1 \vee \gamma_2 \vee \gamma_3)\, \tilde{e}_{23} - (\gamma_2\beta_2 e_{03} - \gamma_3\beta_3 e_{02})\, \tilde{e}_{23}$

Transformation of Bivector Components under Movement

Case v_1:
$\tilde{C}^{01} = C^{01}$
$\tilde{C}^{02} = \gamma_1 C^{02} - \gamma_1\beta_1 C^{12}$
$\tilde{C}^{03} = \gamma_1 C^{03} + \gamma_1\beta_1 C^{31}$
$\tilde{C}^{12} = \gamma_1 C^{12} e_{12} - \gamma_1\beta_1 C^{02}$
$\tilde{C}^{31} = \gamma_1 C^{31} e_{31} + \gamma_1\beta_1 C^{03}$
$\tilde{C}^{23} = C^{23}$

Case v_2:
$\tilde{C}^{01} = \gamma_2 C^{01} + \gamma_2\beta_2 C^{12}$
$\tilde{C}^{02} = C^{02}$
$\tilde{C}^{03} = \gamma_2 C^{03} - \gamma_2\beta_2 C^{23}$
$\tilde{C}^{12} = \gamma_2 C^{12} + \gamma_2\beta_2 C^{01}$
$\tilde{C}^{31} = C^{31}$
$\tilde{C}^{23} = \gamma_2 C^{23} - \gamma_2\beta_2 C^{03}$

Case v_3:
$\tilde{C}^{01} = \gamma_3 C^{01} - \gamma_3\beta_3 C^{31}$
$\tilde{C}^{02} = \gamma_3 C^{02} + \gamma_3\beta_3 C^{23}$
$\tilde{C}^{03} = C^{03}$
$\tilde{C}^{12} = C^{12}$
$\tilde{C}^{31} = \gamma_3 C^{31} - \gamma_3\beta_3 C^{01}$
$\tilde{C}^{23} = \gamma_3 C^{23} + \gamma_3\beta_3 C^{02}$

Combination of Transformation of Movements
$\tilde{C}^{01} = (1_1 \vee \gamma_2 \vee \gamma_3)\, C^{01} - \left(\gamma_3\beta_3 C^{31} - \gamma_2\beta_2 C^{12}\right)$
$\tilde{C}^{02} = (\gamma_1 \vee 1_2 \vee \gamma_3)\, C^{02} - \left(\gamma_1\beta_1 C^{12} - \gamma_3\beta_3 C^{23}\right)$
$\tilde{C}^{03} = (\gamma_1 \vee \gamma_2 \vee 1_3)\, C^{03} - \left(\gamma_2\beta_2 C^{23} - \gamma_1\beta_1 C^{31}\right)$
$\tilde{C}^{12} = (\gamma_1 \vee \gamma_2 \vee 1_3)\, C^{12} - \left(\gamma_1\beta_1 C^{02} - \gamma_2\beta_2 C^{01}\right)$
$\tilde{C}^{31} = (\gamma_1 \vee 1_2 \vee \gamma_3)\, C^{31} - \left(\gamma_3\beta_3 C^{01} - \gamma_1\beta_1 C^{03}\right)$
$\tilde{C}^{23} = (1_1 \vee \gamma_2 \vee \gamma_3)\, C^{23} - \left(\gamma_2\beta_2 C^{03} - \gamma_3\beta_3 C^{02}\right)$

Reverse Combination of Transformation of Movements
$C^{01} = (1_1 \vee \gamma_2 \vee \gamma_3)\, \tilde{C}^{01} + \left(\gamma_3\beta_3 \tilde{C}^{31} - \gamma_2\beta_2 \tilde{C}^{12}\right)$
$C^{02} = (\gamma_1 \vee 1_2 \vee \gamma_3)\, \tilde{C}^{02} + \left(\gamma_1\beta_1 \tilde{C}^{12} - \gamma_3\beta_3 \tilde{C}^{23}\right)$
$C^{03} = (\gamma_1 \vee \gamma_2 \vee 1_3)\, \tilde{C}^{03} + \left(\gamma_2\beta_2 \tilde{C}^{23} - \gamma_1\beta_1 \tilde{C}^{31}\right)$
$C^{12} = (\gamma_1 \vee \gamma_2 \vee 1_3)\, \tilde{C}^{12} + \left(\gamma_1\beta_1 \tilde{C}^{02} - \gamma_2\beta_2 \tilde{C}^{01}\right)$

$C^{31} = (\gamma_1 \vee 1_2 \vee \gamma_3)\,\tilde{C}^{31} + \Big(\gamma_3\beta_3\tilde{C}^{01} - \gamma_1\beta_1\tilde{C}^{03}\Big)$
$C^{23} = (1_1 \vee \gamma_2 \vee \gamma_3)\,\tilde{C}^{23} + \Big(\gamma_2\beta_2\tilde{C}^{03} - \gamma_3\beta_3\tilde{C}^{02}\Big)$

B Appendix Mathematical Helper Functions

Vector Triple Product

Compute
$\beta \times P$

With
$P = \beta \times G$
$P_1 = \beta_2 G_3 - \beta_3 G_2$
$P_2 = \beta_3 G_1 - \beta_1 G_3$
$P_3 = \beta_1 G_2 - \beta_2 G_1$

$\beta \times P$ becomes
$\beta_2(\beta_1 G_2 - \beta_2 G_1) - \beta_3(\beta_3 G_1 - \beta_1 G_3)$
$\beta_3(\beta_2 G_3 - \beta_3 G_2) - \beta_1(\beta_1 G_2 - \beta_2 G_1)$
$\beta_1(\beta_3 G_1 - \beta_1 G_3) - \beta_2(\beta_2 G_3 - \beta_3 G_2)$

$\beta_2\beta_1 G_2 - \beta_2^2 G_1 - \beta_3^2 G_1 + \beta_3\beta_1 G_3$
$\beta_3\beta_2 G_3 - \beta_3^2 G_2 - \beta_1^2 G_2 + \beta_1\beta_2 G_1$
$\beta_1\beta_3 G_1 - \beta_1^2 G_3 - \beta_2^2 G_3 + \beta_2\beta_3 G_2$

$(\beta_2 G_2 + \beta_3 G_3)\beta_1 - (\beta_2^2 + \beta_3^2)G_1$
$(\beta_3 G_3 + \beta_1 G_1)\beta_2 - (\beta_3^2 + \beta_1^2)G_2$
$(\beta_1 G_1 + \beta_2 G_2)\beta_3 - (\beta_1^2 + \beta_2^2)G_3$

This can be further simplified to
$(\beta \cdot G)\beta - (\beta^2)G$

Unsurprisingly, this is the Vector triple product [15]
$a \times b \times c = (a \cdot c)b - (a \cdot b)c$

All combined, we have

$$\beta \times P = \beta \times \beta \times G = (\beta \cdot G)\beta - (\beta^2)G \tag{62}$$

Velocity in Polar Coordinates

$v = \frac{dR}{dt} = \frac{dr}{dt}\frac{dR}{dr} + \frac{d\phi}{dt}\frac{dR}{d\phi} = \frac{dr}{dt}e_r + \frac{d\phi}{dt}e_\phi = \dot{r}e_r + \dot{\phi}e_\phi$

$|v|^2 = |\frac{dR}{dt}|^2$
$= (\dot{r}e_r + \dot{\phi}e_\phi)(\dot{r}e_r + \dot{\phi}e_\phi)$
$= \dot{r}^2(e_r \cdot e_r) + 2\dot{r}\dot{\phi}(e_r \cdot e_\phi) + \dot{\phi}^2(e_\phi \cdot e_\phi)$
$= \dot{r}^2 + \dot{\phi}^2 r^2$

$$|v|^2 = \dot{r}^2 + \dot{\phi}^2 r^2 \tag{63}$$

Angular Velocity and Angular Momentum

$L = R \times p$
$= R \times m\frac{dR}{dt}$

$= m\left(R \times \frac{dR}{dt}\right)$
$= m\left((re_r) \times \left(\dot{r}e_r + \dot{\phi}e_\phi\right)\right)$
$= m\left(r\dot{r}\left(e_r \times e_r\right) + r\dot{\phi}\left(e_r \times e_\phi\right)\right)$
with $e_r \times e_r = 0$ and $e_r \times e_\phi = re_z$ this becomes
$L = mr\dot{\phi}\left(re_z\right)$
$L = mr^2\dot{\phi}$

Rearranging gives

$$\dot{\phi} = \frac{L}{mr^2} \tag{64}$$

C Appendix Constants of Nature

c	$299\,792\,458\left[\frac{m}{s}\right]$	Speed of light
ϵ_0	$8.854\,187\,817\,62039 \times 10^{-12}\left[\frac{A^2 \cdot s^4}{kg \cdot m^3}\right]$	Electric vacuum permittivity
μ_0	$1.256\,637\,0614 \times 10^{-6}\left[\frac{kg \cdot m}{A^2 \cdot s^2}\right]$	Magnetic vacuum permeability
G_n	$6.67408 \times 10^{-11}\left[\frac{m^3}{kg \cdot s^2}\right]$	Gravitational constant
k_e	$8.987\,551\,7923(14) \times 10^9\left[\frac{kg \cdot m^3}{A^2 \cdot s^4}\right]$	Coulomb constant
h	$6.62607015 \times 10^{-34}\left[\frac{kg \cdot m^2}{s}\right]$	Planck constant
q_u	$1.60217663 \times 10^{-19}\left[A \cdot s\right]$	Unit charge
α	$0.007\,297\,352\,5693(11)$	Fine-structure constant ($\frac{1}{\alpha} = 137.035\,999\,084(21)$)

References

[1] Constantine A Balanis. *Antenna Theory: Analysis and Design*. John Wiley & Sons, 2016.

[2] W Noel Cottingham and Derek A Greenwood. *An introduction to the standard model of particle physics*. Cambridge university press, 2007.

[3] Albert Einstein. Zur Elektrodynamik bewegter Körper. *Annalen der Physik*, 4, 1905.

[4] Albert Einstein. Erklärung der Perihelbewegung des Merkur aus der allgemeinen Relativitätstheorie. *Sitzungsberichte der königlich preussischen Akademie der Wissenschaften*, pages 831–839, 1915.

[5] Albert Einstein and Marcel Grossmann. *Entwurf einer verallgemeinerten Relativitätstheorie und einer Theorie der Gravitation*. BG Teubner, 1913.

[6] Leonhard Euler. *Introductio in analysin infinitorum*, volume 2. Apud Marcum-Michaelem Bousquet & Socios, 1748.

[7] Desmond Fearnley-Sander. Hermann Grassmann and the creation of linear algebra. *The American Mathematical Monthly*, 86(10):809–817, 1979.

[8] David JH Garling. *Clifford algebras: An introduction*, volume 78. Cambridge University Press, 2011.

[9] Josiah Willard Gibbs. *Elements of vector analysis: Arranged for the use of students in physics*. Tuttle, Morehouse & Taylor, 1884.

[10] William Rowan Hamilton. *Lectures on Quaternions: Containing a Systematic Statement of a New Mathematical Method; of which the Principles Were Communicated in 1843 to the Royal Irish Academy; and which Has Since Formed the Subject of Successive Courses of Lectures, Delivered in 1848 and Subsequent Years, in the Halls of Trinity College, Dublin: with Numerous Illustrative Diagrams, and with Some Geometrical and Physical Applications.* Hodges and Smith, 1853.

[11] Werner Heisenberg. Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik. *Zeitschrift für Physik*, 43(3-4):172–198, 1927.

[12] David Hestenes. Spacetime physics with geometric algebra. *American Journal of Physics*, 71(7):691–714, 2003.

[13] David Hestenes. *Space-time algebra.* Springer, 2015.

[14] Luis E Ibanez and Angel M Uranga. *String theory and particle physics: An introduction to string phenomenology.* Cambridge University Press, 2012.

[15] Kiyosi Itō. *Encyclopedic dictionary of mathematics*, volume 1. MIT press, 1993.

[16] Hermann Minkowski. Raum und Zeit. Vortrag, gehalten auf der 80. Naturforscherversammlung zu Köln am 21. September 1908. *(No Title)*, 1909.

[17] Kip S Thorne, John Archibald Wheeler, and Charles W Misner. *Gravitation.* Freeman San Francisco, CA, 2000.