Fukaya Categories, Reidemeister Moves, and the Novikov Ring

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Abstract

We construct a Novikov-type ring and generalize Dirac's δ function into a functor. By applying the generalized δ-functor, we are able to reconstruct the Fukaya category $\mathscr{B}_0$.

Preamble

Notation 0.1. *By "category," we mean A_∞-category, unless the category is otherwise specified.*

Let $\mathfrak{C}$ be a chain complex:

$$... \to H_n(X) \to ... \to H_{n+k}(Y \sim X) \to ...$$

and select a suitably large category $\mathscr{C}$ such that there is a canonical embedding $(X/\sim) \hookrightarrow \mathscr{C}$.

Example 0.1. *The natural numbers, $\mathbb{N}$, are a large category with a canonical embedding $\mathbb{N}/\sim$, where $\sim$ is the relationship $n = n' + k$ for every $n, k \in \mathbb{N}$*

Example 0.2. *Suppose X is the space $[0, 1]$ and the equivalence relationship is*

$$x \sim y \iff |x - y| < \varepsilon$$

for a fixed, small ε. Then, the quotient $X/\sim$ has only one element, which corresponds to the category of points.

Example 0.3. *If the equivalence relationship is the equality*

$$x \sim y \iff x = y$$

then $X/\sim$ is isomorphic to X itself, and the corresponding category is the category of locally constant functors.

We can arbitrarily form sufficiently small (i.e., *honestly small* subcategories of $\mathscr{C}$ (call them $\mathscr{C}|_k$) such that $H_n(X)$ is initial and H_{n+k} terminal.

Definition 0.1. *A subcategory of $\mathscr{C}$ is said to be "honestly small" if it admits only finite limits and colimits.*

Proposition 0.1. *Every (sufficiently) large category is closed under permutation of every honestly small category.*

Proof. Let G be a Lie groupoid. Then, we can form the map

$$\mathscr{C}|_k \xrightarrow{G_s \times_t G} (\mathscr{C}|k)^n$$

which is invariant under a change of the natural number n. $\square$

Theorem 0.1. *Let $C = (\mathscr{C}, \omega)$ be a Fukaya category and $C|_k$ an honestly small subcategory of C. Every permutation of $C|_k$ obtained by deformations of holomorphic discs with boundary conditions is closed under the inclusion functor*

$$C|_k \hookrightarrow C$$

Proof. Consider the inclusion functor

$$i : C|_k \longrightarrow C$$

By definition of a subcategory, i embeds the objects and morphisms of $C|_k$ into C.

Let P be a permutation of $C|_k$ induced by a deformation of holomorphic discs with boundary condition.

Since $C|_k$ is honestly small, it has finitely many objects and morphisms. Denote the objects by $X_1, ..., X_n$ and the morphisms by $\phi_1, ..., \phi_n$. Define the automorphism functor $F : C|_k \to C|_k$ as follows: For each object X_i in $C|_k$, assign the object $P(X_i) \in C|_k$ under P. For each morphism $\phi_j : X_a \to X_b$ in $C|_k$, assign it the morphism

$$P(\phi_j) : P(X_a) \to P(X_b)$$

in the permuted subcategory.

$$\begin{tikzcd} C|_k \arrow[r, "i"] \arrow[d, "F"'] & C \arrow[d, "F"] \\ P(C|_k) \arrow[r, "i"'] & P(C) \end{tikzcd}$$

We have shown that the above diagram commutes, which is enough. $\square$

Warning 0.1. *This proof relies on the assumption that deformations of holomorphic discs can be captured by automorphism functors within the context of the specific Fukaya category being considered.*

That is to say, choosing an honestly small category from a suitably large one is tantamount to choosing a suitable basis for the topology in which $\mathscr{C}|_k$ is instantiated. Our proposition allows us to take arbitrary cup products of objects in the restricted category, and map them into a suitably nice cover.

$$H_n(X) \vee ... \vee H_{n+k}(X) = \bigcup_{n,k} X = X|_k$$

As is seen, there is a lot of flexibility with regards to the construction of the cover. The question then behooves us: what exactly do we mean by a "suitably nice cover?" For a bare topological space, all we ask for is that the usual axioms of topology (i.e., distributivity of intersections and unions, etc.) are obeyed, and so long as they are, we are free to claim that the space represents the category $\mathscr{C}|_k$ is *represented* by the space $X|_k$.

However, this data is often insufficient for practical purposes. For instance, we may want to construct Čech nerves, Lefschetz pencils, etc. For this reason, we must define the notion of "suitability" of a cover on a case-by-case basis. We do so here for Fukaya categories.

Let $\mathscr{C}_{Fin} = \mathcal{F}$ be a finite partioning of a category (really, ∞-category), and construct the morphism

$$f(\mathcal{F}) : ker(\mathcal{F}^0) \to im(\mathcal{F}^{n+k})$$

which ranges over a categorically small chain complex.

In this paper, we will not explicitly define Fukaya categories, namely because this is a notoriously difficult task. The focus of the present paper will be to construct a link between Fukaya categories, and the Novikov ring, using a topological version dependent type-theory. To do this, we will build a category which mimics the Novikov ring via a Dirac δ-type functor.

Contents

1 Sums

An *n-sum* is a sum

$$\Sigma_n(k)$$

where k is a field (or possibly a ring). The parts of each k are said to be *coupled* together if $n = 2$. Actually, in a superalgebra, we have

$$k^{\pm} = k^{+} \star k^{-}$$

where $ker(k^{+}) = \Sigma_1$ and $ker(k^{-}) = \Sigma_2 \setminus \Sigma_1$.

These are repetitively applied characters, and are defined so that

$$im(k^{\pm}) = (k^{\pm})^{\dagger}$$

and so $f : ker(k^{\pm}) \to im(k^{\pm})$ has an inverse which is its own identity. We are allowed to say this (in fact we must) if we work with a groupoid of the form

$$G\ {}_s\times_t\ G \to G \times G$$

and we can in that case just replace s by f and t by $g = f^{-1}$

Each $(k^{\pm})^{\dagger}$ decomposes into a partition of unity. In a category $\mathscr{C}$, this looks like a decomposition of the condensed fiber from initial to terminal object into n^k-ary paths.

Notation 1.1. *When we are in the habit of working with n-sums, we oftentimes drop the n and just say "sum." However, other mathematicians may be more explicit.*

Definition 1.1. *When $n = 2$, the sum is known as a* binor.

1.1 Dependent Sums

If a sum has as a character a functor f, then we say that sum is dependent upon the smallest such Σ that contains f. To recognize one of these, we must be able to list the components of $\Sigma_{inf|f\in\Sigma}$, at least up to an arbitrary f, but other than that, we can actually discard all of the other information about the list. For later convenience, we will refer set

$$\Sigma_{inf|f\in\Sigma)=\dot{\Sigma}}$$

Proposition 1.1. *For every $\dot{\Sigma}$, there is a constructible dependent sum.*[1]

[1] The usage of the word "constructible" here is rather trivial. In practice, it is not always feasible to deduce whether the independent part of a sum is "noise" or not. Even more lackadaisically, there is reason to argue that *all* of a dependent sum's independent data is noisy.

This is obtained by adding noise.

Proof. We only must prove that we can extend $\dot{\Sigma}$, all we have to do is *construct* an arbitrary extension:

$$\Sigma_e = \dot{\Sigma} \bigcup (e \sim \emptyset)$$

then we can show that Σ_e depends on $\dot{\Sigma}$ because $\Sigma_e \setminus \dot{\Sigma}$ is homeomorphic to the empty set. $\square$

This is actually an easy proof, because we start here with the smallest object containing the desired functor anyway. However, proving extendibility for larger classes is difficult, if not impossible to do without induction.

Dependent sums allow us to attach extra data to points using a classifier with a pre-specified arity.

Example 1.1. *Let $\mathcal{M}$ be an exotic manifold with submanifolds of mixed types. Then, we can partion $\mathcal{M}$ into two set: $\mathcal{M}_{\mathcal{L}}$ and $\mathcal{M}_{\neg\mathcal{L}} = \mathcal{M} \setminus \mathcal{M}_{\mathcal{L}}$, into Lagrangian or non-Lagrangian outputs. Then, every point within each neighborhood of $\mathcal{M}$ has this additional data attached to it.*

Such classifications with additional data are useful in various contexts, including Fukaya categories where Lagrangians play a crucial role.

1.1.1 Urysohn's Lemma

Mechanically speaking, dependency of a functor (or a function) f is actually a great explicit property to be able to manipulate, because it means that, as long as we can construct a topology $\mathcal{T}$ for a proof relying on some properties of $G \times G$, then we can actually work smoothly with that topology.

This can be shown using Urysohn's lemma. We can exploit nested sums to achieve this goal, because they allow us to construct a sequence of increasingly finer partitions. By Urysohn's Lemma, on a compact space, such a sequence converges to a continuous function. In our case, this continuous function defines the smooth fibers we desire.

Let $\Sigma_2 = \Sigma_e \cup (\Sigma_e \setminus \dot{\Sigma})$ Then, for every $\alpha \in (\Sigma_e \setminus \dot{\Sigma})$, there is a map $\alpha \to [0, 1]$, and the infima are the independent parts of the sum. Urysohn tells us that for every two extremely disconnected points in a set, the set can be divided into n-many decompositions; since we have been working with a coupling model, n has been set to 2.

This smoothness allows us decompose each decomposition into a further n decomposition. Thus, we have n^k-many total decompositions. Technically, we have just imposed a filter on the smooth space with which we are working, but the filter becomes perfectly smooth as the height of its net approaches infinity. So we have actually found a way of transforming a discrete filter into a smooth space.

In a more rigorous way, we can say

Proposition 1.2. *For every binor,* $\mathfrak{B}$

$$\underset{\rightarrow}{lim}\ k \in \mathfrak{B} = \infty$$

This is saying, in more plain language, that if we add a lot of noise to $\mathfrak{B}$, then it will eventually contain infinite information. Thus, the following

Proposition 1.3. *For every binor,* $\mathfrak{B}$

$$\underset{i\rightarrow\infty}{lim}\,\delta_i(k) = \infty$$

which means that every binor is infinitely differentiable.

However, not every binor is integrable. For instance, take

$$\mathfrak{B}^\infty$$

we are not guarunteed a map $\mathfrak{B}^\infty \longrightarrow \mathbb{1}$, so when we have one it is a luxury, and we can write

$$\{\int \Delta = 1 \mid \Delta = [\delta_i] = \int \delta_{[i]}\}$$

2 Dirac's δ functor

We will construct now the functor δ_{Dir}, otherwise known as *Dirac's delta functor.* These are maps

$$\delta_{Dir}(S) = \begin{cases} s \rightarrow 0 \ \text{ if} \mid s \in A \\ s \rightarrow \infty \text{ if} \mid s \in \{S \setminus A\} \end{cases}$$

for every element $s \in S$ of an arbitrary set with subsets A and $\neg A$.

The purpose of this functor is to map a specific subset of a set onto a unitary object, typically a Hermitian one as well. So, $\delta_{Dir}(S)$ encodes information about how to embed each character s using a 1-cell in S or an overcategory. If S is a locally ringed space (see [1]), then transform any cell complex (for instance the chain complex in the preamble), and create an affine space using this data. The Dirac functor tells us if we are going to get a nullhomotopy if we move from one point in this space to another.

By [4], we have that every $x \in X$ on which the functor acts is sent to its zeroth homology sheaf.

Axiom 2.1. *Let A and B be two apartments of a Euclidean building. Then, if there is some point $*$ in A with parameters $\{a, b\}$, then a point $*' \in B$ is diffeomorphic if, and only if,*

$$\{a, b\}(B) = \{a, b\}(A)$$

This means that if there is an earthquake $\mathscr{E} : A \to A$, there is a diffeomorphic earthquake $\tilde{\mathscr{E}} : B \to B$ if an only if the maps on parameters are the same for A and B. For any two diffeomorphic earthquakes in a building, the Dirac δ functor at the representative point of each apartment will have an identical output. Thus,

Proposition 2.1.

$$\mathscr{E} \to \tilde{\mathscr{E}} \neq 0 \to \infty$$

$$and$$

$$\mathscr{E} \to \tilde{\mathscr{E}} \neq \infty \to 0$$

In other words, no expansion or contraction occurs when replacing an earthquake for a diffeomorphic one.

Proposition 2.2. *Two points α and β of an apartment A are isotopic if and only if they are related by an earthquake $\mathscr{E}$*

Proposition 2.3. *A space equipped with the Dirac functor is stratified.*

We make no special claim as to the type of the stratification, but essentially δ_{Dir} takes one property of our choosing, and separates objects into rooms depending on whether they possess said property. The easiest such stratification in this cases is

$$Strat_{Dir}(S) = s \to \{A, S \setminus A\} \ \ \forall s \in S$$

and is actually a binor, classifying each s based upon whether or not they inhabit the apartment A.

2.1 Fukaya categories

Let $Pen(\mathscr{S})$ be a pencil of Lefschetz hyperplanes attached to a space $\mathscr{S}$ with the Dirac stratification. Let ω be a symplectic form on $\mathscr{S}$.

Definition 2.1. *A point $* \in Pen(\mathscr{S})$ is said to be ω-convergent if*

$$\lim_{n \to \infty} f^n(*) = \omega$$

and the associated chain of functors is said to be an ω-convergent chain.

We have just constructed a category (call it $\mathscr{B}_\omega$) whose functors are all embedded into an ω-convergent chain.

Proposition 2.4.

$$\mathscr{B}_\omega = Hol(\mathscr{B}_\bullet)$$

The symplectic form controls most of the holonomy data for a Fukaya category. More precisely, the holonomy of $\mathscr{B}_\bullet$ is dependent on ω. Loosely speaking, this means that δ_{Dir} is implicated in the process of modeling planar motion.

$$\underset{\bullet}{colim}\ \delta_{Dir}(\mathscr{B}_\bullet) \overset{\sim}{\twoheadrightarrow} \mathscr{B}_\omega$$

Restated,

Proposition 2.5. *A monotonic chain consisting of a uniformly nested series of dependent sums asymptotically approximates a Fukaya category.*

Proof. Take the limit

$$\lim_{i\to\infty} \delta_i(S)$$

where each $i \to i+1$ is injective. Then, there is a surjection:

$$\pi : i+1 \overset{\supset}{\twoheadrightarrow} i$$

which extends inductively to

$$\pi_\infty : sup([i]) \overset{\supset}{\twoheadrightarrow} i \;\; \forall i$$

As we progress by raising i successively, we extend the domain of the inverse functor:

$$\lim_{n\to\infty} \pi_n^{-1} = \pi^{-\infty} \sim 0$$

so we obtain the desired quasi-ismorphism, $0 \sim \bullet$, and our category is the Fukaya category $\mathscr{B}_0$ as constructed by Paul Seidel. □

Remark 2.1. *Cataldo and Migliorini [5] may have discovered this trick sooner. However, they did not recognize Fukaya categories. Their approach involves pairing flags $\mathfrak{F}$ over the n-dimensional projective space, and mapping them to iteratively nested subsets of a subspace $Y \subset \mathbb{P}^n$*

They defined the flags to be in good position *when each Λ_{p-1} meets Λ_p transversally, where Λ_p is a flag on a Riemannian manifold with codimension p. Specifically, Λ_0 is the Novikov ring.[2] For our purposes, we restate the above by saying that, for a Fukaya category $\mathscr{B}_\bullet$, the codimension of a Lefschetz pencil in good position is exactly p.*

The Fukaya category we have just constructed is effectively the category of "sufficiently large" chain complexes. Thus, there is an isomorphism

$$\mathscr{B}_\omega \simeq \{\mathscr{C} \mid \mathscr{C} \text{ is sufficiently large}\}$$

3 Reidemeister Moves

The three Reidemester moves, are:

1. $\mathcal{R}_1$: twist
2. $\mathcal{R}_2$: overlap
3. $\mathcal{R}_3$: flip

[2] See [6]

For a more digestible (visual) representation, see [4].

Let $\mathfrak{F}_q$ be a flag of codimension q, and let

$$\mathfrak{F}_q \simeq Pen(\mathscr{B}_q)$$

Then,

Proposition 3.1. *The Reidemeister moves of $\mathfrak{F}_q$ can be written*

$$\theta Pen(\mathscr{B}_q)$$

where $\$\theta : Pen(\mathscr{B}_q) \to Pen(\mathscr{B}_q)$ is an endomorphism of a Lefschetz pencil, and q is the codimension of the typical fiber

$$fib : (\bullet \sim 0) \longrightarrow z$$

for any $z \in GS$

This is actually not so important. What matters more to us, instead, is that $\theta^q \in [\theta]$ for all q. This means that, as we transform our ring-like categories into Fukaya categories, moves on flags of codimension $< q$ equate to injections $(q_{i-k}) \hookrightarrow q_i$.

Proposition 3.2. *The Reidemeister moves of any honestly small category, $\mathscr{C}|_k$ are contained in the overcategory $\mathscr{C}$.*

Proof. For every $\mathscr{C}|_k$, we have a map

$$res_{\mathscr{C},k} : \mathscr{C} \longrightarrow \mathscr{C}|_k$$

We can construct a homeomorphic map

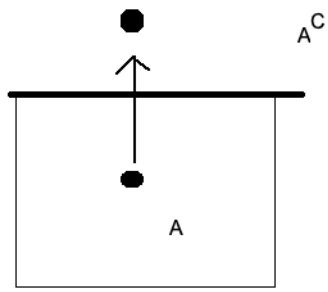
$$res_{\mathscr{C}|_{q_i},k} : \mathscr{C}_{q_i} \longrightarrow \mathscr{C}_{q_i}|_{q_{i-k}}$$

Since the kernel of a Reidemeister move is the image of the above map, we can construct an inverse map $(res_{\mathscr{C}|_{q_i}),k})^{-1}$ which is the inclusion of a flag acted upon by any of the listed Reidemeister moves into the ambient space. □

Example 3.1. *Consider a Reidemeister move $\mathcal{R}_3(A)$ for A an apartment. Suppose that a point in the domain of A (modelled by a rectangle) is reflected outside of A. Then, there is an injection*

$$A \hookrightarrow A \cup A^C$$

where A^C is the complement of A.



A Fukaya categories

The recommended reference on Fukaya categories is [7]. There, we are given some great examples of different algebraic objects, i.e., of Lagrangian submanifolds, along with some very useful visualizations.

We will work with the sphere spectrum of a point, so that any action of a Fukaya category is an $\mathbb{S}$-graded action of a group G. Therefore, we have an isomorphism

$$G \times \mathbb{S} \simeq \dot{g} \in Fuk_{Wr}$$

where Fuk_{Wr} is a wrapped Fukaya category. By the double boundary theorem, we have $\dddot{g} = \emptyset$, so our 2-form is in fact closed, and is indeed a symplectic form. The left G-action lifts the monodromy points of our manifold to the Lefschetz-Picard group $\tilde{\mathcal{G}}$.

Proposition A.1. *Let (A, ω) be a Fukaya category and X, Y be objects in A. If $f : X \to Y$ is a morphism in A and β is a binor in the Fukaya category, then there exists a unique morphism $f \circ \beta : X \to Y$ in A such that:*

1. *For any other morphism $g : X \to Y$ satisfying $g \circ \beta = f$, we have $g = f \circ \beta$. (Uniqueness)*
2. *The composition operation satisfies the following associativity property: For any other morphisms $h : W \to X$ and α a binor,*

$$(f \circ \beta) \circ (\alpha \circ h) = (f \circ (\beta \circ \alpha)) \circ h$$

Proof. We have

$$e \circ (f \circ \Sigma_2 : \Sigma_1 \to \Sigma_2) = (e \circ f) \circ \Sigma_2 : \Sigma_1 \to \Sigma_2$$

with $e \cdot k = Id_k$ for some desired operation. Further,

$$F(\Sigma_n) = ((\Sigma_1 \times \Sigma_2) \times \Sigma_3) \times ... \times \Sigma_n \to \Sigma_n$$

which shows associativity and uniqueness of $f \circ \beta$.

□

Proposition A.2. *The category $\mathscr{B}_{\geq 0}$ of small Fukaya categories whose associated Dirac operators are always real forms a pre-Abelian category.*

Proof. Define direct sum in $\mathscr{B}_{\geq 0}$ as follows:

$$A \oplus B = ([a] \in A) \times ([b] \in B) \ \ s.t. \ \{a, b\} \in \mathscr{B}_{\geq 0}$$

Note that we can compose morphisms by $(f \oplus g)(s) = f(s) \oplus g(s)$.

$$\begin{tikzcd} \mathscr{B}_{\geq 0} \arrow[r, "res_1"] \arrow[d, "res_0"'] & B \arrow[d, "g(B)"] \\ A \arrow[r, "f(A)"'] & \mathscr{B}_{\geq 0} \end{tikzcd}$$

As we can see, $g(B) \circ res_1 = f(A) \circ res_0$. We also have $g(B) \circ f(A) = f(A) \circ g(B)$. This shows that addition of objects is associative and commutative, and carries up to functoriality. This also makes $\mathscr{B}_{\geq 0}$ into a zero-object, since it is both the push-out and pullback.

Further, since this object includes the Novikov ring, which is unital, every object a has identity $id_a = e \circ a = a \circ e$. Thus, we have

$$ker(\{A, B\}) = e \circ \mathscr{B}_{\geq 0} = id_{\mathscr{B}_{\geq 0}} = im(\{A, B\})$$

□

B Spectra

Let $\mathscr{C}$ be a category. Using Urysohn's lemma, we construct a topological category (called the spectral category):

$$\Psi(\mathscr{C}) = \lim_{n} \ \mathscr{C}^n \setminus \mathscr{C}^{n-1}$$

Then, for any chain: $... \to n-1 \to n \to ...$, we can extend the chain by inserting a morphism in between $n-1$ and n, and write: $n-1 \to ... \to n$. However, we can only do this so long as representing the morphisms as a path leaves the endpoints fixed.

Let $f = (f_0, f_1, ..., f_n) : X \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \to ... \xrightarrow{f_{n-1}} X_n \xrightarrow{f_n} Y$ where X_i are objects in $\mathscr{C}$ for all $0 \leq i \leq n \in \mathbb{N}$. Composition of morphisms is defined by concatenation of sequences: $f = (f_0, ..., f_n) : X \to Y$, $g = (g_0, ..., g_m) : Y \to Z$ and their composition is $g \circ f = (g_0, ..., g_m) \circ (f_0, ..., f_n) = (f_0, ..., f_n, g_0, ..., g_m) = X \to Z$

B.1 Knotted Spectra

We can use the machinery of spectral categories to model Reidemeister moves. For instance, take the diagram

$$\begin{tikzcd}
\dots \arrow[r] & c_{n-1} \arrow[r] \arrow[d, leftrightarrow] & c_n \arrow[r] \arrow[d, leftrightarrow] & c_{n+1} \arrow[r] \arrow[d, leftrightarrow] & \dots \\
\dots \arrow[r] & c^*_{n-1} \arrow[r] & c^*_n \arrow[r] & c^*_{n+1} \arrow[r] & \dots
\end{tikzcd}$$

consisting of objects $c \in \mathscr{C}$ and $c^* \in \Psi(\mathscr{C})$. We obtain the following triangulations:

$$\begin{tikzcd}
\dots \arrow[r] & c_{n-1} \arrow[r] & c_n \arrow[r] \arrow[dr] & c_{n+1} \arrow[r] & \dots \\
\dots \arrow[r] & c^*_{n-1} \arrow[r] \arrow[ur] & c^*_n \arrow[r] & c^*_{n+1} \arrow[r] & \dots
\end{tikzcd}$$

and

$$\begin{tikzcd}
\dots \arrow[r] & c_{n-1} \arrow[r] \arrow[dr] & c_n \arrow[r] & c_{n+1} \arrow[r] & \dots \\
\dots \arrow[r] & c^*_{n-1} \arrow[r] & c^*_n \arrow[r] \arrow[ur] & c^*_{n+1} \arrow[r] & \dots
\end{tikzcd}$$

We can overlay these triangulations to obtain a “knot:”

$$\begin{tikzcd}
\dots \arrow[r] & c_{n-1} \arrow[r] \arrow[dr] & c_n \arrow[r] \arrow[dr] & c_{n+1} \arrow[r] & \dots \\
\dots \arrow[r] & c^*_{n-1} \arrow[r] \arrow[ur] & c^*_n \arrow[r] \arrow[ur] & c^*_{n+1} \arrow[r] & \dots
\end{tikzcd}$$

Write c_n^+ for c_n and c_n^- for c_n^*. Then, we have:

$$\forall c_n^\pm \;\; suc(c_n^\pm) = c_{n+1}^\pm$$

Notice, however, that our c_n^- components were all selected from the spectral category, which was constructed using Urysohn's lemma. Thus, the maps $c_n^- \to c_n^\pm$ are always smooth. However, since we are taking our n values in the natural numbers, we have already discretized our space of functors. Thus, the Reidemeister move $\mathcal{R}_1(\Psi(\mathscr{C}))$ is a rough analogy of a discrete action network overlayed atop a smooth spacetime.

The Reidemeister move $\mathcal{R}_2(\Psi(\mathscr{C}))$ represents an unsplitting of the triangulation into a linear sequence, and $\mathcal{R}_3(\Psi(\mathscr{C}))$ would involve the construction of a new dual chain of the same parity as the starting chain. In a 2-chain setup, this would look like a simple qubit swap. Topologically, it is actually a reflection of a mapping cone into its dual space.

B.2 Other applications

Underneath the hood, the machinery of spectral categories is actually quite simple. You take a string (or more generally, a brane) with fixed dimensions and endpoints, and deform it continuously while preserving all its properties. This is more than just an esoteric mathematical construct. I envisage this line of thinking could lead insight into population dynamics (or other dynamical systems) as well as identity philosophy. For instance, take the following system:

$$\mathfrak{S} = \Psi(\mathscr{C}) \cup \delta_i(x)$$

where x is a controlled quantity, $i \in I$ an index, and δ a sufficiently small member of the indexed set.

This equation relates the overall state of the system with its continuous deformations of parts, as well as controllable states. We are thus impelled to think of the implications for control at the personal level: how do small, but subtle, changes in our environment influence our decision-making at the higher level?

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