AVERAGING VACUUM SOLUTIONS

Mikhail Batanov-Gaukhman[1], Julian Cruz[2]

(1) Moscow Aviation Institute (National Research University), Institute No. 2 “Aircraft and rocket engines and power plants”,st. Volokolamsk highway, 4, Moscow – Russia, 125993(e-mail: alsignat@yandex.ru)

(2) Moscow Aviation Institute (National Research University), Institute No. 6 “Aerospace"”, st. Volokolamsk highway, 4, Moscow – Russia, 125993 (e-mail: Julianperez0300@gmail.com)

ABSTRACT

This article discusses known and additional solutions of Einstein's vacuum equations without a lambda term and with a lambda term, with signatures (+ – – –) and (– + + +). The possibility of averaging these solutions is investigated. It is shown that the averaging of metrics-solutions of Einstein's vacuum equations can be used as the basis for metric-dynamic models of stable vacuum formations of the corpuscular type. Ways to solve the problems that arose in this case related to spatial singularities and spherical voids are proposed.

Keywords: vacuum, Einstein's vacuum equation, signature, solutions to the vacuum equation.

INTRODUCTION

This article considers well-known and additional spherically symmetric, stationary solutions of the Einstein vacuum equation without the lambda term

$$R_{ik} = 0, \tag{1}$$

and with the lambda term $R_{ik} \pm \Lambda\, g_{ik} = 0$,

where

$$R_{ik} = \frac{\partial \Gamma^l_{ik}}{\partial x^l} - \frac{\partial \Gamma^l_{il}}{\partial x^k} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{mk} \quad \text{is the Ricci tensor;} \tag{2}$$

$$\Gamma^\lambda_{ik} = \frac{1}{2} g^{\lambda\mu}\left(\frac{\partial g_{\mu k}}{\partial x^i} + \frac{\partial g_{i\mu}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^\mu}\right) \quad \text{is the Christoffel symbols;} \tag{3}$$

g_{ik} and $g^{\lambda\mu}$ are covariant and contravariant components of the metric tensor of a curved 4-dimensional space with the metric

$$ds^2 = g_{ik}\, dx^i dx^k. \tag{4}$$

Eq. (1) has been considered in many scientific publications on modern differential geometry and general relativity, for example, in [1, 2, 3, 4, 5, 6, 7, 8]. However, none of the books and articles known to the author shows a complete set of solutions to this equation, and the relationship between these solutions is not discussed. Therefore, we repeat the solutions to Eq. (1) in detail.

Solutions to the Einstein vacuum equation (1) for the stationary case are sought in a spherical coordinate system

$$(x_0, x_1, x_2, x_3) = (ct, r, \theta, \varphi), \text{ where } c \text{ is the speed of light in } vacuum, \tag{5}$$

in the form of metrics:

$$ds^{(+)2} = e^\nu c^2dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + sin^2\theta\, d\varphi^2) \quad \text{with signature } (+ - - -), \tag{6}$$

or

$$ds^{(-)2} = - e^\nu c^2dt^2 + e^\lambda dr^2 + r^2(d\theta^2 + sin^2\theta\, d\varphi^2) \quad \text{with signature } (- + + +), \tag{7}$$

where ν and λ are the required functions of time t and distance r;

In this article, "4-dimensional pseudo-metrics" will be called "metrics" for simplicity.

In metric (6), the nonzero components of the metric tensor are equal to

$$g_{00} = e^{\nu}, \quad g_{11} = -e^{\lambda}, \quad g_{22} = -r^2, \quad g_{33} = -r^2 \sin^2\theta, \tag{8}$$

and their contravariant components are equal

$$g^{00} = e^{-\nu}, \quad g^{11} = -e^{-\lambda}, \quad g^{22} = -r^{-2}, \quad g^{33} = -r^{-2} \sin^{-2}\theta. \tag{9}$$

Substituting the components of metric tensors (8) and (9) into Eqs. (3), the Christoffel symbols are calculated (the prime means differentiation with respect to *r*, and the dot above the letter means differentiation with respect to *ct*) [1]:

$$\begin{aligned}
&\Gamma^1_{11} = \frac{\lambda'}{2}, && \Gamma^0_{10} = \frac{\nu'}{2}, && \Gamma^2_{33} = -\sin\theta\cos\theta, \\
&\Gamma^0_{11} = \frac{\dot{\lambda}}{2} e^{\lambda-\nu}, && \Gamma^1_{22} = -re^{-\lambda}, && \Gamma^1_{00} = \frac{\nu'}{2} e^{\nu-\lambda}, \\
&\Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r}, && \Gamma^3_{23} = ctg\theta, && \Gamma^0_{00} = \frac{\dot{\nu}}{2}, \\
&\Gamma^1_{10} = \frac{\dot{\lambda}}{2}, && \Gamma^1_{33} = -re^{-\lambda}\sin^2\theta.
\end{aligned} \tag{10}$$

The remaining Christoffel symbols Γ_{kl}^i (except for those that differ by permutation of the indices *k* and *l*) are equal to zero.

It is widely known that when substituting the Christoffel symbols (10) into the vacuum equation (1) for the stationary case (i.e. for ν = const and λ = const), the following system of differential equations is obtained [1]:

$$R_{00} = R_{11} = \nu'' + \nu'^2 + 2\nu'/r = 0, \tag{11}$$

$$R_{22} = e^{-\lambda} (\lambda'/r - 1/r^2) + 1/r^2 = 0, \tag{12}$$

$$R_{33} = e^{-\lambda} (\nu'/r + 1/r^2) - 1/r^2 = 0, \tag{13}$$

$$\nu = -\lambda.$$

Eqs. (11), (12) and (13) each have three identical solutions:

$$e^{-\lambda} = e^{\nu} = (1 + r_0/r), \quad e^{-\lambda} = e^{\nu} = (1 - r_0/r), \quad e^{-\lambda} = e^{\nu} = 1, \tag{14}$$

or $\quad -\lambda = \nu = \ln (1 + r_0/r), \quad -\lambda = \nu = \ln (1 + r_0/r), \quad -\lambda = \nu = \ln 1,$ (15)

where r_0 is the integration constant (in particular, the radius of the sphere, the meaning of which will be clarified below).

It is easy to verify that each of the three Eqs. (14) is a solution to Eqs. (11), (12), and (13) by alternately substituting these solutions into these equations.

Substituting three possible solutions (14) into metric (6), we obtain three metric-solutions to the vacuum equation (1) with the same signature (+ – – –):

$$ds_1^{(+)2} = \left(1 - \frac{r_o}{r}\right) c^2dt^2 - \frac{1}{\left(1 - \frac{r_o}{r}\right)} dr^2 - r^2d\theta^2 - r^2 \sin^2\theta \, d\phi^2, \tag{16}$$

$$ds_2^{(+)2} = \left(1 + \frac{r_o}{r}\right) c^2dt^2 - \frac{1}{\left(1 + \frac{r_o}{r}\right)} dr^2 - r^2d\theta^2 - r^2 \sin^2\theta \, d\phi^2, \tag{17}$$

$$ds_3^{(+)2} = c^2dt^2 - dr^2 - r^2d\theta^2 - r^2 \sin^2\theta \, d\phi^2. \tag{18}$$

Performing similar operations with the components of the metric tensor from metric (7)

$$g_{00} = -e^{\nu}, \quad g_{11} = e^{\lambda}, \quad g_{22} = r^2, \quad g_{33} = r^2\sin^2\theta, \tag{19}$$

and their contravariant components

$$g^{00} = -e^{-\nu}, \quad g^{11} = e^{-\lambda}, \quad g^{22} = r^{-2}, \quad g^{33} = r^{-2} sin^{-2}\theta, \tag{20}$$

we obtain three more metrics-solutions of the vacuum equation (1) with the opposite signature (– + + +):

$$ds_1^{(-)2} = -\left(1 - \frac{r_o}{r}\right) c^2 dt^2 + \frac{1}{\left(1 - \frac{r_o}{r}\right)} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2, \tag{21}$$

$$ds_2^{(-)2} = -\left(1 + \frac{r_o}{r}\right) c^2 dt^2 + \frac{1}{\left(1 + \frac{r_o}{r}\right)} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2, \tag{22}$$

$$ds_3^{(-)2} = -c^2 dt^2 + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2. \tag{23}$$

Note that when $r_0 = 0$, metrics (16) and (17) become metric (18), and metrics (21) and (22) become metric (23).

It is generally accepted that identical metrics with opposite signatures (+ – – –) and (– + + +) are isomorphic. However, this is not always the case. This issue is studied in detail in a series of articles [12, 13, 14, 15]. In this article, we only note the following: if we conditionally accept that metrics with signatures (+ – – –) define metric-dynamic models of "convex" vacuum formations, then similar metrics with the opposite signature (– + + +) define metric-dynamic models of exactly the same, but "concave" vacuum formations. Identical "convex" and "concave" stable vacuum formations completely compensate for each other's manifestations, maintaining the vacuum balance (+ – – –) + (– + + +) = 0.

All metrics (16) – (18) and (21) – (23) are solutions to the vacuum equation (1), but only the quadratic form (16) is called the Schwarzschild metric, provided

$$r_0 = r_g = 2GM/c^4, \tag{24}$$

where M is the mass of the celestial body, G is the gravitational constant ($G \approx 6.67430 \cdot 10^{-11}$ m^3 s^{-2} kg^{-1}).

According to Birkhoff's direct theorem and Israel's inverse theorem, there are no other exact spherically symmetric static solutions of the vacuum equation (1), except for metrics (16) – (18) and (21) – (23), which at infinity tend to the Minkowski metric (i.e. to the metric of flat pseudo-Euclidean space).

However, in general relativity, due to the fact that Eq. (1) is generally covariant, there are many possibilities for choosing other coordinate systems. Of particular interest are the coordinate transformations: Kruskal-Szekeres coordinate; Eddington - Finkelstein coordinates; Lemaître coordinates; Gullstrand - Painlevé coordinates; Isotropic coordinates; Harmonic coordinates, since these transformations make it possible to exclude or shift the spatial singularity to the center at $r_0 = r$ in metrics (16) – (17) and (21) – (22). However, when $r_0 = r$, the temporary singularity in these metrics cannot be excluded.

MATERIALS AND METHOD

1 Averaging metrics-solutions of the vacuum equation

In the introduction, metrics-solutions to the vacuum equation (1) were given, which are well-known to specialists in the field of general relativity and Riemann differential geometry. New results will be presented in this section and below.

Let's consider three metrics (16) – (18):

$$ds_1^{(+)2} = \left(1 - \frac{r_o}{r}\right) c^2 dt^2 - \frac{1}{\left(1 - \frac{r_o}{r}\right)} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2, \tag{16'}$$

$$ds_2^{(+)2} = \left(1 + \frac{r_o}{r}\right) c^2 dt^2 - \frac{1}{\left(1 + \frac{r_o}{r}\right)} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2, \tag{17'}$$

$$ds_3^{(+)2} = c^2 dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2. \tag{18'}$$

Metric (18) is a special case of the first two metrics (16) and (17) for $r_0 = 0$, and describes the state of the original (i.e., uncurved) Einsteinian vacuum.

Metrics (16) and (17) describe different curved states of the Einsteinian vacuum, but they are equivalent and one cannot give preference to any of them without losing information about the region of space under consideration. Therefore, we formulate the hypothesis that both metrics (16) and (17) jointly describe the metric-dynamic state of the same region of the Einsteinian vacuum (hereinafter *vacuum*), and consider the result of their averaging

$$ds_{12}^{(+)2} = \frac{1}{2}\left(ds_1^{(+)2} + ds_2^{(+)2}\right) = c^2dt^2 - \frac{r^2}{r^2-r_0^2}dr^2 - r^2d\theta^2 - r^2 \sin^2\theta \, d\phi^2. \quad (25)$$

In general relativity, the distance between two events with different r, but with the same other coordinates, is determined by the integral [1]

$$\xi = \int_{r_1}^{r_2} \sqrt{-g_{11}^{(+)}}\, dr. \quad (26)$$

If $g_{11}^{(+)} = -(1-r_0/r)^{-1}$ from metric (16) or $g_{11}^{(+)} = -(1+r_0/r)^{-1}$ from metric (17) substitute into integral (26), then such an integral is not taken in elementary functions.

If $g_{11}^{(+)} = -(1-r_0/r)^{-1}$ from metric (16) or $g_{11}^{(+)} = -(1+r_0/r)^{-1}$ from metric (17) substitute into integral (26), then such an integral is not taken in elementary functions.

Whereas when substituting the component $g_{(12)11}^{(+)} = -\frac{r^2}{r^2-r_0^2}$ from the averaged metric (25) into the integral (26), it is possible to find an analytical solution

$$\xi = \int_{r_1}^{r_2} \frac{rdr}{\sqrt{r^2-r_0^2}} = \sqrt{r^2-r_0^2}\,\Big|_{r_1}^{r_2}. \quad (27)$$

Let's first find the size of the segment between the points $r_1 = 0$ and $r_2 = r_0$:

$$\sqrt{r^2-r_0^2}\,\Big|_0^{r_0} = -\sqrt{-r_0^2} = -\sqrt{-1}r_0 = -ir_0. \quad (28)$$

The length of this segment is equal to the radius of the cavity r_0, and the imaginary nature of this result suggests that the averaged metric (25) does not describe the properties of the *vacuum* inside a spherical cavity with radius r_0. In other words, the domain of applicability of metric (25) starts from r_0 and extends to $r_2 = \infty$. In this case we have

$$\sqrt{r^2-r_0^2}\,\Big|_{r_0}^{\infty} = \sqrt{\infty^2 - r_0^2}\,. \quad (29)$$

Here we present the symbol '∞' in an abuse of notation to indicate the corresponding calculations via limits.

If the *vacuum* region under study were not deformed, then the distance between the points $r_2 = \infty$ and $r_1 = r_0$ would be equal to $r_2 - r_1 = \infty - r_0$, and in our case it is equal to value (29), subtracting one from the other, we find

$$\sqrt{\infty^2 - r_0^2} - (\infty - r_0) = r_0, \quad (30)$$

since the limit calculation leads to this result

$$\lim_{x\to\infty} \sqrt{x^2 - r_0^2} - (x - r_0) = r_0\,.$$

The result obtained shows that the vacuum is compressed by an amount ~ r_0 in all radial directions, and the reason for such compression is due to the fact that it is “displaced” from a cavity with radius r_0. This looks like an air bubble in a liquid (see Figure 1).

The distortions of the *vacuum* region under study will be judged by its relative elongation [9]

$$l^{(+)} = \frac{ds^{(+)} - ds_0^{(+)}}{ds_0^{(+)}} = \frac{ds^{(+)}}{ds_0^{(+)}} - 1. \quad (31)$$

In this case, the relative elongation for each coordinate is determined by the equation [9]

$$l_i^{(+)} = \sqrt{1 + \frac{g_{ii}^{(+)} - g_{ii0}^{(+)}}{g_{ii0}^{(+)}}} - 1, \quad (32)$$

where

$g_{ii}^{(+)}$ are the components of the metric tensor of the curved region of the *vacuum*;

$g_{ii0}^{(+)}$ are the components of the metric tensor of the same region of *vacuum* before curvature (i.e. in the absence of its curvature).

Let's substitute into Eqs. (32) the components $g_{ii}^{(+)}$ from the averaged metric (25), and the components $g_{ii0}^{(+)}$ from the original metric (18), as a result we obtain

$$l_r^{(+)} = \frac{\Delta r}{r} = \sqrt{\frac{r^2}{r^2 - r_0^2}} - 1, \qquad l_\theta^{(+)} = 0, \qquad l_\phi^{(+)} = 0. \quad (33)$$

The graph of the function $l_r^{(+)} = \Delta r/r$, with $r_0 = 1$, is shown in Figure 1. At $r = r_0$, this function tends to infinity $\Delta r/r = \infty$, and at $r < r_0$ it becomes imaginary, which once again confirms the "empty bubble in a liquid" model. We will call such an empty bubble a "spherical Schwarzschild cavity."

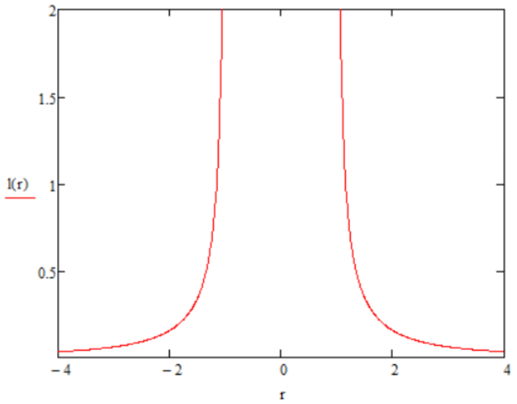

Fig. 1: Graph of a function (35) $l_r^{(+)} = \frac{\Delta r}{r}$

If we now average metrics (21) and (22)

$$ds_{12}^{(-)2} = \frac{1}{2}\left(ds_1^{(-)2} + ds_2^{(-)2}\right) = -c^2dt^2 + \frac{r^2}{r^2 - r_0^2}dr^2 + r^2d\theta^2 + r^2 \sin^2\theta\, d\phi^2 \text{ with signature } (-+++), \quad (34)$$

and perform similar actions (26) – (33) taking into account the original metric (23), we obtain a metric-dynamic model of exactly the same (see Figure 1), but opposite, stable, conditionally "concave" vacuum formation of the type "spherical anti-Schwarzschild cavity" with relative elongation

$$l_r^{(-)} = \frac{\Delta r}{r} = \sqrt{\frac{r^2}{r^2 - r_0^2}} - 1, \qquad l_\theta^{(-)} = 0, \qquad l_\phi^{(-)} = 0. \quad (35)$$

Thus, averaging metrics (16) and (17), as well as averaging metrics (21) and (22), leads to metric-dynamic models of mutually opposite stable vacuum formations such as "spherical Schwarzschild cavity" and "spherical anti-Schwarzschild cavity". Whereas separately metrics (16), (17), (21) and (22) do not lead to such results. This confirms the validity of the hypothesis about the possibility of averaging different metrics-solutions of the same vacuum equation.

The possibility of averaging metrics-solutions of the vacuum equation (1) is also supported by the fact that averaging all six metrics (16) – (18) and (21) – (23) leads to two more trivial (i.e. zero) pseudo-metrics-solutions of this equation

$$\frac{1}{6}\left(ds_1^{(+)2} + ds_2^{(+)2} + ds_3^{(+)2} + ds_1^{(-)2} + ds_2^{(-)2} + ds_3^{(-)2}\right) = \pm 0 \cdot c^2dt^2 \mp 0 \cdot dr^2 \mp 0 \cdot r^2d\theta^2 \mp 0 \cdot r^2 \sin^2\theta\, d\phi^2. \quad (36)$$

.

A significant advantage of the considered averaged metric-dynamic models of "cavity" and "anti-cavity" is the fact that the zero component of the metric tensor in the averaged metrics (25) and (34) is equal to one ($g_{00}^{(+)} = 1$). This means that in these models the time t is global, so these stable vacuum formations can coexist in the same global space with a single time. In addition, there is no temporal singularity in these averaged metrics.

In this paragraph, all known solutions of Einstein's vacuum equation (1) were used and this led to averaged metric-dynamic models of a mutually opposite pair of vacuum formations "cavity" – "anti-cavity". However, this raised three problems:

1] The averaged metrics (25) and (34) turned out to be not the Schwarzschild metric, which, according to the entire scientific community, has been reliably tested and experimentally confirmed in the lower orders of approximation of general relativity to Newtonian theory (i.e. for the case of weak gravitational fields).
2] Relative vacuum elongations (33) and (35) at $r = r_0$ tend to infinity ($\Delta r/r \to \infty$). The presence of such a singularity is a clear indicator of the incompleteness of the mathematical model under consideration.
3] If the vacuum is displaced from a spherical region with radius r_0, then it is not clear what is inside such an empty cavity.

Possible solutions to these problems are suggested below.

2 Qualitative discussion of the singularity problem

At $r = r_0$, both functions of the relative vacuum elongation (33) and (35) tend to infinity ($\Delta r/r \to \infty$, see Figure 2). It is obvious that within the framework of Riemann differential geometry, the problem of the presence of singularities in solutions of the vacuum equation (1) using the above metrics is in principle unsolvable. Perhaps this problem will be solved as a result of increasing the capabilities of differential geometry, for example, by taking into account not only curvature, but also torsions, displacements and other distortions of space.

In this article we note only one circumstance that can help solve this problem.

Let's recall the property of the "Koch curve" fractal.

This fractal has two extraordinary properties: 1) any iteration of the Koch curve is an example of a continuous line to which it is impossible to draw a tangent at any point (i.e., these lines are not differentiable); 2) if the length of the initial Koch segment is 1, then the length of the n-th iteration of this fractal is equal to $(4/3)^{n-1}$. Therefore, when $n = \infty$ the length of the Koch curve tends to infinity.

Let's return to the problem of singularities in averaged metrics (25) and (35). It should be expected that in the region of a sphere with radius r_0, an increase in the length of radial segments occurs due to a decrease in the scale of their brokenness (Figure 2), similar to a decrease in the scale of brokenness of the "Koch curve" as the number increases iterations. This is similar to how when the Reynolds number is exceeded, fluid flow changes from laminar to turbulent. Moreover, as we approach r_0, the elongation of such broken, or bent, or wound, etc., segments can tend to infinity.

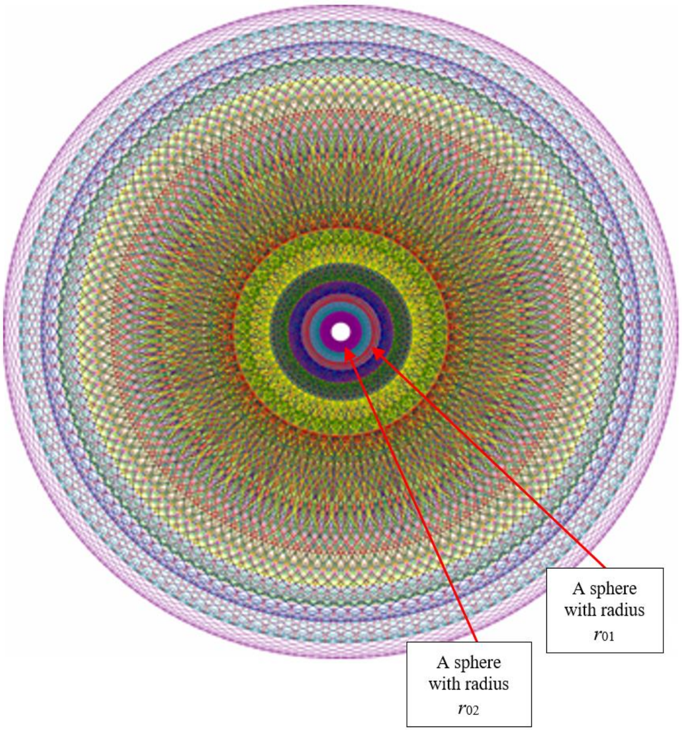

Fig. 2: Increase in brokenness of lines as they approach the central cavity (Prokhorov-Lebedev drawing)

3 Schwarzschild-like averaged metric

Let's consider the case when in the region surrounding a spherical cavity, there is not one boundary sphere, but two (as shown in Figure 2) with radii r_{01} and r_{02} such that

$$r_{01} \approx r_{02} \approx r_0 \quad \text{and} \quad r_{01} \geq r_{02}. \tag{37}$$

In this case, metrics (16) and (17) take the form

$$ds_1^{(+)2} = \left(1 - \frac{r_{o1}}{r}\right) c^2 dt^2 - \frac{1}{\left(1 - \frac{r_{o1}}{r}\right)} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2, \tag{38}$$

$$ds_2^{(+)2} = \left(1 + \frac{r_{o2}}{r}\right) c^2 dt^2 - \frac{1}{\left(1 + \frac{r_{o2}}{r}\right)} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2. \tag{39}$$

Let's average metrics (54) and (55) taking into account conditions (53) (i.e., small differences between r_{01} and r_{02})

$$ds_{12}^{(+)2} = \frac{1}{2}\left(ds_1^{(+)2} + ds_2^{(+)2}\right) \approx \left(1 + \frac{r_{o2} - r_{o1}}{2r}\right) c^2 dt^2 - \frac{r^2}{r^2 - r_0^2} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2. \tag{40}$$

The zero component of the metric tensor in the averaged metric (56) is equal to

$$g_{00} = \left(1 + \frac{r_{o2} - r_{o1}}{2r}\right) = \left(1 - \frac{r_g}{r}\right), \tag{41}$$

where $r_g = \frac{r_{o1} - r_{o2}}{2}$ is a value that can be interpreted as the average Schwarzschild radius of a stable corpuscular vacuum formation.

Taking into account Eq. (41), metric (40) can be represented in a Schwarzschild-like form

$$ds_{(12)}^{(+)2} \approx \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{1}{1 - \frac{r_0^2}{r^2}} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2. \tag{42}$$

For example, it is known that the planet Earth has $r_{gE} \approx 0.9$ cm. Then, according to Eq. (41), if our assumptions are correct, then inside our planet there are two boundary spheres with a difference in radii

$r_{o1E} - r_{o2E} = 2r_{gE} = 1{,}8$ cm.

Then all experimentally confirmed weak gravitational effects in the vicinity of our planet remain in force.

It can also be assumed that the average radius of the Earth's spherical cavity r_0 corresponds to the radius of its solid inner core $r_{0E} \approx 1220$ km.

Thus, within the framework of the considered averaged models of stable vacuum formations, the problem of the Schwarzschild-like gravitational potential is easily solved.

4 Checking the possibility of finding other solutions to Einstein's vacuum equation

To understand the question: "What is inside the spherical cavity and anti-cavity (see Figure 1)?" we made an attempt to look for solutions to the vacuum equation (1) in the form of metrics

$$ds_*^{(+)2} = e^{-\nu} c^2 dt^2 - e^{-\lambda} dr^2 - r^2(d\theta^2 + \sin^2\theta \, d\varphi^2) \text{ with signature } (+ - - -), \tag{43}$$

and

$$ds_*^{(-)2} = - e^{-\nu} c^2 dt^2 + e^{-\lambda} dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\varphi^2) \text{ with signature } (- + + +). \tag{44}$$

These metrics are in some sense antisymmetric with respect to metrics (6) and (7), so we will call them "inverted".

In the "inverted" metric (43), the nonzero components of the metric tensor are equal

$$g_{00} = e^{-\nu}, \quad g_{11} = - e^{-\lambda}, \quad g_{22} = - r^2, \quad g_{33} = - r^2 \sin^2\theta, \tag{45}$$

and their contravariant components are equal

$$g^{00} = e^{\nu}, \quad g^{11} = - e^{\lambda}, \quad g^{22} = - r^{-2}, \quad g^{33} = - r^{-2} \sin^{-2}\theta. \tag{46}$$

Substituting the components of the metric tensors (45) and (46) into Eq. (3), we calculate the Christoffel symbols (the calculations were performed by Cruz Perez Julian Arturo, the calculation is presented in Appendix 1), the prime means differentiation with respect to r, and the dot above the letter means differentiation with respect to ct:

$$\Gamma^1_{11} = -\frac{\lambda'}{2}, \qquad \Gamma^0_{10} = -\frac{\nu'}{2}, \qquad \Gamma^2_{33} = -sin\theta cos\theta, \tag{47}$$

$$\Gamma^0_{11} = -\frac{\dot{\lambda}}{2}e^{\nu-\lambda}, \qquad \Gamma^1_{22} = -re^{\lambda}, \qquad \Gamma^1_{00} = -\frac{\nu'}{2}e^{\lambda-\nu},$$

$$\Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r}, \qquad \Gamma^3_{23} = ctg\theta, \qquad \Gamma^0_{00} = -\frac{\dot{\nu}}{2},$$

$$\Gamma^1_{10} = -\frac{\dot{\lambda}}{2}, \qquad \Gamma^1_{33} = -re^{\lambda}sin^2\theta.$$

The remaining Christoffel symbols Γ^i_{kl} (except for those that differ by permutation of the indices k and l) are equal to zero.

Exactly the same Christoffel symbols Γ^i_{kl} are obtained by using the components of the metric tensor from metric (44).

Thus, in the case when metrics (43) with signature (+ – – –) or (44) with signature (– + + +) are taken as initial ones, instead of the system of Eqs. (11) – (13) we obtain a system of equations (see Appendix 2)

$$R_{00} = R_{11} = \nu'' - \nu'^2 + 2\nu'/r = 0, \tag{48}$$

$$R_{22} = e^{\lambda}(\lambda'/r + 1/r^2) - 1/r^2 = 0, \tag{49}$$

$$R_{33} = e^{\lambda}(\nu'/r - 1/r^2) + 1/r^2 = 0, \tag{50}$$

$$\lambda = -\nu.$$

Eqs. (48), (49) and (50) each have three identical solutions:

$$e^{\lambda} = e^{-\nu} = (1 + r_0/r), \qquad e^{\lambda} = e^{-\nu} = (1 - r_0/r), \qquad e^{\lambda} = e^{-\nu} = 1, \tag{51}$$

or $\lambda = -\nu = ln\,(1 + r_0/r), \quad \lambda = -\nu = ln\,(1 + r_0/r), \quad \lambda = -\nu = ln\,1.$ (52)

When substituting solutions (51) into metrics (43) and (44), we obtain metrics-solutions to the vacuum equation (1), which completely coincide with metrics-solutions (16) – (18) and (21) – (23).

Thus, we are convinced that when using the "inverted" initial metrics (43) and (44), exactly the same metrics-solutions (16) – (18) and (21) – (23) are obtained (the result is quite expected), and the problem of filling spherical cavities and anti-cavities remains unresolved.

Note that not only metrics (6) and (7) or "inverted" metrics (43) and (44), but also metrics with complex components of the metric tensor can be taken as initial ones

$$ds_{**}^{(+)2} = e^{i\nu}c^2dt^2 - e^{i\lambda}dr^2 - r^2(d\theta^2 + sin^2\theta\, d\varphi^2) \quad \text{with signature } (+ - - -), \tag{53}$$

and

$$ds_{**}^{(-)2} = -e^{i\nu}c^2dt^2 + e^{i\lambda}dr^2 + r^2(d\theta^2 + sin^2\theta\, d\varphi^2) \quad \text{with signature } (- + + +). \tag{54}$$

In the case of the initial metric (53), we have

$$\Gamma^1_{11} = \frac{i\lambda'}{2}, \qquad \Gamma^0_{10} = \frac{i\nu'}{2}, \qquad \Gamma^2_{33} = -sin\theta cos\theta,$$

$$\Gamma^0_{11} = i\frac{\dot{\lambda}}{2}e^{i(\lambda-\nu)}, \qquad \Gamma^1_{22} = -re^{-i\lambda}, \qquad \Gamma^1_{00} = \frac{\nu'}{2}e^{i(\lambda-\nu)},$$

$$\Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r}, \qquad \Gamma^3_{23} = ctg\theta, \qquad \Gamma^0_{00} = \frac{i\dot{\nu}}{2},$$

$$\Gamma^1_{10} = \frac{i\dot{\lambda}}{2}, \qquad \Gamma^1_{33} = -re^{-i\lambda}sin^2\theta.$$

Then for the stationary case (i.e., for ν = const and λ = const), the vacuum equation (1) is represented as a system of complex equations

$R_{00} = R_{11} = \nu'' + i\nu'^2 + 2\nu'/r = 0$,

$R_{22} = e^{-i\lambda}(i\lambda'/r - 1/r^2) + 1/r^2 = 0$,

$R_{33} = e^{-i\lambda}(i\nu'/r + 1/r^2) - 1/r^2 = 0$,

$\nu = -\lambda$.

Solutions to this system of equations are

$e^{-i\lambda} = e^{i\nu} = (1 + r_0/r)$, $\quad e^{-i\lambda} = e^{i\nu} = (1 - r_0/r)$, $\quad e^{-i\lambda} = e^{i\nu} = = 1$,

or $\lambda = -\nu = i\,ln\,(1 + r_0/r)$, $\quad \lambda = -\nu = i\,ln\,(1 + r_0/r)$, $\quad \lambda = -\nu = i\,ln\,1$.

Consistently substituting these solutions into the original metric (53), we again obtain three metrics-solutions (16) – (18). Similar actions with the original metric (54) lead to metrics-solutions (21) – (23).

Thus, all attempts to find other solutions to the vacuum equation (1) were unsuccessful.

5 The Schwarzschild - de Sitter world and anti-world

Above, an attempt was made to find additional solutions to the Einstein vacuum equation (1) in order to solve the problem of filling spherical cavities. However, these studies only strengthened the confidence that such solutions are not contained in Eq. (1).

Therefore, let's consider solutions to the vacuum Einstein equation with the Λ-term

$$R_{ik} \pm \Lambda\, g_{ik} = 0, \tag{55}$$

where $\Lambda = 3/r_a^2$, the physical meaning of the radius ra will be clarified later.

There are five metric solutions to equation (55) with signature (+ – – –)

$$\text{I} \quad ds_1^{(+)2} = \left(1 - \frac{r_b}{r} + \frac{r^2}{r_a^2}\right)c^2dt^2 - \frac{dr^2}{\left(1 - \frac{r_b}{r} + \frac{r^2}{r_a^2}\right)} - r^2(d\theta^2 + sin^2\,\theta\, d\phi^2), \tag{56}$$

$$\text{H} \quad ds_2^{(+)2} = \left(1 + \frac{r_b}{r} - \frac{r^2}{r_a^2}\right)c^2dt^2 - \frac{dr^2}{\left(1 + \frac{r_b}{r} - \frac{r^2}{r_a^2}\right)} - r^2(d\theta^2 + sin^2\,\theta\, d\phi^2), \tag{57}$$

$$\text{V} \quad ds_3^{(+)2} = \left(1 - \frac{r_b}{r} - \frac{r^2}{r_a^2}\right)c^2dt^2 - \frac{dr^2}{\left(1 - \frac{r_b}{r} - \frac{r^2}{r_a^2}\right)} - r^2(d\theta^2 + sin^2\,\theta\, d\phi^2), \tag{58}$$

$$\text{H}' \quad ds_4^{(+)2} = \left(1 + \frac{r_b}{r} + \frac{r^2}{r_a^2}\right)c^2dt^2 - \frac{dr^2}{\left(1 + \frac{r_{b4}}{r} + \frac{r^2}{r_a^2}\right)} - r^2(d\theta^2 + sin^2\,\theta\, d\phi^2), \tag{59}$$

$$i \quad ds_5^{(+)2} = c^2dt^2 - dr^2 - r^2(d\theta^2 + sin^2\,\theta\, d\phi^2); \tag{60}$$

and five metric solutions of the same equation with signature (– + + +)

$$\text{H}' \quad ds_1^{(-)2} = -\left(1 - \frac{r_b}{r} + \frac{r^2}{r_a^2}\right)c^2dt^2 + \frac{dr^2}{\left(1 - \frac{r_b}{r} + \frac{r^2}{r_a^2}\right)} + r^2(d\theta^2 + sin^2\,\theta\, d\phi^2), \tag{61}$$

$$\text{V} \quad ds_2^{(-)2} = -\left(1 + \frac{r_b}{r} - \frac{r^2}{r_a^2}\right)c^2dt^2 + \frac{dr^2}{\left(1 + \frac{r_b}{r} - \frac{r^2}{r_a^2}\right)} + r^2(d\theta^2 + sin^2\,\theta\, d\phi^2), \tag{62}$$

$$\text{H} \quad ds_3^{(-)2} = -\left(1 - \frac{r_b}{r} - \frac{r^2}{r_a^2}\right)c^2dt^2 + \frac{dr^2}{\left(1 - \frac{r_b}{r} - \frac{r^2}{r_a^2}\right)} + r^2(d\theta^2 + sin^2\,\theta\, d\phi^2), \tag{63}$$

$$\text{I} \quad ds_4^{(-)2} = -\left(1 + \frac{r_b}{r} + \frac{r^2}{r_a^2}\right)c^2dt^2 + \frac{dr^2}{\left(1 + \frac{r_b}{r} + \frac{r^2}{r_a^2}\right)} + r^2(d\theta^2 + sin^2\,\theta\, d\phi^2), \tag{64}$$

$$i \quad ds_5^{(-)2} = -\,c^2dt^2 + dr^2 + r^2(d\theta^2 + sin^2\,\theta\, d\phi^2). \tag{65}$$

Friedrich Kottler first wrote down the Kottler metric of the form (58)

$$ds^2_{Kottler} = \left(1-\frac{r_b}{r}-\frac{r^2}{r_a^2}\right)c^2dt^2 - \frac{dr^2}{\left(1-\frac{r_b}{r}-\frac{r^2}{r_a^2}\right)} - r^2(d\theta^2 + sin^2\,\theta\,d\phi^2), \qquad (58')$$

in article [10], which was published in March 1918, almost immediately after the publication of Einstein's general relativity. In the case: $r_a = \infty$ *and* $r_b \neq 0$*, the Kottler metric (58) becomes the Schwarzschild metric*

$$ds^2_{\text{Schwarzschild}} = \left(1-\frac{r_b}{r}\right)c^2dt^2 - \frac{dr^2}{\left(1-\frac{r_b}{r}\right)} - r^2(d\theta^2 + sin^2\,\theta\,d\phi^2).$$

In another limiting case: $r_a \neq \infty$ *and* $r_b = 0$*, the Kottler metric (58) becomes the de Sitter metric*

$$ds^2_{\text{de Sitter}} = \left(1-\frac{r^2}{r_a^2}\right)c^2dt^2 - \frac{dr^2}{\left(1-\frac{r^2}{r_a^2}\right)} - r^2(d\theta^2 + sin^2\,\theta\,d\phi^2).$$

In the third case: $r_a = \infty$ *and* $r_b = 0$*, the Kottler metric (143) takes the form of the Minkowski metric*

$$ds^2_{\text{Minkowski}} = c^2dt^2 - dr^2 - r^2(d\theta^2 + sin^2\,\theta\,d\phi^2).$$

Therefore, the metrics-solution (56) – (60) and (61) – (65) of the second Einstein vacuum equation (140) will be called the Kottler - de Sitter- Schwarzschild metrics or, in short, KdSSh-metrics.

Let's average metrics (56) – (60) with signature (+ – – –) and metrics (61) – (65) with signature (– + + +)

$$ds_{1\text{-}4}^{(+)2} = \frac{1}{4}\,(ds_1^{(+)2} + ds_2^{(+)2} + ds_3^{(+)2} + ds_4^{(+)2}).$$

$$ds_{1\text{-}4}^{(-)2} = \frac{1}{4}\,(ds_1^{(-)2} + ds_2^{(-)2} + ds_3^{(-)2} + ds_4^{(-)2}).$$

As a result, we obtain averaged metrics

$$ds^{(+)2}_{1-4} = \ c^2dt^2 - g^{(+)}_{11}(r)dr^2 - r^2(d\theta^2 + sin^2\,\theta\,d\phi^2), \qquad (66)$$

$$ds^{(-)2}_{1-4} = -\,c^2dt^2 + g^{(-)}_{11}(r)dr^2 + r^2(d\theta^2 + sin^2\,\theta\,d\phi^2), \qquad (67)$$

where

$$g^{(+)}_{11}(r) = g^{(-)}_{11}(r) = \frac{1}{4}\left[\frac{1}{\left(1-\frac{r_b}{r}+\frac{r^2}{r_a^2}\right)} + \frac{1}{\left(1+\frac{r_b}{r}-\frac{r^2}{r_a^2}\right)} + \frac{1}{\left(1-\frac{r_b}{r}-\frac{r^2}{r_a^2}\right)} + \frac{1}{\left(1+\frac{r_b}{r}+\frac{r^2}{r_a^2}\right)}\right]. \qquad (68)$$

We substitute the components $g^{(+)}_{ii}$ of the averaged metric (67) or the components $g^{(-)}_{ii}$ of the averaged metric (67) into the expressions for the relative elongation (32)

$$l^{(+)}_i = \sqrt{1+\frac{g^{(+)}_{ii}-g^{(+)}_{ii0}}{g^{(+)}_{ii0}}} - 1, \qquad l^{(-)}_i = \sqrt{1+\frac{g^{(-)}_{ii}-g^{(-)}_{ii0}}{g^{(-)}_{ii0}}} - 1,$$

where the components $g^{(+)}_{ii0}$ are taken from the non-curved metric (60), and the components $g^{(-)}_{ii0}$ are taken from the non-curved metric (65).

As a result, we get

$$l_r^{(\pm)} = \frac{\Delta r}{r} = \sqrt{g_{11}^{(\pm)}(r)} - 1 = \tag{69}$$

$$= \sqrt{\frac{1}{4}\left[\frac{1}{\left(1-\frac{r_b}{r}+\frac{r^2}{r_a^2}\right)} + \frac{1}{\left(1+\frac{r_b}{r}-\frac{r^2}{r_a^2}\right)} + \frac{1}{\left(1-\frac{r_b}{r}-\frac{r^2}{r_a^2}\right)} + \frac{1}{\left(1+\frac{r_b}{r}+\frac{r^2}{r_a^2}\right)}\right]} - 1,$$

$$l_t^{(\pm)} = 0, \qquad l_\theta^{(\pm)} = 0, \qquad l_\phi^{(\pm)} = 0.$$

The graph, for example, of the function $l_r^{(+)}$ (69) with $r_a = 60$ and $r_b = 1.5$, which determines the relative elongation of vacuum in the radial direction, is shown in Figure 3. From this graph it is clear that the result is an almost hollow ball (i.e., de Sitter space) with compacted edges, inside of which there is a spherical Schwarzschild cavity, which is described by metrics (16) – (18), more precisely, by the averaged metric (25).

Indeed, if in metrics (56) – (58) we direct r_a to infinity ($r_a \to \infty$), i.e., for example, assume that ra is the radius of the Universe, then in the vicinity of a small cavity with radius $r_{b,} = r_0$, which is commensurate, for example, with the gravitational radius of the "black hole", the deformed state of the vacuum will be described by the averaged metric (25).

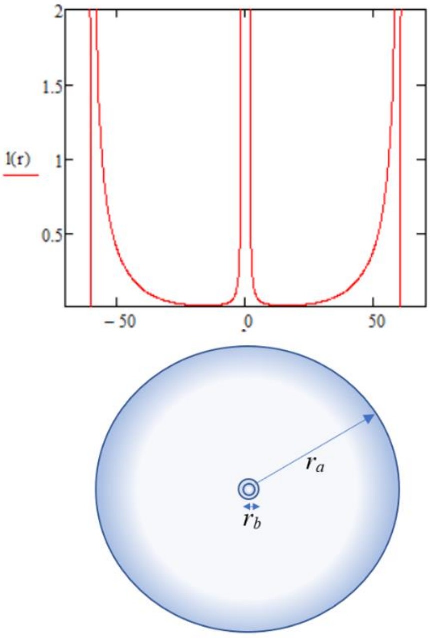

Fig. 3: Graph of the relative elongation function $l_r^{(+)}$ (69), which determines the relative elongation of the *vacuum* in the radial direction

Shown in Figure 3, the vacuum formation resembles a biological cell with an outer shell and an internal nucleolus, so we will call it a Schwarzschild-de Sitter cell.

Performing similar operations with metrics - solutions (61) – (65) with the opposite signature (– + + +), we obtain exactly the same, but opposite Schwarzschild - de Sitter anti-cell.

Averaging all ten metrics-solutions (56) – (65) of the second vacuum equation (55) leads to two more trivial (i.e. zero) pseudo-metric-solutions of this equation

$$\frac{1}{10}\left(\sum_{k=1}^{5} ds_k^{(+)2} + \sum_{k=1}^{5} ds_k^{(-)2}\right) = \pm 0 \cdot c^2dt^2 \mp 0 \cdot dr^2 \mp 0 \cdot r^2 d\theta^2 \mp 0 \cdot r^2 \sin^2\theta \, d\phi^2.$$

It is obvious that Einstein's vacuum equation with the Λ-term (55) also does not allow us to solve the problem of filling the "spherical Schwarzschild cavity" and the "spherical anti-Schwarzschild cavity", which in this case find themselves inside of the de Sitter space or of the anti-de Sitter space, respectively.

6 Einstein's third vacuum equation

As shown above, solutions to the first and second Einstein vacuum equations (1) and (55) make it possible to construct metric-dynamic models of a mutually opposite pair of single stable vacuum formations, but do not allow solving the problem of filling spherical cavities and anti-cavities inside these formations. In addition, these equations lack the potential to describe many stable spherical objects. In this regard, it is proposed to consider the possibility of expanding the vacuum equation (55).

Let's recall that in order to write down Eq. (55), Einstein used the following property of the metric tensor and the Einstein tensor:

$$\Lambda\nabla_j g_{ik} = \nabla_j \Lambda g_{ik} = 0, \qquad \nabla_j\left(R_{ik} - \tfrac{1}{2}R g_{ik}\right) = 0. \tag{70}$$

However, it is obvious that the covariant derivative of the infinite series is also equal to zero

$$\nabla_j(\Lambda_1\, g_{ik} + \Lambda_2 g_{ik} + \Lambda_3 g_{ik} + \ldots + \Lambda_\infty g_{ik}) = \Lambda_1\, \nabla_j g_{ik} + \Lambda_2 \nabla_j g_{ik} + \ldots + \Lambda_\infty \nabla_j g_{ik} = 0, \tag{71}$$

where $\Lambda_1, \Lambda_2, \ldots, \Lambda_\infty$ – are constants that can take both positive ($\Lambda_i > 0$) and negative ($\Lambda_j < 0$) values.

We use the same method that Einstein used to introduce the Λ-term into Eq. (55), and write the extended vacuum equation

$$R_{ik} - \frac{1}{2} R g_{ik} + \Lambda_1 g_{ik} + \Lambda_2 g_{ik} + \Lambda_3 g_{ik} + \ldots + \Lambda_\infty g_{ik} = R_{ik} - \frac{1}{2} R g_{ik} + g_{ik} \sum_{k=1}^{\infty} \pm \Lambda_k = 0, \tag{72}$$

where according to the expression $\Lambda_k = 3/r_{aj}^2$ or $-3/r_{aj}^2$, here r_{aj} is the radius of the j-th spherical formation.

The covariant and ordinary partial derivatives of the tensor on the left side of Eq. (72) are equal to zero:

$$\nabla_j\left(R_{ik} - \frac{1}{2} R g_{ik} + g_{ik} \sum_{k=1}^{\infty} \Lambda_k\right) = \frac{\partial\left(R_{ik} - \frac{1}{2} R g_{ik} + g_{ik} \sum_{k=1}^{\infty} \Lambda_k\right)}{\partial x^j} = 0, \tag{73}$$

therefore, this equation is an expression of conservation laws, just like vacuum equations (1) and (55).

The extended equation (72) will be called Einstein's third vacuum equations.

Possible solutions to Eq. (72) are proposed in article [11].

In the following articles it is intended to show that Einstein's third vacuum equations (72) allows for the description of a set of interacting stable vacuum formations of different sizes. It is possible that this will make it possible to solve the problems formulated in this article and develop the vacuum theory of elementary particles and propose a corpuscular cosmological model.

CONCLUSION

This article is based on the author's conviction that there is nothing superfluous in mathematics, especially in Einstein's vacuum equations (1) and (55)

$$R_{ik} = 0, \tag{1′}$$

$$R_{ik} \pm \Lambda\, g_{ik} = 0. \tag{55′}$$

Almost all the fundamental epistemological principles of modern science are concentrated in this equations:

1] The principle of general covariance (i.e., the independence of the form of the equation and invariants from the choice of coordinate system or reference system; in essence, the tensor nature of the equations);
2] The principle of coordinate invariance (i.e., the independence of the laws of physics from the choice of coordinate system);
3] The principle of equivalence (i.e. local distortions, movements and accelerations are put into correspondence with local curved reference systems). The concept of "influence of force" is replaced by inertial motion in curved space-time;
4] The principle of independence of the speed of light from the reference system (i.e., the unification of space and time into a single space-time continuum with a metric of the form $ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 = 0$);
5] The principle of causality (i.e. any event can have a cause-and-effect impact only on those events that occur later than it, i.e. inside a circle with a radius of no more than $l = cdt$, where dt is the time interval between events);
6] The principle of extremum of action (i.e. the geodesic lines of a curved 4-dimensional space are extremal);
7] The principle of symmetry (i.e., the conditions of non-variability, from which conservation laws follow);
8] The principle of relativity (i.e., the equations include only relative quantities, including time).

Therefore, each solution to vacuum equations (1) and (55) is important. In addition, it is obvious that since all these solutions determine the metric-dynamic state of the same volume of space, they must be combined into a single system.

This article attempts to show that averaging (i.e., the arithmetic mean) of the metrics-solutions of vacuum equations (1) and (55) is not meaningless, and leads to metric-dynamic models of mutually opposite stable corpuscular vacuum formations of the "spherical Schwarzschild cavity" (25) and "spherical anti-Schwarzschild cavity" (34), as well as "Schwarzschild-de Sitter cell" (66) and "Schwarzschild-de Sitter anti-cell" (67).

However, as noted by mathematician David Reid, it is possible that useful information may be contained in other types of averaging of metrics-solutions of the vacuum equation (1), for example, in their: the geometric mean, or the harmonic mean, or the quadratic mean, or the cubic mean.

Averaged metrics (25), (34), (66) and (67) have clear advantages, because in them the zero component of the metric tensor is equal to one ($g_{00}^{(\pm)} = 1$). This means that in these models time t is global and there are no time singularities. In addition, within the framework of averaged models (25) and (34), the problem of weak Schwarzschild-like gravitational potential is easily solved (see §3).

However, spatial singularities in the averaged metrics (25), (34), (66) and (67) are preserved. The article proposes to associate the limitless stretching of the Einstein vacuum, near a spherical cavity with a radius $r_{b,} = r_0$ (Figure 2 and 5), with the fact that in this region the vacuum begins to "boil", i.e. strongly branches, twists, curls, etc. (see §2).

At the same time, in all "convex" and "concave" stable metric-dynamic models, which are described by averaged metrics (25), (34), (66) and (67), there is a spherical cavity from which the Einstein vacuum is displaced. Therefore, it is completely unclear what this cavity is filled with? In addition, it is not clear what is beyond the upper boundary of the Schwarzschild - de Sitter cell with radius r_a (see Figure 3)?

It is obvious that metric-dynamic models of stable vacuum formations based on averaged solutions of Einstein's vacuum equations (1) and (55) are not complete. Therefore, it is proposed to consider the possibility of using the extended Einstein third vacuum equation (72)

$$R_{ik} - \frac{1}{2} R g_{ik} + g_{ik} \sum_{k=1}^{\infty} \Lambda_k = 0.$$

The solution to this equation has already been partially proposed in [11] and will be refined in subsequent articles.

Today, the metric (42) can be verified experimentally

$$ds_{(12)}^{(+)2} \approx \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{1}{1 - \frac{r_0^2}{r^2}} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2. \tag{42'}$$

whose zero component $g_{(12)00}^{(+)} = \left(1 - \frac{r_g}{r}\right)$ coincides with the zero component of the Schwarzschild metric (16)

$$ds_{(1)}^{(+)2} = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{1}{\left(1 - \frac{r_g}{r}\right)} dr^2 - r^2 d\theta^2 - r^2 \sin^2\theta rt \, d\phi^2. \tag{16'}$$

Therefore, metric (42) is suitable for describing all known weak gravitational effects.

However, the component $g_{(12)11}^{(+)} = \left(1 - \frac{r_0^2}{r^2}\right)^{-1}$ of metric (42) and the component $g_{(1)11}^{(+)} = \left(1 - \frac{r_g}{r}\right)^{-1}$ of metric (16) are different. If we assume that planet Earth has $r_g = (r_{o1} - r_{o2})/2 \approx$ 0,9 cm and $r_0 \approx 1220$ km (see §3), then the difference between metrics (42) and (16) should be so significant, which may well be discovered experimentally.

However, observables in the case of the averaged metric (42) should be geodesic lines, which are determined by the equation (see § 1.1 in [15])

$$\frac{d^2 x^l}{ds^2} + \frac{1}{\sqrt{2}} \left(\Gamma_{ij}^{l(+)} + i\Gamma_{ij}^{l(-)}\right) \frac{dx^i}{ds} \frac{dx^j}{ds} = 0,$$

where

$\Gamma_{ij}^{l(+)} = \frac{1}{2} g^{l\mu(+)} \left(\frac{\partial g_{\mu i}^{(+)}}{\partial x^j} + \frac{\partial g_{\mu j}^{(+)}}{\partial x^i} - \frac{\partial g_{ij}^{(+)}}{\partial x^\mu} \right)$ is Christoffel symbols corresponding to metric (38),

$\Gamma_{ij}^{l(-)} = \frac{1}{2} g^{l\mu(-)} \left(\frac{\partial g_{\mu i}^{(-)}}{\partial x^j} + \frac{\partial g_{\mu j}^{(-)}}{\partial x^i} - \frac{\partial g_{ij}^{(-)}}{\partial x^\mu} \right)$ is Christoffel symbols corresponding to metric (39).

Even if it is confirmed that the radius of the spherical cavity r_0 of our planet is about 1000 km, then this will be significant evidence of the validity of the hypothesis proposed in this article about the possibility of averaging metrics-solutions of Einstein's vacuum equations.

ACKNOWLEDGMENTS

I express my gratitude to Dr. Levi T.S. for support and assistance. Constructive comments and recommendations from mathematician David Reid significantly improved the article. D. Reed's contribution is so significant that he may well be included among the co-authors. Useful editing and a large amount of calculations related to §4 were performed by Cruz Perez Julian Arturo.

REFERENCES

[1] Landau L.D., Lifshitz E.M. (1971). The Classical Theory of Fields / Course of theoretical phys-ics, V. 2 Translated from the Russian by Hamermesh M. University of Minnesota – Pergamon Press Ltd. Oxford, New York, Toronto, Sydney, Braunschwei, p. 387.

[2] Stephani, H.; Kramer, D.; MacCallum, M.; Hoenselaers, C.; Herlt, E. (2003). Exact Solutions of Einstein's Field Equations. Cambridge University Press. ISBN 0-521-46136-7.

[3] Belinski, V.; Verdaguer, E. (2001). Gravitational solitons. Cambridge University Press. ISBN 0-521-80586-4.

[4] Wheeler, J.A.; Misner, C.; Thorne, K.S. (1973). Gravitation. W.H. Freeman & Co. ISBN 0-7167-0344-0.

[5] Brown, Harvey (2005). Physical Relativity. Oxford University Press. p. 164. ISBN 978-0-19-927583-0.

[6] Malcolm, A.H. MacCallum (2013). "Exact Solutions of Einstein's equations". Scholarpedia. **8** (12): 8584. doi:10.4249/scholarpedia.8584.

[7] Bičák, J. (2000). "Selected Solutions of Einstein's Field Equations: Their Role in General Relativity and Astrophysics". Einstein's Field Equations and Their Physical Implications. Lecture Notes in Physics. Vol. 540. pp. 1–126. arXiv:gr-qc/0004016. doi:10.1007/3-540-46580-4_1. ISBN 978-3-540-67073-5. S2CID 119449917. An excellent modern survey.

[8] Bonnor, W. B. (1992). "Physical interpretation of vacuum solutions of Einstein's equations. Part I. Time-independent solutions". Gen. Rel. Grav. **24** (5): 551–573. doi:10.1007/BF00760137. S2CID 122301194.

[9] Sedov, L.I. (1994). "Continuum mechanics. T.1". – Moscow: Nauka, [in Russian]. (Available in English "A course in continuum mechanics", translation from the Russian, ed. by J. R. M. Radok).

[10] Kottler, F. (1918) Uber die physikalischen Grundlagen der Einsteinschen Gravitationstheorie// Annalen der Physik, Vol. 56, pp. 401-462. doi:10.1002/andp.19183611402.

[11] Batanov-Gaukhman, M. (2023) "Multilayer and Multilevel Cosmological Models Based on Solutions of the Extended Einstein Field Equations" Doi: 10.20944/preprints202302.0353.v1, https://www.preprints.org/manuscript/202302.0353/v1

[12] Batanov-Gaukhman, M. (2023). Geometrized Vacuum Physics. Part I. Algebra of Stignatures. Avances en Ciencias e Ingeniería, 14 (1), 1-26, https://www.executivebs.org/publishing.cl/avances-en-ciencias-e-ingenieria-vol-14-nro-1-ano-2023-articulo-1/ ; and Preprints, 2023060765. https://doi.org/10.20944/preprints202306.0765.v3.

[13] Batanov-Gaukhman, M. (2023). Geometrized Vacuum Physics. Part II. Algebra of Signatures. Avances en Ciencias e Ingeniería, 14 (1), 27-55, https://www.executivebs.org/publishing.cl/avances-en-ciencias-e-ingenieria-vol-14-nro-1-ano-2023-articulo-2/: and Preprints, 2023070716, https://doi.org/10.20944/preprints202307.0716.v1.

[14] Batanov-Gaukhman, M. (2023). Geometrized Vacuum Physics. Part III. Curved Vacuum Area. Avances en Ciencias e Ingeniería Vol. 14 nro 2 año 2023 Articulo 5, https://www.executivebs.org/publishing.cl/avances-en-ciencias-e-ingenieria-vol-14-nro-2-ano-2023-articulo-5/; and Preprints 2023, 2023080570. https://doi.org/10.20944/preprints202308.0570.v4.

[15] Batanov-Gaukhman, M., (2023). Geometrized Vacuum Physics. Part IV: Dynamics of Vacuum Layers. Preprints 2023, 2023, 101244. https://doi.org/10.20944/preprints202310.1244.v3.

Appendix 1

A.1 Calculation of Christoffel symbols in the case of an inverted metric

Consider the inverted original metric (43)

$ds_*^{(-)2} = e^{-\nu}c^2dt^2 - e^{-\lambda}dr^2 - r^2(d\theta^2 + sin^2\theta\, d\varphi^2)$ with signature $(+ - - -)$, (43′)

to this metric the non-zero components of the metric tensor are equal to

$g_{00} = e^{-\nu}$, $g_{11} = - e^{-\lambda}$, $g_{22} = - r^2$, $g_{33} = - r^2 sin^2\theta$,

and their contravariant components are equal

$g^{00} = \mathrm{e}^{\nu}$, $g^{11} = - e^{\lambda}$, $g^{22} = - r^{-2}$, $g^{33} = - r^{-2} sin^{-2}\theta$.

Let's calculate non-zero Christoffel symbols (3)

$$\Gamma^p_{ik} = \frac{1}{2} g^{p\mu}\left(\frac{\partial g_{\mu k}}{\partial x^i} + \frac{\partial g_{i\mu}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^\mu}\right) \quad \text{(A.1.1)}$$

$$\Gamma^p_{ik} = \frac{1}{2} g^{p\mu}\left(\frac{\partial g_{\mu k}}{\partial x^i} + \frac{\partial g_{i\mu}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^\mu}\right)$$
$$\Gamma^1_{11} = \frac{1}{2} g^{10}(\dots) + \frac{1}{2} g^{11}(\dots) + \frac{1}{2} g^{12}(\dots) + \frac{1}{2} g^{13}(\dots)$$
$$\Gamma^1_{11} = \frac{1}{2} g^{11}(\dots) = \frac{1}{2} g^{11}\left(\frac{\partial g_{11}}{\partial x^1} + \frac{\partial g_{11}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^1}\right)$$
$$\Gamma^1_{11} = \frac{1}{2} g^{11}\left(\frac{\partial g_{11}}{\partial x^1}\right) = -\frac{e^\lambda}{2}\left(\frac{\partial - e^\lambda}{\partial r}\right) = -\frac{\lambda'}{2}$$
$$\Gamma^1_{11} = -\frac{\lambda'}{2}.$$

$$\Gamma^p_{ik} = \frac{1}{2} g^{p\mu}\left(\frac{\partial g_{\mu k}}{\partial x^i} + \frac{\partial g_{i\mu}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^\mu}\right)$$
$$\Gamma^0_{10} = \frac{1}{2} g^{00}(\dots) + \frac{1}{2} g^{01}(\dots) + \frac{1}{2} g^{02}(\dots) + \frac{1}{2} g^{03}(\dots)$$
$$\Gamma^0_{10} = \frac{1}{2} g^{00}(\dots) = \frac{1}{2} g^{00}\left(\frac{\partial g_{00}}{\partial x^1} + \frac{\partial g_{10}}{\partial x^0} - \frac{\partial g_{10}}{\partial x^0}\right)$$
$$\Gamma^0_{10} = \frac{1}{2} g^{00}\left(\frac{\partial g_{00}}{\partial x^1}\right) = \frac{e^\nu}{2}\left(\frac{\partial e^{-\nu}}{\partial r}\right) = \frac{e^\nu e^{-\nu}}{2}\left(\frac{\partial - \nu}{\partial r}\right) = -\frac{\nu'}{2}$$
$$\Gamma^0_{10} = -\frac{\nu'}{2}.$$

$$\Gamma^p_{ik} = \frac{1}{2} g^{p\mu}\left(\frac{\partial g_{\mu k}}{\partial x^i} + \frac{\partial g_{i\mu}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^\mu}\right)$$
$$\Gamma^2_{33} = \frac{1}{2} g^{20}(\dots) + \frac{1}{2} g^{21}(\dots) + \frac{1}{2} g^{22}(\dots) + \frac{1}{2} g^{23}(\dots)$$
$$\Gamma^2_{33} = \frac{1}{2} g^{22}(\dots) = \frac{1}{2} g^{22}\left(\frac{\partial g_{23}}{\partial x^3} + \frac{\partial g_{32}}{\partial x^3} - \frac{\partial g_{33}}{\partial x^2}\right)$$
$$\Gamma^2_{33} = \frac{1}{2} g^{22}\left(-\frac{\partial g_{33}}{\partial x^2}\right) = \frac{1}{2} g^{22}\left(-\frac{\partial g_{33}}{\partial \theta}\right) = \frac{-r^{-2}}{2}\left(-\frac{\partial - r^2 sin^2\theta}{\partial \theta}\right) = -sin\theta\, cos\theta$$
$$\Gamma^2_{33} = -sin\theta\, cos\theta.$$

$$\Gamma^p_{ik} = \frac{1}{2} g^{p\mu}\left(\frac{\partial g_{\mu k}}{\partial x^i} + \frac{\partial g_{i\mu}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^\mu}\right)$$
$$\Gamma^0_{11} = \frac{1}{2} g^{00}(\dots) + \frac{1}{2} g^{01}(\dots) + \frac{1}{2} g^{02}(\dots) + \frac{1}{2} g^{03}(\dots)$$
$$\Gamma^0_{11} = \frac{1}{2} g^{00}(\dots) = \frac{1}{2} g^{00}\left(\frac{\partial g_{01}}{\partial x^1} + \frac{\partial g_{10}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^0}\right)$$
$$\Gamma^0_{11} = \frac{1}{2} g^{00}\left(-\frac{\partial g_{11}}{\partial x^0}\right) = \frac{e^\nu}{2}\left(-\frac{\partial - e^{-\lambda}}{\partial ct}\right) = \frac{e^{\nu-\lambda}}{2}\left(\frac{\partial - \lambda}{\partial ct}\right) = -\frac{\dot{\lambda}}{2} e^{\nu-\lambda}$$
$$\Gamma^0_{11} = -\frac{\dot{\lambda}}{2} e^{\nu-\lambda}.$$

$$\Gamma^{p}_{ik} = \frac{1}{2} g^{p\mu}\left(\frac{\partial g_{\mu k}}{\partial x^i} + \frac{\partial g_{i\mu}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^\mu}\right)$$
$$\Gamma^1_{22} = \frac{1}{2} g^{10}(\dots) + \frac{1}{2} g^{11}(\dots) + \frac{1}{2} g^{12}(\dots) + \frac{1}{2} g^{13}(\dots)$$
$$\Gamma^1_{22} = \frac{1}{2} g^{11}(\dots) = \frac{1}{2} g^{11}\left(\frac{\partial g_{12}}{\partial x^2} + \frac{\partial g_{21}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^1}\right)$$
$$\Gamma^1_{22} = \frac{1}{2} g^{11}\left(-\frac{\partial g_{22}}{\partial x^1}\right) = \frac{-e^{\lambda}}{2}\left(-\frac{\partial -r^2}{\partial r}\right) = -r\, e^{\lambda}$$
$$\Gamma^1_{22} = -r\, e^{\lambda}.$$

$$\Gamma^{p}_{ik} = \frac{1}{2} g^{p\mu}\left(\frac{\partial g_{\mu k}}{\partial x^i} + \frac{\partial g_{i\mu}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^\mu}\right)$$
$$\Gamma^1_{00} = \frac{1}{2} g^{10}(\dots) + \frac{1}{2} g^{11}(\dots) + \frac{1}{2} g^{12}(\dots) + \frac{1}{2} g^{13}(\dots)$$
$$\Gamma^1_{00} = \frac{1}{2} g^{11}(\dots) = \frac{1}{2} g^{11}\left(\frac{\partial g_{10}}{\partial x^0} + \frac{\partial g_{01}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^1}\right)$$
$$\Gamma^1_{00} = \frac{1}{2} g^{11}\left(-\frac{\partial g_{00}}{\partial x^1}\right) = \frac{-e^{\lambda}}{2}\left(-\frac{\partial e^{-\nu}}{\partial r}\right) = -\frac{e^{\lambda-\nu}}{2}\left(\frac{\partial -\nu}{\partial r}\right) = -\frac{\nu'}{2}\, e^{\lambda-\nu}$$
$$\Gamma^1_{00} = -\frac{\nu'}{2}\, e^{\lambda-\nu}.$$

$$\Gamma^{p}_{ik} = \frac{1}{2} g^{p\mu}\left(\frac{\partial g_{\mu k}}{\partial x^i} + \frac{\partial g_{i\mu}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^\mu}\right)$$
$$\Gamma^2_{12} = \frac{1}{2} g^{20}(\dots) + \frac{1}{2} g^{21}(\dots) + \frac{1}{2} g^{22}(\dots) + \frac{1}{2} g^{23}(\dots)$$
$$\Gamma^2_{12} = \frac{1}{2} g^{22}(\dots) = \frac{1}{2} g^{22}\left(\frac{\partial g_{22}}{\partial x^1} + \frac{\partial g_{12}}{\partial x^2} - \frac{\partial g_{12}}{\partial x^2}\right)$$
$$\Gamma^2_{12} = \frac{1}{2} g^{22}\left(\frac{\partial g_{22}}{\partial x^1}\right) = \frac{-r^{-2}}{2}\left(\frac{\partial -r^2}{\partial r}\right) = \frac{1}{r}$$
$$\Gamma^2_{12} = \frac{1}{r}.$$

$$\Gamma^{p}_{ik} = \frac{1}{2} g^{p\mu}\left(\frac{\partial g_{\mu k}}{\partial x^i} + \frac{\partial g_{i\mu}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^\mu}\right)$$
$$\Gamma^3_{13} = \frac{1}{2} g^{30}(\dots) + \frac{1}{2} g^{31}(\dots) + \frac{1}{2} g^{32}(\dots) + \frac{1}{2} g^{33}(\dots)$$
$$\Gamma^3_{13} = \frac{1}{2} g^{33}(\dots) = \frac{1}{2} g^{33}\left(\frac{\partial g_{33}}{\partial x^1} + \frac{\partial g_{13}}{\partial x^3} - \frac{\partial g_{13}}{\partial x^3}\right)$$
$$\Gamma^3_{13} = \frac{1}{2} g^{33}\left(\frac{\partial g_{33}}{\partial x^1}\right) = \frac{-r^{-2} sin^{-2}\theta}{2}\left(-\frac{\partial -r^2 sin^2\theta}{\partial r}\right) = \frac{1}{r}$$
$$\Gamma^3_{13} = \frac{1}{r}.$$

$$\Gamma^{p}_{ik} = \frac{1}{2} g^{p\mu}\left(\frac{\partial g_{\mu k}}{\partial x^i} + \frac{\partial g_{i\mu}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^\mu}\right)$$
$$\Gamma^3_{23} = \frac{1}{2} g^{30}(\dots) + \frac{1}{2} g^{31}(\dots) + \frac{1}{2} g^{32}(\dots) + \frac{1}{2} g^{33}(\dots)$$
$$\Gamma^3_{23} = \frac{1}{2} g^{33}(\dots) = \frac{1}{2} g^{33}\left(\frac{\partial g_{33}}{\partial x^2} + \frac{\partial g_{23}}{\partial x^3} - \frac{\partial g_{23}}{\partial x^3}\right)$$
$$\Gamma^3_{23} = \frac{1}{2} g^{33}\left(\frac{\partial g_{33}}{\partial x^2}\right) = -\frac{1}{2} r^{-2} sin^{-2}\theta\left(\frac{\partial -r^2 sin^2\theta}{\partial \theta}\right) = \frac{1}{2} sin^{-2}\theta\, 2 sin\theta \cos\theta = \frac{\cos\theta}{sin\theta} = tg\theta$$
$$\Gamma^3_{23} = tg\theta.$$

$$\Gamma^{p}_{ik} = \frac{1}{2} g^{p\mu}\left(\frac{\partial g_{\mu k}}{\partial x^i} + \frac{\partial g_{i\mu}}{\partial x^k} - \frac{\partial g_{ik}}{\partial x^\mu}\right)$$
$$\Gamma^0_{00} = \frac{1}{2} g^{00}(\dots) + \frac{1}{2} g^{01}(\dots) + \frac{1}{2} g^{02}(\dots) + \frac{1}{2} g^{03}(\dots)$$
$$\Gamma^0_{00} = \frac{1}{2} g^{00}(\dots) = \frac{1}{2} g^{00}\left(\frac{\partial g_{00}}{\partial x^0} + \frac{\partial g_{00}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^0}\right)$$
$$\Gamma^0_{00} = \frac{1}{2} g^{00}\left(\frac{\partial g_{00}}{\partial x^0}\right) = \frac{1}{2} e^{\nu}\left(\frac{\partial e^{-\nu}}{\partial ct}\right) = -\frac{\dot{\nu}}{2}$$
$$\Gamma^0_{00} = -\frac{\dot{\nu}}{2}.$$

$$\Gamma^{p}_{ik} = \frac{1}{2} g^{p\mu} \left(\frac{\partial g_{\mu k}}{\partial x^{i}} + \frac{\partial g_{i\mu}}{\partial x^{k}} - \frac{\partial g_{ik}}{\partial x^{\mu}} \right)$$
$$\Gamma^{1}_{10} = \frac{1}{2} g^{10}(\dots) + \frac{1}{2} g^{11}(\dots) + \frac{1}{2} g^{12}(\dots) + \frac{1}{2} g^{13}(\dots)$$
$$\Gamma^{1}_{10} = \frac{1}{2} g^{11}(\dots) = \frac{1}{2} g^{11} \left(\frac{\partial g_{10}}{\partial x^{1}} + \frac{\partial g_{11}}{\partial x^{0}} - \frac{\partial g_{00}}{\partial x^{1}} \right)$$
$$\Gamma^{1}_{10} = \frac{1}{2} g^{11} \left(\frac{\partial g_{11}}{\partial x^{0}} \right) = \frac{1}{2} - e^{\lambda} \left(\frac{\partial - e^{-\lambda}}{\partial ct} \right) = -\frac{\dot{\lambda}}{2}$$
$$\Gamma^{1}_{10} = -\frac{\dot{\lambda}}{2}\,.$$

$$\Gamma^{p}_{ik} = \frac{1}{2} g^{p\mu} \left(\frac{\partial g_{\mu k}}{\partial x^{i}} + \frac{\partial g_{i\mu}}{\partial x^{k}} - \frac{\partial g_{ik}}{\partial x^{\mu}} \right)$$
$$\Gamma^{1}_{33} = \frac{1}{2} g^{10}(\dots) + \frac{1}{2} g^{11}(\dots) + \frac{1}{2} g^{12}(\dots) + \frac{1}{2} g^{13}(\dots)$$
$$\Gamma^{1}_{33} = \frac{1}{2} g^{11}(\dots) = \frac{1}{2} g^{11} \left(\frac{\partial g_{13}}{\partial x^{3}} + \frac{\partial g_{31}}{\partial x^{3}} - \frac{\partial g_{33}}{\partial x^{1}} \right)$$
$$\Gamma^{1}_{33} = \frac{1}{2} g^{11} \left(-\frac{\partial g_{33}}{\partial x^{1}} \right) = \frac{1}{2} - e^{\lambda} \left(-\frac{\partial - r^{2} sin^{2}\theta}{\partial r} \right) = -re^{\lambda} sin^{2}\theta$$
$$\Gamma^{1}_{33} = -re^{\lambda} sin^{2}\theta.$$

Let's collect the calculation results:

$$\Gamma^{1}_{11} = -\frac{\lambda'}{2}, \qquad \Gamma^{0}_{10} = -\frac{\nu'}{2}, \qquad \Gamma^{2}_{33} = -sin\theta cos\theta,$$
$$\Gamma^{0}_{11} = -\frac{\dot{\lambda}}{2} e^{\nu-\lambda}, \qquad \Gamma^{1}_{22} = -re^{\lambda}, \qquad \Gamma^{1}_{00} = -\frac{\nu'}{2} e^{\lambda-\nu},$$
$$\Gamma^{2}_{12} = \Gamma^{3}_{13} = \frac{1}{r}, \qquad \Gamma^{3}_{23} = ctg\theta, \qquad \Gamma^{0}_{00} = -\frac{\dot{\nu}}{2},$$
$$\Gamma^{1}_{10} = -\frac{\dot{\lambda}}{2}, \qquad \Gamma^{1}_{33} = -re^{\lambda} sin^{2}\theta. \tag{A.1.2}$$

Similar Christoffel symbols are obtained by using the metric tensor components from the original inverted metric (44) with the opposite signature

$ds_{*}^{(+)2} = -e^{-\nu}c^{2}dt^{2} + e^{-\lambda}dr^{2} + r^{2}(d\theta^{2} + sin^{2}\theta\, d\varphi^{2})$ with signature (– + + +).

Let's check with several examples that the remaining Christoffel symbols are equal to zero:

$$\Gamma^{p}_{ik} = \frac{1}{2} g^{p\mu} \left(\frac{\partial g_{\mu k}}{\partial x^{i}} + \frac{\partial g_{i\mu}}{\partial x^{k}} - \frac{\partial g_{ik}}{\partial x^{\mu}} \right)$$
$$\Gamma^{1}_{31} = \frac{1}{2} g^{10}(\dots) + \frac{1}{2} g^{11}(\dots) + \frac{1}{2} g^{12}(\dots) + \frac{1}{2} g^{13}(\dots)$$
$$\Gamma^{1}_{31} = \frac{1}{2} g^{11}(\dots) = \frac{1}{2} g^{11} \left(\frac{\partial g_{13}}{\partial x^{3}} + \frac{\partial g_{31}}{\partial x^{3}} - \frac{\partial g_{31}}{\partial x^{1}} \right) = 0$$
$$\Gamma^{1}_{31} = 0.$$

$$\Gamma^{p}_{ik} = \frac{1}{2} g^{p\mu} \left(\frac{\partial g_{\mu k}}{\partial x^{i}} + \frac{\partial g_{i\mu}}{\partial x^{k}} - \frac{\partial g_{ik}}{\partial x^{\mu}} \right)$$
$$\Gamma^{1}_{32} = \frac{1}{2} g^{10}(\dots) + \frac{1}{2} g^{11}(\dots) + \frac{1}{2} g^{12}(\dots) + \frac{1}{2} g^{13}(\dots)$$
$$\Gamma^{1}_{32} = \frac{1}{2} g^{11}(\dots) = \frac{1}{2} g^{11} \left(\frac{\partial g_{12}}{\partial x^{3}} + \frac{\partial g_{31}}{\partial x^{2}} - \frac{\partial g_{33}}{\partial x^{1}} \right) = 0$$
$$\Gamma^{1}_{32} = 0.$$

$\Gamma^{p}_{ik} = \frac{1}{2}g^{p\mu}\left(\frac{\partial g_{\mu k}}{\partial x^{i}} + \frac{\partial g_{i\mu}}{\partial x^{k}} - \frac{\partial g_{ik}}{\partial x^{\mu}}\right)$
$\Gamma^{0}_{32} = \frac{1}{2}g^{00}(\dots) + \frac{1}{2}g^{01}(\dots) + \frac{1}{2}g^{12}(\dots) + \frac{1}{2}g^{13}(\dots)$
$\Gamma^{1}_{32} = \frac{1}{2}g^{00}(\dots) = \frac{1}{2}g^{00}\left(\frac{\partial g_{02}}{\partial x^{3}} + \frac{\partial g_{30}}{\partial x^{3}} - \frac{\partial g_{32}}{\partial x^{0}}\right) = 0$
$\Gamma^{1}_{32} = 0.$

$\Gamma^{p}_{ik} = \frac{1}{2}g^{p\mu}\left(\frac{\partial g_{\mu k}}{\partial x^{i}} + \frac{\partial g_{i\mu}}{\partial x^{k}} - \frac{\partial g_{ik}}{\partial x^{\mu}}\right)$
$\Gamma^{3}_{31} = \frac{1}{2}g^{30}(\dots) + \frac{1}{2}g^{31}(\dots) + \frac{1}{2}g^{32}(\dots) + \frac{1}{2}g^{33}(\dots)$
$\Gamma^{3}_{31} = \frac{1}{2}g^{33}(\dots) = \frac{1}{2}g^{33}\left(\frac{\partial g_{31}}{\partial x^{3}} + \frac{\partial g_{33}}{\partial x^{1}} - \frac{\partial g_{31}}{\partial x^{3}}\right) = 0$
$\Gamma^{1}_{32} = 0.$

$\Gamma^{p}_{ik} = \frac{1}{2}g^{p\mu}\left(\frac{\partial g_{\mu k}}{\partial x^{i}} + \frac{\partial g_{i\mu}}{\partial x^{k}} - \frac{\partial g_{ik}}{\partial x^{\mu}}\right)$
$\Gamma^{2}_{31} = \frac{1}{2}g^{20}(\dots) + \frac{1}{2}g^{21}(\dots) + \frac{1}{2}g^{22}(\dots) + \frac{1}{2}g^{23}(\dots)$
$\Gamma^{2}_{31} = \frac{1}{2}g^{22}(\dots) = \frac{1}{2}g^{22}\left(\frac{\partial g_{21}}{\partial x^{3}} + \frac{\partial g_{32}}{\partial x^{1}} - \frac{\partial g_{31}}{\partial x^{2}}\right) = 0$
$\Gamma^{2}_{31} = 0.$

$\Gamma^{p}_{ik} = \frac{1}{2}g^{p\mu}\left(\frac{\partial g_{\mu k}}{\partial x^{i}} + \frac{\partial g_{i\mu}}{\partial x^{k}} - \frac{\partial g_{ik}}{\partial x^{\mu}}\right)$
$\Gamma^{3}_{20} = \frac{1}{2}g^{30}(\dots) + \frac{1}{2}g^{31}(\dots) + \frac{1}{2}g^{32}(\dots) + \frac{1}{2}g^{33}(\dots)$
$\Gamma^{3}_{20} = \frac{1}{2}g^{33}(\dots) = \frac{1}{2}g^{33}\left(\frac{\partial g_{30}}{\partial x^{3}} + \frac{\partial g_{23}}{\partial x^{0}} - \frac{\partial g_{20}}{\partial x^{3}}\right) = 0$
$\Gamma^{3}_{20} = 0.$

…

Appendix 2

A.2 Einstein's system of vacuum equations without the Λ-term in the case of an inverted metric

Let's write the right-hand sides of Einstein's first vacuum equation (1)

$$R_{ik} = \frac{\partial \Gamma^{l}_{ik}}{\partial x^{l}} - \frac{\partial \Gamma^{l}_{il}}{\partial x^{k}} + \Gamma^{l}_{ik}\Gamma^{m}_{lm} - \Gamma^{m}_{il}\Gamma^{l}_{mk} = 0, \quad \text{(A.2.1)}$$

for the stationary case, i.e. when all components of the inverted metric tensor (43) do not depend on time *t*.
In this case, *ν* and *λ* do not depend on time, therefore from the Christoffel symbols (A.1.2) remain

$$\Gamma^{1}_{11} = -\frac{\lambda'}{2}, \quad \Gamma^{0}_{10} = -\frac{\nu'}{2}, \quad \Gamma^{2}_{33} = -sin\theta cos\theta, \quad \Gamma^{1}_{22} = -re^{\lambda}, \quad \Gamma^{1}_{00} = -\frac{\nu'}{2}e^{\lambda-\nu}, \quad \text{(A.2.2)}$$
$$\Gamma^{2}_{12} = \Gamma^{3}_{13} = \frac{1}{r}, \quad \Gamma^{3}_{23} = ctg\theta, \quad \Gamma^{1}_{33} = -re^{\lambda}sin^{2}\theta.$$

Substituting the Christoffel symbols (A.2.2) into equations (A.2.1), as a result for the stationary case we obtain

$$R_{11} = R_{00} = \frac{\partial \Gamma^{1}_{00}}{\partial x^{1}} + \Gamma^{1}_{00}\Gamma^{1}_{11} + \Gamma^{1}_{00}\Gamma^{2}_{12} + \Gamma^{1}_{00}\Gamma^{3}_{13} - \Gamma^{0}_{01}\Gamma^{1}_{00} = 0,$$

$$R_{11} = R_{00} = \frac{\partial}{\partial x^{1}}\left(-\frac{\nu'}{2}e^{\lambda-\nu}\right) + \left(-\frac{\nu'}{2}e^{\lambda-\nu}\right)\left[-\frac{\lambda'}{2} + \frac{1}{r} + \frac{1}{r} + \frac{\nu'}{2}\right] = 0,$$

$$R_{11} = R_{00} = -\frac{\nu''}{2}e^{\lambda-\nu} - \frac{\nu'}{2}e^{\lambda-\nu}(\lambda' - \nu') + \left(-\frac{\nu'}{2}e^{\lambda-\nu}\right)\left[\nu' + \frac{2}{r}\right] = 0,$$

$$R_{11} = R_{00} = -e^{\lambda-v}\left[\frac{v''}{2} + \frac{v'}{2}(\lambda' - v') + \frac{v'^2}{2} + \frac{v'}{r}\right] = 0,$$

$$R_{11} = R_{00} = v'' - 2v'^2 + v'^2 + \frac{2v'}{r} = 0,$$

$$R_{11} = R_{00} = v'' - v'^2 + \frac{2v'}{r} = 0. \tag{A.2.3}$$

$$R_{22} = \frac{\partial\Gamma^1_{22}}{\partial x^1} - \frac{\partial\Gamma^3_{23}}{\partial x^2} + \Gamma^1_{22}\Gamma^0_{10} + \Gamma^1_{22}\Gamma^1_{11} + \Gamma^1_{22}\Gamma^2_{12} + \Gamma^1_{22}\Gamma^3_{13} - \Gamma^2_{21}\Gamma^1_{22} - \Gamma^1_{22}\Gamma^2_{12} - \Gamma^3_{23}\Gamma^3_{32} = 0,$$

$$R_{22} = \frac{\partial}{\partial x^1}\left(-re^{\lambda}\right) - \frac{\partial}{\partial x^2}(\cot\theta) + \left(-re^{\lambda}\right)\left[-\frac{v'}{2} - \frac{\lambda'}{2} + \frac{1}{r} + \frac{1}{r} - \frac{1}{r} - \frac{1}{r}\right] - \cot^2\theta = 0,$$

$$R_{22} = -e^{\lambda} - re^{\lambda}\lambda' + \csc^2\theta - \cot^2\theta = 0,$$

$$R_{22} = -e^{\lambda}(1 + r\lambda') + 1 = 0,$$

$$R_{22} = e^{\lambda}\left(\frac{\lambda'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = 0. \tag{A.2.4}$$

$$R_{33} = \frac{\partial\Gamma^1_{33}}{\partial x^1} + \frac{\partial\Gamma^2_{33}}{\partial x^2} + \Gamma^1_{33}\Gamma^0_{10} + \Gamma^1_{33}\Gamma^1_{11} + \Gamma^1_{33}\Gamma^2_{12} - \Gamma^1_{33}\Gamma^3_{13} - \Gamma^3_{32}\Gamma^2_{33} = 0,$$

$$R_{33} = \frac{\partial}{\partial x^1}\left(-r\sin^2\theta\, e^{\lambda}\right) + \frac{\partial}{\partial x^2}(-\sin\theta\cos\theta) + \left(-r\sin^2\theta\, e^{\lambda}\right)\left[-\frac{v'}{2} - \frac{\lambda'}{2} + \frac{1}{r} - \frac{1}{r}\right] - \cot\theta\,(-\sin\theta\cos\theta) = 0,$$

$$R_{33} = -\sin^2\theta\, e^{\lambda} - r\sin^2\theta\, e^{\lambda}\lambda' - \cos^2\theta + \sin^2\theta + \cos^2\theta = 0,$$

$$R_{33} = \sin^2\theta\left[e^{\lambda}(-1 - r\lambda') + 1\right] = 0,$$

$$R_{33} = e^{\lambda}(v'r - 1) + 1,$$

$$R_{33} = e^{\lambda}\left(\frac{v'}{r} - \frac{1}{r^2}\right) + \frac{1}{r^2} = 0. \tag{A.2.5}$$