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Expanded full paper on the Geometric Entity Dualization operation J_e introduced first in the research monograph *Geometric Entity Dualization and Dual Quaternion Geometric Algebra in PGA G(3,0,1) with Double PGA G(6,0,2) for General Quadrics* dated 2023-12-16.

Geometric Entity Dualization in the Geometric Algebra PGA G(3,0,1)

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Abstract. In Geometric Algebra, the degenerate-metric algebra $\mathcal{G}_{3,0,1}$ is known as the Projective Geometric Algebra (PGA) for 3D space (3DPGA). In PGA, there is a point-based geometric algebra (point-based PGA) and a plane-based geometric algebra (plane-based PGA). Both algebras have homogeneous geometric entities for points, lines, and planes. The two algebras of PGA are dual to each other through a new geometric entity dualization operation J_e, which is introduced in this paper as its main subject and contribution. The new dualization J_e is an anti-involution with inverse $-J_e = D_e$. Using J_e, the dual of a point-based PGA entity is its corresponding plane-based PGA entity representing the same geometry (point, line, or plane) with the same orientation. Using $D_e = -J_e$, the inverse dual (undual) of a plane-based PGA entity is its corresponding point-based PGA entity with the same orientation. The new dualization operation J_e maintains the correct orientation of an entity. J_e is defined by a table of duals that are found empirically by observation to maintain correct entity orientation through the dualization. We define a Hodge star dualization operation to be purely an involution, or else purely an anti-involution, between all basis blades and their dual basis blades. As an anti-involution, J_e is also implemented by algebraic methods using Hodge star dualizations in non-degenerate algebras that correspond to PGA. In the prior literature, there are other definitions for the duals in PGA that may not maintain the correct entity orientation and are different than J_e.

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1. Introduction

In Geometric Algebra [8][3][11](§4.1.1), the degenerate-metric algebra $\mathcal{G}_{3,0,1}$ is known as the Projective Geometric Algebra (PGA) for 3D space (3DPGA) [6][7][5][2]. In PGA, there is a point-based geometric algebra (point-based

PGA) and a plane-based geometric algebra (plane-based PGA). Both algebras have homogeneous geometric entities for points, lines, and planes. The two algebras of PGA are dual to each other through a new *geometric entity dualization operation* $J_e(\mathbf{A}) = \mathbf{A}^\star$ (§4), which is introduced in this paper as its main subject and contribution in Section 4.

In $\mathcal{G}_{3,0,1}$, the basis vectors are $\{\mathbf{e}_0,\, \mathbf{e}_1,\, \mathbf{e}_2,\, \mathbf{e}_3\}$ with metric $[g_{ij}] = [\mathbf{e}_i \cdot \mathbf{e}_j] = \mathrm{diag}(0, 1, 1, 1)$. We use the hat notation $\hat{\mathbf{A}} = \mathbf{A}/\sqrt{|\mathbf{A}^2|}$ for the unit of any k-vector $\mathbf{A}$. The unit pseudoscalar of the subalgebra $\mathcal{G}_3$ is $\mathbf{I}_3 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$. The unit pseudoscalar of $\mathcal{G}_{3,0,1}$ is $\mathbf{I}_4 = \mathbf{e}_0\mathbf{I}_3$. The metric is called degenerate since $\mathbf{e}_0^2 = 0$ and $\mathbf{I}_4^2 = 0$, and $\mathbf{I}_4^{-1}$ does not exist. For multivector $A \in \mathcal{G}_3$, the dual of A in the subalgebra $\mathcal{G}_3$ is $A^* = A/\mathbf{I}_3 = A \cdot \mathbf{I}_3^{-1}$. In $\mathcal{G}_{3,0,1}$, the unit pseudoscalar $\mathbf{I}_4$, being degenerate, cannot be used for dualization in the general case, except for the special case where $A \in \mathcal{G}_3$ and $A^\star = \mathbf{I}_4 A = -\mathbf{e}_0 A^*$. For $A \in \mathcal{G}_{3,0,1}$ in the point-based PGA, the dual of A is $A^\star = J_e(A)$ (§4) in the plane-based PGA. For $A^\star \in \mathcal{G}_{3,0,1}$ in the plane-based PGA, the undual of $A^\star$ is $(A^\star)^{-\star} = A^{-\star\star} - J_e(A^\star) = D_e(A^\star) = A$ (§4) in the point-based PGA. Before we can discuss J_e in detail, we must review PGA in the point-based (§2) and plane-based algebras (§3), and review concepts of dualization (§4). In the remainder of this section, we begin with an overview of the point-based and plane-based algebras of PGA, and then describe the structure and contents of the rest of the paper.

In the point-based algebra of PGA, a homogeneous point $(w = 1, x, y, z)$ is embedded as the 1-blade (vector) $\mathbf{P}_\mathbf{t} = \mathbf{e}_0 + \mathbf{t}$, where $\mathbf{t} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$. We consider $\mathbf{P}_\mathbf{t}$ to be the symbolic "**t**est" point entity. Other points, such as $\mathbf{P} = \mathbf{P}_\mathbf{p}$, embedding point $\mathbf{p} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3$, are considered to be numerical or non-symbolic points. In the point-based algebra, the wedge product $\wedge$ acts as the *join product* of 1-blade points. The join product of points represents the span, or linear combination, of the points. The join product of two points $\{\mathbf{P}_1, \mathbf{P}_2\}$ forms the 2-blade line $\mathbf{L} = \mathbf{P}_2 \wedge \mathbf{P}_1$. The linear combination, or pencil, of two points $\{\mathbf{P}_1, \mathbf{P}_2\}$ is all points $\mathbf{P} = (1-t)\mathbf{P}_2 + t\mathbf{P}_1$ with real parameter t along the line $\mathbf{L}$ between $\{\mathbf{P}_1,\, \mathbf{P}_2\}$. The span of $\{\mathbf{P}_1,\, \mathbf{P}_2\}$ includes any point $\mathbf{P} = (1-t)\mathbf{P}_2 + t\mathbf{P}_1$ on the line $\mathbf{L}$ such that $\mathbf{P} \wedge \mathbf{L} = 0$. We call the line $\mathbf{L}$ an *outer product null space* (OPNS) geometric entity [11]. Point $\mathbf{P}_\mathbf{t}$ is on line $\mathbf{L}$ if and only if $\mathbf{P}_\mathbf{t} \wedge \mathbf{L} = 0$. Line $\mathbf{L}$ represents the set (or pencil) of points $\{\mathbf{P} : \mathbf{P} \wedge \mathbf{L} = 0\}$ called the OPNS of $\mathbf{L}$. The join product of three points $\{\mathbf{P}_1,\, \mathbf{P}_2,\, \mathbf{P}_3\}$ forms the 3-blade plane $\mathbf{\Pi} = \mathbf{P}_3 \wedge \mathbf{P}_2 \wedge \mathbf{P}_1$. The plane $\mathbf{\Pi}$ is also an OPNS geometric entity. The plane $\mathbf{\Pi}$ represents the span, or bundle, of points $\mathbf{P} = (1-s)((1-t)\mathbf{P}_2 + t\mathbf{P}_1) + s\mathbf{P}_3$ on the plane. Point $\mathbf{P}_\mathbf{t}$ is on plane $\mathbf{\Pi}$ if and only if $\mathbf{P}_\mathbf{t} \wedge \mathbf{\Pi} = 0$. The plane $\mathbf{\Pi}$ represents the OPNS set (or bundle) of points $\{\mathbf{P} : \mathbf{P} \wedge \mathbf{\Pi} = 0\}$. The point $\mathbf{P}_\mathbf{t}$ is also an OPNS geometric entity. Point $\mathbf{P}_\mathbf{t}$ represents the same point as point $\mathbf{P}$ if and only if $\mathbf{P}_\mathbf{t} \wedge \mathbf{P} = 0$. In the point-based algebra of PGA, the 1-blade point $\mathbf{P}$, 2-blade line $\mathbf{L}$, and 3-blade plane $\mathbf{\Pi}$ geometric entities are OPNS geometric entities. We also call the point-based geometric algebra of PGA the OPNS PGA. In Section 2, we carefully derive these entities and discuss them further, paying close attention to their orientation, which is important for defining the operation J_e (§4).

In the plane-based algebra of PGA, a homogeneous plane with normal vector $\mathbf{n} = n_x\mathbf{e}_1 + n_y\mathbf{e}_2 + n_z\mathbf{e}_3$ through point $\mathbf{p} = p_x\mathbf{e}_1 + p_y\mathbf{e}_2 + p_z\mathbf{e}_3$ is embedded as the 1-blade (vector) $\boldsymbol{\pi} = \mathbf{n} + (\mathbf{p}\cdot\mathbf{n})\mathbf{e}_0$. If $\mathbf{n} = \hat{\mathbf{n}} = \mathbf{n}/\sqrt{\mathbf{n}^2}$ (a unit vector), then $\boldsymbol{\pi} = \hat{\boldsymbol{\pi}}$ is a unit plane, where $\boldsymbol{\pi}^2 = 1$ and $d = \mathbf{p}\cdot\hat{\mathbf{n}}$ is the distance of the plane from the origin along the direction $\hat{\mathbf{n}}$. In the plane-based algebra, the wedge product $\wedge$ acts as the *meet product* of 1-blade planes. The meet product of planes represents their intersection. The meet product of two planes $\{\boldsymbol{\pi}_1, \boldsymbol{\pi}_2\}$ forms the 2-blade line $\boldsymbol{l} = \boldsymbol{\pi}_2 \wedge \boldsymbol{\pi}_1$. The linear combination, or pencil, of planes $\{\boldsymbol{\pi}_1, \boldsymbol{\pi}_2\}$ are planes $\boldsymbol{\pi} = (1-t)\boldsymbol{\pi}_2 + t\boldsymbol{\pi}_1$ also intersecting line $\boldsymbol{l}$ such that $\boldsymbol{\pi}\wedge\boldsymbol{l} = 0$. The line $\boldsymbol{l}$ represents the pencil of planes as an OPNS geometric entity. The meet product of three planes $\{\boldsymbol{\pi}_1, \boldsymbol{\pi}_2, \boldsymbol{\pi}_3\}$ forms the 3-blade point $\boldsymbol{p} = \boldsymbol{\pi}_1\wedge\boldsymbol{\pi}_2\wedge\boldsymbol{\pi}_3$. The point $\boldsymbol{p}$ represents the bundle of planes $\boldsymbol{\pi} = (1-s)((1-t)\boldsymbol{\pi}_2 + t\boldsymbol{\pi}_1) + s\boldsymbol{\pi}_3$ passing through point $\boldsymbol{p}$ such that $\boldsymbol{\pi}\wedge\boldsymbol{p} = 0$. For $\{\boldsymbol{\pi}_x = \mathbf{e}_1 + x\mathbf{e}_0,\ \boldsymbol{\pi}_y = \mathbf{e}_2 + y\mathbf{e}_0,\ \boldsymbol{\pi}_z = \mathbf{e}_3 + z\mathbf{e}_0\}$, $\boldsymbol{p}_{\mathbf{t}} = \boldsymbol{\pi}_x\wedge\boldsymbol{\pi}_y\wedge\boldsymbol{\pi}_z$ embeds the point $\mathbf{t} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$. Point $\boldsymbol{p}_{\mathbf{t}}$ is on plane $\boldsymbol{\pi}$ if and only if $\boldsymbol{p}_{\mathbf{t}}\wedge\boldsymbol{\pi} = \boldsymbol{p}_{\mathbf{t}}\times\boldsymbol{\pi} = 0$. The product $\times$ is the commutator product, defined for any two multivectors $\{A, B\}$ as $A\times B = (AB - BA)/2$. Point $\boldsymbol{p}_{\mathbf{t}}$ is on line $\boldsymbol{l}$ if and only if $\boldsymbol{p}_{\mathbf{t}}\times\boldsymbol{l} = 0$. Point $\boldsymbol{p}_{\mathbf{t}}$ represents the same point as $\boldsymbol{p}$ if and only if $\boldsymbol{p}_{\mathbf{t}}\times\boldsymbol{p} = 0$. In the plane-based geometric algebra of PGA, the 1-blade plane $\boldsymbol{\pi}$, 2-blade line $\boldsymbol{l}$, and 3-blade point $\boldsymbol{p}$ geometric entities are *commutator product null space* (CPNS) geometric entities. We also call the plane-based geometric algebra of PGA the CPNS PGA. Plane $\boldsymbol{\pi}$ represents the CPNS set of points $\{\boldsymbol{p}_{\mathbf{t}} : \boldsymbol{p}_{\mathbf{t}}\times\boldsymbol{\pi} = 0\}$. Line $\boldsymbol{l}$ represents the CPNS set of points $\{\boldsymbol{p}_{\mathbf{t}} : \boldsymbol{p}_{\mathbf{t}}\times\boldsymbol{l} = 0\}$. In Section 3, we carefully derive these entities and discuss them further, playing close attention to their orientation, which is important for defining the operation J_e (§4).

The orientation of the geometric entities is important for applications. An orientation is just a plus or minus sign. For example, the plane $\boldsymbol{\pi}$ and $-\boldsymbol{\pi}$ have opposite facing sides. The most important contribution of this paper is that all of the entities, the dualization operation J_e, and all of the other operations are defined or derived such that the geometric entities maintain their correct orientations through all operations. The new dualization operation J_e is different than the dualization operations J [5] and $\star$ [2] in prior literature by having different signs such that J_e maintains the correct orientation of entities and their duals.

The point-based and plane-based entities are related to each other as duals. The dual of a point-based grade k entity $\mathbf{A}\in\mathcal{G}^k_{3,0,1}$ is its dual plane-based grade $4-k$ entity $J_e(\mathbf{A}) = \mathbf{A}^\star = \boldsymbol{a}\in\mathcal{G}^{4-k}_{3,0,1}$. The dual of the point-based 1-blade point $\mathbf{P}$ is the plane-based 3-blade point $\boldsymbol{p} = J_e(\mathbf{P}) = \mathbf{P}^\star$. The dual of the point-based 2-blade line $\mathbf{L}$ is the plane-based 2-blade line $\boldsymbol{l} = \mathbf{L}^\star$. The dual of the point-based 3-blade plane $\mathbf{\Pi}$ is the plane-based 1-blade plane $\boldsymbol{\pi} = \mathbf{\Pi}^\star$. The new operation J_e is an anti-involution, where $J_e(J_e(\mathbf{A})) = -\mathbf{A}$. The inverse $J_e^{-1} = -J_e$ dualizes a plane-based grade $4-k$

entity $\mathbf{A}^\star = \boldsymbol{a} \in \mathcal{G}^{4-k}_{3,0,1}$ to its dual point-based grade k entity $-J_e(\mathbf{A}^\star) = \mathbf{A}^{-\star\star} = \boldsymbol{a}^{-\star} = \mathbf{A} \in \mathcal{G}^{k}_{3,0,1}$. We also define the alias $D_e = J_e^{-1} = -J_e$ so that $D_e(\boldsymbol{a}) = \boldsymbol{a}^{-\star} = \mathbf{A}$ is the dualization operation on an entity $\boldsymbol{a}$ in the plane-based algebra to its corresponding dual entity $\boldsymbol{a}^{-\star} = \mathbf{A}$ in the point-based algebra. The corresponding dual entities represent the same geometry with the same orientation when J_e and $D_e = -J_e$ are used correctly, using J_e to dualize from point-based to plane-based, and $D_e = -J_e$ to dualize from plane-based to point-based.

The point $\boldsymbol{p}_{\mathbf{t}} = \mathbf{P}^\star_{\mathbf{t}}$ is the dual of point $\mathbf{P}_{\mathbf{t}}$ and represents the same point with the same orientation. Dual entities represent the same geometry with the same orientation. The OPNS of line $\mathbf{L}$ is $\{\mathbf{P}_{\mathbf{t}} : \mathbf{P}_{\mathbf{t}} \wedge \mathbf{L} = 0\}$, where $\mathbf{P}_{\mathbf{t}} \wedge \mathbf{L}$ is grade 3. The CPNS of line $\boldsymbol{l} = \mathbf{L}^\star$ is $\{\boldsymbol{p}_{\mathbf{t}} : \boldsymbol{p}_{\mathbf{t}} \times \boldsymbol{l} = 0\}$, where $\boldsymbol{p}_{\mathbf{t}} \times \boldsymbol{l}$ is also grade 3 and should match $\mathbf{P}_{\mathbf{t}} \wedge \mathbf{L}$ with the same orientation. The OPNS of plane $\mathbf{\Pi}$ is $\{\mathbf{P}_{\mathbf{t}} : \mathbf{P}_{\mathbf{t}} \wedge \mathbf{\Pi} = 0\}$, where $\mathbf{P}_{\mathbf{t}} \wedge \mathbf{\Pi}$ is grade 4. The CPNS of plane $\boldsymbol{\pi} = \mathbf{\Pi}^\star$ is $\{\boldsymbol{p}_{\mathbf{t}} : \boldsymbol{p}_{\mathbf{t}} \times \boldsymbol{\pi} = 0\}$, where $\boldsymbol{p}_{\mathbf{t}} \times \boldsymbol{\pi} = \boldsymbol{p}_{\mathbf{t}} \wedge \boldsymbol{\pi}$ is also grade 4 and should match $\mathbf{P}_{\mathbf{t}} \wedge \mathbf{\Pi}$ with the same orientation. The OPNS of point $\mathbf{P}$ is $\{\mathbf{P}_{\mathbf{t}} : \mathbf{P}_{\mathbf{t}} \wedge \mathbf{P} = 0\}$, where $\mathbf{P}_{\mathbf{t}} \wedge \mathbf{P}$ is grade 2. The CPNS of point $\boldsymbol{p} = \mathbf{P}^\star$ is $\{\boldsymbol{p}_{\mathbf{t}} : \boldsymbol{p}_{\mathbf{t}} \times \boldsymbol{p} = 0\}$, where $\boldsymbol{p}_{\mathbf{t}} \times \boldsymbol{p}$ is also grade 2 and should match $\mathbf{P}_{\mathbf{t}} \wedge \mathbf{P}$ with the same orientation. These matching grades and orientations are the requirements that must be satisfied by the new geometric entity dualization operation J_e. In Section 4, these requirements are carefully observed to derive and define J_e empirically as a table duals for all $2^4 = 16$ basis blades in $\mathcal{G}_{3,0,1}$. We then find that J_e is an anti-involution that can be implemented as a Hodge dual in three different non-degenerate algebras $\mathcal{G}_{p,q,0} \in \{\mathcal{G}_{4,0,0}, \mathcal{G}_{1,3,0}, \mathcal{G}_{1,3,0}\}$ having basis blades that correspond to the basis blades of $\mathcal{G}_{3,0,1}$.

In PGA, there is a rotation operator $R = \exp(\theta\hat{\mathbf{n}}^*/2)$ that applies to all entities in both the point-based and plane-based algebras. R rotates any entity A, as $A' = RAR^{-1}$, counterclockwise by angle θ around the axis $\hat{\mathbf{n}}$ centered on the origin. In the plane-based algebra, there is also a translation operator $T = \exp(\mathbf{e}_0\mathbf{d}/2)$, for translation by vector displacement $\mathbf{d}$, that applies as $\boldsymbol{a}' = T\boldsymbol{a}T^{-1}$ only for $\boldsymbol{a} \in \{\boldsymbol{\pi}, \boldsymbol{l}, \boldsymbol{p}\}$ in the plane-based algebra. In the plane-based algebra, there are also operations for reflections, projections and rejections.

The plane-based algebra of PGA is similar to CGA $\mathcal{G}_{4,1}$ [3][11] and its *inner product null space* (IPNS) entities. In CGA, the unit pseudoscalar is $\mathbf{I}_5 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4\mathbf{e}_5$. and dualization of any CGA entity A is $A^* = A/\mathbf{I}_5$, and the undual is $A = A^*\mathbf{I}_5$. In the OPNS CGA, the OPNS point is $\mathbf{P}_{\mathbf{t}}$, the OPNS line is $\mathbf{L}$, and the OPNS of $\mathbf{L}$ is $\{\mathbf{P}_{\mathbf{t}} : \mathbf{P}_{\mathbf{t}} \wedge \mathbf{L} = 0\}$. In CGA, we can dualize $\mathbf{P}_{\mathbf{t}} \wedge \mathbf{L}$ as $(\mathbf{P}_{\mathbf{t}} \wedge \mathbf{L})/\mathbf{I}_5 = \mathbf{P}_{\mathbf{t}} \cdot (\mathbf{L}/\mathbf{I}_5) = \mathbf{P}_{\mathbf{t}} \cdot \mathbf{L}^*$, where $\mathbf{L}^*$ is the dual IPNS line and the conformal point $\mathbf{P}_{\mathbf{t}} = \mathbf{t} + \mathbf{t}^2\mathbf{e}_\infty/2 + \mathbf{e}_o$ is not dualized. In PGA, we cannot dualize using the unit pseudoscalar $\mathbf{I}_4$ since it is degenerate and has no inverse, and instead we use the new operation J_e. In PGA, we must also dualize the homogeneous OPNS point $\mathbf{P}_{\mathbf{t}} = \mathbf{e}_0 + \mathbf{t}$ as the CPNS point $\boldsymbol{p}_{\mathbf{t}} = \mathbf{P}^\star_{\mathbf{t}}$. In CGA, the dual of an OPNS entity is its dual IPNS entity.

In PGA, the dual of an OPNS entity is its dual CPNS entity, similar to the IPNS CGA entity. The CGA element $\mathbf{e}_\infty$, representing *the* point at infinity, has no corresponding element in PGA, but its algebraic role is often fulfilled by the PGA element $\mathbf{e}_0$ in the plane-based algebra. In [10] and [9], the PGA point-based and plane-based algebras are seen as two subalgebras of CGA and a dualization operation between these CGA-based subalgebras is given. The approach in this paper for J_e is different than in [10] and [9] and other prior literature on PGA. The new operation J_e is not defined in terms of CGA or within CGA. The new PGA dualization operation J_e is defined by empirical observation of orientation between dual entities, and we obtain J_e as an anti-involution different from all prior literature but more like the CGA dualization operation, which is also an anti-involution since $\mathbf{I}_5^2 = -1$.

The paper is organized as follows. In Section 2, we review the point-based geometric algebra of PGA and carefully derive the point-based point $\mathbf{P}$, line $\mathbf{L}$, and plane $\mathbf{\Pi}$ geometric entities. In Section 3, we review the plane-based geometric algebra of PGA and carefully derive the plane-based plane $\boldsymbol{\pi}$, line $\boldsymbol{l}$, and point $\boldsymbol{p}$ geometric entities. In Section 4, we review dualization and introduce the new geometric entity dualization operation J_e. In Section 5, we conclude the paper.

2. The Point-based Geometric Algebra of PGA

In this section, we define or derive each entity in the point-based algebra of PGA, also called the OPNS PGA. The orientation of each entity is important for defining the new geometric entity dualization operation J_e in Section 4.

We use many geometric algebra identities, including $\mathbf{I}_4 = -\mathbf{I}_3\mathbf{e}_0$, $\mathbf{b}^* = \mathbf{b}/\mathbf{I}_3 = -\mathbf{b}\mathbf{I}_3$ (for the 2-blade *dual* $\mathbf{b}^*$ of 1-blade $\mathbf{b}$ in $\mathcal{G}_3$), $\mathbf{aB} = \mathbf{a}\cdot\mathbf{B} + \mathbf{a}\wedge\mathbf{B} = \frac{1}{2}(\mathbf{aB} - (-1)^k\mathbf{Ba}) + \frac{1}{2}(\mathbf{aB} + (-1)^k\mathbf{Ba})$ (for the product $\mathbf{aB}$ of 1-vector $\mathbf{a}$ and k-vector $\mathbf{B}$), $(\mathbf{a}\cdot\mathbf{b})\mathbf{I}_3 = \frac{1}{2}(\mathbf{ab}+\mathbf{ba})\mathbf{I}_3 = -\frac{1}{2}(\mathbf{ab}^* + \mathbf{b}^*\mathbf{a}) = -\mathbf{a}\wedge\mathbf{b}^*$, $(\mathbf{a}\cdot\mathbf{b})\mathbf{I}_4 = \mathbf{a}\wedge\mathbf{b}^*\wedge\mathbf{e}_0$, and $(\mathbf{a}\cdot\mathbf{b}^*)\mathbf{I}_4 = -\frac{1}{2}(\mathbf{ab}^* - \mathbf{b}^*\mathbf{a})\mathbf{I}_3\mathbf{e}_0 = \mathbf{a}\wedge\mathbf{e}_0\wedge\mathbf{b}$.

2.1. OPNS PGA Geometric Entities

The point-based geometric entities are the OPNS 1-blade point $\mathbf{P}$, the OPNS 2-blade line $\mathbf{L}$, and the OPNS 3-blade plane $\mathbf{\Pi}$.

2.1.1. OPNS PGA 1-blade Point Geometric Entity. The OPNS PGA 1-blade point entity $\mathbf{P}_\mathbf{t}$, embedding vector point $\mathbf{t} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$, is defined in standard form and orientation as

$$\mathbf{P}_\mathbf{t} = \mathbf{e}_0 + \mathbf{t} = D_e(\boldsymbol{p}_\mathbf{t}) = \boldsymbol{p}_\mathbf{t}^{-\star}. \tag{1}$$

The point $\mathbf{P}_\mathbf{t}$ represents the homogeneous coordinates $(w = 1, x, y, z)$. The dual is $J_e(\mathbf{P}_\mathbf{t}) = \mathbf{P}_\mathbf{t}^\star = \boldsymbol{p}_\mathbf{t}^{-\star\star} = \boldsymbol{p}_\mathbf{t}$, which is the plane-based point $\boldsymbol{p}_\mathbf{t}$ representing the same point with the same orientation. A directed point at infinity is defined as

$$\mathbf{P}_{\infty\hat{\mathbf{t}}} = \lim_{\|\mathbf{t}\|\to\infty} \frac{\mathbf{P}_\mathbf{t}}{\|\mathbf{t}\|} = \hat{\mathbf{t}}. \tag{2}$$

More generally, $\mathbf{P}_{\infty\mathbf{t}} = \mathbf{t}$, since points are homogeneous and can be scaled by any non-zero scalar $\|\mathbf{t}\| \neq 0$. For finite point $\mathbf{P}_\mathbf{t}$, vector $\mathbf{t}$ can be projected as

$$\mathbf{t} = \mathbf{I}_3(\mathbf{P}_\mathbf{t} \wedge \mathbf{e}_0)^\star/(\mathbf{I}_3 \wedge \mathbf{P}_\mathbf{t})^\star. \tag{3}$$

2.1.2. OPNS PGA 2-blade Line Geometric Entity. The OPNS PGA 2-blade line $\mathbf{L}$ spanning points $\mathbf{P}_1$ and $\mathbf{P}_2$, or through point $\mathbf{P} = \mathbf{P}_1 = \mathbf{e}_0+\mathbf{p}_1 = \mathbf{e}_0+\mathbf{p}$ with direction $\mathbf{d} = \mathbf{P}_2 - \mathbf{P}_1 = \mathbf{p}_2 - \mathbf{p}_1$, is

$$\mathbf{L} = \mathbf{L}_{\mathbf{p},\mathbf{d}} = \mathbf{d} \wedge \mathbf{P} = \mathbf{P}_2 \wedge \mathbf{P}_1 = D_e(\boldsymbol{l}_{\mathbf{p},\mathbf{d}}) = \boldsymbol{l}_{\mathbf{p},\mathbf{d}}^{-\star}. \tag{4}$$

$\mathbf{L}_{\mathbf{p},\mathbf{d}}$ and $J_e(\mathbf{L}_{\mathbf{p},\mathbf{d}}) = \mathbf{L}_{\mathbf{p},\mathbf{d}}^\star = \boldsymbol{l}_{\mathbf{p},\mathbf{d}}$ represent the same line with the same orientation. If $\mathbf{d} = \hat{\mathbf{d}}$, then $J_e(\mathbf{L}_{\mathbf{p},\hat{\mathbf{d}}}) = \hat{\boldsymbol{l}}_{\mathbf{p},\mathbf{d}} = \hat{\boldsymbol{l}}$ is a plane-based unit line, where $\hat{\boldsymbol{l}}^2 = -1$.

The line $\mathbf{L}$ is derived as follows: Given two 3D points on the line, $\mathbf{p}_1$ and $\mathbf{p}_2$, the direction and orientation of the line is defined by $\mathbf{d} = \mathbf{p}_2 - \mathbf{p}_1$, toward $\mathbf{p}_2$. Given any third point $\mathbf{p}_3 = \mathbf{t}$ on the line, $\mathbf{t}-\mathbf{p}_1$ should be parallel to $\mathbf{d}$ and the projection $((\mathbf{t}-\mathbf{p}_1)\cdot\mathbf{d}^*)\mathbf{d}^{*-1} = -((\mathbf{t}-\mathbf{p}_1)\cdot\mathbf{d}^*)\hat{\mathbf{d}}^*\|\mathbf{d}\|^{-1}$ should be 0. We abridge $\hat{\mathbf{d}}^*\|\mathbf{d}\|^{-1}$ and take $(\mathbf{p}_1 - \mathbf{t}) \cdot \mathbf{d}^* = 0$ as the *vector-valued* condition for $\mathbf{t}$ to be on the line. We did not abridge the minus sign since that is the orientation of the projection. This condition is essentially the Plücker coordinates $(\mathbf{d}, \mathbf{m} = \mathbf{p}_1 \times \mathbf{d})$ condition $\mathbf{t} \times \mathbf{d} = \mathbf{m}$ for a line. We dualize the vector-valued condition into the 3-blade condition $\mathbf{P}_\mathbf{t} \wedge \mathbf{L} = ((\mathbf{p}_1 - \mathbf{t})\cdot\mathbf{d}^*)\mathbf{I}_4$. Using the identity $(\mathbf{a}\cdot\mathbf{b}^*)\mathbf{I}_4 = \mathbf{a}\wedge\mathbf{e}_0\wedge\mathbf{b}$, then $(\mathbf{e}_0+\mathbf{t})\wedge\mathbf{L} = (\mathbf{p}_1-\mathbf{t})\wedge(\mathbf{e}_0\wedge\mathbf{d}) = -\mathbf{e}_0\wedge\mathbf{p}_1\wedge\mathbf{d} - \mathbf{t}\wedge\mathbf{e}_0\wedge\mathbf{d}$. Let $\mathbf{L} = -\mathbf{p}_1\wedge\mathbf{d} - \mathbf{e}_0\wedge\mathbf{d} = -(\mathbf{e}_0+\mathbf{p}_1)\wedge\mathbf{d} = \mathbf{d}\wedge\mathbf{P}_1$. If $\mathbf{t}$ is on the line, then $-\mathbf{t}\wedge\mathbf{p}_1\wedge\mathbf{d} = 0$, so this term can be ignored. With this derivation of $\mathbf{L}$, we have established a certain orientation, $\mathbf{L} = \mathbf{d}\wedge\mathbf{P} = \mathbf{P}_{\infty\mathbf{d}}\wedge\mathbf{P}$. This orientation is such that the line acts as an axis of rotation when we dualize it to the CPNS PGA 2-blade line $\boldsymbol{l} = J_e(\mathbf{L})$, where $R_{\boldsymbol{l}} = \exp(\theta\hat{\boldsymbol{l}}/2)$ is a rotor for counterclockwise rotation around $\boldsymbol{l}$ by angle θ with $\hat{\mathbf{d}}$ as the axis of rotation through $\boldsymbol{l}$ in the sense of the right-hand rule.

2.1.3. OPNS PGA 3-blade Plane Geometric Entity. The OPNS PGA 3-blade plane entity $\mathbf{\Pi}$ through point $\mathbf{P} = \mathbf{e}_0 + \mathbf{p}$ with normal $\mathbf{n}$ is

$$\mathbf{\Pi} = \mathbf{\Pi}_{\mathbf{p},\mathbf{n}} = \mathbf{P} \wedge \mathbf{n}^* = D_e(\boldsymbol{\pi}_{\mathbf{p},\mathbf{n}}) = \boldsymbol{\pi}_{\mathbf{p},\mathbf{n}}^{-\star}. \tag{5}$$

$\mathbf{\Pi}_{\mathbf{p},\mathbf{n}}$ and $J_e(\mathbf{\Pi}_{\mathbf{p},\mathbf{n}}) = \mathbf{\Pi}_{\mathbf{p},\mathbf{n}}^\star = \boldsymbol{\pi}_{\mathbf{p},\mathbf{n}}$ represent the same plane with same orientation. If $\mathbf{n} = \hat{\mathbf{n}}$, then $J_e(\mathbf{\Pi}_{\mathbf{p},\hat{\mathbf{n}}}) = \hat{\boldsymbol{\pi}}_{\mathbf{p},\mathbf{n}} = \hat{\boldsymbol{\pi}}$ is a plane-based unit plane, where $\hat{\boldsymbol{\pi}}^2 = 1$.

The plane spanning three points $\{\mathbf{P}_1, \mathbf{P}_2, \mathbf{P}_3\}$ arranged *clockwise* on the plane is

$$\mathbf{\Pi}_{\circlearrowright} = \mathbf{P}_1 \wedge \mathbf{P}_2 \wedge \mathbf{P}_3. \tag{6}$$

If the points are arranged *counterclockwise* on the plane, then the plane is

$$\mathbf{\Pi}_{\circlearrowleft} = \mathbf{P}_3 \wedge \mathbf{P}_2 \wedge \mathbf{P}_1. \tag{7}$$

The plane with normal $\hat{\mathbf{n}}$ through $\mathbf{p}$, or distance $d = \mathbf{p}\cdot\hat{\mathbf{n}}$ from origin, is

$$\mathbf{\Pi}_{d,\hat{\mathbf{n}}} = \mathbf{e}_0 \wedge \hat{\mathbf{n}}^* + \mathbf{p} \wedge \hat{\mathbf{n}}^* = \mathbf{e}_0 \wedge \hat{\mathbf{n}}^* - (\mathbf{p}\cdot\hat{\mathbf{n}})\mathbf{I}_3 = D_e(\hat{\boldsymbol{\pi}}_{d,\mathbf{n}}) = \hat{\boldsymbol{\pi}}_{d,\mathbf{n}}^{-\star}. \tag{8}$$

The plane $\mathbf{\Pi}$ is derived as follows: Given any point $\mathbf{p}$ on the plane and its normal vector $\mathbf{n}$, then any other point $\mathbf{t}$ on the plane must satisfy the *scalar-valued* condition $(\mathbf{t}-\mathbf{p})\cdot\mathbf{n}=0$. We dualize as the pseudoscalar condition $\mathbf{P_t}\wedge\mathbf{\Pi}=((\mathbf{t}-\mathbf{p})\cdot\mathbf{n})\mathbf{I}_4=0$. Using the identity $(\mathbf{a}\cdot\mathbf{b})\mathbf{I}_4=\mathbf{a}\wedge\mathbf{b}^*\wedge\mathbf{e}_0$, then $(\mathbf{e}_0+\mathbf{t})\wedge\mathbf{\Pi}=(\mathbf{t}-\mathbf{p})\wedge\mathbf{n}^*\wedge\mathbf{e}_0=\mathbf{t}\wedge\mathbf{e}_0\wedge\mathbf{n}^*+\mathbf{e}_0\wedge\mathbf{p}\wedge\mathbf{n}^*$. Let $\mathbf{\Pi}=\mathbf{e}_0\wedge\mathbf{n}^*+\mathbf{p}\wedge\mathbf{n}^*=\mathbf{P_p}\wedge\mathbf{n}^*=\mathbf{n}^*\wedge\mathbf{P}$. With $\mathbf{P}=\mathbf{P}_1$ and $\mathbf{n}^*=(\mathbf{P}_3-\mathbf{P}_1)\wedge(\mathbf{P}_2-\mathbf{P}_1)$, then $\mathbf{\Pi}=\mathbf{P}_3\wedge\mathbf{P}_2\wedge\mathbf{P}_1$, where the three points are arranged counterclockwise on the face side of the plane, which is into the direction $\mathbf{n}$.

2.2. Point-based PGA Operations

2.2.1. Rotation Operation. In the point-based algebra, the 2-versor rotor (rotation operator) $R=\exp(\theta\hat{\mathbf{n}}^*/2)=\cos(\theta/2)+\sin(\theta/2)\hat{\mathbf{n}}^*$ can be applied to any multivector or entity A as $A'=RAR^{-1}$ to rotate A around axis $\hat{\mathbf{n}}$ counterclockwise by angle θ (by right-hand rule) centered on the origin.

The rotor R can be derived as successive reflections in two planes through the origin, represented by their normal vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$, with angle $\theta/2$ between them. Then, A is reflected successively in each plane as $A'=-\hat{\mathbf{b}}(-\hat{\mathbf{a}}A\hat{\mathbf{a}}^{-1})\hat{\mathbf{b}}^{-1}$, or $R=\hat{\mathbf{b}}\hat{\mathbf{a}}=\hat{\mathbf{b}}\cdot\hat{\mathbf{a}}+\hat{\mathbf{b}}\wedge\hat{\mathbf{a}}=\cos(\theta/2)+\sin(\theta/2)\hat{\mathbf{n}}^*$ and $A'=RAR^{-1}$. Each reflection is called a 1-versor operation, where vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ are called 1-versors. The rotor R is called a 2-versor, which is the product of two 1-versors (vectors).

2.2.2. Join Operation. As discussed in Section 1, in the point-based geometric algebra of PGA, the wedge product $\wedge$ of points is the *join product*, producing an entity that represents the span of the joined points. The join of two points is their line $\mathbf{L}=\mathbf{P}_2\wedge\mathbf{P}_1$ (4) and the join of three points is their plane $\mathbf{\Pi}=\mathbf{P}_3\wedge\mathbf{P}_2\wedge\mathbf{P}_1$ (5). The join product of points is why it is called the point-based geometric algebra.

3. The Plane-based Geometric Algebra of PGA

In this section, we define or derive each entity in the plane-based algebra of PGA, also called the CPNS PGA. The orientation of each entity is important for defining the new geometric entity dualization operation J_e in Section 4.

3.1. CPNS PGA Geometric Entities

The plane-based geometric entities are the CPNS 1-blade plane $\boldsymbol{\pi}$, the CPNS 2-blade line $\boldsymbol{l}$, and the CPNS 3-blade point $\boldsymbol{p}$.

3.1.1. CPNS 1-blade Plane Geometric Entity. The CPNS PGA 1-blade plane entity $\boldsymbol{\pi}=\boldsymbol{\pi}_{\mathbf{p},\mathbf{n}}$ with normal $\mathbf{n}$ through $\mathbf{p}$ is

$$\boldsymbol{\pi}=\boldsymbol{\pi}_{\mathbf{p},\mathbf{n}}=\mathbf{n}+(\mathbf{p}\cdot\mathbf{n})\mathbf{e}_0=J_e(\mathbf{\Pi}_{\mathbf{p},\mathbf{n}})=\mathbf{\Pi}^{\star}_{\mathbf{p},\mathbf{n}}. \tag{9}$$

If $\mathbf{n}=\hat{\mathbf{n}}$, then $d=\mathbf{p}\cdot\mathbf{n}$ is the distance from the origin and $\boldsymbol{\pi}=\hat{\boldsymbol{\pi}}=\hat{\boldsymbol{\pi}}_{d,\mathbf{n}}$ is a unit plane, where $\hat{\boldsymbol{\pi}}^2=1$ and $D_e(\hat{\boldsymbol{\pi}})=D_e(\hat{\boldsymbol{\pi}}_{d,\mathbf{n}})=\hat{\boldsymbol{\pi}}^{-\star}_{d,\mathbf{n}}=\mathbf{\Pi}_{d,\hat{\mathbf{n}}}$. The join of three points is $\boldsymbol{\pi}=(\boldsymbol{p}_3^{-\star}\wedge\boldsymbol{p}_2^{-\star}\wedge\boldsymbol{p}_1^{-\star})^{\star}=\boldsymbol{p}_3\vee\boldsymbol{p}_2\vee\boldsymbol{p}_1$.

The plane $\boldsymbol{\pi}$ is derived as follows: The plane is defined by any point $\mathbf{p}$ on the plane and its normal vector $\mathbf{n}$. Any point $\mathbf{t}$ on the plane must satisfy the scalar-valued condition $(\mathbf{t}-\mathbf{p})\cdot\mathbf{n}=0$, the same as for $\mathbf{\Pi}$ (5). We dualize as the pseudoscalar condition $\boldsymbol{p}_{\mathbf{t}}\times\boldsymbol{\pi}=\boldsymbol{p}_{\mathbf{t}}\wedge\boldsymbol{\pi}=((\mathbf{t}-\mathbf{p})\cdot\mathbf{n})\mathbf{I}_4=0$ and solve for the 1-blade plane $\boldsymbol{\pi}$ and 3-blade point $\boldsymbol{p}_{\mathbf{t}}$. Using the identity $(\mathbf{a}\cdot\mathbf{b})\mathbf{I}_4=\mathbf{b}\wedge\mathbf{a}^*\wedge\mathbf{e}_0$, then $(\mathbf{t}\cdot\mathbf{n}-\mathbf{p}\cdot\mathbf{n})\mathbf{I}_4=-\mathbf{t}^*\wedge\mathbf{e}_0\wedge\mathbf{n}+\mathbf{I}_3(\mathbf{p}\cdot\mathbf{n})\mathbf{e}_0=\boldsymbol{p}_{\mathbf{t}}\wedge\boldsymbol{\pi}$. Let $\boldsymbol{p}_{\mathbf{t}}=-\mathbf{t}^*\wedge\mathbf{e}_0+\mathbf{I}_3=(1+\mathbf{t}^*\mathbf{I}_4)\mathbf{I}_3$ and $\boldsymbol{\pi}=\mathbf{n}+(\mathbf{p}\cdot\mathbf{n})\mathbf{e}_0$, so we have derived the point $\boldsymbol{p}_{\mathbf{t}}$ and plane $\boldsymbol{\pi}$ together. We see that $\mathbf{P}_{\mathbf{t}}\wedge\mathbf{\Pi}=\boldsymbol{p}_{\mathbf{t}}\wedge\boldsymbol{\pi}$, so that they represent the same condition with the same orientation, which is important for observing the duals and determining J_e in Section 4.

3.1.2. CPNS 2-blade Line Geometric Entity. The CPNS PGA 2-blade line entity $\boldsymbol{l}$ is the meet of two planes as

$$\boldsymbol{l}=\boldsymbol{\pi}_2\wedge\boldsymbol{\pi}_1. \tag{10}$$

The line through $\mathbf{p}$ in direction $\mathbf{d}$ is

$$\boldsymbol{l}=\boldsymbol{l}_{\mathbf{p},\mathbf{d}}=\mathbf{d}^*-(\mathbf{p}\cdot\mathbf{d}^*)\mathbf{e}_0=J_e(\mathbf{L}_{\mathbf{p},\mathbf{d}})=\mathbf{L}^{\star}_{\mathbf{p},\mathbf{d}}. \tag{11}$$

If $\mathbf{d}=\hat{\mathbf{d}}$, then $\boldsymbol{l}=\hat{\boldsymbol{l}}$ is a unit line, where $\hat{\boldsymbol{l}}^2=-1$ and $D_e(\hat{\boldsymbol{l}})=\hat{\boldsymbol{l}}^{-\star}=\mathbf{L}_{\mathbf{p},\hat{\mathbf{d}}}$. The join of two points is $\boldsymbol{l}=(\boldsymbol{p}_2^{-\star}\wedge\boldsymbol{p}_1^{-\star})^{\star}=\boldsymbol{p}_2\vee\boldsymbol{p}_1$.

The line $\boldsymbol{l}$ is derived as follows: The line is defined by any point $\mathbf{p}$ on the line and the direction $\mathbf{d}$ through $\mathbf{p}$. Any point $\mathbf{t}$ on the line must satisfy the vector-valued condition $(\mathbf{p}-\mathbf{t})\cdot\mathbf{d}^*=0$, the same as for $\mathbf{L}$ (4). We dualize the vector-valued condition into the 3-blade condition $\boldsymbol{p}_{\mathbf{t}}\times\boldsymbol{l}=((\mathbf{p}-\mathbf{t})\cdot\mathbf{d}^*)\mathbf{I}_4=\langle(\mathbf{I}_3-\mathbf{t}^*\mathbf{e}_0)\boldsymbol{l}\rangle_3$, with 3-blade point $\boldsymbol{p}_{\mathbf{t}}=\mathbf{I}_3-\mathbf{t}^*\mathbf{e}_0$ and 2-blade line $\boldsymbol{l}$, solving for $\boldsymbol{l}$. We expand geometric products as $((\mathbf{p}-\mathbf{t})\mathbf{d}^*)\mathbf{I}_4=\mathbf{p}\mathbf{d}^*\mathbf{I}_4-\mathbf{t}\mathbf{d}^*\mathbf{I}_4=\mathbf{I}_3\mathbf{e}_0\mathbf{p}\mathbf{d}^*-\mathbf{e}_0\mathbf{t}^*\mathbf{d}^*$. Factoring out $\boldsymbol{p}_{\mathbf{t}}$, we let $\boldsymbol{l}=\langle\mathbf{e}_0\mathbf{p}\mathbf{d}^*\rangle_2+\mathbf{d}^*=\mathbf{e}_0(\mathbf{p}\cdot\mathbf{d}^*)+\mathbf{d}^*$. We now find that $\boldsymbol{p}_{\mathbf{t}}\times\boldsymbol{l}=(\mathbf{I}_3-\mathbf{e}_0\mathbf{t}^*)\times(\mathbf{d}^*+\mathbf{e}_0(\mathbf{p}\cdot\mathbf{d}^*))=\mathbf{I}_3\times(\mathbf{e}_0(\mathbf{p}\cdot\mathbf{d}^*))-(\mathbf{e}_0\mathbf{t}^*)\times\mathbf{d}^*$ is the grade 3 part. We see that $\boldsymbol{p}_{\mathbf{t}}\times\boldsymbol{l}=\mathbf{P}_{\mathbf{t}}\wedge\mathbf{L}$, so that they represent the same condition with the same orientation, which is important for observing the duals and determining J_e in Section 4.

3.1.3. CPNS 3-blade Point Geometric Entity. The CPNS PGA 3-blade point entity $\boldsymbol{p}_{\mathbf{t}}$, embedding $\mathbf{t}=x\mathbf{e}_1+y\mathbf{e}_2+z\mathbf{e}_3$, is

$$\boldsymbol{p}_{\mathbf{t}}=(1+\mathbf{e}_0\mathbf{t})\mathbf{I}_3=\mathbf{I}_3+\mathbf{I}_4\mathbf{t}=\mathbf{I}_3-\mathbf{e}_0\mathbf{t}^*=J_e(\mathbf{P}_{\mathbf{t}})=\mathbf{P}^{\star}_{\mathbf{t}}. \tag{12}$$

In dual quaternions, $1+\mathbf{e}_0\mathbf{t}=1+\mathbf{t}^*\mathbf{I}_4=p_{\mathbf{t}}$ is a homogeneous point. The product of two points $\boldsymbol{p}_{\mathbf{t}}\boldsymbol{p}_{\mathbf{p}}=-p_{\mathbf{t}}\overline{p}_{\mathbf{p}}=-p_{\mathbf{t}-\mathbf{p}}=-(1+\mathbf{e}_0(\mathbf{t}-\mathbf{p}))$ represents their difference, and the grade 2 *bivector-valued* part is given by $\boldsymbol{p}_{\mathbf{t}}\times\boldsymbol{p}_{\mathbf{p}}$. Points of form $\boldsymbol{p}_{\infty\mathbf{t}}=\mathbf{I}_4\mathbf{t}=J_e(\mathbf{t})=J_e(\mathbf{P}_{\infty\mathbf{t}})$ represent directed infinite points at infinity. The meet of three planes is the point

$$\boldsymbol{p}=\boldsymbol{\pi}_3\wedge\boldsymbol{\pi}_2\wedge\boldsymbol{\pi}_1=\boldsymbol{\pi}\wedge\boldsymbol{l}. \tag{13}$$

For $\{\boldsymbol{\pi}_x=\mathbf{e}_1+x\mathbf{e}_0,\ \boldsymbol{\pi}_y=\mathbf{e}_2+y\mathbf{e}_0,\ \boldsymbol{\pi}_z=\mathbf{e}_3+z\mathbf{e}_0\}$, $\boldsymbol{p}_{\mathbf{t}}=\boldsymbol{\pi}_x\wedge\boldsymbol{\pi}_y\wedge\boldsymbol{\pi}_z$ embeds the point $\mathbf{t}=x\mathbf{e}_1+y\mathbf{e}_2+z\mathbf{e}_3$. For finite point $\boldsymbol{p}_{\mathbf{t}}$, vector $\mathbf{t}$ can be projected as

$$\mathbf{t}=\mathbf{I}_3(\boldsymbol{p}_{\mathbf{t}}^{-\star}\wedge\mathbf{e}_0)^{\star}/(\mathbf{I}_3\wedge\boldsymbol{p}_{\mathbf{t}}^{-\star})^{\star}. \tag{14}$$

3.2. Plane-based PGA Operations

3.2.1. Meet Operation. As discussed in Section 1, in the plane-based geometric algebra of PGA, the wedge product $\wedge$ of planes is their *meet product*, producing an entity that represents their intersection. The meet of two planes is their intersection line $\boldsymbol{l} = \boldsymbol{\pi}_2 \wedge \boldsymbol{\pi}_1$ (11) and the meet of three planes is their intersection point $\boldsymbol{p} = \boldsymbol{\pi}_3 \wedge \boldsymbol{\pi}_2 \wedge \boldsymbol{\pi}_1$ (12). The meet product of planes is why it is called the plane-based geometric algebra.

3.2.2. Translation Operation. In the plane-based algebra, the point $\boldsymbol{p}_{\mathbf{t}} = (1 + \mathbf{t}^* \mathbf{I}_4)\mathbf{I}_3 = p_{\mathbf{t}}\mathbf{I}_3$ is based on the homogeneous dual quaternion point $p_{\mathbf{t}}$. The product of two dual quaternion points $p_{\mathbf{t}}$ and $p_{\mathbf{d}}$ is commutative and $p_{\mathbf{t}} p_{\mathbf{d}} = p_{\mathbf{t}+\mathbf{d}}$, which is a translation operation on $p_{\mathbf{t}}$ into $p_{\mathbf{t}+\mathbf{d}}$. We can use

$$T = T_{\mathbf{d}} = p_{\mathbf{d}/2} = 1 + \mathbf{d}^* \mathbf{I}_4/2 = 1 + \mathbf{e}_0 \mathbf{d}/2 = \exp(\mathbf{e}_0 \mathbf{d}/2) \tag{15}$$

as a translator (translation operator). Acting on $\boldsymbol{p}_{\mathbf{t}}$, we have $\boldsymbol{p}_{\mathbf{t}'} = T \boldsymbol{p}_{\mathbf{t}} T^{-1} = T p_{\mathbf{t}} T \mathbf{I}_3 = p_{\mathbf{t}+\mathbf{d}} \mathbf{I}_3 = \boldsymbol{p}_{\mathbf{t}+\mathbf{d}}$, where T acts as a 2-versor translation operator on any point $\boldsymbol{p}$. Since $\boldsymbol{p} = \boldsymbol{\pi}_3 \wedge \boldsymbol{\pi}_2 \wedge \boldsymbol{\pi}_1$, then by outermorphism, T also acts on each plane $\boldsymbol{\pi}_i$, translating it correctly, and therefore translating the point correctly. Since T also correctly translates any plane $\boldsymbol{\pi}$, it also correctly translates any line $\boldsymbol{l} = \boldsymbol{\pi}_2 \wedge \boldsymbol{\pi}_1$ by outermorphism.

The translator T can only be used in the plane-based algebra on the plane-based entities; it cannot be used in the point-based algebra. To translate entities in the point-based algebra, they have to be dualized to plane-based entities, translated, and then dualized back to point-based entities.

3.2.3. Reflection Operation. We can reflect any general plane $\boldsymbol{\pi}_1 = \mathbf{n}_1 + d_1 \mathbf{e}_0$ in another plane $\boldsymbol{\pi}_2 = \mathbf{n}_2$ through the origin with normal $\mathbf{n}_2$ as $\boldsymbol{\pi}_1' = -\boldsymbol{\pi}_2 \boldsymbol{\pi}_1 \boldsymbol{\pi}_2^{-1} = -\hat{\boldsymbol{\pi}}_2 \boldsymbol{\pi}_1 \hat{\boldsymbol{\pi}}_2 = -\hat{\mathbf{n}}_2(\mathbf{n}_1 + d_1 \mathbf{e}_0)\hat{\mathbf{n}}_2 = -\hat{\mathbf{n}}_2 \mathbf{n}_1 \hat{\mathbf{n}}_2 + d_1 \mathbf{e}_0$, where $\hat{\boldsymbol{\pi}}_2 = \hat{\mathbf{n}}_2 = \boldsymbol{\pi}_2/\sqrt{\boldsymbol{\pi}_2^2}$. If $\boldsymbol{\pi}_2 = \mathbf{n}_2 + d_2 \mathbf{e}_0$ is also a general plane, then we can use the translator $T = T_{d_2 \mathbf{n}_2}$ to translate it to the origin, reflect in it, then translate back as $\boldsymbol{\pi}_1' = T T^{-1}(-\boldsymbol{\pi}_2 \boldsymbol{\pi}_1 \boldsymbol{\pi}_2^{-1}) T T^{-1} = -\boldsymbol{\pi}_2 \boldsymbol{\pi}_1 \boldsymbol{\pi}_2^{-1}$. The translations cancel, showing that we can reflect general planes in general planes.

In the plane-based algebra, the point $\boldsymbol{p} = \boldsymbol{\pi}_3 \wedge \boldsymbol{\pi}_2 \wedge \boldsymbol{\pi}_1$ and line $\boldsymbol{l} = \boldsymbol{\pi}_2 \wedge \boldsymbol{\pi}_1$ are the meet of planes, and by outermorphism they can also be reflected in any general plane $\boldsymbol{\pi}$ as $\boldsymbol{p}' = \pm \boldsymbol{\pi} \boldsymbol{p} \boldsymbol{\pi}^{-1}$ and $\boldsymbol{l}' = (-\boldsymbol{\pi} \boldsymbol{\pi}_2 \boldsymbol{\pi}^{-1}) \wedge (-\boldsymbol{\pi} \boldsymbol{\pi}_1 \boldsymbol{\pi}^{-1}) = \boldsymbol{\pi} \boldsymbol{l} \boldsymbol{\pi}^{-1}$. For $\boldsymbol{p}'$ we can choose the sign. We choose $\boldsymbol{p}' = \boldsymbol{\pi} \boldsymbol{p} \boldsymbol{\pi}^{-1}$ to reflect the point as non-oriented so that $\boldsymbol{p}' = \mathbf{e}_0 + \mathbf{p}'$ remains in standard form and orientation with term $+\mathbf{e}_0$. We choose $\boldsymbol{p}' = -\boldsymbol{\pi} \boldsymbol{p} \boldsymbol{\pi}^{-1}$ to reflect the point as oriented so that $\boldsymbol{p}' = -(\mathbf{e}_0 + \mathbf{p}')$ has non-standard orientation. The choice of sign depends on the application and how the reflected point is to be used. When reflecting the three points of a plane $\mathbf{\Pi}_1 = \mathbf{P}_3 \wedge \mathbf{P}_2 \wedge \mathbf{P}_1$ in another plane $\boldsymbol{\pi}_2$, we choose oriented reflection $\mathbf{P}_i' = (-\boldsymbol{\pi}_2 \mathbf{P}_i^\star \boldsymbol{\pi}_2^{-1})^{-\star}$ and $\mathbf{\Pi}_1' = \mathbf{P}_3' \wedge \mathbf{P}_2' \wedge \mathbf{P}_1'$. The orientation of the reflected line $\boldsymbol{l}'$ is the negative of $\boldsymbol{l}$, so that if $\boldsymbol{l}$ is an axis of counterclockwise rotation, then $\boldsymbol{l}'$ is an axis of clockwise rotation in the reflected mirror image.

Successive reflections in two parallel unit planes, $\boldsymbol{\pi}_1 = \hat{\mathbf{n}} + d_1\mathbf{e}_0$ and $\boldsymbol{\pi}_2 = \hat{\mathbf{n}} + d_2\mathbf{e}_0$, separated by $\mathbf{d}/2 = (d_2 - d_1)\hat{\mathbf{n}}$ is the translation operator $T = \boldsymbol{\pi}_2\boldsymbol{\pi}_1 = \exp(\mathbf{e}_0\mathbf{d}/2)$.

3.2.4. Rotation Operation. Successive reflections in two non-parallel unit planes, $\boldsymbol{\pi}_1 = \hat{\mathbf{n}}_1 + d_1\mathbf{e}_0$ and $\boldsymbol{\pi}_2 = \hat{\mathbf{n}}_2 + d_2\mathbf{e}_0$, meeting in line $\boldsymbol{l} = \boldsymbol{\pi}_2 \wedge \boldsymbol{\pi}_1$ with angle $\theta/2$ between them is the rotor $R_{\boldsymbol{l}} = \boldsymbol{\pi}_2\boldsymbol{\pi}_1 = \boldsymbol{\pi}_2 \cdot \boldsymbol{\pi}_1 + \boldsymbol{\pi}_2 \wedge \boldsymbol{\pi}_1 = \cos(\theta/2) + \sin(\theta/2)\hat{\boldsymbol{l}} = \exp(\theta\hat{\boldsymbol{l}}/2)$ that rotates around the line $\boldsymbol{l}$ by angle θ counterclockwise around the direction $\mathbf{d} = (\hat{\mathbf{n}}_2 \wedge \hat{\mathbf{n}}_1)\mathbf{I}_3$ of the line by right-hand rule. The rotor $R_{\boldsymbol{l}}$ generalizes the rotor R and can also be formed as $R_{\boldsymbol{l}} = T_{\mathbf{p}}RT_{\mathbf{p}}^{-1}$, where $R = \exp(\theta\hat{\mathbf{d}}^*/2)$ and $\mathbf{p}$ is the new center of rotation or is any point on the unit line $\hat{\boldsymbol{l}} = \hat{\mathbf{d}}^* - (\mathbf{p} \cdot \hat{\mathbf{d}}^*)\mathbf{e}_0$. While R can rotate any PGA entity, the rotor $R_{\boldsymbol{l}}$ can only be used in the plane-based algebra since it uses the plane-based translator T.

3.2.5. Projection Operation. In the plane-based algebra, we can project a lower dimensional geometric object onto a higher dimensional geometric object. A point is 0-dim, a line is 1-dim, and a plane is 2-dim. A point $\boldsymbol{p}$ can be projected onto a plane $\boldsymbol{\pi}$ as $\boldsymbol{p}' = (\boldsymbol{p} \cdot \boldsymbol{\pi})\boldsymbol{\pi}^{-1}$ or onto a line $\boldsymbol{l}$ as $\boldsymbol{p}' = (\boldsymbol{p} \cdot \boldsymbol{l})\boldsymbol{l}^{-1}$, and a line $\boldsymbol{l}$ can be projected onto a plane $\boldsymbol{\pi}$ as $\boldsymbol{l}' = (\boldsymbol{l} \cdot \boldsymbol{\pi})\boldsymbol{\pi}^{-1}$.

3.2.6. Rejection Operation. In the plane-based algebra, we can reject a plane $\boldsymbol{\pi}$ from a line $\boldsymbol{l}$ as $\boldsymbol{\pi}' = (\boldsymbol{\pi} \wedge \boldsymbol{l})\boldsymbol{l}^{-1}$ or reject a line $\boldsymbol{l}$ from plane $\boldsymbol{\pi}$ as $\boldsymbol{l}' = (\boldsymbol{l} \wedge \boldsymbol{\pi})\boldsymbol{\pi}^{-1}$. The rejected line $\boldsymbol{l}'$ is orthogonal to $\boldsymbol{\pi}$ with the same meet $\boldsymbol{l}' \wedge \boldsymbol{\pi} = \boldsymbol{l} \wedge \boldsymbol{\pi}$. The rejected plane $\boldsymbol{\pi}'$ is orthogonal to $\boldsymbol{l}$ with the same meet $\boldsymbol{\pi}' \wedge \boldsymbol{l} = \boldsymbol{\pi} \wedge \boldsymbol{l}$.

4. Geometric Entity Dualization Operation

This section introduces the new geometric entity dualization operation $J_e(\mathbf{A}) = \mathbf{A}^\star = \boldsymbol{a}$ and its inverse (undual) $D_e(\boldsymbol{a}) = -J_e(\boldsymbol{a}) = \boldsymbol{a}^{-\star} = \mathbf{A}^{-\star\star} = \mathbf{A}$ for PGA $\mathcal{G}_{3,0,1}$. For notation, an element $\mathbf{A} \in \{\mathbf{P}, \mathbf{L}, \mathbf{\Pi}\}$ is in the point-based algebra, and an element $\boldsymbol{a} \in \{\boldsymbol{p}, \boldsymbol{l}, \boldsymbol{\pi}\}$ is in the plane-based algebra.

In Section 2, we reviewed the point-based algebra of PGA, also called the OPNS PGA with outer product null space (OPNS) geometric entities for point $\mathbf{P}$, line $\mathbf{L}$, and plane $\mathbf{\Pi}$. We defined the OPNS 1-blade point $\mathbf{P} = \boldsymbol{p}^{-\star}$ (1) having a standard form and orientation. We derived the OPNS 2-blade line $\mathbf{L} = \boldsymbol{l}^{-\star}$ (4) and OPNS 3-blade plane $\mathbf{\Pi} = \boldsymbol{\pi}^{-\star}$ (5) such that they each have a well-defined orientation. The point-based OPNS entities $\mathbf{A} \in \{\mathbf{P}, \mathbf{L}, \mathbf{\Pi}\}$ will be found to be "undual" to the plane-based CPNS entities $\boldsymbol{a} \in \{\boldsymbol{p}, \boldsymbol{l}, \boldsymbol{\pi}\}$ such that dual entities represent the same geometry with the same orientation.

In Section 3, we reviewed the plane-based algebra of PGA, also called the CPNS PGA with commutator product null space (CPNS) geometric entities for point $\boldsymbol{p}$, line $\boldsymbol{l}$, and plane $\boldsymbol{\pi}$. We derived the CPNS 1-blade plane $\boldsymbol{\pi} = \mathbf{\Pi}^\star$ (9), CPNS 2-blade line $\boldsymbol{l} = \mathbf{L}^\star$ (11), and CPNS 3-blade point $\boldsymbol{p} = \mathbf{P}^\star$ (12) such that they each have a well-defined orientation. The plane-based CPNS entities

$\boldsymbol{a} \in \{\boldsymbol{p}, \boldsymbol{l}, \boldsymbol{\pi}\}$ are dual to the point-based OPNS entities $\mathbf{A} \in \{\mathbf{P}, \mathbf{L}, \mathbf{\Pi}\}$ such that dual entities represent the same geometry with the same orientation.

The geometric entity dualization operation J_e is to be defined such that the dual entities represent the same geometry with the same orientation through the dualization operation, without any incorrect sign (orientation) changes. Using the entities as defined, we shall compare various pairs of dual entities to observe empirically how each basis blade in $\mathcal{G}_{3,0,1}$ should dualize to maintain the requirement that corresponding dual entities in the point-based and plane-based algebras represent the same geometry with the same orientation. We shall define J_e by empirical observation, making no attempt to generalize the dualization. We are only concerned with dualization in $\mathcal{G}_{3,0,1}$. We make no initial assumptions about the duals or the dualization operation J_e, but we shall find empirically that J_e turns out to be an anti-involution with inverse $J_e^{-1} = -J_e$.

This section is organized as follows. In Section 4.1, we review concepts of dualization in geometric algebra. In Section 4.2, we empirically observe dual entities representing the same geometry with the same orientation to determine and define the new geometric entity dualization operation $J_e(\mathbf{A}) = \mathbf{A}^\star$ as the observed duals of basis blades that are collected in Table 1. In Section 4.3, we implement J_e by the algebraic methods of Table 2 using algebras $\{\mathcal{G}_{4,0,0}, \mathcal{G}_{3,1,0}, \mathcal{G}_{1,3,0}\}$ with a non-degenerate metric that correspond to PGA $\mathcal{G}_{3,0,1}$ with a degenerate metric.

4.1. Review of Dualization in Geometric Algebra

This section, which may be skipped on a first reading and read later, provides supporting theory for the methodology used in Sections 4.2 and 4.3 to define and implement the new geometric entity dualization operation J_e for PGA.

In Section 4.1.1, we begin with a brief review of the basic concepts of geometric algebra that we need to discuss dualization. In Section 4.1.2, we review the basic concepts of dualization in geometric algebra. In Section 4.1.3, we generalize to more advanced concepts of dualization, including forms of Hodge star $\star$ dualizations for degenerate metric algebras.

In Section 4.2, we use the basic concepts of dualization to empirically observe and determine the duals of basis blades in PGA $\mathcal{G}_{3,0,1}$ as J_e in Table 1. In Section 4.3, we use the more advanced concepts of dualization to formulate the implementations of J_e by the algebraic methods of Table 2.

4.1.1. Review of Basic Concepts of Geometric Algebra. The geometric algebra $\mathcal{G}_n$ has a set of n basis vectors $\{\mathbf{e}_i : 0 \leq i < n\}$. The $n \times n$ matrix of inner products $g_{ij} = [\mathbf{e}_i \cdot \mathbf{e}_j]$ defines the *metric* (or metric tensor) of $\mathcal{G}_n$. The notation $\mathcal{G}_{p,q,r}$, $n = p + q + r$, usually defines a form of diagonal metric, where $\mathbf{e}_i \cdot \mathbf{e}_j = 0$ for $i \neq j$ and we specify only the diagonal entries $g_{ii} = \mathrm{diag}(\mathbf{e}_0 \cdot \mathbf{e}_0 = \mathbf{e}_0^2, \mathbf{e}_1^2, \ldots, \mathbf{e}_{n-1}^2)$ with any p, q, and r of $\mathbf{e}_i^2$ equal to $+1$, -1, and 0, respectively. The numbers (p, q, r) indicate the metric *signature* of $\mathcal{G}_{p,q,r}$. If $r \neq 0$, then the metric is called degenerate, and otherwise non-degenerate for $r = 0$. PGA $\mathcal{G}_{3,0,1}$ has the set of basis vectors $\{\mathbf{e}_i : 0 \leq i < 4\}$ and the degenerate metric $g_{ij} = [\mathbf{e}_i \cdot \mathbf{e}_j] = \mathrm{diag}(0, 1, 1, 1)$.

A linear combination $\mathbf{v} = \sum a^i \mathbf{e}_i = a^i \mathbf{e}_i$ of basis vectors $\mathbf{e}_i$ is a vector $\mathbf{v}$. An outer product $\mathbf{A} = \bigwedge^k \mathbf{v}_i = \mathbf{v}_1 \wedge \mathbf{v}_2 \wedge \cdots \wedge \mathbf{v}_k$ of k different vectors $\mathbf{v}_i$ is called a k-blade, or simple k-vector, and is said to have grade k. The basis vectors $\mathbf{e}_i$ are also called basis 1-blades. The product of any two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ is the geometric product $\mathbf{v}_1\mathbf{v}_2 = \mathbf{v}_1 \cdot \mathbf{v}_2 + \mathbf{v}_1 \wedge \mathbf{v}_2$. For $i \neq j$, then $\mathbf{e}_i\mathbf{e}_j = \mathbf{e}_i \wedge \mathbf{e}_j$, and we say that products of basis vectors are blades.

In $\mathcal{G}_{p,q,r}$, $n = p + q + r$, there is a set of 2^n *basis blades* of the form $\mathcal{B} = \left\{\mathbf{e}_0^{b_0}\mathbf{e}_1^{b_1} \ldots \mathbf{e}_{n-1}^{b_{n-1}} : b_i = 0|1\right\}$ with $\mathbf{e}_i^0 = 1$ and $\mathbf{e}_i^1 = \mathbf{e}_i$, where $b_0, b_1, \ldots, b_{n-1}$ is one of the 2^n patterns of n-bit binary strings for the exponents. There are basis blades with grades 0 to n, with the grade being the number of bits $b_i = 1$. If $b_i = 0$ for all i, then $\mathbf{e}_0^{b_0}\mathbf{e}_1^{b_1} \ldots \mathbf{e}_{n-1}^{b_{n-1}} = 1$, which is called the basis 0-blade. If $b_i = 1$ for all i, then $\mathbf{e}_0\mathbf{e}_1 \ldots \mathbf{e}_{n-1} = \mathbf{I}_n$ is the basis n-blade, which is also called the unit pseudoscalar. In $\mathcal{G}_{p,q,r}$, there are $\frac{n!}{k!(n-k)!}$ different basis k-blades, or different basis $(n-k)$-blades, as combinations of basis vectors multiplied in ascending index order. A scalar multiple a of a basis k-blade $\mathbf{E} \in \mathcal{B}$ is called a simple k-blade $a\mathbf{E}$.

A linear combination $\mathbf{A}_k = \sum a^i \mathbf{E}_i = a^i \mathbf{E}_i$ of basis k-blades $\mathbf{E}_i$ is called a k-vector $\mathbf{A}_k$, which is also said to be homogeneous of grade k and may or may not be a simple k-vector or simple k-blade. A linear combination $A = \sum a^i \mathbf{A}_i$ of basis blades $\mathbf{A}_i$ of various grades is called a *multivector* A. The grade k part $\mathbf{A}_k = \langle A \rangle_k$ of A is taken using the grade part operator $\langle\rangle_k$. For r-vector $\mathbf{A}_r$ and s-vector $\mathbf{B}_s$, $r \leq s$, we have the identity $\mathbf{A}_r \cdot \mathbf{B}_s = (-1)^{r(s-1)}\mathbf{B}_s \cdot \mathbf{A}_r$. For r-vector $\mathbf{A}_r$ and s-vector $\mathbf{B}_s$, $r + s \leq n$, we have the identity $\mathbf{A}_r \wedge \mathbf{B}_s = (-1)^{rs}\mathbf{B}_s \wedge \mathbf{A}_r$.

The geometric product $V = \mathbf{v}_1\mathbf{v}_2 \ldots \mathbf{v}_k$ of k vectors $\mathbf{v}_i$ with inverses $\mathbf{v}_i^{-1}$ is called a k-versor and is an operator for k successive reflections in the hyperplanes orthogonal to the vectors $\mathbf{v}_i$ as $A' = (-1)^k VAV^{-1}$, called a versor "sandwich" product or outermorphism of A. For $k = 2$, the 2-versor $V = R = \mathbf{v}_1\mathbf{v}_2$ is called a rotation operator or rotor.

4.1.2. Basic Concepts of Dualization. For the basis k-blade $\mathbf{e}_0^{b_0}\mathbf{e}_1^{b_1} \ldots \mathbf{e}_{n-1}^{b_{n-1}}$, having $b_i = 1$ k times and $b_i = 0$ $n-k$ times, its *dual* is the $(n-k)$-blade $\pm\mathbf{e}_0^{\neg b_0}\mathbf{e}_1^{\neg b_1} \ldots \mathbf{e}_{n-1}^{\neg b_{n-1}}$, having $\neg b_i = 0$ k times and $\neg b_i = 1$ $n-k$ times, where $\neg$ is the bitwise NOT (complement) operation on the n exponents b_i of the basis k-blade. The sign $\pm$ depends on how the dualization operation is formed or defined. For notation, the dual of basis k-blade $\mathbf{A}$ is the simple $(n-k)$-blade denoted $\mathbf{A}^*$. The dualization operation $*$ is considered to be a *linear operation* on any sum of terms and the dual of any multivector A is denoted $A^* = \sum a^i \mathbf{A}_i^*$, dualizing each basis blade $\mathbf{A}_i$ in A to its dual $\mathbf{A}_i^*$ in A^*. For all k-vectors, we always have $\mathbf{A}_k \cdot \mathbf{A}_k^* = 0$, and we say that $\mathbf{A}_k$ and its dual $\mathbf{A}_k^*$ are orthogonal. The operation $*$ could be defined as a table of duals for each basis blade $\mathbf{A}$ to its dual $\mathbf{A}^*$ and implemented as a table-based algorithm in software, or it could be implemented as a linear operator by an algebraic method. The signs $\pm$ on duals are often required or found to be either purely an involution $(\mathbf{A}^*)^* = \mathbf{A}^{**} = \mathbf{A}$ or purely an anti-involution $\mathbf{A}^{**} = -\mathbf{A}$ for all basis blades, which may allow the linear operation $*$ to be implemented by a basic algebraic method.

The basic algebraic method of implementation for the dual in a non-degenerate algebra $\mathcal{G}_{p,q,0}$ is $A^* = A/\mathbf{I}_n = A\mathbf{I}_n^{-1} = A \cdot \mathbf{I}_n^{-1}$, which is a linear operation on any multivector A. Multiplication with the pseudoscalar $\mathbf{I}_n$ is always an inner product. If $\mathbf{I}_n^2 = 1$, then the dualization is an involution, where $(\mathbf{A}^*)^* = \mathbf{A}^{**} = \mathbf{A}\mathbf{I}_n^2 = \mathbf{A}$. If $\mathbf{I}_n^2 = -1$, then the dualization is an anti-involution, where $\mathbf{A}^{**} = \mathbf{A}\mathbf{I}_n^2 = -\mathbf{A}$ and we define an inverse dual "undual" operation $(\mathbf{A}^*)^{-*} = \mathbf{A}^{-**} = (\mathbf{A}\mathbf{I}_n^{-1})\mathbf{I}_n = \mathbf{A}$. More generally, to take the dual of k-blade $\mathbf{A}$, we can choose either the right-hand side (RHS) dualization $\mathbf{A}^* = \mathbf{A}(\pm\mathbf{I}_n)$ or the left-hand side (LHS) dualization $\mathbf{A}^* = (\pm\mathbf{I}_n)\mathbf{A} = (-1)^{k(n-1)}\mathbf{A}(\pm\mathbf{I}_n)$. For n even and k odd, then the RHS and LHS dualizations differ by sign. With the choice of RHS or LHS and the choice of the orientation sign $\pm$ of the pseudoscalar, there are four possible forms of the basic algebraic method for the dualization operation that may give four distinct sets of duals of the basis blades.

As examples, recall the dualizations in $\mathcal{G}_3$ and CGA $\mathcal{G}_{4,1}$. For $\mathcal{G}_3$, $n = 3$ is odd and the RHS and LHS forms are not distinct dualization operations. The same is true for CGA $\mathcal{G}_{4,1}$ with $n = 5$ odd. For $\mathcal{G}_3$ and $\mathcal{G}_{4,1}$, there are still the choices of the orientation signs $\pm$ of their unit pseudoscalars, and both require an undual operation since their unit pseudoscalars square to -1, making their dualizations anti-involutions. In CGA, we also have OPNS entities for point $\mathbf{P_t}$, line $\mathbf{L}$, and plane $\mathbf{\Pi}$. For example, we test point $\mathbf{P_t}$ on line $\mathbf{L}$ as $\mathbf{P_t} \wedge \mathbf{L} = 0$. We can dualize as $(\mathbf{P_t} \wedge \mathbf{L})\mathbf{I}_5 = \mathbf{P_t} \cdot (\mathbf{L} \cdot \mathbf{I}_5) = \mathbf{P_t} \cdot \mathbf{L}^* = 0$. The point $\mathbf{P_t}$ is not dualized, and just $\mathbf{L}$ is dualized as the CGA IPNS line $\mathbf{L}^* = \boldsymbol{l}$. CGA OPNS entities dualize to CGA IPNS entities and vice versa. The CGA point entity $\mathbf{P_t} = \mathbf{t} + \mathbf{t}^2\mathbf{e}_\infty/2 + \mathbf{e}_o$ is usually the same in OPNS and IPNS, though it can be dualized to an IPNS grade 4 point.

For PGA $\mathcal{G}_{3,0,1}$, $n = 4$ is even and the RHS and LHS forms are distinct dualization operations that give different signs for the duals of basis blades with odd grade k. However, $\mathcal{G}_{3,0,1}$ is an algebra with a degenerate metric, where $\mathbf{I}_4^2 = 0$ and $\mathbf{I}_4^{-1}$ does not exist. Therefore, the basic algebraic method for the implementation of the dual, as $A^* = A\mathbf{I}_4^{-1}$ or similar forms ($\pm$ LHS or RHS), cannot be used directly in PGA $\mathcal{G}_{3,0,1}$. For the dualization operation for PGA, we have to consider some more advanced concepts of dualization.

4.1.3. Advanced Concepts of Dualization. For the RHS form of the basic algebraic dualization operation, $\mathbf{A}^{*_R} = \mathbf{A}(\pm\mathbf{I}_n)$ on $\mathbf{A} \in \mathcal{G}_{p,q,0}$, the *defining relation* is $\mathbf{A} \wedge \mathbf{B}^{*_R} = \mathbf{B} \wedge \mathbf{A}^{*_R} = (\mathbf{A} \cdot \mathbf{B})^{*_R}$ for k-vectors $\mathbf{A}$ and $\mathbf{B}$ (both grade k). For the LHS form of the basic algebraic dualization operation, $\mathbf{A}^{*_L} = (\pm\mathbf{I}_n)\mathbf{A}$ on $\mathbf{A} \in \mathcal{G}_{p,q,0}$, the defining relation is $\mathbf{A}^{*_L} \wedge \mathbf{B} = \mathbf{B}^{*_L} \wedge \mathbf{A} = (\mathbf{A} \cdot \mathbf{B})^{*_L}$ for k-vectors $\mathbf{A}$ and $\mathbf{B}$. These relations are also called the Hodge star $*$ defining relations in a non-degenerate algebra $\mathcal{G}_{p,q,0}$, $n = p + q$.

While the RHS dual $\mathbf{A}^{*_R} = \mathbf{A}(\pm\mathbf{I}_n)$ and LHS dual $\mathbf{A}^{*_L} = (\pm\mathbf{I}_n)\mathbf{A}$ are distinct for n even, the RHS and LHS defining relations are actually equivalent (or hold the same) for all n since, for example, $\mathbf{A}\wedge\mathbf{B}^{*_R} = \mathbf{A}\wedge(\mathbf{B}(\pm\mathbf{I}_n)) = (-1)^{k(n-1)}(-1)^{k(n-k)}((\pm\mathbf{I}_n)\mathbf{B}) \wedge \mathbf{A} = \mathbf{B}^{*_L} \wedge \mathbf{A}$ and $(\mathbf{A} \wedge \mathbf{B}^{*_R})(\pm\mathbf{I}_n) = (\pm\mathbf{I}_n)(\mathbf{B}^{*_L} \wedge \mathbf{A}) = (\mathbf{A}\cdot\mathbf{B})\mathbf{I}_n^2$. However, $\mathbf{A} \wedge \mathbf{B}^{*_R} \neq \mathbf{B}^{*_R} \wedge \mathbf{A}$, so we must still use one defining relation or the other. For RHS, we must use $\mathbf{A} \wedge \mathbf{B}^* = \mathbf{B} \wedge \mathbf{A}^* = (\mathbf{A}\cdot\mathbf{B})^*$. For LHS, we must use $\mathbf{A}^* \wedge \mathbf{B} = \mathbf{B}^* \wedge \mathbf{A} = (\mathbf{A}\cdot\mathbf{B})^*$.

Using the RHS form, the inner product $\mathbf{A}\cdot\mathbf{B}$ is defined by the metric as $\mathbf{A}\cdot\mathbf{B} = (\mathbf{A}\wedge\mathbf{B}^*)^{-*} = \mathbf{A}\cdot\mathbf{B}^{-**}$. For example, with $\mathbf{B}^* = \mathbf{B}\mathbf{I}_n$ and $\mathbf{I}_n^2 = -1$, then $(\mathbf{A}\wedge\mathbf{B}^*)^{-*} = -(\mathbf{A}\wedge(\mathbf{B}\mathbf{I}_n))\mathbf{I}_n = -\mathbf{A}\cdot((\mathbf{B}\mathbf{I}_n)\cdot\mathbf{I}_n) = -\mathbf{A}\cdot(\mathbf{B}\mathbf{I}_n^2) = \mathbf{A}\cdot\mathbf{B}$. Using the LHS form with $\mathbf{A}^* = \mathbf{I}_n\mathbf{A}$ and $\mathbf{I}_n^2 = -1$, then $(\mathbf{A}^*\wedge\mathbf{B})^{-*} = -\mathbf{I}_n((\mathbf{I}_n\mathbf{A})\wedge\mathbf{B}) = -(\mathbf{I}_n\cdot(\mathbf{I}_n\mathbf{A}))\cdot\mathbf{B} = -(\mathbf{I}_n^2\mathbf{A})\cdot\mathbf{B} = \mathbf{A}\cdot\mathbf{B}$. For $\mathbf{I}_n^2 = 1$, the undual operation $-*$ is the same as $*$, which is an involution. For $\mathbf{I}_n^2 = -1$, the undual operation $-*$ is $(\mathbf{A}^*)^{-*} = \mathbf{A}^{-**} = \mathbf{A}$, using $\mp\mathbf{I}_n$ with opposite sign such that $(\pm\mathbf{I}_n)(\mp\mathbf{I}_n) = 1$ on a RHS or LHS form.

Using the RHS form, the inner product $\mathbf{A}\cdot\mathbf{B}$ can be defined by dualization (instead of by the metric) as $\mathbf{A}\cdot\mathbf{B} = (\mathbf{A}\wedge\mathbf{B}^*)^{-*} = (\mathbf{B}\wedge\mathbf{A}^*)^{-*}$. Using the LHS form, the inner product $\mathbf{A}\cdot\mathbf{B}$ can be defined by dualization as $\mathbf{A}\cdot\mathbf{B} = (\mathbf{A}^*\wedge\mathbf{B})^{-*} = (\mathbf{B}^*\wedge\mathbf{A})^{-*}$. The definition of $\mathbf{A}\cdot\mathbf{B}$ by dualization using the outer product is independent of the metric, but is instead dependent on the definition of the duals and unduals, which may be given by a table or implemented by an algebraic method. Then, the metric is said to be defined by the duals. In a degenerate metric algebra $\mathcal{G}_{p,q,r\neq 0}$, we define the inner product by dualization, using the outer product and the definition of the duals.

The defining relations for the basic Hodge star $*$ dualization in $\mathcal{G}_{p,q,0}$ continue to hold when modified by an outermorphism operation using any basis blade $\mathbf{E} \in \mathcal{B}$. For example, the RHS relations become $\mathbf{E}\mathbf{A}^*(\pm\mathbf{E}) = \mathbf{E}\mathbf{A}(\pm\mathbf{I}_n)(\pm\mathbf{E}) = \mathbf{A}^{*\prime}$ and $\mathbf{E}(\mathbf{A}\wedge\mathbf{B}^*)(\pm\mathbf{E}) = \mathbf{E}(\mathbf{B}\wedge\mathbf{A}^*)(\pm\mathbf{E}) = \mathbf{E}(\mathbf{A}\cdot\mathbf{B})^*(\pm\mathbf{E})$. The outermorphism is similar for the LHS relations. The orientation sign $\pm$ on $\pm\mathbf{E}$ is another choice. The outermorphism operation using basis blade $\mathbf{E}$, also called a sandwich product, may modify the signs $\pm$ on duals $\mathbf{A}^{*\prime} = \pm\mathbf{A}^*$. An outermorphism using a general k-versor $V = \mathbf{v}_1\mathbf{v}_2\ldots\mathbf{v}_k$ does not produce *dual* elements, so V must be limited to any basis k-blade $\mathbf{E} \in \mathcal{B}$. While $\mathbf{A}^*$ may be an involution (or anti-involution), $\mathbf{A}^{*\prime}$ may change to be the opposite as an anti-involution (or involution). The outermorphism generalizes the basic Hodge star $*$ dualization $\mathbf{A}^*$ as $\mathbf{A}^{*\prime} = \mathbf{A}^\star$. Beginning with the generalization of the basic Hodge star $*$ by outermorphism using any basis blade, we introduce the notation $\mathbf{A}^\star$ for the dual, which we may call the advanced Hodge star $\star$ dualization $\mathbf{A}^\star$. We allow $\mathbf{A}$ to be any multivector, since dualization operations are linear operations. For $\mathbf{A} \in \mathcal{G}_3$, we define $\mathbf{A}^* = \mathbf{A}/\mathbf{I}_3$.

The advanced Hodge star $\star$ dualization $\mathbf{A}^\star$ also further generalizes dualization to degenerate metric algebras $\mathcal{G}_{p,q,r\neq 0}$. For a degenerate metric algebra $\mathcal{G}_{p,q,r\neq 0}$, $n = p+q+r$, with basis vectors $\{\mathbf{e}_i : 0 \le i < n\}$, we choose a corresponding non-degenerate metric algebra $\mathcal{G}_{p',q',0}$, $n = p' + q'$ with basis vectors $\{\boldsymbol{e}_i : 0 \leq i < n\}$ in which the basic dual $\boldsymbol{A}^* \in \mathcal{G}_{p',q',0}$, or modified "advanced" dual $\boldsymbol{A}^{*\prime} = \boldsymbol{A}^\star \in \mathcal{G}_{p',q',0}$, corresponds to the dual $\mathbf{A}^\star \in \mathcal{G}_{p,q,r\neq 0}$. The correspondence is denoted $\mathbf{A}^\star \hat{=} \boldsymbol{A}^\star$. We use upright bold letters for $\mathbf{A} \in \mathcal{G}_{p,q,r\neq 0}$ in the degenerate algebra, and italic bold letters for $\boldsymbol{A} \in \mathcal{G}_{p',q',0}$ in the corresponding non-degenerate algebra. There are 2^n corresponding basis blades $(\mathbf{E}_i = \mathbf{e}_0^{b_0}\mathbf{e}_1^{b_1}\ldots\mathbf{e}_{n-1}^{b_{n-1}}) \hat{=} (\boldsymbol{e}_0^{b_0}\boldsymbol{e}_1^{b_1}\ldots\boldsymbol{e}_{n-1}^{b_{n-1}} = \boldsymbol{E}_i)$. To compute the dual $\mathbf{A}^\star$, $\mathbf{A} = a^i\mathbf{E}_i$ is transferred to a corresponding element $\boldsymbol{A} = a^i\boldsymbol{E}_i$ by copying its scalar components a^i onto the corresponding basis blades $\boldsymbol{E}_i \hat{=} \mathbf{E}_i$. In the non-degenerate algebra, $\boldsymbol{A}^\star = a^i\boldsymbol{E}_i^\star$ is computed using a basic Hodge star $*$ operation, which may be followed by an outermorphism using any basis blade $\boldsymbol{E}$. Then, $\boldsymbol{A}^\star$ is transferred back to the corresponding element $\mathbf{A}^\star = a^i\mathbf{E}_i^\star$ as the dual of $\mathbf{A}$.

The choices for the corresponding non-degenerate algebra $\mathcal{G}_{p',q',0}$, form of basic Hodge star $*$ operation, and outermorphism (or choice of form of advanced Hodge star $\star$ operation) can be such that the duals, produced for the degenerate algebra $\mathcal{G}_{p,q,r\neq 0}$ using the corresponding algebra $\mathcal{G}_{p',q',0}$, may match a table of duals that have been predetermined by empirical method (observation of duals) or by some other requirements placed on the signs of the duals. However, the advanced Hodge star $\star$ is limited to being an involution or anti-involution on the signs of duals, and the dualization mapping is always a one-to-one and onto (bijective) mapping of basis blades to basis blades with signs $\pm$.

It may require some experimentation to form an advanced Hodge star $\star$ operation that matches a given table of duals that are an involution or anti-involution. For PGA $\mathcal{G}_{3,0,1}$, it was not difficult to find the correct forms in $\mathcal{G}_{4,0,0}$, $\mathcal{G}_{3,1,0}$, and $\mathcal{G}_{1,3,0}$ as the algebraic methods of Table 2 for the new geometric entity dualization operation J_e of Table 1.

In a degenerate algebra $\mathcal{G}_{p,q,r\neq 0}$, there is a non-degenerate subalgebra $\mathcal{G}_{p,q,0}$. For any element $\mathbf{A} \in \mathcal{G}_{p,q,0}$, it is possible to take the dual of $\mathbf{A}$ as the RHS dual $\mathbf{A}^\star = \mathbf{A}(\pm\mathbf{I}_n)$ or LHS dual $\mathbf{A}^\star = (\pm\mathbf{I}_n)\mathbf{A}$ without using another corresponding non-degenerate algebra $\mathcal{G}_{p',q',0}$. In PGA $\mathcal{G}_{3,0,1}$ with $\mathbf{I}_4 = \mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$, the dual of $\mathbf{A} \in \mathcal{G}_{3,0,0}$ can be computed directly in PGA as $\mathbf{A}^\star = \mathbf{I}_4\mathbf{A} = \mathbf{e}_0\mathbf{A}\mathbf{I}_3 = -\mathbf{e}_0(\mathbf{A}/\mathbf{I}_3) = -\mathbf{e}_0\mathbf{A}^*$. For $\mathbf{A}^\star \in \mathcal{G}_{3,0,0}$, the undual is directly computed in PGA as $\mathbf{A}^{-\star\star} = \mathbf{A} = -\mathbf{I}_4\mathbf{A}^\star = -\mathbf{e}_0\mathbf{A}^\star\mathbf{I}_3$. The undual $\mathbf{A}^{-\star\star} = \mathbf{A} \in \mathcal{G}_{3,0,0}$ cannot be computed directly in PGA, so the directly computed dual of $\mathbf{A} \in \mathcal{G}_{3,0,0}$ (or undual of $\mathbf{A}^\star \in \mathcal{G}_{3,0,0}$) in PGA is one-way. For $\mathbf{A} \in \mathcal{G}^k_{3,0,0}$, where $\mathbf{A}$ is grade k, then $\mathbf{A}^\star = \mathbf{I}_4\mathbf{A} = (-1)^k\mathbf{A}\mathbf{I}_4$. For the line entity $\mathbf{L}$, its vector-valued (grade $k = 1$) Plücker line condition $\mathbf{A} = (\mathbf{p} - \mathbf{t}) \cdot \mathbf{d}^* = 0$ is dualized directly in PGA as $((\mathbf{p} - \mathbf{t}) \cdot \mathbf{d}^*)\mathbf{I}_4 = \mathbf{I}_4((\mathbf{t} - \mathbf{p}) \cdot \mathbf{d}^*) = \mathbf{P}_\mathbf{t} \wedge \mathbf{L}$, maintaining the directional orientation of the line.

In PGA $\mathcal{G}_{3,0,1}$, we have OPNS entities for point $\mathbf{P}_\mathbf{t}$, line $\mathbf{L}$, and plane $\boldsymbol{\Pi}$. Similar to CGA, we also test point $\mathbf{P}_\mathbf{t}$ on line $\mathbf{L}$ as $\mathbf{P}_\mathbf{t} \wedge \mathbf{L} = 0$. However, we cannot dualize in PGA the same way as in CGA since the PGA metric is degenerate. Instead, we dualize in PGA as $(\mathbf{P}_\mathbf{t} \wedge \mathbf{L} \hat{=} ((\mathbf{P}_\mathbf{t}^\star \cdot \mathbf{L})\mathbf{I}_4 =$

$\langle\mathbf{P}_{\mathbf{t}}^{\star}\mathbf{L}\mathbf{I}_4\rangle_3)) = \langle\mathbf{P}_{\mathbf{t}}^{\star}\mathbf{L}^{\star}\rangle_3 = \mathbf{P}_{\mathbf{t}}^{\star} \times \mathbf{L}^{\star} = \boldsymbol{p}_{\mathbf{t}} \times \boldsymbol{l}$. We also have $(\mathbf{P}_{\mathbf{t}} \wedge \mathbf{\Pi} \hat{=} ((\mathbf{P}_{\mathbf{t}}^{\star} \cdot \mathbf{\Pi})\mathbf{I}_4 = \langle\mathbf{P}_{\mathbf{t}}^{\star}\mathbf{\Pi}\mathbf{I}_4\rangle_4)) = \langle\mathbf{P}_{\mathbf{t}}^{\star}\mathbf{\Pi}^{\star}\rangle_4 = \boldsymbol{p}_{\mathbf{t}} \wedge \boldsymbol{\pi} = \boldsymbol{p}_{\mathbf{t}} \times \boldsymbol{\pi}$, and $(\mathbf{P}_{\mathbf{t}} \wedge \mathbf{P} \hat{=} ((\mathbf{P}_{\mathbf{t}}^{\star} \cdot \mathbf{P})\mathbf{I}_4 = \langle\mathbf{P}_{\mathbf{t}}^{\star}\mathbf{P}\mathbf{I}_4\rangle_2)) = \langle\mathbf{P}_{\mathbf{t}}^{\star}\mathbf{P}^{\star}\rangle_2 = \boldsymbol{p}_{\mathbf{t}} \times \boldsymbol{p}$. These are the defining relations for our new entity dualization operation J_e for PGA $\mathcal{G}_{3,0,1}$, which we shall observe in Section 4.2 to define J_e as Table 1. *The inner products cannot be computed directly in the degenerate metric of* $\mathcal{G}_{3,0,1}$ *and are defined by dualization and the dual outer products*. For example, the inner product $(\mathbf{P}_{\mathbf{t}}^{\star} \cdot \mathbf{L})\mathbf{I}_4 = \langle\mathbf{P}_{\mathbf{t}}^{\star}\mathbf{L}\mathbf{I}_4\rangle_3$ corresponds to (and is defined by) the dual outer product $\mathbf{P}_{\mathbf{t}} \wedge \mathbf{L} = \langle\mathbf{P}_{\mathbf{t}}^{\star}\mathbf{L}^{\star}\rangle_3 = \mathbf{P}_{\mathbf{t}}^{\star} \times \mathbf{L}^{\star} = \boldsymbol{p}_{\mathbf{t}} \times \boldsymbol{l}$. Only a correspondence ($\hat{=}$) exists from the inner products to the dual outer products. The degenerate inner product $\mathbf{P}_{\mathbf{t}}^{\star} \cdot \mathbf{L}$ also depends on the dualization, and the dualization $\mathbf{L}^{\star} = \mathbf{L}\mathbf{I}_4 = \mathbf{L} \cdot \mathbf{I}_4$ cannot be computed directly in PGA since the inner product is degenerate. The correct non-degenerate corresponding geometric expressions for the inner products are $\mathbf{P}_{\mathbf{t}}^{\star} \cdot \mathbf{L} \hat{=} (\mathbf{p} - \mathbf{t}) \cdot \mathbf{d}^*$ (*vector-valued* Plücker line condition), $\mathbf{P}_{\mathbf{t}}^{\star} \cdot \mathbf{\Pi} \hat{=} (\mathbf{t} - \mathbf{p}) \cdot \mathbf{n}$ (*scalar-valued* plane condition), and $(\mathbf{P}_{\mathbf{t}}^{\star} \cdot \mathbf{P})\mathbf{I}_4 \hat{=} (\mathbf{P}_{\mathbf{t}} \wedge \mathbf{P} = -\mathbf{e}_0(\mathbf{t} - \mathbf{p}))$ (*bivector-valued* point difference condition in dual form, where $\mathbf{P}_{\mathbf{t}}^{\star} \cdot \mathbf{P} \hat{=} \mathbf{p}^* - \mathbf{t}^*$), which were derived in Sections 2 and 3 as we reviewed and derived the PGA geometric entities $\mathbf{L}$ (4), $\mathbf{\Pi}$ (5), and $\boldsymbol{p}_{\mathbf{t}}$ (12).

In Section 4.2, we use the basic concepts of dualization and the defining relations for the PGA duals to empirically observe the duals of basis blades and define J_e for PGA as Table 1. In Section 4.3, we use the advanced concepts of dualization to implement J_e for PGA three different ways by the algebraic methods of Table 2.

The basic and advanced concepts of dualization can also be used for other degenerate algebras. For the degenerate algebra $\mathcal{G}_{1,3,1}$, which we have called Space Time PGA (STPGA) [4], a geometric entity dualization operation J_e has also been empirically defined as a table of duals. Two algebraic methods, in $\mathcal{G}_{5,0,0}$ and $\mathcal{G}_{1,4,0}$, as two different advanced Hodge star $\star$ dualizations, have been found to implement J_e for STPGA $\mathcal{G}_{1,3,1}$. The advanced concepts of dualization, such as the advanced Hodge star $\star$, may be useful to find algebraic methods for geometric entity dualization operations for other degenerate metric algebras that may be studied in the future. The advanced concepts of dualization for degenerate algebras could be researched further to simplify or generalize the concepts, including the correspondence relationships.

4.2. Determination of Duals by Empirical Method

In this section, we empirically observe dual entities representing the same geometry with the same orientation to determine and define the new geometric entity dualization operation J_e. In Sections 4.2.1, 4.2.2, and 4.2.3, we compare dual entities and observe the duals of the four basis 1-blades, six basis 2-blades, and four basis 3-blades, respectively. For each grade k of basis blades, we compare grade k entities in OPNS PGA to their corresponding dual grade $4-k$ entities in CPNS PGA and directly observe the corresponding dual basis blades. In Section 4.2.4, we observe the duals of the basis 0-blade

1 and the basis 4-blade pseudoscalar $\mathbf{I}_4$. In Section 4.2.5, we collect all of the basis blade duals into Table 1, which defines the new geometric entity dualization operation J_e for PGA $\mathcal{G}_{3,0,1}$.

4.2.1. Duals of the Four Basis 1-blades. We compare the OPNS PGA 1-blade point $\mathbf{P}_\mathbf{t} = \mathbf{e}_0 + \mathbf{t}$ with the CPNS PGA 3-blade point $\boldsymbol{p}_\mathbf{t} = \mathbf{I}_3 - \mathbf{e}_0\mathbf{t}^* = \mathbf{I}_3 + \mathbf{e}_0\mathbf{t}\mathbf{I}_3$. The points are considered to be in standard unit point form and orientation. We compare the three points $\mathbf{x} = \mathbf{e}_1$, $\mathbf{y} = \mathbf{e}_2$, and $\mathbf{z} = \mathbf{e}_3$. The basis 1-blades should dualize to basis 3-blades as follows:

For $\mathbf{x}$, we have $\mathbf{P}_\mathbf{x} = \mathbf{e}_0 + \mathbf{e}_1$ and $\boldsymbol{p}_\mathbf{x} = \mathbf{I}_3 - \mathbf{e}_0\mathbf{e}_1^* = \mathbf{I}_3 - \mathbf{e}_0\mathbf{e}_3\mathbf{e}_2$. We observe the duals $J_e(\mathbf{e}_0) = \mathbf{I}_3$ and $J_e(\mathbf{e}_1) = -\mathbf{e}_0\mathbf{e}_3\mathbf{e}_2 = \mathbf{e}_0\mathbf{e}_2\mathbf{e}_3$.

For $\mathbf{y}$, we have $\mathbf{P}_\mathbf{y} = \mathbf{e}_0 + \mathbf{e}_2$ and $\boldsymbol{p}_\mathbf{y} = \mathbf{I}_3 - \mathbf{e}_0\mathbf{e}_2^* = \mathbf{I}_3 - \mathbf{e}_0\mathbf{e}_1\mathbf{e}_3$. We observe the duals $J_e(\mathbf{e}_0) = \mathbf{I}_3$ and $J_e(\mathbf{e}_2) = -\mathbf{e}_0\mathbf{e}_1\mathbf{e}_3$.

For $\mathbf{z}$, we have $\mathbf{P}_\mathbf{z} = \mathbf{e}_0 + \mathbf{e}_3$ and $\boldsymbol{p}_\mathbf{z} = \mathbf{I}_3 - \mathbf{e}_0\mathbf{e}_3^* = \mathbf{I}_3 - \mathbf{e}_0\mathbf{e}_2\mathbf{e}_1$. We observe the duals $J_e(\mathbf{e}_0) = \mathbf{I}_3$ and $J_e(\mathbf{e}_3) = -\mathbf{e}_0\mathbf{e}_2\mathbf{e}_1 = \mathbf{e}_0\mathbf{e}_1\mathbf{e}_2$.

4.2.2. Duals of the Six Basis 2-blades. For observing the correct basis 2-blade duals, we will look at entities for three lines along different directions, and for each line we compare the OPNS PGA 2-blade line entity $\mathbf{L}_{\mathbf{p},\hat{\mathbf{d}}} = \hat{\mathbf{d}} \wedge \mathbf{P}_\mathbf{p}$ with its corresponding dual CPNS PGA 2-blade line entity $\boldsymbol{l}_{\mathbf{p},\hat{\mathbf{d}}} = \hat{\mathbf{d}}^* - (\mathbf{p}\cdot\hat{\mathbf{d}}^*)\mathbf{e}_0$. First, we always check that the two dual entities have the same geometric null space entity and orientation. Also, for the OPNS PGA line entity, we can ignore the third pseudoscalar term of the null space entity that represents $\mathbf{t}\wedge\mathbf{p}\wedge\hat{\mathbf{d}}$, which is 0 for any point $\mathbf{t}$ on the line through $\mathbf{p}$ in direction $\hat{\mathbf{d}}$.

For $\mathbf{p} = \mathbf{e}_1$, $\hat{\mathbf{d}} = \mathbf{e}_2$, we have the 3-blade null space entities with same orientation:

$$\begin{aligned}
\mathbf{P}_\mathbf{t} \wedge \mathbf{L}_{\mathbf{p},\hat{\mathbf{d}}} &= (x-1)\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2 - z\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3 - z\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 && (16)\\
\boldsymbol{p}_\mathbf{t} \times \boldsymbol{l}_{\mathbf{p},\hat{\mathbf{d}}} &= (x-1)\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2 - z\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3 && (17)\\
\mathbf{L}_{\mathbf{p},\hat{\mathbf{d}}} &= -\mathbf{e}_0\mathbf{e}_2 - \mathbf{e}_1\mathbf{e}_2 && (18)\\
\boldsymbol{l}_{\mathbf{p},\hat{\mathbf{d}}} &= \mathbf{e}_0\mathbf{e}_3 + \mathbf{e}_1\mathbf{e}_3. && (19)
\end{aligned}$$

We observe the basis 2-blade duals: $J_e(\mathbf{e}_0\mathbf{e}_2) = -\mathbf{e}_1\mathbf{e}_3$ and $J_e(\mathbf{e}_1\mathbf{e}_2) = -\mathbf{e}_0\mathbf{e}_3$.

For $\mathbf{p} = \mathbf{e}_2$, $\hat{\mathbf{d}} = \mathbf{e}_3$, we have the 3-blade null space entities with same orientation:

$$\begin{aligned}
\mathbf{P}_\mathbf{t} \wedge \mathbf{L}_{\mathbf{p},\hat{\mathbf{d}}} &= x\mathbf{e}_0\mathbf{e}_1\mathbf{e}_3 + (y-1)\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3 - x\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 && (20)\\
\boldsymbol{p}_\mathbf{t} \times \boldsymbol{l}_{\mathbf{p},\hat{\mathbf{d}}} &= x\mathbf{e}_0\mathbf{e}_1\mathbf{e}_3 + (y-1)\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3 && (21)\\
\mathbf{L}_{\mathbf{p},\hat{\mathbf{d}}} &= -\mathbf{e}_0\mathbf{e}_3 - \mathbf{e}_2\mathbf{e}_3 && (22)\\
\boldsymbol{l}_{\mathbf{p},\hat{\mathbf{d}}} &= \mathbf{e}_0\mathbf{e}_1 - \mathbf{e}_1\mathbf{e}_2. && (23)
\end{aligned}$$

We observe the basis 2-blade duals: $J_e(\mathbf{e}_0\mathbf{e}_3) = \mathbf{e}_1\mathbf{e}_2$ and $J_e(\mathbf{e}_2\mathbf{e}_3) = -\mathbf{e}_0\mathbf{e}_1$.

For $\mathbf{p} = \mathbf{e}_3$, $\hat{\mathbf{d}} = \mathbf{e}_1$, we have the 3-blade null space entities with same orientation:

$$\begin{aligned}
\mathbf{P_t} \wedge \mathbf{L}_{\mathbf{p},\hat{\mathbf{d}}} &= -y\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2 + (1-z)\mathbf{e}_0\mathbf{e}_1\mathbf{e}_3 - y\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 && (24)\\
\boldsymbol{p}_\mathbf{t} \times \boldsymbol{l}_{\mathbf{p},\hat{\mathbf{d}}} &= -y\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2 + (1-z)\mathbf{e}_0\mathbf{e}_1\mathbf{e}_3 && (25)\\
\mathbf{L}_{\mathbf{p},\hat{\mathbf{d}}} &= -\mathbf{e}_0\mathbf{e}_1 + \mathbf{e}_1\mathbf{e}_3 && (26)\\
\boldsymbol{l}_{\mathbf{p},\hat{\mathbf{d}}} &= \mathbf{e}_0\mathbf{e}_2 - \mathbf{e}_2\mathbf{e}_3. && (27)
\end{aligned}$$

We observe the basis 2-blade duals: $J_e(\mathbf{e}_0\mathbf{e}_1) = \mathbf{e}_2\mathbf{e}_3$ and $J_e(\mathbf{e}_1\mathbf{e}_3) = \mathbf{e}_0\mathbf{e}_2$.

We can further observe that these six basis 2-blade empirical duals appear to support an anti-involution, $J_e(J_e(\mathbf{A})) = -\mathbf{A}$. We assume that J_e is a linear operator so that $J_e(a\mathbf{A} + b\mathbf{B}) = aJ_e(\mathbf{A}) + bJ_e(\mathbf{B})$.

4.2.3. Duals of the Four Basis 3-blades. We compare the OPNS PGA 3-blade plane $\mathbf{\Pi}_{\mathbf{p},\hat{\mathbf{n}}} = \mathbf{P_p} \wedge \hat{\mathbf{n}}^*$ with its dual CPNS PGA 1-blade plane $\boldsymbol{\pi}_{\mathbf{p},\hat{\mathbf{n}}} = \hat{\mathbf{n}} + (\mathbf{p}\cdot\hat{\mathbf{n}})\mathbf{e}_0$. We compare the three planes, $x = 1$, $y = 1$, and $z = 1$. First, we compare the 4-blade geometric null space entities to make sure they have the same scale and orientation, then we observe the duals.

For $x = 1$ ($\mathbf{p} = \mathbf{e}_1$, $\hat{\mathbf{n}} = \mathbf{e}_1$), we have the 4-blade null space entities with same orientation:

$$\begin{aligned}
\mathbf{P_t} \wedge \mathbf{\Pi}_{\mathbf{p},\hat{\mathbf{n}}} &= (x-1)\mathbf{I}_4 && (28)\\
\boldsymbol{p}_\mathbf{t} \times \boldsymbol{\pi}_{\mathbf{p},\hat{\mathbf{n}}} &= (x-1)\mathbf{I}_4 && (29)\\
\mathbf{\Pi}_{\mathbf{p},\hat{\mathbf{n}}} &= -\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3 - \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 && (30)\\
\boldsymbol{\pi}_{\mathbf{p},\hat{\mathbf{n}}} &= \mathbf{e}_0 + \mathbf{e}_1. && (31)
\end{aligned}$$

We observe the duals of the basis 3-blades: $J_e(\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3) = -\mathbf{e}_1$ and $J_e(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3) = -\mathbf{e}_0$.

For $y = 1$ ($\mathbf{p} = \mathbf{e}_2$, $\hat{\mathbf{n}} = \mathbf{e}_2$), we have the 4-blade null space entities with same orientation:

$$\begin{aligned}
\mathbf{P_t} \wedge \mathbf{\Pi}_{\mathbf{p},\hat{\mathbf{n}}} &= (y-1)\mathbf{I}_4 && (32)\\
\boldsymbol{p}_\mathbf{t} \times \boldsymbol{\pi}_{\mathbf{p},\hat{\mathbf{n}}} &= (y-1)\mathbf{I}_4 && (33)\\
\mathbf{\Pi}_{\mathbf{p},\hat{\mathbf{n}}} &= \mathbf{e}_0\mathbf{e}_1\mathbf{e}_3 - \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 && (34)\\
\boldsymbol{\pi}_{\mathbf{p},\hat{\mathbf{n}}} &= \mathbf{e}_0 + \mathbf{e}_2. && (35)
\end{aligned}$$

We observe the duals of the basis 3-blades: $J_e(\mathbf{e}_0\mathbf{e}_1\mathbf{e}_3) = \mathbf{e}_2$ and $J_e(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3) = -\mathbf{e}_0$.

For $z = 1$ ($\mathbf{p} = \mathbf{e}_3$, $\hat{\mathbf{n}} = \mathbf{e}_3$), we have the 4-blade null space entities with same orientation:

$$\begin{aligned}
\mathbf{P_t} \wedge \mathbf{\Pi}_{\mathbf{p},\hat{\mathbf{n}}} &= (z-1)\mathbf{I}_4 && (36)\\
\boldsymbol{p}_\mathbf{t} \times \boldsymbol{\pi}_{\mathbf{p},\hat{\mathbf{n}}} &= (z-1)\mathbf{I}_4 && (37)\\
\mathbf{\Pi}_{\mathbf{p},\hat{\mathbf{n}}} &= -\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2 - \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 && (38)\\
\boldsymbol{\pi}_{\mathbf{p},\hat{\mathbf{n}}} &= \mathbf{e}_0 + \mathbf{e}_3. && (39)
\end{aligned}$$

We observe the duals of the basis 3-blades: $J_e(\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2) = -\mathbf{e}_3$ and $J_e(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3) = -\mathbf{e}_0$. We further observe that, between the duals of 3-blades and 1-blades, we also have an anti-involution for J_e.

4.2.4. Duals of the Basis 0-blade and Basis 4-blade. $\mathbf{P_t} \wedge \mathbf{\Pi_{p,n}} = ((\mathbf{t}-\mathbf{p})\cdot\mathbf{n})\mathbf{I}_4 = 0$ represents the scalar-valued condition $(\mathbf{t}-\mathbf{p})\cdot\mathbf{n} = 0$ for point $\mathbf{P_t}$ on plane $\mathbf{\Pi_{p,n}}$ in a well-defined orientation, which is dualized into the pseudoscalar-valued condition $((\mathbf{t}-\mathbf{p})\cdot\mathbf{n})\mathbf{I}_4 = 0$. Therefore, we observe the dual of the basis 0-blade 1 as $J_e(1) = \mathbf{I}_4$. Since J_e is found to be an anti-involution with inverse $J_e^{-1} = -J_e$, then $-J_e(\mathbf{I}_4) = 1$ and the dual of the basis 4-blade $\mathbf{I}_4$ is $J_e(\mathbf{I}_4) = -1$.

4.2.5. Definition of Geometric Entity Dualization Operation. In the prior sections (4.2.1, 4.2.2, 4.2.3, and 4.2.4), we found the dual for each of the 16 basis blades in $\mathcal{G}_{3,0,1}$ by direct empirical observation of point-based and plane-based dual entities representing the same geometry with the same orientation. The observed duals define the new geometric entity dualization operation J_e for PGA $\mathcal{G}_{3,0,1}$.

$\mathbf{A}$	$\mathbf{e}_0$	$\mathbf{e}_1$	$\mathbf{e}_2$	$\mathbf{e}_3$	$\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2$	$\mathbf{e}_0\mathbf{e}_1\mathbf{e}_3$	$\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3$	$\mathbf{I}_3 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$
$J_e(\mathbf{A})$	$\mathbf{I}_3 = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$	$\mathbf{e}_0\mathbf{e}_2\mathbf{e}_3$	$-\mathbf{e}_0\mathbf{e}_1\mathbf{e}_3$	$\mathbf{e}_0\mathbf{e}_1\mathbf{e}_2$	$-\mathbf{e}_3$	$\mathbf{e}_2$	$-\mathbf{e}_1$	$-\mathbf{e}_0$

$\mathbf{A}$	1	$\mathbf{e}_0\mathbf{e}_1$	$\mathbf{e}_0\mathbf{e}_2$	$\mathbf{e}_0\mathbf{e}_3$	$\mathbf{e}_1\mathbf{e}_2$	$\mathbf{e}_1\mathbf{e}_3$	$\mathbf{e}_2\mathbf{e}_3$	$\mathbf{I}_4 = \mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$
$J_e(\mathbf{A})$	$\mathbf{I}_4 = \mathbf{e}_0\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$	$\mathbf{e}_2\mathbf{e}_3$	$-\mathbf{e}_1\mathbf{e}_3$	$\mathbf{e}_1\mathbf{e}_2$	$-\mathbf{e}_0\mathbf{e}_3$	$\mathbf{e}_0\mathbf{e}_2$	$-\mathbf{e}_0\mathbf{e}_1$	-1

TABLE 1. Geometric Entity Dualization Operation $J_e(\mathbf{A}) = \mathbf{A}^\star$ on OPNS PGA grade k basis blade $\mathbf{A}$ (in the point-based algebra) to its dual CPNS PGA grade $4-k$ basis blade $J_e(\mathbf{A}) = \mathbf{A}^\star$ (in the plane-based algebra).

Table 1 defines the new geometric entity dualization operation J_e for PGA $\mathcal{G}_{3,0,1}$. Table 1 shows the dual $J_e(\mathbf{A}) = \mathbf{A}^\star$ for each basis blade $\mathbf{A}$ in $\mathcal{G}_{3,0,1}$, dualizing from the point-based OPNS PGA to the plane-based CPNS PGA, summarizing our observations of what the dual $J_e(\mathbf{A})$ should be for each basis blade $\mathbf{A}$ so that entities dualize to corresponding entities representing the same geometric null space with the same orientation.

To dualize from the plane-based CPNS PGA to the point-based OPNS PGA, the inverse dual $J_e^{-1} = -J_e$ ("undual") should be used instead to maintain the correct entity orientation. If the dualization direction or orientation of $-J_e$ is preferred, then an alias could be used, such as $D_e = -J_e$, for the geometric entity dualization operation D_e from the plane-based CPNS PGA to the point-based OPNS PGA with $D_e^{-1} = -D_e$.

4.3. Implementation by Algebraic Methods

We can implement the new geometric entity dualization operation $J_e(\mathbf{A}) = \mathbf{A}^\star$, defined by Table 1 for PGA $\mathcal{G}_{3,0,1}$, by corresponding algebraic methods $\mathcal{J}_e(\boldsymbol{A}) = \boldsymbol{A}^\star$ in non-degenerate geometric algebras $\mathcal{G}_{p,q,0} \in \{\mathcal{G}_{4,0,0}, \mathcal{G}_{3,1,0}, \mathcal{G}_{1,3,0}\}$ that correspond to $\mathcal{G}_{3,0,1}$.

$\mathcal{G}_{p,q,0}$	$\mathcal{G}_{4,0,0}$	$\mathcal{G}_{3,1,0}$	$\mathcal{G}_{1,3,0}$
$\mathcal{J}_e(\boldsymbol{A})$	$\boldsymbol{I}_3\boldsymbol{I}_4\boldsymbol{A}\boldsymbol{I}_3 = \boldsymbol{e}_0\boldsymbol{A}\boldsymbol{I}_3$	$\boldsymbol{I}_4\boldsymbol{A}$	$\boldsymbol{A}\boldsymbol{I}_4$

TABLE 2. Entity dualization $\mathcal{J}_e(\boldsymbol{A})$ in non-degenerate geometric algebras $\mathcal{G}_{p,q,0}$.

Table 2 gives the dualization operations $\mathcal{J}_e(\boldsymbol{A}) = \boldsymbol{A}^\star \in \mathcal{G}_{p,q,0}$ that have been found to implement $J_e(\mathbf{A}) = \mathbf{A}^\star \in \mathcal{G}_{3,0,1}$ in non-degenerate algebras $\mathcal{G}_{p,q,0}$ that correspond to $\mathcal{G}_{3,0,1}$. The coefficients on the basis blades in the dual $\boldsymbol{A}^\star \in \mathcal{G}_{p,q,0}$ are transferred onto corresponding basis blades in $\mathbf{A}^\star \in \mathcal{G}_{3,0,1}$. The complete geometric entity dualization operation is $J_e(\mathbf{A}) = \mathbf{A}^\star = \mathcal{G}_{3,0,1}(\mathcal{J}_e(\mathcal{G}_{p,q,0}(\mathbf{A})))$, where $\mathcal{G}_{p,q,0}(\mathbf{A}) = \boldsymbol{A} \in \mathcal{G}_{p,q,0}$ and $\mathcal{G}_{3,0,1}(\boldsymbol{A}^\star) = \mathbf{A}^\star \in \mathcal{G}_{3,0,1}$ denote the operations that transfer coordinates between corresponding algebras with different metrics.

The algebras $\mathcal{G}_{4,0,0}$ and $\mathcal{G}_{3,1,0}$ have the same metric as $\mathcal{G}_{3,0,1}$ for the subalgebra $\mathcal{G}_3$. For $\mathbf{A} \in \mathcal{G}_3$, its dual can be directly computed in $\mathcal{G}_{3,0,1}$ as $\mathbf{A}^\star = \mathbf{e}_0\mathbf{A}\mathbf{I}_3 = \mathbf{I}_4\mathbf{A} = -\mathbf{e}_0\mathbf{A}^*$.

The following three Python functions using $\mathcal{GA}$lgebra [1] for SymPy each implement $J_e(\mathbf{A})$ of Table 1 in one of the non-degenerate algebras $\mathcal{G}_{p,q,0} \in \{\mathcal{G}_{4,0,0}, \mathcal{G}_{3,1,0}, \mathcal{G}_{1,3,0}\}$ of Table 2. Only one function is needed, and all three produce Table 1 as required.

```
# Create the algebras.
g301 = Ga('e*0|1|2|3',g=[ 0, 1, 1, 1])
g400 = Ga('e*0|1|2|3',g=[ 1, 1, 1, 1])
g310 = Ga('e*0|1|2|3',g=[-1, 1, 1, 1])
g130 = Ga('e*0|1|2|3',g=[ 1,-1,-1,-1])
```

```
# Get the basis for PGA G(3,0,1).
(e0,e1,e2,e3) = g301.mv()
# Create the unit pseudoscalars.
I3 = e1^e2^e3; I4 = e0^I3
```

```
# Entity Dualization Operation Je in G(4,0,0)
def Je_g400(A):
  EA = g400.mv(A); EI3 = g400.mv(I3); EI4 = g400.mv(I4)
  return g301.mv(EI3*EI4*EA*EI3)
```

```
# Entity Dualization Operation Je in G(3,1,0)
def Je_g310(A):
  EA = g310.mv(A); EI4 = g310.mv(I4)
  return g301.mv(EI4*EA)
```

```
# Entity Dualization Operation Je in G(1,3,0)
def Je_g130(A):
  EA = g130.mv(A); EI4 = g130.mv(I4)
  return g301.mv(EA*EI4)
```

5. Conclusion

In Section 1, we gave an overview of PGA $\mathcal{G}_{3,0,1}$ and the new geometric entity dualization operation J_e. In Section 2, we reviewed the point-based geometric algebra of PGA and carefully derived the point-based point $\mathbf{P}$, line $\mathbf{L}$, and plane $\mathbf{\Pi}$ geometric entities. In Section 3, we reviewed the plane-based geometric algebra of PGA and carefully derived the plane-based plane $\boldsymbol{\pi}$, line $\boldsymbol{l}$, and point $\boldsymbol{p}$ geometric entities. In Section 4, we reviewed dualization and introduced the new geometric entity dualization operation J_e, which is defined by Table 1 and implemented by the algebraic methods of Table 2 using non-degenerate algebras.

As we conclude, we can now compare the new geometric entity dualization operation J_e to some other definitions for the duals in PGA as given in prior literature, all of which seem to be different than J_e of Table 1.

In [10] and [9], the point-based and plane-based algebras of PGA $\mathcal{G}_{3,0,1}$ are each represented as a different subalgebra of CGA $\mathcal{G}_{4,1}$, and a pair of dualization operations are defined that dualize between the two subalgebras. By representing PGA within CGA, [10] and [9] are basically using CGA for only points, lines, and planes. This may defeat part of the advantage of PGA, that it requires only $2^4 = 16$ basis blades, while CGA needs $2^5 = 32$ basis blades. One of the possible advantages of PGA, its smaller algebra size, may be lost by using CGA. The table of duals given in [9] shows that the dualization operation in [9] is not an involution or anti-involution, being an involution for some basis blades and an anti-involution for other basis blades. We found that our new dualization operation J_e is an anti-involution, so it seems unlikely that the dualization in [9] would give correct orientations. The dualization in [10] is claimed to be an involution, but it is actually a pair of two different dualization operations, each to dualize from one algebra to the other in the two different directions. An any case, the dualization in [10] must also be different than our new dualization operation J_e, which is an anti-involution and is not implemented within CGA.

In [5], a dualization operation for PGA, denoted $J(\mathbf{e})$, is defined by a table of duals, showing that $J(\mathbf{e})$ is not an involution or anti-involution. This again differs from our new dualization operation J_e, which is an anti-involution defined specifically to maintain orientation through the dualization operation.

In [2], a dualization operation for PGA, denoted using Hodge star $\star$ notation as $\star\mathbf{A}$, is given by a table of duals similar to in [5]. The dual $\star\mathbf{A}$ is not an involution or anti-involution, so it is different than our new operation J_e, which is defined specifically to maintain orientation through the dualization operation.

Unlike the prior literature, we use our basic and advanced concepts of dualization to derive the PGA entities and their duals representing the same geometry in the same orientation, empirically observe duals to define J_e as Table 1, and then implement J_e by the algebraic methods of Table 2 using non-degenerate algebras. In most of the prior literature, the dual is not an involution or anti-involution and cannot be implemented by any algebraic method in the form of a basic Hodge star $*$ or advanced Hodge star $\star$ as we have defined them in Section 4. According to our theory of the advanced Hodge star $\star$ dualization for degenerate algebras $\mathcal{G}_{p,q,r\neq 0}$, any form of the dualization for PGA $\mathcal{G}_{3,0,1}$ should be either an involution or an anti-involution, and we find it to be the anti-involution J_e that maintains the correct entity orientation through the dualization $\mathbf{A}^{\star} = J_e(\mathbf{A}) = \boldsymbol{a}$, from point-based entity $\mathbf{A}$ to its dual plane-based entity $\boldsymbol{a}$, or its inverse "undual" operation $\boldsymbol{a}^{-\star} = -J_e(\boldsymbol{a}) = D_e(\boldsymbol{a}) = \mathbf{A}$, from plane-based entity $\boldsymbol{a}$ to its undual point-based entity $\mathbf{A}$.

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