Generalized soft likelihood functions in combining evidence

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Abstract

Information fusion is an important topic in scientific research. Soft likelihood function is a common method of fusing evidence from multiple sources. However, when the combined evidence contains equally important decision information, the fusion results obtained using existing methods do not reflect the attitudinal characteristics of decision makers. To address this problem, a novel generalised soft likelihood function is developed in this paper. First, a new notion of decision maker (DM) pair is defined, which is used to characterise the outcome of the decision as well as the reliability of the evidence. Then, a series of algorithms for correcting the initial evidence set data are formulated. Eventually, a generic soft likelihood function for fusing compatible evidence information is proposed. Numerical examples are used to illustrate the effectiveness of the proposed methodology.

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1. Introduction

In the field of information fusion, measurement of uncertain information and processing of uncertain information are two crucial issues [1]. Like many tools for dealing with uncertain information, such as fuzzy sets [2, 3, 4], Z numbers [5, 6, 7, 8], D numbers [9], evidence theory [10, 11, 12, 13, 14], evidential reasoning [15], R numbers [16], entropy-based approaches [17, 18], and quantum-based approaches [19, 20, 21], likelihood function is also an essential way of dealing with uncertain information by combining multiple compatible pieces of evidence.

It is well known that the initial likelihood function is the product of multiple probabilities. When there are probabilistic events in the evidence set that are mutually exclusive with other propositions, using the likelihood function to fuse this data information can produce results that are counter-intuitive to human intuition [22]. Therefore, to overcome this problem, Yager et al. [22] initially developed a novel algorithmic model in 2017, referred to as the soft likelihood function. Compared to the initial likelihood function, this soft likelihood function uses ordered weighted averaging (OWA) aggregation [23, 24] to assign weights to the different likelihood function terms, which allows the fusion of evidence information to remove the influence of conflicting evidence. Moreover, as the OWA weights take into account the different attitudinal characteristics of the decision makers, this also makes the soft likelihood function more flexible when combining information from multiple sources of evidence. With these advantages, the soft likelihood function proposed by Yager et al. has been extensively studied by many scholars in different fields in recent years. In general, these studies can be distinguished into two main categories, i.e., extending this soft likelihood function in other domains of knowledge, and compensating for the shortcomings of this soft likelihood function.

For instance, in the first group, Jiang and Hu [25] explored the value of the soft likelihood function in combined belief structures based on the context of Dempster-Shafer belief structures [26, 27]. Fei [28] discussed the practical application of this soft likelihood function to interval-valued fuzzy decision making. Based on the background of evidence theory and the soft likelihood function, Fei et al. [29], and Li and Fei [30] respectively proposed a new rule for combining evidential information in evidence theory. Fei et al. [31, 32] also studied the soft likelihood function for decision making in intuitionistic fuzzy and pythagorean fuzzy environments. The role of the soft likelihood function to uncertain information processing based on D numbers and Z numbers are investigated in our previous work [33, 34]. In addition, based on the soft likelihood function, a novel multi-sensor fusion algorithm is designed, and its application in target recognition systems is examined by the author [35]. In the second group, by drawing inspiration from the model proposed by Yager et al. Song and Deng [36] proposed a new soft likelihood function based on the power OWA (POWA) operator [37, 38]. Recently, the author in [39] discovered that the method proposed by Song and Deng did not express the optimism of decision makers well, so they revised the method from a theoretical perspective and proposed a new soft likelihood function. In terms of the development of the soft likelihood function, these aforementioned studies have greatly filled in the development of the theory of the soft likelihood function and can be regarded as a driving force for the theory to move towards practice. Note that this paper is an improvement of the algorithm of the soft likelihood function itself, and thus we focus on the study of the second class of strategies.

By reviewing the existing studies on soft likelihood functions, we have discovered that weights in soft likelihood functions are assigned to the likelihood term of each piece of evidence. However, when there is evidence of equal probability in the evidence set, it is clear that the existing weighting approach is not reasonable. This is because intuitively, evidence of the same probability has equal importance, which means that they should have the same priority. Moreover, we are aware that the above problem can create the issue that the fusion results of soft likelihood functions do not reflect the preferences of decision makers. Therefore, the goal of this paper is to find a general soft likelihood function that can overcome the above-mentioned drawbacks. In summary, the main contributions of this paper are summarised as follows. (1) Firstly, a new concept of a decision maker (DM) pair is defined, which is used to characterise the outcome of the decision and the reliability of the evidence. (2) Then, based on the decision pair, a series of algorithms for correcting the initial evidence set data are formulated. (3) Ultimately, a generic soft likelihood function for fusing compatible evidence information is proposed. (4) In addition, experiments show that the proposed method can better reflect the relevance of decision preferences and fusion outcomes than existing methods.

The organizational structure of this paper is as follows. Section 2 introduces some background information for this study. Section 3 reviews the existing research on soft likelihood functions. Section 4 presents a new generalised soft likelihood function. Section 5 illustrates the generality and superiority of the proposed method by comparison. Finally, a conclusion of this paper in Section 6.

2. Preliminaries

In this section, we review some basic concepts, such as likelihood function, and some aggregation type operators.

2.1. Likelihood function [22]

The likelihood function is of great value in the evaluation of evidence, allowing us to make further use of limited resources and thus focus our attention on potential objects with greater likelihood. In a criminal justice case, the more likely a suspect is, the more willing the police will be to investigate that person.

Definition 1. For a given object V_τ, assume that evidence support is collected from ϑ independent sources, denoted as $\mathbb{P}_{\tau\kappa}$. Then the initial likelihood function is the product of these compatible probabilities, defined by

$$\mathbb{L}_\tau = \prod_{\kappa=1}^{\vartheta} \mathbb{P}_{\tau\kappa} \tag{1}$$

Definition 2. Let $\pi_\tau(\kappa)$ be the index of the κth maximum compatible probability associated with the candidate C_τ such that $Prod_\tau(q)$ is the product of the qth maximum probability, then the likelihood function is defined by

$$Prod_{\tau}(q) = \prod_{\kappa=1}^{q} p_{\tau\pi_{\tau}(\kappa)} \tag{2}$$

2.2. Ordered weighted averaging (OWA) operator [23, 24]

Since its introduction by Yager [23], the OWA aggregation operator has achieved many practical applications to date, such as data mining [40, 41], decision making [42, 43], information fusion [22, 44], etc. A brief introduction to this operator is defined as follows.

Definition 3. Let an OWA aggregation operator with dimension N satisfy a mapping: $\Re^n \to \Re$. It has an associated N dimensional vector, expressed as

$$\vec{W} = \begin{bmatrix} w_1 & \cdot & \cdot & \cdot & w_N \end{bmatrix}^T \tag{3}$$

such that

$$\sum_{i=1}^{N} w_i = 1; 0 \leq w_i \leq 1, \forall i = 1, ..., N \tag{4}$$

in which

$$\mathbb{OWA}(a_1, \cdots, a_N) = \sum_{q=1}^{N} w_q v_q \tag{5}$$

and v_q is the qth largest of the a_x $(x = 1, \cdots, N)$.

In the above equations, $\vec{W}$ represents the OWA weighting vector, and its component w_q are called the OWA weights. If let π be an index function and π_x be the index of xth largest argument value, then the OWA operator can be denoted by

$$OWA(a_1, ..., a_n) = \sum_{x=1}^{n} w_x a_{\pi_x} \tag{6}$$

In particular, a functional method that characterizes the attitude of the decision makers is defined to obtain the OWA weights, denoted as

$$w_x = (\frac{x}{N})^{\frac{1-\alpha}{\alpha}} - (\frac{x-1}{N})^{\frac{1-\alpha}{\alpha}} \tag{7}$$

in which $0 \leq \alpha \leq 1$. α is used as a parameter representing attitude preference. The larger the α, the more optimistic. Normally, α is taken as $0.1, ..., 1$.

2.3. Power ordered weighted averaging (POWA) operator [37, 38]

Since the OWA weights depend only on the independent variables, and there is no link between the candidate variables. To overcome this issue, by introducing the idea of support functions in the power average (PA) operator, Yager [37] designed a unique power OWA (called the POWA) operator defined as follows.

Definition 4. Let $\mathbb{F}$ be a basic unitinterval monotonic (BUM) function, then a POWA operator can be defined by

$$\mathbb{POWA}_{\mathbb{F}}(c_1, ..., c_n) = \sum_{q=1}^{N} v_q c_{index(q)} \tag{8}$$

in which

$$v_q = \mathbb{F}\left(\frac{U_q}{TI}\right) - \mathbb{F}\left(\frac{U_{q-1}}{TI}\right) \tag{9}$$

$$U_q = \sum_{q=1}^{N} I_{index(q)} \tag{10}$$

$$I_{index(q)} = 1 + L(c_{index(q)}) \tag{11}$$

$$L(c_{index(q)}) = \sum_{p=1}^{N} Sup(c_{index(q)}, c_{index(p)}) \tag{12}$$

In the above formulas, U_{q-1} is set equal to 0. $Sup(\cdot)$ function represents the support that c_q get from c_p, and it contains the following properties:

- $Sup(a,b) \in [0,1]$
- $Sup(a,b) = Sup(b,a)$
- $Sup(a,b) \geq Sup(x,y)$, if and $only$ if $|a-b| < |x-y|$

According to the above, and then the POWA weights $w_j^{'}$ are defined as

$$w_j^{'} = \frac{U_j}{TI} - \frac{U_{j-1}}{TI} \tag{13}$$

in which

$$U_i = \sum_{j=1}^{i} I_i \tag{14}$$

with

$$I_i = w_j \left(1 + \sum_{j=1, j\neq i}^{n} w_j Sup(a_i, a_j)\right) \tag{15}$$

$$TI = \sum_{i=1}^{n} w_j \left(1 + \sum_{j=1, j\neq i}^{n} w_j Sup(a_i, a_j)\right) \tag{16}$$

In the above formulas, w_j represents the OWA weights.

3. Existing soft likelihood functions

It is well known that the initial likelihood function can produce counter-intuitive results when combining conflicting compatible evidence. Thus, soft likelihood function was created precisely for this problem. In this section, by surveying the literature, we review the existing research on algorithms for soft likelihood functions per se. Broadly speaking, the existing research on soft likelihood functions can be distinguished into two main categories in the following subsections, namely, the soft likelihood function based on OWA operators and the soft likelihood function based on OWA operators.

3.1. Soft likelihood function based on OWA operator [22]

Initially, Yager et al. [22] suggested a new soft likelihood function by assigning OWA weights to each product term in the likelihood function, defined as follows.

Definition 5. In connection with the OWA weights, the soft likelihood function is formally defined as

$$L_{i,\omega}^{OWA} = \sum_{j=1}^{q} w_j Prod_i(j) \tag{17}$$

where w_j indicates the OWA weights and $Prod_i(j)$ represents the likelihood function values.

Next, Yager et al. [22] proposed a way to compute a reliable soft likelihood associated with each p_{ij} by considering the product of the probabilities associated with candidate events x_i and the normalized reliability.

Definition 6. For each possible event x_i, its normalized reliability r_{ij} is calculated by

$$r_{ij} = \frac{R_{ij}}{TR_i} \tag{18}$$

where TR_i represents the total reliability associated with x_i, i.e., $TR_i = \sum_{j=1}^{q} R_{ij}$. For each x_i, then $\sum_{j=1}^{q} r_{ij} = 1$.

Considering the index function σ_i, $\sigma_i(k)$ is the kth largest index of these products. $p_{i\sigma_i(k)} \times r_{i\sigma_i(k)}$ is the kth largest product of these $p \times r$ products. For the x_i, the ordering in the evidence is based on the product of the evidence's compatibility probability and the reliability of evidence. Therefore, since the index function, the likelihood function is newly expressed by

$$Prod_i^r(j) = \prod_{k=1}^{j} p_{i\sigma_i(k)} \tag{19}$$

and S_{ij} is denoted as

$$S_{ij} = \sum_{k=1}^{j} r_{i\sigma_i(k)} \tag{20}$$

in which S_{ij} is the sum of the normalized reliability associated with the j largest $p \times r$ products of the candidate x_i.

If let $f(\cdot)$ represent a weight generation function for implementing a soft likelihood function, the OWA weights of $j = 1$ to q associated with x_i is denoted by

$$w_{ij} = f(S_{ij}) - f(S_{i(j-1)}) \tag{21}$$

Further, considering $f(x) = x^D$ with $D = (1 - \alpha)/(\alpha)$, where α is expressed as optimism degree for decision makers. Finally, the weights of the reliable soft likelihood function are defined as

$$w_{ij} = (S_{ij})^{\frac{1-\alpha}{\alpha}} - (S_{i(j-1)})^{\frac{1-\alpha}{\alpha}} \tag{22}$$

Using the above weights, finally, the reliable soft likelihood function value of

x_i can be obtained by

$$L^r_{i,\alpha} = \sum_{j=1}^{q} \left(\left(S_{ij} \right)^{\frac{1-\alpha}{\alpha}} \right) - \left(S_{i(j-1)} \right)^{\frac{1-\alpha}{\alpha}} \right) \right) Prod^r_i(j) \tag{23}$$

In addition, Tian et al. [33, 34] recently extended the Yager et al.'s soft likelihood function based on OWA operator to the domain of Z numbers and D numbers to deal with uncertainty information characterized by knowledge of different frameworks.

Remarks: As the methods proposed in this paper focus on the shortcomings of existing soft likelihood function algorithms themselves, the extensions mentioned above are not the focus of our discussion. Here we only briefly list the work they have done; for more information please refer to Refs. [33, 34].

3.2. Soft likelihood function based on POWA operator [36, 39]

Following Yager et al.'s approach of assigning weights based on the OWA operator, Song and Deng [36] subsequently proposed a soft likelihood function based on the POWA operator. As described in the previous section, the POWA operator, as an improvement of the OWA operator, better takes into account the correlation between the probabilities of evidence. However, the author in [39] recently found that the method proposed by song et al. does not reflect the attitudinal characteristics of decision makers. Therefore, to address this problem, they developed a modified soft likelihood function, defined as follows.

Definition 7. With the POWA weights, this soft likelihood function is defined as

$$L_{i,\omega'}^{POWA} = \sum_{t=1}^{p} w_t' Prod_i(t) \tag{24}$$

where w_t' indicates the OWA weights and $Prod_i(t)$ represents the likelihood function values.

4. Proposed a generalized soft likelihood function

Based on the background knowledge in the previous two sections, it is clear that the essence of the soft likelihood function is actually to assign different weights to the product terms obtained from each likelihood function to solve the problem of conflicting data fusion. Yet when the same probabilities occur in a set of compatible evidence sets, the existing soft likelihood function assigns different weights to them. Clearly, it is not reasonable to assign the same weights in this case, as evidence with the same probability of support is intuitively of "equal importance", i.e. equal priority. We recognise that these existing weight allocation methods may result in a final fusion that does not reflect the level of decision makers' preferences at different levels of positivity. Furthermore, let us note that the phenomenon of "equally important probabilistic information" in the evidence set is a common occurrence in real-world data fusion processes. Thus, our motivation here is to find a way to cleverly handle this data information prior to fusion using a soft likelihood function to better represent the decision maker's level of preference.

In this section, we first define a novel representation of decision information, which we call the decision maker (DM) pair, and which is seen as a "container" that takes into account the decision maker's decision information as well as its own reliability. As "equal" evidence emerges from the

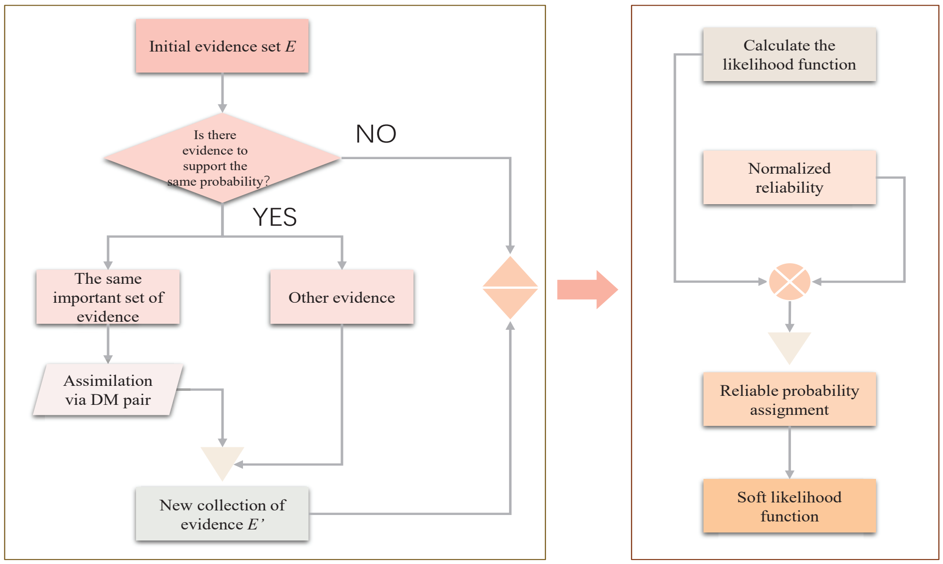

Figure 1: The hierarchical flow chart of the proposed negation method.

fused information, it is assimilated by the DM pair into an integrated body of evidence, ultimately forming a new evidence set. The reliability of the evidence information itself is then imported by means of a defined reliability algorithm. Finally the evidence information from multiple sources is fused through the proposed new likelihood function model. The flow chart of the proposed generalized algorithm is shown in Fig. 1.

Definition 8. For the jth decision maker (DM), let $< v_{ij}, r_{ij} >$ denote the DM pair associated with him. Where v_{ij} represents the decision result of the DM, and r_{ij} represents the reliability of the DM, i.e., the ability of the expert to provide a correct assessment or solution to a given problem domain. Then, suppose a suspect x_i is considered, and its support degree compatible with n evidence sources is defined by

$$E^{'} = \left\{ \underbrace{< v_{i1}, r_{i1} >}_{DM-1}, ..., \underbrace{< v_{in}, r_{in} >}_{DM-n} \right\},\ 0 \leq r_{ij} \leq 1 \tag{25}$$

where, the ordering of $< v_{ij}, r_{ij} >$ depends on the first variable v_{ij}.

Next, we consider the representation of compatible evidence in a special case. Assume there are $q = n$ sources of evidence. Consider a suspect x_i, whose probability of being compatible with the n sources of evidence is $E^{'}$. If $p_{i1} = \cdots = p_{iq}$ $(q \leq n)$ in $E^{'}$, then the compatible evidence set is represent as E. That is

$$E^{'} = \left\{ \underbrace{< p_{i1}, r_{i1} >}_{DM-1}, \underbrace{< p_{i2}, r_{i2} >}_{DM-2}, ..., \underbrace{< p_{in}, r_{in} >}_{DM-n} \right\} \tag{26}$$

$$\Downarrow$$

$$E = \left\{ \underbrace{< p, r >}_{DM-1\ and\ \cdots\ and\ DM-q}, ..., \underbrace{< p_{in}, r_{in} >}_{DM-n} \right\} \tag{27}$$

in which $p = \sum_{j=1}^{q} p_{ij}/q$ and $r = \sum_{j=1}^{q} r_{ij}/q$.

Combining the index function α_i, the likelihood function is defined as follows

$$Prod_i(j)^{'} = \prod_{k=1}^{j} p_{i\alpha_i(k)} \tag{28}$$

where the sort is determined by p_i in $< p_i, r_i >$, $\alpha_i(k)$ is the kth largest index in the product of these p_i, and the ordering is induced by $\alpha_i(k)$. Here $Prod_i(j)^{'}$ represents the product of the first j ordered probabilities.

Then, by the following formula, we calculate the normalized reliability of each DM in the q evidence sources

$$\mu_{ij} = \frac{r_{ij}}{r_i} \tag{29}$$

where r_i represents the sum of the reliability associated with x_i, i.e., $r_i = \sum_{j=1}^{q} r_{ij}$. We can see that for x_i, there is $\sum_{j=1}^{q} \mu_{ij} = 1$.

We further let β_{ij} is the sum of the normalized reliabilities associated with the j largest $p \times \mu$ ($p_{i\alpha_i(k)} \times \mu_{i\alpha_i(k)}$) products for candidate x_i, which is defined as follows

$$\beta_{ij} = \sum_{k=1}^{j} \mu_{i\alpha_i(k)} \tag{30}$$

In particular, if we consider the weight of the $f(x) = x^D$ function to generate the soft likelihood function, then we have

$$w_{ij} = (\beta_{ij})^{\frac{1-\gamma}{\gamma}} - (\beta_{i(j-1)})^{\frac{1-\gamma}{\gamma}} \tag{31}$$

where, the degree of optimsim is γ and $D = (1-\gamma)/\gamma$.

Finally, combined with the weight w_{ij}, the soft likelihood function is expressed as follows

$$\widetilde{L}_{i,\gamma} = \sum_{j=1}^{q} \left((\beta_{ij})^{\frac{1-\gamma}{\gamma}} - (\beta_{i(j-1)})^{\frac{1-\gamma}{\gamma}} \right) Prod_i(j)' \tag{32}$$

5. Numerical comparisons

Usually, in a soft likelihood function, if the degree of optimism is more correlated with the soft likelihood function value, it indicates that this soft likelihood is more reasonable [22, 39]. That is, the role of decision preferences in this case is well captured by the results of the fusion of information from the data. In this section, we shall demonstrate the computational process of the proposed method using some numerical examples, and show the superiority of the method by comparing it with the existing soft likelihood function in each different case.

Remarks: Examples 1 and 2 are used to indicate the usability of the proposed method. Where Example 1 is designed in the context that the initial evidence set does not contain evidence of equal probability and does not consider the reliability of the decision maker. Example 2 is designed under the condition that the initial evidence set does not contain evidence of the same probability and the reliability of the decision maker is considered. Examples 3 and 4 are used to point out the superiority of the proposed method. Where Example 3 is designed in the context that the initial evidence set contains evidence of the same probability and contains conflicting evidence in this evidence set. Example 4 is designed under the condition that the initial evidence set contains evidence of the same probability and that this evidence set does not contain conflicting evidence. Furthermore, as the soft likelihood function proposed by the author [39] (denoted in the figure as mi et al.) does not take into account the reliability of the evidence sources, this part is not discussed in the examples shown below.

5.1. Example 1 (without considering reliability)

Assume we have $q = 5$ sources of evidence. Consider a suspect x_i whose probability of compatibility with the five sources of evidence is

$$E^{'} = \{p_{i1} = 0.5, p_{i2} = 1, p_{i3} = 0.3, p_{i4} = 0.8, p_{i5} = 0.7\}$$

5.1.1. Results of the proposed soft likelihood function

We simply recall that in the proposed method, the weight is expressed as $w_{ij} = (\beta_{ij})^{(1-\gamma)/\gamma} - (\beta_{i(j-1)})^{(1-\gamma)/\gamma}$, and $Prod_i(j)' = \prod_{k=1}^{j} p_{i\alpha_i(k)}$. Since the reliability of the decision maker is not considered, in other words, the reliability of the DM is equally important. In this case, we take $r_{i1} = \cdots = r_{i5} = \delta$ $(0 \leq \delta \leq 1)$. Then, the compatible evidence is expressed as

$$E = \left\{ \underbrace{< 0.5, \delta >}_{DM-1}, \underbrace{< 1, \delta >}_{DM-2}, \underbrace{< 0.3, \delta >}_{DM-3}, \underbrace{< 0.8, \delta >}_{DM-4}, \underbrace{< 0.7, \delta >}_{DM-5} \right\}$$

The normalized reliability of each of the five evidence sources is expressed as follows

$$\mu_{ij} = \frac{r_{ij}}{r_i} = \frac{\delta}{5\delta} = 0.2$$

Then, we calculate the value of $p_{ij} \times \mu_{ij}$ $(p \times \mu)$, and the results are shown in Table 1.

Table 1: The probability-reliability products in Example 1.

j	r_{ij}	Probability (p_{ij})	Reliability (μ_{ij})	$p_{ij} \times \mu_{ij}$	Index order (α_i)
1	δ	0.5	0.2	0.1	4
2	δ	1	0.2	0.2	1
3	δ	0.3	0.2	0.06	5
4	δ	0.8	0.2	0.16	2
5	δ	0.7	0.2	0.14	3

As we can see from Table 1, using the index function α_i, the order of probability is expressed as $\alpha_i(1) = 2, \alpha_i(2) = 4, \alpha_i(3) = 5, \alpha_i(4) = 1, \alpha_i(5) = 3$. Then according to the index value, we calculate $Prod_i(j)' = \prod_{k=1}^{j} p_{i\alpha_i(k)}$, which can be expressed as $Prod_i(j)' = Prod_i(j-1)' p_{i\alpha_i(k)}$, and the calculation results are shown in Table 2.

Table 2: The probability products in Example 1.

Ordered probability	$Prod_i(j)'$
$p_{i\alpha_i(1)} = p_{i2} = 1$	$Prod_i(1)' = 1$
$p_{i\alpha_i(2)} = p_{i4} = 0.8$	$Prod_i(2)' = 1 \times 0.8 = 0.8$
$p_{i\alpha_i(3)} = p_{i5} = 0.7$	$Prod_i(3)' = 0.8 \times 0.7 = 0.56$
$p_{i\alpha_i(4)} = p_{i1} = 0.5$	$Prod_i(4)' = 0.56 \times 0.5 = 0.28$
$p_{i\alpha_i(5)} = p_{i3} = 0.3$	$Prod_i(5)' = 0.28 \times 0.3 = 0.084$

Then, ordering the normalized reliabilities based on the index α_i, we can calculate the sum of the normalization probabilities, and the results are shown in Table 3.

Table 3: The sum of the probabilities of normalized reliability in Example 1.

j	1	2	3	4	5
$\mu_{i\alpha_i(j)}$	0.5	1	0.3	0.8	0.7
β_{ij}	0.2	0.2	0.2	0.2	0.2

Next, we show the procedure for computing the values of the proposed soft likelihood function under different preferences. In particular, taking $\gamma = 0.5$ as an example and using Eq. (32), we can obtain the results as shown in Table 4.

Table 4: The soft likelihood function under $\gamma = 0.5$ in Example 1.

j	$\beta_{i(j)}$	$\beta_{i(j-1)}$	w_{ij}	$Prod_i(j)'$	$w_{ij}Prod_i(j)'$
1	0.2	0	0.2	1	0.2
2	0.4	0.2	0.2	0.8	0.16
3	0.6	0.4	0.2	0.56	0.112
4	0.8	0.6	0.2	0.28	0.056
5	1	0.8	0.2	0.084	0.0168

From Table 4, we can see that $\sum_j w_{ij} = 1$. In this case where $\gamma = 0.5$ the soft likelihood value is 0.5448. Further, we consider the soft likelihood function values under $\gamma = 0.1, ..., 1$, and the results are shown in Table 5.

Table 5: The values of the proposed soft likelihood function under different levels of optimism in Example 1.

Function value	Degree of optimism									
	γ=0.1	γ=0.2	γ=0.3	γ=0.4	γ=0.5	γ=0.6	γ=0.7	γ=0.8	γ=0.9	γ=1
$\widetilde{L}_{i,\gamma}$	0.1132	0.2070	0.3184	0.4330	0.5448	0.6508	0.7495	0.8404	0.9238	1.0000

5.1.2. Comparisons - without considering reliability

We review the soft likelihood function based on OWA operator proposed by Yager et al. as $L_{i,\alpha} = \sum_{j=1}^{5} \left((\frac{j}{5})^{(1-\alpha)/\alpha} - (\frac{j-1}{5})^{(1-\alpha)/\alpha} \right) \prod_{k=1}^{j} p_{i\lambda_i(k)}$. Using the index function $\lambda_i(k)$, the order of probability is expressed as $\lambda_i(1) = 2, \lambda_i(2) = 4, \lambda_i(3) = 5, \lambda_i(4) = 1, \lambda_i(5) = 3$. Then, we consider the soft

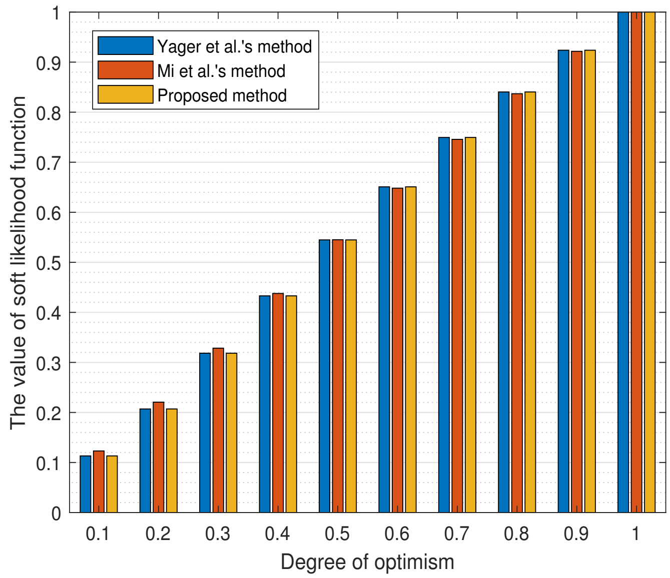

Figure 2: The comparison of different methods in Example 1.

likelihood function values under $\alpha = 0.1, ..., 1$, and the results are shown in Table 6. Similarly, we calculate the soft likelihood function values based on the POWA operator proposed by the author under the same conditions, and the results are also shown in Table 6. Figure 2 visualises the results of the comparison between the proposed method, the Yager et al.'s method and the author's method.

Table 6: The comparison of different methods in Example 1.

Method	Degree of optimism									
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Yager et al.'s model [22]	0.1132	0.2070	0.3184	0.4330	0.5448	0.6508	0.7495	0.8404	0.9238	1.0000
The author's model in [39]	0.1231	0.2206	0.3285	0.4378	0.5451	0.6482	0.7457	0.8369	0.9216	1.0000
Proposed model	0.1132	0.2070	0.3184	0.4330	0.5448	0.6508	0.7495	0.8404	0.9238	1.0000

5.2. Example 2 (consider reliability)

Assume we have $q = 5$ sources of evidence. Consider a suspect x_i whose probability of compatibility with the five sources of evidence is

$$E^{'} = \{p_{i1} = 0.5, p_{i2} = 1, p_{i3} = 0.3, p_{i4} = 0.8, p_{i5} = 0.7\}$$

and their evidence reliabilities are as follows

$$R = \{r_{i1} = 1, r_{i2} = 0.7, r_{i3} = 0.5, r_{i4} = 0.6, r_{i5} = 0.4\}$$

5.2.1. Results of the proposed soft likelihood function

Now we simply recall that in the proposed method, the weight is expressed as $w_{ij} = (\beta_{ij})^{(1-\gamma)/\gamma} - (\beta_{i(j-1)})^{(1-\gamma)/\gamma}$, and $Prod_i(j)^{'} = \prod_{k=1}^{j} p_{i\gamma_i(k)}$. Then, the compatible evidence is expressed as

$$E = \left\{ \underbrace{< 0.5, 1 >}_{DM-1}, \underbrace{< 1, 0.7 >}_{DM-2}, \underbrace{< 0.3, 0.5 >}_{DM-3}, \underbrace{< 0.8, 0.6 >}_{DM-4}, \underbrace{< 0.7, 0.4 >}_{DM-5} \right\}$$

The normalized reliability of each of the five evidence sources is expressed as follows

$$\mu_{ij} = \frac{r_{ij}}{r_i} = \frac{r_{ij}}{3.2}$$

Then, we calculate the value of $p_{ij} \times \mu_{ij}$ $(p \times \mu)$, and the results are shown in Table 7.

Table 7: The probability-reliability products in Example 2.

j	r_{ij}	Probability (p_{ij})	Reliability (μ_{ij})	$p_{ij} \times \mu_{ij}$	Index order (α_i)
1	1	0.5	0.3125	0.1563	2
2	0.7	1	0.2188	0.2188	1
3	0.5	0.3	0.1563	0.0469	5
4	0.6	0.8	0.1875	0.1500	3
5	0.4	0.7	0.1250	0.0875	4

Using the index function α_i, the order of probability is expressed as $\alpha_i(1) = 2, \alpha_i(2) = 1, \alpha_i(3) = 4, \alpha_i(4) = 5, \alpha_i(5) = 3$. Then according to the index value, we calculate $Prod_i(j)' = \prod_{k=1}^{j} p_{i\alpha_i(k)}$, which can be expressed as $Prod_i(j)' = Prod_i(j-1)' p_{i\alpha_i(k)}$, and the calculation results are shown in Table 8.

Table 8: The probability products in Example 2.

Ordered probability	$Prod_i(j)'$
$p_{i\alpha_i(1)} = p_{i2} = 1$	$Prod_i(1)' = 1$
$p_{i\alpha_i(2)} = p_{i1} = 0.5$	$Prod_i(2)' = 1 \times 0.5 = 0.5$
$p_{i\alpha_i(3)} = p_{i4} = 0.8$	$Prod_i(3)' = 0.5 \times 0.8 = 0.4$
$p_{i\alpha_i(4)} = p_{i5} = 0.7$	$Prod_i(4)' = 0.4 \times 0.7 = 0.28$
$p_{i\alpha_i(5)} = p_{i3} = 0.3$	$Prod_i(5)' = 0.28 \times 0.3 = 0.084$

Then, ordering the normalized reliabilities based on the index α_i, we can calculate the sum of the normalization probabilities, and the results are shown in Table 9.

Table 9: The sum of the probabilities of normalized reliability in Example 2.

j	1	2	3	4	5
$\mu_{i\alpha_i(j)}$	0.2188	0.3125	0.1875	0.1250	0.1563
β_{ij}	0.2188	0.5313	0.7188	0.8438	1

Next, we show the procedure for computing the values of the proposed soft likelihood function under different preferences. In particular, taking $\gamma = 0.5$ as an example and using Eq. (32), we can obtain the results as shown in Table 10.

Table 10: The soft likelihood function under $\gamma = 0.5$ in Example 2.

j	$\beta_{i(j)}$	$\beta_{i(j-1)}$	w_{ij}	$Prod_i(j)'$	$w_{ij}Prod_i(j)'$
1	0.2188	0	0.2188	1	0.2188
2	0.5313	0.2188	0.3125	0.5	0.1563
3	0.7188	0.5313	0.1875	0.4	0.0750
4	0.8438	0.7188	0.1250	0.28	0.0350
5	1	0.8438	0.1562	0.084	0.0131

From Table 10, we can see that $\sum_j w_{ij} = 1$. In this case where $\gamma = 0.5$ the soft likelihood value is 0.4982. Further, we consider the soft likelihood function values under $\gamma = 0.1, ..., 1$, and the results are shown in Table 11.

Table 11: The values of the proposed soft likelihood function under different levels of optimism in Example 2.

Function value	Degree of optimism									
	γ=0.1	γ=0.2	γ=0.3	γ=0.4	γ=0.5	γ=0.6	γ=0.7	γ=0.8	γ=0.9	γ=1
$\widetilde{L}_{i,\gamma}$	0.1330	0.2245	0.3087	0.3989	0.4982	0.6025	0.7074	0.8097	0.9075	1.0000

5.2.2. Comparisons - consider reliability

First, we review the method proposed by Yager et al. to include the reliability of evidence in the soft likelihood function as $L^r_{i,\alpha} = \sum_{j=1}^{5} \left((S_{ij})^{(1-\alpha)/\alpha} \right) - \left((S_{i(j-1)})^{(1-\alpha)/\alpha} \right) Prod^r_i(j)$. Then using the index function σ_i, we have $\sigma_i(1) = 2, \sigma_i(2) = 1, \sigma_i(3) = 4, \sigma_i(4) = 5, \sigma_i(5) = 3$.

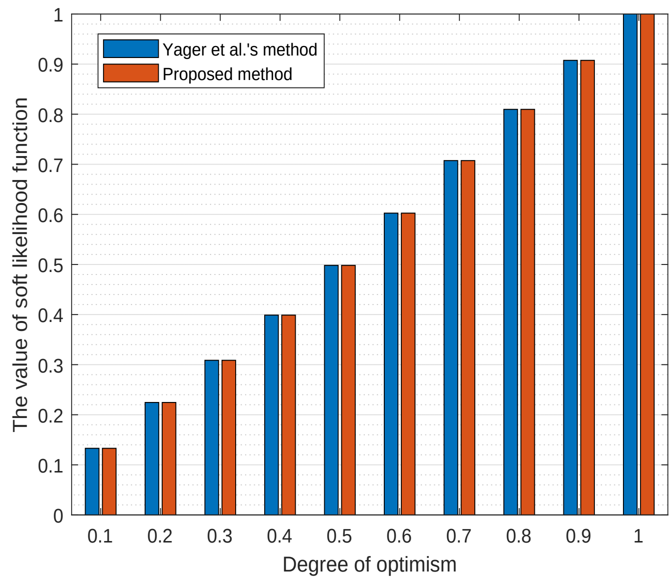

Figure 3: The comparison of different methods in Example 2.

We consider the soft likelihood function values under $\alpha = 0.1, ..., 1$, and the results are shown in Table 12. Figure 3 shows the comparison between the proposed method and Yager et al.'s method.

Table 12: The comparison of different methods in Example 2.

Method	Degree of optimism									
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Yager et al.'s model [22]	0.1330	0.2245	0.3087	0.3989	0.4982	0.6025	0.7074	0.8097	0.9075	1.0000
Proposed model	0.1330	0.2245	0.3087	0.3989	0.4982	0.6025	0.7074	0.8097	0.9075	1.0000

5.3. Summary of Examples 1 and 2

In the above two examples, by comparing the proposed method with existing methods, the results show that:

(1): (Example 1) When the reliability of the decision maker is not considered, the proposed method and the Yager et al.'s method have the same soft likelihood function values at different levels of optimism. Furthermore, we also observe that the soft likelihood function values of the proposed method and the author's method in [39] exhibit slight variability under the same conditions, due to the use of POWA generating function weights by the author in [39]

(2): (Example 2) When considering the reliability of the decision maker, the proposed method and the Yager et al.'s method show exactly the same trend of variation at different levels of optimism, i.e., both methods have the same soft likelihood function value.

Therefore, based on the above results, we can conclude that the proposed method degrades to approximately the same results as the existing method, considering as well as not considering reliability, when the compatible evidence set supporting the occurrence of an event does not contain the same probabilities. In this case, the preference parameters in both the proposed and existing methods reflect the degree of optimism of the decision maker in the fusion results of the soft likelihood function. Thus Examples 1 and 2 illustrate the usability of the proposed method. Next, we shall illustrate the superiority of the proposed method compared to the existing method by comparison and analysis through two additional examples.

5.4. Example 3 (contains conflicting evidence)

Assume we have $q = 3$ sources of evidence. Consider a suspect x_i whose probability of compatibility with the three sources of evidence is

$$E' = \{p_{i1} = 1, p_{i2} = 1, p_{i3} = 0\}$$

Obviously, we can see that the third compatible probability corresponds to a conflict evidence.

5.4.1. Comparisons - without considering reliability

In this case, we take $r_{i1} = r_{i2} = r_{i3} = \delta$ $(0 \leq \delta \leq 1)$. Then, the evidence of compatibility after conversion using the proposed method is expressed as

$$E = \left\{ \underbrace{< 1, \delta >}_{DM-1\ and\ DM-2}, \underbrace{< 0, \delta >}_{DM-3} \right\}$$

As in the previous steps, the proposed method is used to calculate the values of the soft likelihood function at different levels of optimism as shown in Table 13. At the same level of optimism, the results calculated by Yager et al.'s method and the author's previous model are also shown in Table 13. Figure 4 shows the comparison between the Yager et al.'s method, the author's method and the proposed method in this paper.

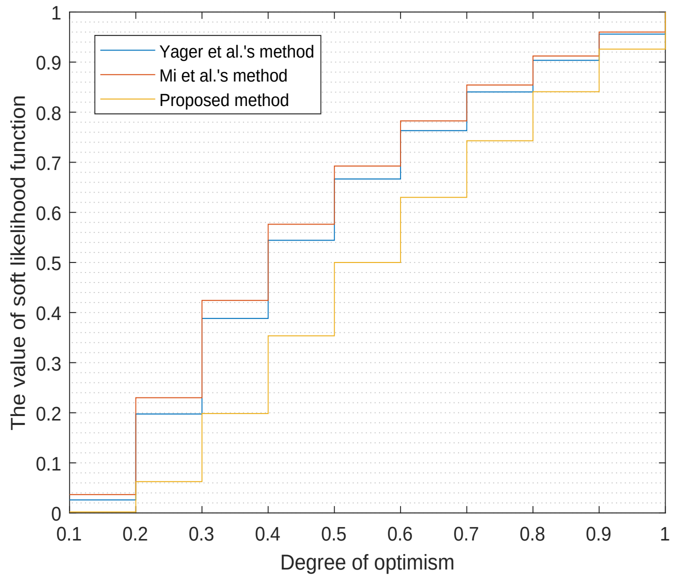

Figure 4: The comparison of different methods without considering reliability in Example 1.

Table 13: The comparison of different methods without considering reliability in Example 1.

Method	Degree of optimism									
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Yager et al.'s model [22]	0.0260	0.1975	0.3883	0.5443	0.6667	0.7632	0.8405	0.9036	0.9559	1.0000
The author's model in [39]	0.0366	0.2300	0.4243	0.5763	0.6925	0.7827	0.8543	0.9122	0.9600	1.0000
Proposed model	0.0020	0.0625	0.1984	0.3536	0.5000	0.6300	0.7430	0.8409	0.9259	1.0000

5.4.2. Comparisons - consider reliability

It is assumed that for these three sources of evidence, namely

$$E^{'} = \{p_{i1} = 1, p_{i2} = 1, p_{i3} = 0\}$$

and that their corresponding reliability is

$$R = \{r_{i1} = 1, r_{i2} = 0.7, r_{i3} = 0.5\}$$

Then, the evidence set of compatibility after conversion using the proposed method is expressed as

$$E = \left\{ \underbrace{< 1, 0.85 >}_{DM-1\ and\ DM-2}, \underbrace{< 0, 0.5 >}_{DM-3} \right\}$$

Similarly, as in the previous steps, the proposed method is used to calculate the values of the soft likelihood function at different levels of optimism as shown in Table 14. At the same level of optimism, the results calculated using Yager et al.'s method are also shown in Table 14. Figure 5 shows the comparison between the proposed method and the Yager et al.'s method.

Table 14: The comparison of different methods under the condition of reliability in Example 3.

Method	Degree of optimism									
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Yager et al.'s model [22]	0.0982	0.3565	0.5479	0.6793	0.7727	0.8421	0.8954	0.9376	0.9718	1.0000
Proposed model	0.0156	0.1572	0.3398	0.4996	0.6296	0.7346	0.8202	0.8909	0.9499	1.0000

5.5. Example 4 (without containing conflicting evidence)

Assume we have $q = 3$ sources of evidence. Consider a suspect x_i whose probability of compatibility with the three sources of evidence is

$$E^{'} = \{p_{i1} = 1, p_{i2} = 1, p_{i3} = 0.5\}$$

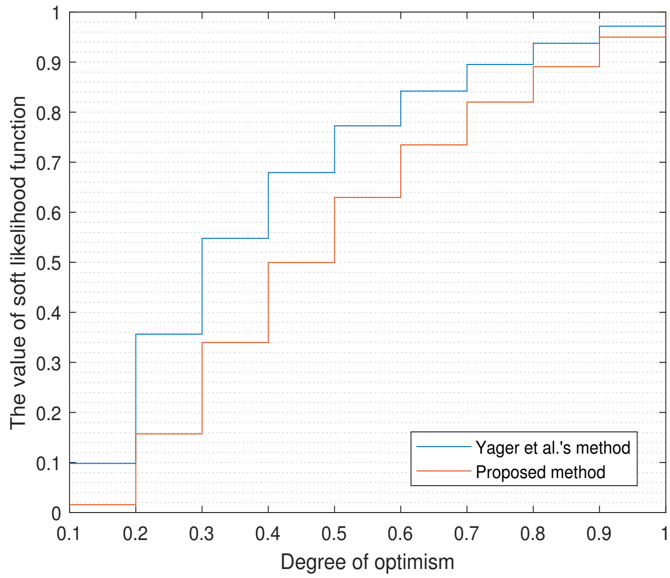

Figure 5: The comparison of different methods under the condition of reliability in Example 3.

5.5.1. Comparisons - without considering reliability

Similarly, in this case, we take $r_{i1} = r_{i2} = r_{i3} = \delta$ $(0 \leq \delta \leq 1)$. Then, the evidence of compatibility after conversion using the proposed method is expressed as

$$E = \left\{ \underbrace{< 1, \delta >}_{DM-1\ and\ DM-2}, \underbrace{< 0.5, \delta >}_{DM-3} \right\}$$

Then, as in the previous steps, the proposed method is used to calculate the values of the soft likelihood function at different levels of optimism as shown in Table 15. At the same level of optimism, the results calculated using Yager's method and the author's previous model [39] are shown in Table 15. Figure 6 shows the comparison between the Yager et al.'s method,

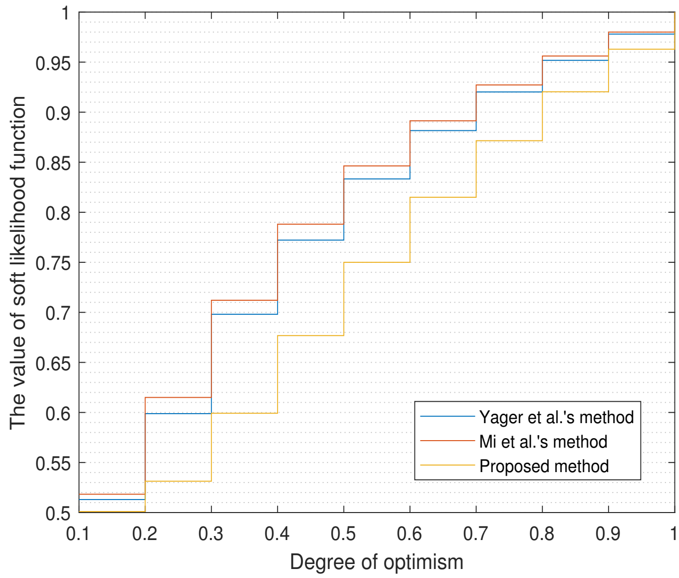

Figure 6: The comparison of different methods without considering reliability in Example 4.

the author's method [39] and the proposed method in this paper.

Table 15: The comparison of different methods without considering reliability in Example 4.

Method	Degree of optimism									
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Yager et al.'s model [22]	0.5130	0.5988	0.6981	0.7722	0.8333	0.8816	0.9202	0.9518	0.9780	1.0000
The author's model in [39]	0.5183	0.6150	0.7121	0.7881	0.8463	0.8914	0.9272	0.9561	0.9800	1.0000
Proposed model	0.5010	0.5313	0.5992	0.6768	0.7500	0.8150	0.8715	0.9204	0.9629	1.0000

5.5.2. Comparisons - consider reliability

Assume we have $q = 3$ sources of evidence. Consider a suspect x whose probability of compatibility with the 3 sources of evidence is

$$E^{'} = \{p_{i1} = 1, p_{i2} = 1, p_{i3} = 0.5\}$$

and their evidence reliabilities are as follows

$$R = \{r_{i1} = 1, r_{i2} = 0.7, r_{i3} = 0.5\}$$

Similarly, the evidence of compatibility after conversion using the proposed method is expressed as

$$E = \left\{ \underbrace{< 1, 0.85 >}_{DM-1 \ and \ DM-2}, \underbrace{< 0.5, 0.5 >}_{DM-3} \right\}$$

Finally, as in the previous steps, the proposed method is used to calculate the values of the soft likelihood functions at different levels of optimism as shown in Table 16. At the same level of optimism, the results calculated using Yager et al.'s method are shown in Table 16. Figure 7 shows the comparison between the Yager et al.'s method and the proposed method.

Table 16: The comparison of different methods in Example 4 under the condition of reliability.

Method	Degree of optimism									
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
Yager et al.'s model [22]	0.5491	0.6783	0.7740	0.8396	0.8864	0.9210	0.9477	0.9688	0.9859	1.0000
Proposed model	0.5078	0.5786	0.6699	0.7498	0.8148	0.8673	0.9101	0.9454	0.9749	1.0000

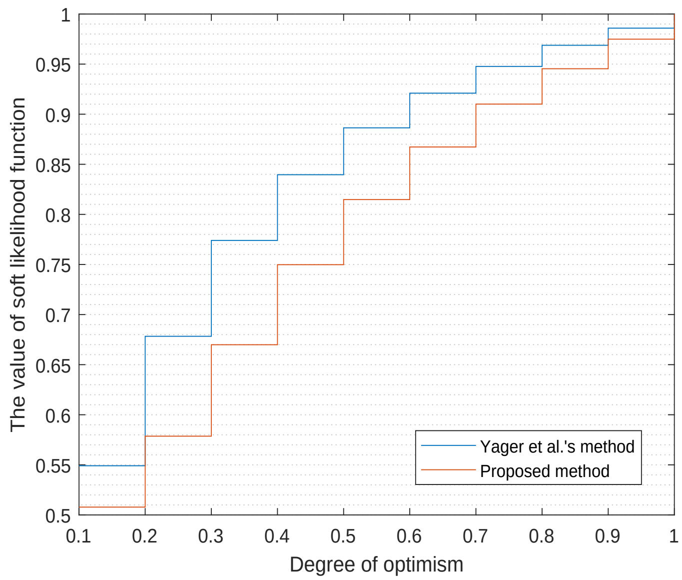

Figure 7: The comparison of different methods under the condition of reliability in Example 4.

5.6. Summary of Examples 3 and 4

Based on the values obtained from the calculation of the two numerical examples above and the visualisation of the results in the graph, we can see that:

Example 3: When there is conflict evidence in the evidence set of compatible probability, in this case:

(1): If the reliability of the decision maker is not taken into account, the variation of the preference parameter with the value of the soft likelihood function in the proposed method is more relevant (i.e., more linear) than in the methods of Yager et al. and the author's approach in [39]

(2): The variation of the preference parameter with the value of the soft likelihood function in the proposed method is also more relevant than the method of Yager et al., if the reliability of the decision maker is considered.

Example 4: In an alternative decision-making setting, where the compatible evidence set does not contain conflicting evidence, in which case:

(1): When the reliability of the decision maker is not considered, the variation of the preference parameter with the value of the soft likelihood function in the proposed method is more relevant than in the methodology of Yager et al. and the author in [39].

(2): When considering the reliability of the decision maker, the variation of the preference parameter with the value of the soft likelihood function in the proposed method is also more relevant than in the Yager et al.'s method.

Thus, with Examples 3 and 4, we illustrate the superiority of the proposed soft likelihood function compared to existing methods for fusing evidence data when compatible evidence sets contain equally important evidence, and in two respects, whether they contain conflicting evidence.

5.7. Discussion and analysis

It is shown by some numerical examples that, in general, the proposed method degrades to Yager et al.'s method when the same probabilities do not exist in a compatible evidence set. When there is "equally important evidence" in the evidence set, the proposed method gives a better convergence effect, i.e., it better reflects the level of decision makers' preferences at different levels of optimism. Thus, in other words, this also confirms that the proposed method is indeed a generalised method in these cases.

6. Conclusion

In this paper a generalised soft likelihood function is proposed. First, a novel concept of DM pair is introduced, which integrates the reliability of decision outcomes and evidence sources. Next, a series of algorithms for correcting the initial set of compatible evidence is formulated. Ultimately, a generic soft likelihood function for fusing compatible evidence information is defined. By using some examples of design, we have shown that the proposed method is indeed a broad approach. Moreover, some general advantages about the proposed method have been verified. More specifically, when the compatible evidence set supporting the occurrence of an event does not contain the same probabilities, the proposed method's calculations approximate the results of existing methods. When the compatible evidence set contains "evidence of equal status", the proposed method better reflects the relevance of decision preferences and fusion outcomes than existing methods.

However, it is worth noting that the proposed approach, while obtaining "consistency" in preferences and results, may underestimate the necessity of the evidence source itself. Therefore, finding a more perfect general soft likelihood function will be the goal of our future exploration. In addition, in the future, we plan to use the proposed method in more scenarios to further explore the practical value of the proposed method.

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