Negation of Atanassov's intuitionistic fuzzy sets from the perspective of maximum entropy

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***Abstract*—In fuzzy systems, how to represent uncertainty is a crucial research topic. Negation is an inherent characteristic of knowledge, and it provides a brand-new perspective of solving problems from the opposite of the events. Intuitionistic fuzzy sets (IFSs), as a generalization of the fuzzy sets, have the ability to better express fuzzy information. However, since the existing methods have not completely broken through the constraints of the first (classical) negation and inconsistent calculation standards, IFSs still have limitations in expressing uncertainty. To address this issue, and strengthen the performance of fuzzy systems to represent uncertain information, this paper proposed a novel method to obtain the negation of the IFS from the perspective of maximum entropy. Some desired theorems and properties are investigated to denote the nature of the negative IFS. Moreover, entropy is used to describe the connection between the IFS and uncertainty in the negation process. Futhermore, based on the negation, this paper designed a new approach to measure the uncertainty of the IFS. Then, a new pattern classification algorithm is developed. Finally, the practical applications show the effectiveness of the negation method.**

***Index Terms*—Intuitionistic fuzzy sets, negation, uncertainty, pattern classification.**

I. Introduction

In the field of information science, fuzzy is an important property characteristic of information. In fact, fuzzy information inevitably exists in social and practical applications. Then, how to characterize and deal with uncertain information becomes very important. By reviewing the literatures, existing research can be divided into two main categories. The first category deals with uncertain information in a specific knowledge framework, which includes evidence theory [1]–[3], fuzzy sets [4], [5], Z-numbers [6]–[8], D-numbers [9], [10], etc. The second category is to process and predict uncertain information in different fields of research through mathematical models, such as soft likelihood function (SLF) [11]–[13], entropy algorithms [14]–[16], distance algorithms [14], [17], and divergence algorithms [18], [19], correlation coefficient [20]–[22], and so on.

It is worth noting that, no matter in any of the above categories, negation is a general problem-solving rule. For an objective problem, the solution method is generally derived from the front. However, when the event is difficult to solve from the front, a strategy worthy of being adopted is to solve it by finding the opposite of the event (i.e., negation). For instance, when a theorem is difficult to prove to be true, if we can find a counterexample, at least we can show that it is not true. Moreover, when the cost of solving the positive problem is greater, the value of obtaining the results of the event from the negation is higher. In order to determine the negation of the probabilistic events, Professor Zadeh first formally formulated a calculation method in his BISC blog. After that, this issue has attracted widespread attention from researchers. In particular, Yager proposed the negation of a probability distribution of maximum entropy in 2015 [23]. Later, some Bayes-based properties [24], [25] and extended methods [26], [27] are studied. In recent years, there has been more and more research on negation in different fields, e.g., evidence theory [28]–[32], Z-numbers theory [33]. As a result, the expression performance of uncertainty in these fields has been greatly enriched and is still evolving [34].

In particular, IFSs [35], [36], as an extension of fuzzy sets, provides a better framework for the representation of fuzzy information, and many fields have been used as the main driving force to apply this theory to practice, such as medical diagnosis [37], decision making [38]–[41], pattern recognition [42], etc. However, the uncertainty expression performance of the IFSs is still not perfect enough. On the one hand, because of the limitation of the IFSs structure framework. In order to expand the knowledge representation range and performance of the IFSs, some technical methods have been adopted, including interval-valued IFSs [43], [44], membership performance improvement [45], and extended algorithms [46], [47]. On the other hand, it lacks a universal negation operation method. Actually, the research on IFSs negation started very early, in order to break through the classic negation behavior defined on IFSs and to defend the IFSs name [48], [49]. In the process of advancement of this type of research, first some negation operations were generated for different implications [50]–[53]. And then Atanassov and some scholars spent a lot of effort to verify whether they met the Law for Excluded Middle (LEM) and De Morgan's Laws (DMLs) to illustrate the non-classical behavioral characteristics of negation [54]–[58]. However, it is very likely that due to complexity and the lack of a unified calculation standard, this will limit the development of the IFSs negation to a certain extent. Frankly speaking, in the past ten years, the realization of the negation of IFSs has not attracted widespread attention from scholars. Last but not least, the discussion on the negation of IFSs is still an open issue, and there is still room for improvement in

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order to enhance the ability to represent fuzzy information.

Next, let us consider an simple example in a knowledge-based system. Assume that two experts are invited to provide opinions on a decision problem. The final decision-making results of the first expert is 70% support, 20% against, and 10% neutral, marked as R_1. However, the results of the other expert are exactly the opposite of the first expert, marked as R_2. Apparently, there is $R_2 = \neg R_1$. For R_1, it is not difficult to generate a possible IFS, $A_1 = \{< x, 0.7, 0.2 > \mid x \in X\}$, according to the IFSs theory. However, for R_2, we cannot directly obtain the corresponding IFS, only $A_2 = \neg A_1$, that is, A_2 is the negation of A_1. It can be seen that negation is very important for processing uncertain information, which is expected to further enhance the reasoning ability of IFSs. Therefore, the research motivation of this parper is to find a general method to characterize the negative IFS information.

A. Contributions

The main contributions of this paper are as follows. First, we propose a general approach to obtain the negation of the IFS, by drawing inspiration from the Yager model and uncertainty measures. Moreover, the proposed method focuses on the number of core elements in the IFS while considering probability, and the negation of core elements is independent of other elements. Second, we investigate the nature and properties of the negation operation, and study the process of the IFS negation through some cases. Experimental results show that the IFS negation method is usually irreversible due to the uncertainty of the IFS. Third, based on this interesting discovery, through distance and negation, we design a novel mathematical model to measure the uncertainty of the IFS. Fourth, in order to reflect the actual utility, we provide a new pattern classification algorithm based on the negation method. The application results show that the proposed method is effective for solving practical problems, which also reflects the huge potential of the proposed negation method in practice.

B. Paper outline

The remainder of the paper is organized as follows. Section II introduces the background knowledge of this research. Section III proposes the negation method of IFSs and two important theorems. Section IV defines some properties of the negation operation. Section V shows the process of negation operation through several cases. Section VI evaluates the proposed negation method theoretically and experimentally, and proposes an IFS uncertainty measure model. Section VII designs an algorithm for pattern classification based on the negation, and illustrates the effectiveness of the method through two application cases. Finally, Section VIII concludes this paper and points out future work.

II. Preliminaries

A. Atanassov's intuitionistic fuzzy sets

In the following, we shall introduce some concepts related to IFSs.

Definition 1. *(Fuzzy set [4]). Let X be a universe of discourse (UOD), a fuzzy set F is defined by*

$$F = \{< x, \mu_F(x) > \mid x \in X\}, \quad (1)$$

with

$$\mu_F(x) : X \to [0,1], \quad (2)$$

where $\mu_F(x)$ represents the membership level of the element x to F.

Atanassov extended the fuzzy sets and defined the following IFSs.

Definition 2. *(Intuitionistic fuzzy set [35]). Let X be a UOD, an IFS F on X is given by*

$$F = \{< x, \mu_F(x), \upsilon_F(x) > \mid x \in X\}, \quad (3)$$

in which

$$\mu_F(x) : X \to [0,1] \ \ and\ \ \upsilon_F(x) : X \to [0,1], \quad (4)$$

subjected to

$$0 \le \mu_F(x) + \upsilon_F(x) \le 1, \ \forall x \in X. \quad (5)$$

In the above formulas, $\mu_F(x)$ and $\upsilon_F(x)$ are the membership and nonmembership degrees of the element x to F, respectively.

For each IFS F in X, the degree of indeterminacy of x to F is defined by

$$\pi_F(x) = 1 - \mu_F(x) - \upsilon_F(x), \ \forall x \in X. \quad (6)$$

B. Distance measure for IFSs

In IFSs theory, distance measurement, used to express the difference between two IFSs, is a key mathematical tool. In recent years, this subject is receiving more and more attention from scholars. Recently, a novel distance formula based on Jensen–Shannon divergence was proposed by xiao [59], which is briefly introduced as follows.

Definition 3. *Suppose there are two IFSs A and B in a UOD Y, where $A = \{< y, \mu_A(y), \upsilon_A(y) > \mid y \in Y\}$ with $\pi_A(y) = 1 - \mu_A(y) - \upsilon_A(y)$, and $B = \{< y, \mu_B(y), \upsilon_B(y) > \mid y \in B\}$ with $\pi_B(y) = 1 - \mu_B(y) - \upsilon_B(y)$. A divergence measure between A and B, indicated by*

$$JS_{IFS}(A,B) = \frac{1}{2}\left[KL\left(A, \frac{A+B}{2}\right) + KL\left(B, \frac{B+A}{2}\right)\right], \quad (7)$$

with

$$KL(A,B) = \mu_A(y)log\frac{\mu_A(y)}{\mu_B(y)} + \upsilon_A(y)log\frac{\upsilon_A(y)}{\upsilon_B(y)} + \pi_A(y)log\frac{\pi_A(y)}{\pi_B(y)}, \quad (8)$$

in which $KL(A,B)$ is called Kullback-Leibler (K-L) divergence.

Then, a distance measure, denoted as $d_x(A,B)$, is defined by

$$d_X(A,B) = \sqrt{JS_{IFS}(A,B)} \quad (9)$$

The distance measure is intuitively understood as, the larger the $d_X(A,B)$, the greater the difference between IFSs A and B.

PROPERTY 1. *The properties satisfied by $d_X(A,B)$ are listed as follows:*
(P1.1) Nondegeneracy: $d_X(A,B)=0$, if and only if $A=B$, for $A,B \in Y$;
(P1.2) Symmetry: $d_X(A,B)=d_X(B,A)$, for $A,B \in Y$;
(P1.3) Inequality: $d_X(A,B)+d_X(B,C) \geq d_X(A,C)$, for $A,B,C \in Y$;
(P1.4) Boundedness: $0 \leq d_X(A,B) \leq 1$, for $A,B \in Y$.

C. Shannon entropy measure

Shannon entropy [60], named after Claude Shannon, was first proposed in 1948. Since then, Shannon entropy has been widely used in information science. Shannon entropy is an uncertainty measure about a random variable.

DEFINITION 4. *Consider a discrete random variable X with possible values (x_1, $\cdots$, x_i, $\cdots$, x_m), the Shannon entropy is defined as*

$$H(X)=H(p_1,\cdots,p_i,\cdots,p_m)=-\sum_{i=1}^{m} p_i \ log_q \ p_i, \quad (10)$$

where

$$p_i = Prod(X=x_i). \quad (11)$$

In Eq. (10), when $q=2$, the unit of information entropy is bit.

PROPERTY 2. *In addition, Shannon entropy attains, but is not limited to, the following properties:*
(P2.1) Boundedness: $0 \leq H(X) \leq log_2 m$;
(P2.2) Symmetry: $H(p_1,p_2,...)=H(p_2,p_1,...)=...$;
(P2.3) Grouping: $H(p_1,p_2,...,p_m)=H(p_1+p_2,p_3,...,p_m)+(p_1+p_2)H(p_1/(p_1+p_2),p_2/(p_1+p_2))$.

III. PROPOSED NEGATION OF IFSS

In many knowledge-based systems, a fact that has been demonstrated is that rare events may seriously affect the performance of the system in some cases, so rare events are very important for research [61]. The proposed negation method provides IFSs with the ability to derive negative information from the positive of knowledge, which is also the first modeling of the rare events in IFSs. Before formally proposing the negative concept of IFSs, below, we shall first define the core elements of an IFS.

For the IFS F on UOD X, we make the following definition. Namely, if

$$\mu_F(x), \upsilon_F(x), \pi_F(x) \ > 0, \ \forall x \in X, \quad (12)$$

then we call it the core element.

Next, a simple intuitive example is given to illustrate the construction ideas of the proposed method. In probability theory, suppose p_k represents the probability of event x_k, then what will its negative event be? Naturally speaking, consider mathematical logic, that is, $\neg x_k = 1-p_k$. However, we cannot simply use this logic to get the negation of the IFS. Furthermore, suppose that for this event, an IFS is given by $F=\{x_k, \mu_F(x), \upsilon_F(x) > | \ x_k \in X\}$. According to the above mathematical logic, for the negation of F (i.e., $\bar{F}$), the result is that $\bar{\mu}_{\bar{F}}(x)=\bar{\upsilon}_{\bar{F}}(x)=\bar{\pi}_{\bar{F}}(x)=\frac{2}{3}$. However, this result is obviously inappropriate, because the sum of all the core elements of F is not equal to 1. Hence, for the negation of the IFS, we should not only consider the probability, but also the influence of the number of core elements in the modeling process.

Below, we shall show how to get the negation of IFSs more reasonably. Note that in this paper, the negation of an IFS F is represented as $\bar{F}$, and the production flow of the negation is listed as follows.

Step 1: For core elements $\mu_F(x)$, $\upsilon_F(x)$ and $\pi_F(x)$ in UOD, we use $1-\mu_F(x)$, $1-\upsilon_F(x)$ and $1-\pi_F(x)$ to replace the original probability assignment $\mu_F(x)$, $\upsilon_F(x)$ and $\pi_F(x)$ respectively. Since for a possible event $Prod(e_i)=p_i$, $1-p_i$ can represent the complementary concept of p_i. Thus we can obtain

$$\bar{F}=\{< x, \bar{\mu}_{\bar{F}}(x), \bar{\upsilon}_{\bar{F}}(x) > | \ x \in X\}, \quad (13)$$

with

$$\bar{\pi}_{\bar{F}}(x)=1-\bar{\mu}_{\bar{F}}(x)-\bar{\upsilon}_{\bar{F}}(x), \ \forall x \in X, \quad (14)$$

where $\bar{\mu}_{\bar{F}}(x)=1-\mu_F(x)$, $\bar{\upsilon}_{\bar{F}}(x)=1-\upsilon_F(x)$ and $\bar{\pi}_{\bar{F}}(x)=1-\bar{\mu}_{\bar{F}}(x)-\bar{\upsilon}_{\bar{F}}(x)=\mu_F(x)+\upsilon_F(x)-1$.

Step 2: Compute the sum φ of all core elements after negation, i.e.,

$$\varphi=\bar{\mu}_{\bar{F}}(x)+\bar{\upsilon}_{\bar{F}}(x)+\bar{\pi}_{\bar{F}}(x). \quad (15)$$

Step 3: It can be inferred that the φ may not be equal to 1. Hence, a necessary step is to normalize the negation of all core elements. Namely,

$$\bar{\mu}_{\bar{F}}(x)=\frac{1-\mu_F(x)}{\varphi}; \quad (16)$$

$$\bar{\upsilon}_{\bar{F}}(x)=\frac{1-\upsilon_F(x)}{\varphi}; \quad (17)$$

as well as

$$\bar{\pi}_{\bar{F}}(x)=\frac{1-\pi_F(x)}{\varphi}=\frac{\mu_F(x)+\upsilon_F(x)}{\varphi}. \quad (18)$$

Step 4: Finally, the general negation formula of IFS F is defined by

$$\bar{F}=\{< x, \bar{\mu}_{\bar{F}}(x), \bar{\upsilon}_{\bar{F}}(x) > | \ x \in X\}, \quad (19)$$

with

$$\bar{\pi}_{\bar{F}}(x)=1-\bar{\mu}_{\bar{F}}(x)-\bar{\upsilon}_{\bar{F}}(x), \ \forall x \in X. \quad (20)$$

And explicitly,

$$\bar{\mu}_{\bar{F}}(x)=\frac{1-\mu_F(x)}{m-1}; \quad (21)$$

$$\bar{\upsilon}_{\bar{F}}(x)=\frac{1-\upsilon_F(x)}{m-1}; \quad (22)$$

and

$$\bar{\pi}_{\bar{F}}(x)=\frac{1-\pi_F(x)}{m-1}=\frac{\mu_F(x)+\upsilon_F(x)}{m-1}. \quad (23)$$

where m $(1 \leq m \leq 3)$ is the number of core elements.

In fact, the proposed method makes use of the complementarity of each core element to generate negative elements. Then, a necessary approach is used to normalize the probability of each core element so that their sum is equal to 1.

The role of the negation method is to equally distribute the probability to each core element. If and only if the IFS contains only two core elements, the negation method will exchange the probability of the two core elements after the negation process. After Section IV, we shall use a few simple examples to illustrate the negative execution flow. But before that, let us introduce two general theorems for the proposed negation method, where Theorem 1 shows the general formula of the proposed negation, and Theorem 2 shows the convergence of the method.

THEOREM 1. *Assume the iterative negation is regarded as a series of sequences, and let α_k represent the IFS of the core element after the $k-th$ negation. Then, the general formula of IFS negation is defined by*

$$\alpha_k = -\frac{1-m\alpha_0}{m(m-1)^{k-1}} + \frac{1}{m}, \tag{24}$$

where α_0 is the initial core element value of the IFS, and m $(1 \leq m \leq 3)$ is the number of core elements.

PROOF. *Consider an arbitrary IFS F in UOD X. For any core element α, based on the previously proposed core element negation formulas, we have*

$$\alpha_{k+1} = \frac{1-\alpha_k}{m-1}. \tag{25}$$

Since $\frac{1}{m}$ is a constant in the iterative process, subtracting the fixed value on both sides of Eq. (25), we have

$$\begin{aligned}\alpha_{k+1} - \frac{1}{m} &= \frac{1-\alpha_k}{m-1} - \frac{1}{m} \\ &= \frac{1}{m-1} - \frac{(m-1)+m\alpha_k}{m(m-1)} \\ &= \frac{1-(\frac{m-1}{m}+\alpha_k)}{m-1} \\ &= \frac{\alpha_i - \frac{1}{m}}{1-m}.\end{aligned}$$

Integrating both sides of the equation, we finally have

$$\alpha_{k+1} - \frac{1}{m} = \frac{\alpha_i - \frac{1}{m}}{1-m}.$$

Next, we simplify the above equation as follows

$$\frac{\alpha_{k+1} - \frac{1}{m}}{\alpha_k - \frac{1}{m}} = \frac{1}{1-m}.$$

It can be observed that the $\alpha_k - \frac{1}{m}$ (marked as β_k) is the common formula of the geometric sequence. Moreover, the first item $\beta_0 = \alpha_0 + \frac{1}{m}$ and the ratio $r = \frac{1}{1-m}$. Then, we can easily deduce the general form of α_k is given by

$$\begin{aligned}\alpha_k &= \beta_k + \frac{1}{m} \\ &= \beta_1 r^{k-1} + \frac{1}{m} \\ &= (\alpha_1 + \frac{1}{m})(\frac{1}{1-n})^{i-1} + \frac{1}{m} \\ &= \frac{m\alpha_0 - 1}{m(m-1)^{k-1}} + \frac{1}{m} \\ &= \frac{m\alpha_0 - 1}{m(m-1)^{k-1}} + \frac{1}{m}.\end{aligned}$$

THEOREM 2. *For an IFS F in UOD X, when the probability is equally assigned to each core element, i.e.,*

$$\mu_F(x) = \upsilon_F(x) = \pi_F(x) = \frac{1}{m}, \ \forall x \in X. \tag{26}$$

And then the proposed negation converges to a value, and the value is equal to $\frac{1}{n}$. Where m $(1 \leq m \leq 3)$ is the number of core elements.

PROOF. *To simplify the representation, the notation is the same as Theorem 1. That is the α_k denotes the IFS of the core element after the $k-th$ negation. According to the Theorem 1, for α_k, we know that*

$$\alpha_k = \frac{m\alpha_0 - 1}{m(m-1)^{k-1}} + \frac{1}{m}. \tag{27}$$

Next, we take the limit of k on both sides of Eq. (27), namely,

$$\lim_{k\to+\infty} \alpha_k = \lim_{k\to+\infty} (\frac{m\alpha_0 - 1}{m(m-1)^{k-1}} + \frac{1}{m}).$$

If restricted to $|m-1| > 1$, we have

$$\lim_{k\to+\infty} \alpha_k = \frac{1}{n}.$$

In the above proof process, if $|m-1| \leq 1$ is mandatory, is there a potential threat to the proposed negative algorithm? This may cause confusion to readers. Along this line of thought, we shall focus on the two special cases below.

Special case one: Assume that for any given IFS F in UOD X, only one core element exists, e.g., $\alpha = 1$ (α denotes the core element). Then the negation of the F is defined by

$$\alpha = 0, \ and \ \emptyset = 1, \ with \ x \in \emptyset. \tag{28}$$

In Eq. (28), $\emptyset$ is used to model the open world, which denotes any set mutually exclusive with the X, and the set does not contain core element α. The negation F (i.e., not F) represents the membership grade of element x to $\emptyset$, and $\emptyset = 1$ represents a complete membership relationship.

Due to the complexity of fuzzy systems and lack of complete knowledge, in some cases, we cannot explain some phenomena with existing knowledge. For example, for the above problem, we don't know which core elements exist, but we at least know that it cannot be α, because it is the negation of α. Moreover, if the prior information is known, namely not α, we cannot store this type of information by adding other more core elements in UOD. Because this simple "addition operation" may lead to a change in the amount of the IFS information, thereby posing a potential threat to the representation of knowledge. Eq. (28) shows that the empty set can also represent the core element as a container for absorbing unknown items. Hence, if we deny an IFS that contains only one core element, the total probability will be assigned to the empty set. In other words, we don't know which specific core element the probability will be assigned to, but at least it is not α.

Excluding a kind of intuitive thinking, note that usually the negation of the IFS does not change the probability assignment of the core elements in the original IFS, but generates a new IFS (different from the original IFS) in the process of negation. Therefore, the above two special cases are reasonable.

Special case two: Assume an IFS $F = \{< x, \mu_F(x), \upsilon_F(x) > \mid x \in X\}$ with $\pi_F(x) = 1 - \mu_F(x) - \upsilon_F(x)$ in UOD X, which contains two core elements. In the case, three cases are considered, e.g., $\mu_F(x) = 0$, $\upsilon_F(x) = 0$ or $\pi_F(x) = 0$. Then let the probability values assigned to the two core elements be δ and τ, and the constraint $\delta + \tau = 1$. And then for F, some possible forms are listed as follows

- $F = \{< x, 0.5, 0.5 > \mid x \in X\}$, $\pi_F(x) = 0$;
- $F = \{< x, 0.5, 0 > \mid x \in X\}$, $\pi_F(x) = 0.5$;
- $F = \{< x, 0, 0.5 > \mid x \in X\}$, $\pi_F(x) = 0.5$.

Correspondingly, the negation of F is

- $\bar{F} = \{< x, 0.5, 0.5 > \mid x \in X\}$, $\bar{\pi}_{\bar{F}}(x) = 0$;
- $\bar{F} = \{< x, 0.5, 0 > \mid x \in X\}$, $\bar{\pi}_{\bar{F}}(x) = 0.5$;
- $\bar{F} = \{< x, 0, 0.5 > \mid x \in X\}$, $\bar{\pi}_{\bar{F}}(x) = 0.5$.

In the above three cases, the negation of IFS will not produce any changes because the probability is evenly assigned to each core element. That is, its negation has returned to itself. This is the only state where IFS cannot be negated.

IV. Some properties of the proposed negation of IFSs

In this section, we show some properties satisfied by IFSs negation through reasoning and proof.

Assume there are two finite UOD X and Y, and two IFSs are respectively

$$P = \{< x, \mu_P(x), \upsilon_P(x) > \mid x \in X\},$$

with

$$\pi_P(x) = 1 - \mu_P(x) - \upsilon_P(x),\ \forall x \in X;$$

as well as

$$Q = \{< y, \mu_Q(y), \upsilon_Q(y) > \mid y \in Y\},$$

with

$$\pi_Q(y) = 1 - \mu_Q(y) - \upsilon_Q(y),\ \forall y \in Y.$$

Then for P, let m $(1 \le m \le 3)$ denote the number of core elements, and α_i $(1 \le i \le m)$ denote the $i-th$ core element. And then for Q, let n $(1 \le n \le 3)$ denote the number of core elements, and β_j $(1 \le j \le n)$ denote the $j-th$ core element, .

The properties of P and Q are deduced as follows.

Property 1. *For P and Q, we have*

$$\alpha_i \bar{\beta}_j = \left(\frac{m-1}{n-1}\right) \bar{\alpha}_i \beta_j + \frac{\alpha_i - \beta_j}{n-1},\ \forall i, j. \tag{29}$$

Proof. *We have*

$$\begin{aligned} \alpha_i \bar{\beta}_j &= \alpha_i \left(\frac{1-\beta_j}{n-1}\right) \\ &= \frac{\alpha_i - \beta_j}{n-1} + \frac{\beta_j(1-\alpha_i)}{n-1} \\ &= \left(\frac{m-1}{n-1}\right) \beta_j \bar{\alpha}_i + \frac{\alpha_i - \beta_j}{n-1}. \end{aligned}$$

Moreover, for $i = 1, 2, 3$; $j = 1, 2, 3$, i.e., $m = n$, we can deduce

$$\alpha_i \bar{\beta}_j = \beta_j \bar{\alpha}_i + \frac{\alpha_i - \beta_j}{m-1}.$$

Property 2. *For P and Q, we have*

$$\alpha_i \bar{\bar{\beta}}_j = \left(\frac{m-1}{n-1}\right)^2 \beta_j \bar{\bar{\alpha}}_i + \frac{\alpha_i(n-2) - \beta_j(m-2)}{(n-1)^2},\ \forall i, j. \tag{30}$$

Proof. *We have*

$$\begin{aligned} \alpha_i \bar{\bar{\beta}}_j &= \alpha_i \frac{(\beta_j + n - 2)}{(n-1)^2} \\ &= \beta_j \frac{(\alpha_i + m - 2)}{(n-1)^2} + \frac{\alpha_i(n-2) - \beta_j(m-2)}{(n-1)^2} \\ &= \left(\frac{m-1}{n-1}\right)^2 \beta_j \bar{\bar{\alpha}}_i + \frac{\alpha_i(n-2) - \beta_j(m-2)}{(n-1)^2}. \end{aligned}$$

Moreover, for $i = 1, 2, 3$; $j = 1, 2, 3$, i.e., $m = n$, we can deduce

$$\alpha_i \bar{\bar{\beta}}_j = \beta_j \bar{\bar{\alpha}}_i + \frac{(\alpha_i - \beta_j)(m-2)}{(m-1)^2}.$$

Property 3. *For P and Q, we have*

$$\overline{\left(\frac{\alpha_i + \beta_i}{2}\right)} = \frac{\bar{\alpha}_i + \bar{\beta}_i}{2}, \tag{31}$$

and

$$\overline{\left(\frac{\alpha_i + (2^\partial - 1)\beta_i}{2^\partial}\right)} = \bar{\alpha}_i + \bar{\beta}_i - \overline{\left(\frac{(2^\partial - 1)(\alpha_i + \beta_i)}{2^\partial}\right)}, \tag{32}$$

in which $\partial = 0, 1, ...$ for $\forall i$.

Proof. *We have*

$$\begin{aligned} \overline{\left(\frac{\alpha_i + \beta_i}{2}\right)} &= \frac{1 - \left(\frac{\alpha_i + \beta_j}{2}\right)}{m-1} \\ &= \frac{(1-\alpha_i) + (1-\beta_i)}{2(m-1)} \\ &= \frac{\bar{\alpha}_i + \bar{\beta}_i}{2}. \end{aligned}$$

Moreover, we can deduce

$$\begin{aligned} \overline{\left(\frac{\alpha_i + (2^\partial - 1)\beta_i}{2^\partial}\right)} &= \frac{1 - \left(\frac{\alpha_i + (2^\partial - 1)\beta_i}{2^\partial}\right)}{m-1} \\ &= \frac{1-\alpha_i}{m-1} + \frac{1-\beta_i}{m-1} - \frac{1 - \left(\frac{\beta_i + (2^\partial - 1)\alpha_i}{2^\partial}\right)}{2^\partial(m-1)} \\ &= \bar{\alpha}_i + \bar{\beta}_i - \overline{\left(\frac{(2^\partial - 1)(\alpha_i + \beta_i)}{2^\partial}\right)}. \end{aligned}$$

PROPERTY 4. *For P and Q, we have*

$$max(\bar{\alpha}_i, \bar{\beta}_i) \geq \frac{max(\alpha_i, \beta_i)}{m-1}, \ \forall i; \tag{33}$$

and

$$min(\bar{\alpha}_i, \bar{\beta}_i) \geq \frac{min(\alpha_i, \beta_i)}{m-1}, \ \forall i. \tag{34}$$

PROOF. *For $\forall i$, we have*

$$\begin{aligned} max(\bar{\alpha}_i, \bar{\beta}_i) &= \frac{(\bar{\alpha}_i + \bar{\beta}_i) + |\bar{\alpha}_i - \bar{\beta}_i|}{2} \\ &= \frac{\left(\frac{1-\alpha_i+1-\beta_i}{m-1}\right) + |\frac{\bar{\alpha}_i - \bar{\beta}_i}{m-1}|}{2} \\ &= \frac{\left(\frac{1-\alpha_i+1-\beta_i}{m-1}\right) + |\frac{\bar{\alpha}_i - \bar{\beta}_i}{m-1}|}{2} \\ &= \frac{max(\alpha_i, \beta_i)}{m-1} + \frac{\overline{\alpha_i + \beta_i}}{2}. \end{aligned}$$

Thus, we can obtain

$$max(\bar{\alpha}_i, \bar{\beta}_i) \geq \frac{max(\alpha_i, \beta_i)}{m-1}.$$

In the same way, we can also prove

$$min(\bar{\alpha}_i, \bar{\beta}_i) \geq \frac{min(\alpha_i, \beta_i)}{m-1}.$$

PROPERTY 5. *For P and Q, we have*

$$\partial\bar{\alpha}_i + (1-\partial)\bar{\beta}_i = \overline{(\partial\alpha_i + (1-\partial)\beta_i)}, \ 0 \leq \partial \leq 1, \ \forall i. \tag{35}$$

PROOF. *For $0 \leq \partial \leq 1$, we have*

$$\sum_{i=1}^{m} \left(\partial\bar{\alpha}_i + (1-\partial)\bar{\beta}_i\right) = 1.$$

Then for $\forall i$, we can deduce

$$\begin{aligned} \partial\bar{\alpha}_i + (1-\partial)\bar{\beta}_i &= \frac{\partial(1-\alpha_i)}{m-1} + \frac{(1-\partial)(1-\beta_i)}{m-1} \\ &= \frac{1-(\partial\alpha_i + (1-\partial)\beta_i)}{m-1} \\ &= \overline{(\partial\alpha_i + (1-\partial)\beta_i)}. \end{aligned}$$

V. NUMERICAL EXAMPLES

In this section, several numerical examples are provided to show how to obtain the IFS negation in accordance with the proposed negation method. Moreover, these examples are also used as the motivation for the follow up research of this paper, and have a positive effect.

EXAMPLE 1. *Consider a given UOD X, and an IFS $F = \{< x, 0.3, 0.6 >\} (x \in X)$. The negative F is calculated as follows*

$$\bar{\mu}_{\bar{F}}(x) = \frac{1-\mu_F(x)}{m-1} = 0.35;$$

$$\bar{\upsilon}_{\bar{F}}(x) = \frac{1-\upsilon_F(x)}{m-1} = 0.20;$$

$$\bar{\pi}_{\bar{F}}(x) = \frac{\mu_F(x)+\upsilon_F(x)}{m-1} = 0.45.$$

In the end we have

$$\bar{F} = \{< x, 0.35, 0.20 >\} \ with \ \bar{\pi}_{\bar{F}}(x) = 0.45.$$

EXAMPLE 2. *Consider a given UOD X, and an IFS $F = \{< x, \frac{1}{3}, \frac{1}{3} >\} (x \in X)$. The negative F is calculated by follows*

$$\bar{\mu}_{\bar{F}}(x) = \frac{1-\mu_F(x)}{m-1} = \frac{1}{3};$$

$$\bar{\upsilon}_{\bar{F}}(x) = \frac{1-\upsilon_F(x)}{m-1} = \frac{1}{3};$$

$$\bar{\pi}_{\bar{F}}(x) = \frac{\mu_F(x)+\upsilon_F(x)}{m-1} = \frac{1}{3}.$$

Finaly we have

$$\bar{F} = \left\{< x, \frac{1}{3}, \frac{1}{3} >\right\} \ with \ \bar{\pi}_{\bar{F}}(x) = \frac{1}{3}.$$

Obviously, the initial IFS F is exactly the same as the negative one, since

$$\mu_F(x) = \upsilon_F(x) = \pi_F(x) = \frac{1}{3}.$$

EXAMPLE 3. *Consider a given UOD X, and an IFS $F = \{< x, 0.2, 0.7 >\} (x \in X)$. The negation of F is calculated as follows*

$$\bar{\mu}_{\bar{F}}(x) = \frac{1-\mu_F(x)}{m-1} = 0.40;$$

$$\bar{\upsilon}_{\bar{F}}(x) = \frac{1-\upsilon_F(x)}{m-1} = 0.15;$$

$$\bar{\pi}_{\bar{F}}(x) = \frac{\mu_F(x)+\upsilon_F(x)}{m-1} = 0.45.$$

Thus we have

$$\bar{F} = \{< x, 0.40, 0.15 >\} \ with \ \bar{\pi}_{\bar{F}}(x) = 0.45.$$

Comparing the original IFS F with its negative $\bar{F}$, we can observe that the core elements with lower support get higher support after being negated. Moreover, if we negate the IFS $\bar{F}$ again, i.e.,

$$\bar{\bar{\mu}}_{\bar{\bar{F}}}(x) = \frac{1-\bar{\mu}_{\bar{F}}(x)}{m-1} = 0.30;$$

$$\bar{\bar{\upsilon}}_{\bar{\bar{F}}}(x) = \frac{1-\bar{\upsilon}_{\bar{F}}(x)}{m-1} = 0.425;$$

$$\bar{\bar{\pi}}_{\bar{\bar{F}}}(x) = \frac{\bar{\mu}_{\bar{F}}(x)+\bar{\upsilon}_{\bar{F}}(x)}{m-1} = 0.275.$$

We finally have

$$\bar{\bar{F}} = \{< x, 0.30, 0.425 >\} \ with \ \bar{\bar{\pi}}_{\bar{\bar{F}}}(x) = 0.275.$$

It is apparent that the initial IFS F does not equal to the one after taking the negation twice. Generally speaking, the process of getting IFS negation is irreversible, i.e.,

$$F_i \neq \bar{\bar{F}}_i, \ \forall i. \tag{36}$$

Note that the above equation is valid if and only if the number of core elements in the IFS is greater than 2.

Now we review the above examples, we find that if the probability of the IFS F is unevenly distributed, the probability assignment after negation is different, as in Example 1. If the probability assignment of IFS F is completely uniform, the probability assignment after negation is the same, as in Example 2. In other words, in these two cases, the initial state and the negative state of the IFS are consistent. Moreover, as in Example 3, in the process of continuously obtaining negatives, it is worth thinking about what has changed in the uncertainty of the IFS, which led to this irreversible phenomenon. It can be seen that the above numerical examples have produced many interesting phenomena. In the next section, we shall focus on exploring the possible causes of these problems.

VI. Discussion and Analysis

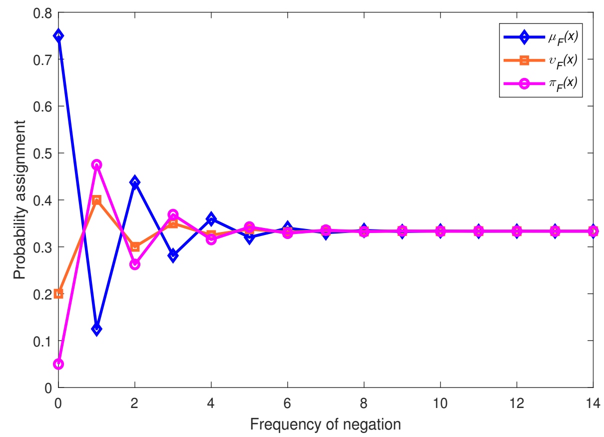
Inevitably, uncertainty can be seen everywhere in fuzzy systems. Therefore, how to effectively deal with uncertainty is a crucial subject, which has important significance for the development of fuzzy systems. To solve this problem, information entropy is used to measure the uncertainty of IFSs. In addition, it is worth savoring that previous studies by scholars have shown a remarkable phenomenon, that is, the negation of probability distribution will achieve the distribution of maximum entropy in some form [23].

Now, let us return to where we are interested. In this section, we shall discuss why the initial probability assignment of the IFS is consistent with the state after negation, and why the negation process is irreversible. Furthermore, how to use the negation method to construct a new model to measure the uncertainty of the IFS.

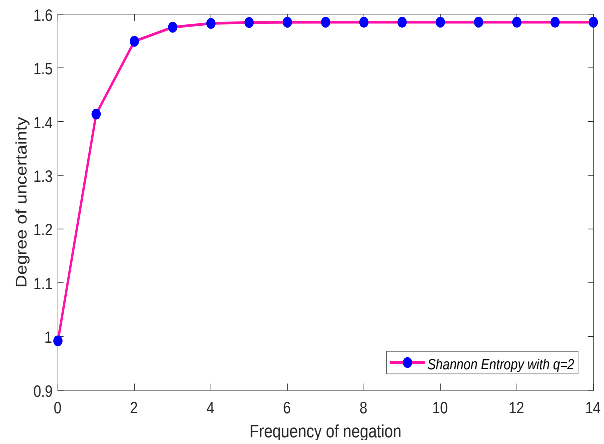
A. From the perspective of entropy

Recalling our previous settings, we define $\bar{F}$ as the negation of any IFS F. We observe that there are two special cases when negating certain IFS. One of the interesting findings is that when some IFSs contain only two core elements, the probability of each core element is exchanged after the IFS is negated. In general, after the negative operation, the uncertainty of the IFS will be changed. Besides, in the negation process, the number of negation iterations depends on the uncertainty of the initial state of the IFS. In the following section, an example is used to demonstrate the above characteristics.

EXAMPLE 4. *Consider a given UOD X, and an IFS* $F = \{< x, 0.75, 0.2 >\}\ (x \in X)$ *with* $\pi_F(x) = 0.05$. *A change in the characteristics of the IFS is shown in Fig. 1.*



(a) The change in the probability assignment of the IFS with the number of negative iterations.



(b) The change in the degree of uncertainty about the IFS with the number of negative iterations.

Fig. 1. The graph about the IFS probability assignment and uncertainty changes in Example 4.

Fig. 1(a) is used to illustrate the change of the IFS after each negation process, and Fig. 1(b) is used to illustrate the change of uncertainty after performing the same operation. For Fig. 1(a), as a whole, we can observe that the probability assignment of the IFS tends to converge as the number of iterations increases. More specifically, the probabilities of adjacent core elements are getting closer and closer, and finally the IFS converges to a state where the probability is evenly distributed to each core element. This phenomenon is also consistent with Theorem 2. Correspondingly, from Fig. 1(b), we can find that in the negation operation, as the number of iterations increases, the uncertainty of the IFS also increases. Finally, when each core element has the same probability assignment degree, the uncertainty of the IFS reaches the maximum. It also shows that the proposed negation method works in a way that changes the uncertainty. We can also infer that if an initial IFS is a state with a large entropy, the number of negation operations will also decrease in the process of converging to a certain value.

Based on the results of our experiments, we find that the negation process will change the IFS (no special cases are considered). The IFS, which takes negation twice, is not equal to the initial IFS. We believe that the reason why this interesting phenomenon can be produced is due to the change in the uncertainty of IFS (quantified by entropy).

We can also explain this phenomenon well from the perspective of physics. As we all know, things in the universe have a tendency to spontaneously become more chaotic, which means that entropy will continue to increase. This is the famous entropy increase principle. The famous second law of thermodynamics also shows that the entropy of an isolated system never automatically decreases. Entropy does not change in the reversible process and increases in the irreversible process.

In addition, we know that energy is always in a state of consumption, and energy becomes more and more inefficient with the frequency of use. For instance, someone spend 100 J of energy to get an object from ground A to ground B. In this process, a lot of energy is not 100% transformed, but part of it is lost in the universe. However, this energy is irreversible, cannot be reused, and is always increasing. The part of the energy that is difficult to reuse is represented, which is represented by entropy. Through the inspiration drawn from the laws of thermodynamics and simulation results, combined with the background of fuzzy systems, we consider that negation is another form of the IFS. We also believe that information is similar to energy, which is constantly consumed. In the process of negation iteration, part of the information is lost and dissipated in the fuzzy systems.

B. Proposed the uncertainty measure of IFS based on distance and negation

In the field of IFSs, existing studies are focusing on the distance between two IFSs [59]. However, how to measure the degree of uncertainty of an IFS is an unresolved subject. To address this issue, based on the negation and the distance function, to the best of our knowledge, a solution is provided to enrich the vacancy of IFS on this subject. The distance measure of IFS represents the difference in the probability assignment of core elements at the same position in two different UODs, and for two IFSs, the smaller the distance, the more similar. The proposed uncertainty measure method is inspired by the distance between two IFSs. As we proved in Theorem 2, when the probability is evenly assigned to each core element, the negation of the IFS will converge to a certain value, namely $\frac{1}{n}$. At this time, the initial IFS and its negation are the same value, and the uncertainty of the IFS has reached the maximum. Therefore, we can first calculate the negative form of the IFS, and then use the distance between the initial IFS and the negative IFS to characterize the uncertainty of the initial IFS. The obtained distance measure represents the relative uncertainty of the IFS elements, which is an effective measurement method for sharing UOD between the two IFSs (i.e., the initial IFS and the negative IFS). Logically speaking, this modeling process is reasonable, because the negative IFS still shares the same UOD with the initial IFS. Therefore, measuring the difference between the initial IFS and the negative IFS is also the correct way to express the uncertainty of the IFS.

According to the above discussion, for a given UOD X and an IFS $A = \{< x, \mu_A(x), \upsilon_A(x) > | \ x \in X\}$ with $\pi_A(x) = 1 - \mu_A(x) - \upsilon_A(x),\ \forall x \in X$, based on the distance formula, the proposed uncertainty measure of the IFS is defined by

$$d_{UM}(A, \bar{A}) = 1 - \sqrt{JS_{IFS}(A, \bar{A})}. \tag{37}$$

And more specifically, $d_{UM}(A, \bar{A})$ can be written by

$$\begin{aligned} d_{UM}(A, \bar{A}) = 1 - \Bigg[\frac{1}{2}\Big(&\mu_A(x) log\frac{2\mu_A(x)}{\mu_A(x) + \bar{\mu}_{\bar{A}}(x)} \\ &+\bar{\mu}_{\bar{A}}(x) log\frac{2\bar{\mu}_{\bar{A}}(x)}{\mu_A(x) + \bar{\mu}_{\bar{A}}(x)} \\ &+\upsilon_A(x) log\frac{2\upsilon_A(x)}{\upsilon_A(x) + \bar{\upsilon}_{\bar{A}}(x)} \\ &+\bar{\upsilon}_{\bar{A}}(x) log\frac{2\bar{\upsilon}_{\bar{A}}(x)}{\upsilon_A(x) + \bar{\upsilon}_{\bar{A}}(x)} \\ &+\pi_A(x) log\frac{2\pi_A(x)}{\pi_A(x) + \bar{\pi}_{\bar{A}}(x)} \\ &+\bar{\pi}_{\bar{A}}(x) log\frac{2\bar{\pi}_{\bar{A}}(x)}{\pi_A(x) + \bar{\pi}_{\bar{A}}(x)} \Big)\Bigg]^{\frac{1}{2}}. \end{aligned} \tag{38}$$

First of all, reviewing the previous definitions, we know that when the number of core elements in IFS is different, the negation of IFS takes different forms. Specifically, when the number of core elements is equal to 1, the probability of the core element is assigned to the empty set, that is, the open world. In addition, when the probability is equally distributed to the core elements, the negation of IFS will remain unchanged. Next, through experimental simulation technology, we shall first discuss the properties satisfied by the proposed IFS uncertainty measure when the number of core elements is equal to 2 or 3. Before that, we first set up some common restrictions for these two cases. The degree of membership and degree of non-membership of the IFS A (i.e., variables $\mu_A(x)$ and $\upsilon_A(x)$) are limited to $(0, 1)$, and they are restricted to $\mu_P(x) + \upsilon_P(x) < 1$, shown in Fig. 2(a) and Fig. 2(d). Then, through Eq. (38), when the number of core elements is different, the uncertainty measure of the IFS A is shown in Fig. 2(b) and Fig. 2(e).

From Figs. 2(a)-2(c), we can observe that when the number of core elements of A is equal to 2, the change of $\mu_A(x)$ and $\upsilon_A(x)$ is subjected to $(0, 1)$, and $d_{UM}(A, \bar{A})$ is always greater than 0.2 but less than 1. Only when $\mu_A(x) = \upsilon_A(x) = \frac{1}{2}$, the uncertainty measure of A is equal to 1. Similarly, when the number of core elements contained in A is equal to 3, we can also observe from Figs. 2(d)-2(f) that the change of $\mu_A(x)$ and $\upsilon_A(x)$ is limited to $(0, 1)$, and $d_{UM}(A, \bar{A})$ is always greater than 0.2 but less than 1. Only when $\mu_A(x) = \upsilon_A(x) = \pi_A(x) = \frac{1}{3}$, the uncertainty measure of A is equal to 1. Moreover, we can calculate that the minimum value of the uncertainty measure of A in the above cases. In particular, when the number of core elements is equal to 2, the minimum value is equal to 0.1722; when the number of core elements is equal to 3, the minimum value equal to 0.1742. This is also an interesting finding, since the negation of the IFS does not completely negate the intersection between core elements (except a special case). Hence, this is an intuitive result. In the following, we shall discuss the possible changes of the uncertainty measure of A in the special case mentioned above. When the number of core elements contained in A is equal to 1, as discussed earlier, the negation of A assigns probability 1 to the empty set. At this time, we calculate $d_{UM}(A, \bar{A})$ by Eq. (38), and the result

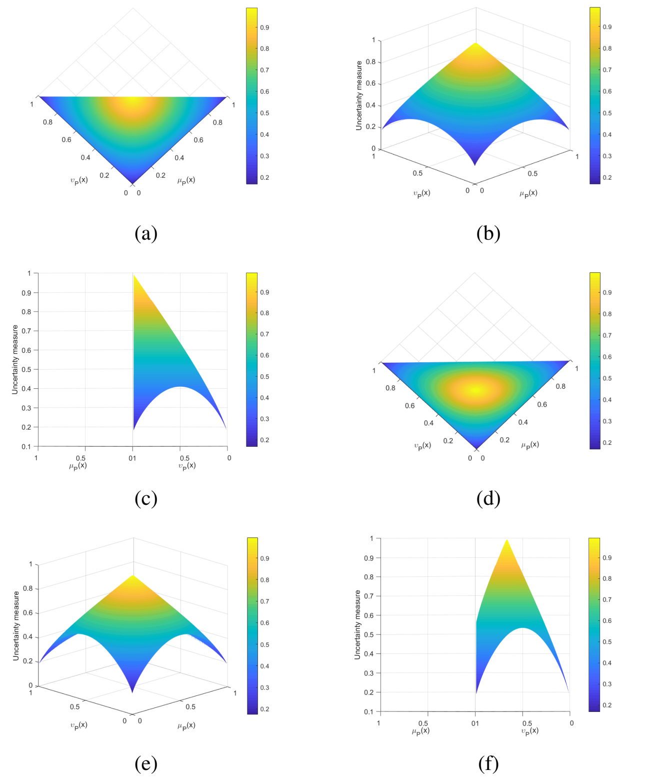


Fig. 2. Figs. 2(a)-2(c) show the changes in variables and uncertainty measures when the number of IFS core elements is equal to 2; Figs. 2(d)-2(f) show the changes in variables and uncertainty measures when the number of IFS core elements is equal to 3.

is equal to 0, which means that in this case, the uncertainty measure of A has reached the minimum value. This result is reliable, since the uncertainty of absolute events is minimal. In summary, simulation experiments verify some desired basic properties of the proposed IFS uncertainty measure, including non-negativity, boundedness, and symmetry, and found some interesting values. This may be the cornerstone of our follow up research.

VII. Algorithm and Applications

Looking back at our previous analysis, based on the entropy point of view, we discuss the reasons for the uncertainty of the proposed negation method as the number of negation iterations changes. And borrowed knowledge from the field of physical sciences, we explain this interesting discovery by drawing inspiration from it. Then, we define a new formula to measure the uncertainty of IFS and verify some basic properties. The simulation results show that the proposed method is an effective mathematical tool to measure the uncertainty of IFS. Naturally, another aspect that readers may be interested in is that there are opposites to events, it stands to reason that solving problems from the opposites of this event should also be a common method. In order to answer this question, in the following, combined with two practical application examples, we shall illustrate the effectiveness of IFS negation in solving the problem. But first, we need a general algorithm as a carrier for application implementation. Based on this distance measure, we propose a new algorithm for pattern classification.

A. Proposed the pattern classification algorithm

Notation and setting: Let a UOD set with m attributes be $A = \{a_1, \cdots, a_i, \cdots, a_m\}$, and $B = \{B_1, \cdots, B_j, \cdots, B_n\}$ denote a set of n patterns represented by IFS $B_j = \{< a_i, \mu_{B_j}(a_i), \upsilon_{B_j}(a_i) > \mid a_i \in A\}$. For the given z samples $C = \{C_1, \cdots, C_k, \cdots, C_z\}$ represented by IFS $C_k = \{< a_i, \mu_{C_k}(a_i), \upsilon_{C_k}(a_i) > \mid a_i \in A\}$, the goal of the algorithm is to classify these samples according to the corresponding patterns.

The execution steps of the algorithm are given as follows. ***Step 1:*** Based on the proposed IFS negation, each pattern B_j and sample C_k characterized by IFSs are calculated as negative forms, namely

$$\bar{B}_j = \left\{< a_i, \bar{\mu}_{\bar{B}_j}(a_i), \bar{\upsilon}_{\bar{B}_j}(a_i) > \mid a_i \in A\right\}, \tag{39}$$

and

$$\bar{C}_k = \left\{< a_i, \bar{\mu}_{\bar{C}_k}(a_i), \bar{\upsilon}_{\bar{\mu}_{\bar{C}_k}}(a_i) > \mid a_i \in A\right\}. \tag{40}$$

Step 2: The distance Eq. (9), is used to measure the difference between the negation of pattern $\bar{B}_j$ and the negation of sample $\bar{C}_k$. For all i, the distance formula is given by

$$\begin{aligned} d_X^{Avg}(\bar{B}_j, \bar{C}_k) &= \frac{1}{m}\sum_{i=1}^{m} d_X(\bar{B}_j, \bar{C}_k) \\ &= \frac{1}{m}\sum_{i=1}^{m}\Bigg[\frac{1}{2}\Big(\ \bar{\mu}_{\bar{B}_j}(a_i) log\frac{2\bar{\mu}_{\bar{B}_j}(a_i)}{\bar{\mu}_{\bar{B}_j}(a_i)+\bar{\mu}_{\bar{C}_k}(a_i)} \\ &\quad +\bar{\mu}_{\bar{C}_k}(a_i) log\frac{2\bar{\mu}_{\bar{C}_k}(a_i)}{\bar{\mu}_{\bar{B}_j}(a_i)+\bar{\mu}_{\bar{C}_k}(a_i)} \\ &\quad +\bar{\upsilon}_{\bar{B}_j}(a_i) log\frac{2\bar{\upsilon}_{\bar{B}_j}(a_i)}{\bar{\upsilon}_{\bar{B}_j}(a_i)+\bar{\upsilon}_{\bar{C}_k}(a_i)} \\ &\quad +\bar{\upsilon}_{\bar{C}_k}(a_i) log\frac{2\bar{\upsilon}_{\bar{C}_k}(a_i)}{\bar{\upsilon}_{\bar{B}_j}(a_i)+\bar{\upsilon}_{\bar{C}_k}(a_i)} \\ &\quad +\bar{\pi}_{\bar{B}_j}(a_i) log\frac{2\bar{\pi}_{\bar{B}_j}(a_i)}{\bar{\pi}_{\bar{B}_j}(a_i)+\bar{\pi}_{\bar{C}_k}(a_i)} \\ &\quad +\bar{\pi}_{\bar{C}_k}(a_i) log\frac{2\bar{\pi}_{\bar{C}_k}(a_i)}{\bar{\pi}_{\bar{B}_j}(a_i)+\bar{\pi}_{\bar{C}_k}(a_i)}\ \Big)\Bigg]^{\frac{1}{2}}. \end{aligned} \tag{41}$$

Step 3: If the distance between $\bar{B}_j$ and $\bar{C}_k$ is the minimum, the following formula will be used to select this distance combination, that is

$$d_X^{Avg}(\bar{B}_\xi, \bar{C}_k) = \min_{1\le j\le n} d_X^{Avg}(\bar{B}_j, \bar{C}_k). \tag{42}$$

Step 4: Finally, sample C_k is classified into pattern B_ξ, which is calculated according to the following formula, i.e.

$$\xi = arg\ \min_{1\le j\le n}\left\{d_X^{Avg}(\bar{B}_j, \bar{C}_k)\right\},\ \bar{C}_k \leftarrow \bar{B}_\xi. \tag{43}$$

A general pattern classification algorithm is provided as shown in Algorithm 1.

B. Applications

Application one: A pattern classification problem is given, which consists of three attributes $A = \{a_1, a_2, a_3\}$, three

Algorithm 1: The proposed pattern classification algorithm.

Input: Attribute set: $A = \{a_1, \cdots, a_i, \cdots, a_m\}$;
Pattern set: $B = \{B_1, \cdots, B_j, \cdots, B_n\}$;
Sample set: $C = \{C_1, \cdots, C_k, \cdots, C_z\}$.
Output: The classification result B_ξ.

```
% * Step 1 * %
for j = 1; 0 ≤ n do
    Compute the negation of B_j by Eqs. (21) - (23) to get B̄_j;
end
for k = 1; 0 ≤ z do
    Calculate the negation of C_k by Eqs. (21) - (23) to get C̄_k;
end
for k = 1; k ≤ z do
    % * Step 2 * %
    for j = 1; j ≤ n do
        Obtain the distance d_X^{Avg}(B̄_j, C̄_k) by Eq. (41);
    end
    % * Step 3 * %
    Filter out the smallest distance d_X^{Avg}(B̄_ξ, C̄_k) by Eq. (42);
    Classify sample C_k into pattern B_ξ by Eq. (43).
end
```

patterns $P = \{P_1, P_2, P_3\}$ and one test sample T. The pattern and test data are described by IFSs as $P_j = \{< a_i, \mu_{P_j}(a_i), \upsilon_{P_j}(a_i) > | \ a_i \in A\}$ $(1 \leq i \leq 3; 1 \leq j \leq 3)$ and $T = \{< a_i, \mu_T(a_i), \upsilon_T(a_i) > | \ a_i \in A\}$ $(1 \leq i \leq 3)$ respectively, which are shown in Table I. For this problem, a desired result is to find a pattern corresponding to the T.

TABLE I
PATTERN CLASSIFICATION PROBLEM WITH THREE-CLASSES AND THREE-ATTRIBUTES IN APPLICATION ONE.

			Attributes		
			a_1	a_2	a_3
Patterns	P_1	$\mu_{P_1}(a)$	1.00	0.80	0.70
		$\upsilon_{P_1}(a)$	0.00	0.00	0.10
	P_2	$\mu_{P_2}(a)$	0.90	1.00	0.90
		$\upsilon_{P_2}(a)$	0.10	0.00	0.00
	P_3	$\mu_{P_3}(a)$	0.60	0.80	1.00
		$\upsilon_{P_3}(a)$	0.20	0.00	0.00
Test sample	T	$\mu_T(a)$	0.50	0.60	0.80
		$\upsilon_T(a)$	0.30	0.20	0.10

Then, combined with the proposed pattern classification algorithm, the recognition process is as follows.
Step 1: The negations $\bar{P}_1$, $\bar{P}_2$, $\bar{P}_3$ of patterns P_1, P_2, P_3, and the negation $\bar{T}_1$ of the test sample T are calculated. Then the results are shown in Table II.

TABLE II
NEGATION OF IFSS IN APPLICATION ONE.

			Attributes		
			a_1	a_2	a_3
Patterns	$\bar{P}_1$	$\bar{\mu}_{\bar{P}_1}(a)$	0.00	0.20	0.15
		$\bar{\upsilon}_{\bar{P}_1}(a)$	0.00	0.00	0.45
	$\bar{P}_2$	$\bar{\mu}_{\bar{P}_2}(a)$	0.10	0.00	0.10
		$\bar{\upsilon}_{\bar{P}_2}(a)$	0.90	0.00	0.00
	$\bar{P}_3$	$\bar{\mu}_{\bar{P}_3}(a)$	0.20	0.20	0.00
		$\bar{\upsilon}_{\bar{P}_3}(a)$	0.40	0.00	0.00
Test sample	$\bar{T}$	$\bar{\mu}_{\bar{T}}(a)$	0.25	0.20	0.10
		$\bar{\upsilon}_{\bar{T}}(a)$	0.35	0.40	0.45

Step 2: The distance $d_X^{Avg}(\cdot)$ between $\bar{P}_1$, $\bar{P}_2$, $\bar{P}_3$ and $\bar{T}$ is gradually computed as follows

$$d_X^{Avg}(\bar{P}_1, \bar{T}) = 0.3546;$$
$$d_X^{Avg}(\bar{P}_2, \bar{T}) = 0.4006;$$
$$d_X^{Avg}(\bar{P}_3, \bar{T}) = 0.2804.$$

Step 3: The minimum distance between $\bar{P} = \{\bar{P}_1, \bar{P}_2, \bar{P}_3\}$ and $\bar{T}$ is obtained by

$$d_X^{Avg}(\bar{P}_3, \bar{T}) = 0.2804.$$

Step 4: The sample T is classified according to

$$\xi = 3, \ \bar{T} \leftarrow \bar{P}_3.$$

Next, in order to illustrate the performance of the proposed algorithm, we compare the results generated by the proposed method with some existing methods. The results are listed in Table III and Fig 3.

From the intuitive graphs and data provided, it can be found that, except for the d_{SW} method that produces counter-intuitive results, since $d_{SW}(P_1, T) = d_{SW}(P_2, T) = 0.11$, the proposed method maintains the same result $dist(P_3, T) < dist(P_1, T) < dist(P_2, T)$ as d_{SK-H}, d_{SK-E}, d_G, d_{W^1}, $d_{W^{\frac{1}{2}}}$, d_P, d_{YF}, d_{H-M}, d_{H-LA}, d_{SM}, d_{LZ}, d_{YC} and d_X.

Application two: Assume that for a pattern classification problem constructed by three attributes $A = \{a_1, a_2, a_3\}$, three patterns $P = \{P_1, P_2, P_3\}$, and a given test sample T, we want to classify T into corresponding patterns, namely P_1, P_2 and P_3. The model and sample data are all formulated using the IFSs standard and are represented as $P_j = \{< a_i, \mu_{P_j}(a_i), \upsilon_{P_j}(a_i) > | \ a_i \in A\}$ $(1 \leq i \leq 3; 1 \leq j \leq 3)$ and $T = \{< a_i, \mu_T(a_i), \upsilon_T(a_i) > | \ a_i \in A\}$ $(1 \leq i \leq 3)$ respectively. These results are listed in Table IV.

TABLE III
THE RESULTS OF DISTANCE MEASURES AND PATTERN CLASSIFICATION IN APPLICATION ONE.

Methods	Literatures	Distance measures			Classification results
		$dist(P_1,T)$	$dist(P_2,T)$	$dist(P_3,T)$	
d_{SK-H}	[62]	0.27	0.30	0.17	P_3
d_{SK-E}	[62]	0.28	0.29	0.16	P_3
d_G	[63]	0.27	0.30	0.17	P_3
d_{W^1}	[64]	0.16	0.18	0.09	P_3
$d_{W^{\frac{1}{2}}}$	[64]	0.22	0.23	0.15	P_3
d_P	[65]	0.27	0.30	0.17	P_3
d_{YF}	[66]	0.27	0.30	0.17	P_3
d_{H-T}	[67]	0.32	0.19	0.15	P_3
d_{H-R}	[67]	0.19	0.17	0.10	P_3
d_{H-L}	[67]	0.11	0.08	0.04	P_3
d_{H-KD}	[67]	0.26	0.25	0.16	P_3
d_{H-M}	[67]	0.21	0.25	0.15	P_3
d_{H-LA}	[67]	0.21	0.27	0.15	P_3
d_{H-G}	[67]	0.41	0.24	0.19	P_3
d_{SW}	[68]	0.11	0.11	0.06	P_3
d_{SM}	[69]	0.21	0.22	0.16	P_3
d_{LZ}	[70]	0.35	0.42	0.24	P_3
d_{YC}	[71]	0.30	0.33	0.22	P_3
d_X	[59]	0.38	0.40	0.28	P_3
Proposed method	-	0.35	0.40	0.21	P_3

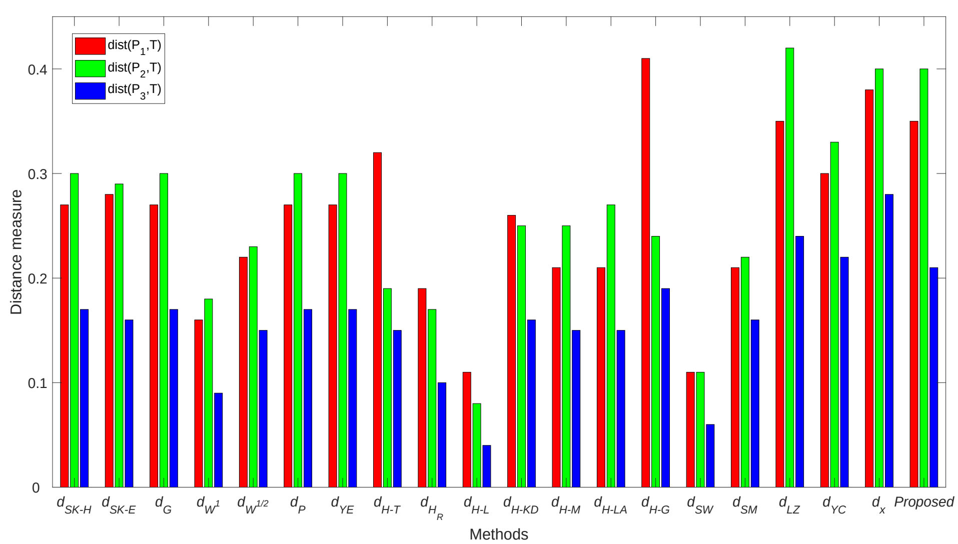


Fig. 3. Comparison of different methods in Application one.

TABLE IV
PATTERN CLASSIFICATION PROBLEM WITH THREE-CLASSES AND THREE-ATTRIBUTES IN APPLICATION TWO.

			Attributes		
			a_1	a_2	a_3
Patterns	P_1	$\mu_{P_1}(a)$	0.15	0.25	0.35
		$\upsilon_{P_1}(a)$	0.25	0.35	0.45
	P_2	$\mu_{P_2}(a)$	0.05	0.15	0.25
		$\upsilon_{P_2}(a)$	0.15	0.25	0.35
	P_3	$\mu_{P_3}(a)$	0.16	0.26	0.36
		$\upsilon_{P_3}(a)$	0.26	0.36	0.46
Test sample	T	$\mu_T(a)$	0.30	0.40	0.50
		$\upsilon_T(a)$	0.20	0.30	0.40

Below we shall implement the proposed new pattern classification algorithm, and the detailed steps are as follows.
Step 1: First, the negations $\bar{P}_1$, $\bar{P}_2$, $\bar{P}_3$ of the three given patterns P_1, P_2, P_3, and the negation $\bar{T}_1$ of the test sample T are computed. And the the results are shown in Table V.

TABLE V
NEGATION OF IFSS IN APPLICATION TWO.

			Attributes		
			a_1	a_2	a_3
Patterns	$\bar{P}_1$	$\bar{\mu}_{\bar{P}_1}(a)$	0.425	0.375	0.325
		$\bar{\upsilon}_{\bar{P}_1}(a)$	0.375	0.325	0.275
	$\bar{P}_2$	$\bar{\mu}_{\bar{P}_2}(a)$	0.475	0.425	0.375
		$\bar{\upsilon}_{\bar{P}_2}(a)$	0.425	0.375	0.325
	$\bar{P}_3$	$\bar{\mu}_{\bar{P}_3}(a)$	0.420	0.370	0.320
		$\bar{\upsilon}_{\bar{P}_3}(a)$	0.370	0.320	0.270
Test sample	$\bar{T}$	$\bar{\mu}_{\bar{T}}(a)$	0.350	0.300	0.250
		$\bar{\upsilon}_{\bar{T}}(a)$	0.400	0.350	0.300

Step 2: The distance $d_X^{Avg}(\cdot)$ from $\bar{T}$ to $\bar{P}_1$, $\bar{P}_2$, $\bar{P}_3$ and $\bar{T}$ is calculated as follows

$$d_X^{Avg}(\bar{P}_1, \bar{T}) = 0.0948;$$
$$d_X^{Avg}(\bar{P}_2, \bar{T}) = 0.1539;$$
$$d_X^{Avg}(\bar{P}_3, \bar{T}) = 0.0912.$$

Step 3: The minimum distance between $\bar{T}$ and $\bar{P} = \{\bar{P}_1, \bar{P}_2, \bar{P}_3\}$ is obtained by

$$d_X^{Avg}(\bar{P}_3, \bar{T}) = 0.0912.$$

Step 4: The sample T is classified according to

$$\xi = 3, \ \bar{T} \leftarrow \bar{P}_3.$$

Then, the results of the proposed method and other existing methods are compared and shown in Table VI and Fig 4.

On the one hand, from the results, it is apparent that the proposed method and d_{SK-H}, d_{SK-E}, d_G, d_{W^1}, d_P, d_{YF}, d_{SW}, d_{LZ}, d_{YC}, and d_X can classify the test sample T as P_3. However, some existing methods cannot determine the classification results, e.g., $d_{W^{\frac{1}{2}}}$, d_{H-T}, d_{H-R}, d_{H-L}, d_{H-KD}, d_{H-M}, d_{H-LA}, d_{H-G}, and d_{SW}. On the other hand, considering the distance measure, we can know that the proposed method and the methods d_{SK-H}, d_{SK-E}, d_G, d_{W^1}, $d_{W^{\frac{1}{2}}}$, d_P, d_{YF}, d_{SM}, d_{LZ}, d_{YC}, and d_X can all get a consistent ranking, i.e., $dist(P_3, T) < dist(P_1, T) < dist(P_2, T)$. However, some existing methods have shown counterintuitive results, e.g., $d_{W^{\frac{1}{2}}}(P_1, T) = d_{W^{\frac{1}{2}}}(P_3, T) = 0.10$; $d_{H-T}(P_1, T) = d_{H-T}(P_3, T) = 0.05$; $d_{H-R}(P_1, T) = d_{H-R}(P_3, T) = 0.05$; $d_{H-L}(P_1, T) = d_{H-T}(P_3, T) = 3.70 \times 10^{-17}$; $d_{H-KD}(P_1, T) = d_{H-KD}(P_3, T) = 0.10$; $d_{H-M}(P_1, T) = d_{H-M}(P_3, T) = 0.10$; $d_{H-LA}(P_1, T) = d_{H-LA}(P_3, T) = 0.07$. Hence, this also shows that these methods cannot be used for this pattern classification problem due to algorithm defects.

The above two application examples illustrate the immense value of IFS negation for practical problems. In fact, the negation method turns the "possible event" characterized by IFS into the "impossible event". Therefore, the solution of the problem also shifts to the opposite of the event.

VIII. CONCLUSION

In this paper, the concept of the IFS core elements is first defined to describe the number of focal elements in the IFS. Then a general method to obtain the negation of the IFS is proposed. Some numerical examples are used to illustrate the negation process, and a phenomenon is shown that when the IFS degenerates into a Bayes structure, the proposed negation method will degenerate into the negation of the probability distribution. Some theorems and properties are investigated to illustrate the nature of negation operations. Next, based on the meaning of physical science and entropy, we discuss the reason why the IFS core elements probability reaches the maximum entropy distribution in the negation operation, and a finding is confirmed, which may be caused by the uncertainty of the IFS. Moreover, based on distance and negation, a novel IFS measurement method is proposed to measure the uncertainty of the IFS. Through experimental simulation, some interesting phenomena are discovered, i.e., for the IFS, when the number of core elements is equal to 2 or 3, the minimum value of the $d_{UM}(A, \bar{A})$ is 0.1722 or 0.1742. In addition, a new pattern classification algorithm is proposed based on the negation and distance, and two application examples are effectively solved. By comparing with many methods, i.e. d_{SK-H}, d_{SK-E}, d_G, d_{W^1}, $d_{W^{\frac{1}{2}}}$, d_P, d_{YF}, d_{H-T}, d_{H-R}, d_{H-L}, d_{H-KD}, d_{H-M}, d_{H-LA}, d_{H-G}, d_{SW}, d_{SM}, d_{LZ}, d_{YC} and d_X, it shows the potential value of the negation method in practice.

This research has established the basic foundation for the IFS negation problem and opened up many interesting research

TABLE VI
THE RESULTS OF DISTANCE MEASURES AND PATTERN CLASSIFICATION IN APPLICATION TWO.

Methods	Literatures	Distance measures			Classification results
		$dist(P_1, T)$	$dist(P_2, T)$	$dist(P_3, T)$	
d_{SK-H}	[62]	0.15	0.30	0.14	P_3
d_{SK-E}	[62]	0.13	0.28	0.12	P_3
d_G	[63]	0.15	0.25	0.14	P_3
d_{W^1}	[64]	0.10	0.20	0.09	P_3
$d_{W^{\frac{1}{2}}}$	[64]	0.10	0.15	0.10	Cannot be identified
d_P	[65]	0.15	0.30	0.14	P_3
d_{YF}	[66]	0.15	0.30	0.14	P_3
d_{H-T}	[67]	0.05	0.12	0.05	Cannot be identified
d_{H-R}	[67]	0.04	0.07	0.04	Cannot be identified
d_{H-L}	[67]	3.70×10^{-17}	3.70×10^{-17}	3.70×10^{-17}	Cannot be identified
d_{H-KD}	[67]	0.10	0.15	0.10	Cannot be identified
d_{H-M}	[67]	0.10	0.15	0.10	Cannot be identified
d_{H-LA}	[67]	0.07	0.08	0.07	Cannot be identified
d_{H-G}	[67]	0.05	0.08	0.05	Cannot be identified
d_{SW}	[68]	0.01	0.05	0.01	Cannot be identified
d_{SM}	[69]	0.14	0.19	0.10	P_3
d_{LZ}	[70]	0.20	0.40	0.19	P_3
d_{YC}	[71]	0.11	0.23	0.10	P_3
d_X	[59]	0.15	0.31	0.14	P_3
Proposed method	-	0.10	0.15	0.09	P_3

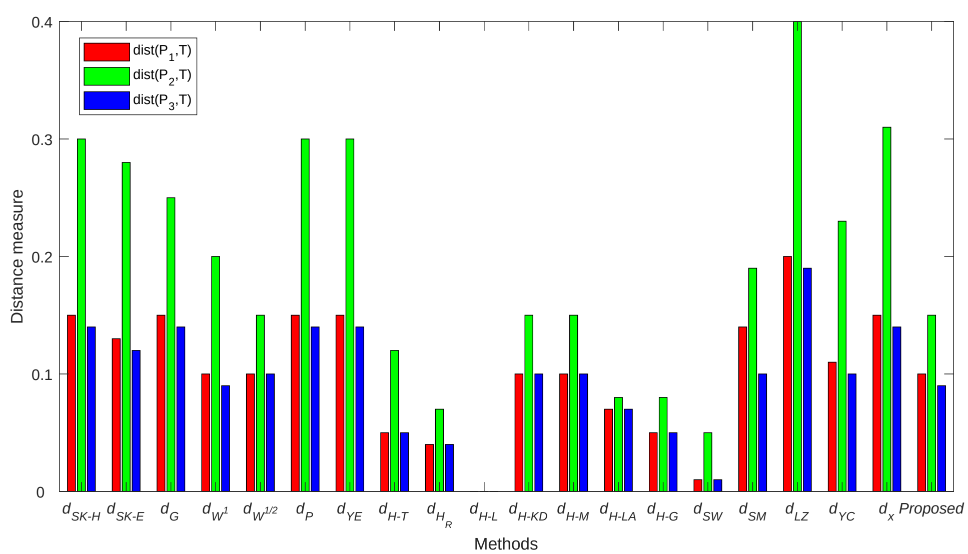


Fig. 4. Comparison of different methods in Application two.

topics. However, it is worth noting that the negative research is an open issue, and the work in this paper is also a preliminary discussion of the IFS negation. How to construct a more effective model is still our future exploration direction.

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CONFLICT OF INTEREST

The authors declare that they have no conflict of interest.

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