Proof of invariance of $\mathrm{d}s^2$ from the constancy of the speed of light

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Abstract

In this short note, two elementary proofs of invariance of distance element $\mathrm{d}s^2$ from the speed of light are given. The proofs should be accessible even to school-going students.

1 Introduction

In special relativity, from the constancy of the speed of light, if $\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$ is zero in one frame, in another frame, $\mathrm{d}s'^2 = c^2\mathrm{d}t'^2 - \mathrm{d}x'^2 - \mathrm{d}y'^2 - \mathrm{d}z'^2$ should also be zero. Landau and Lifshitz [2][page 4] say they must be proportional to each other. Einstein[1] says, "This quantity might be transformed with a factor. This depends upon the fact that the right-hand side of (29) might be multiplied by a factor λ, independent of v." He also observes that "this condition is satisfied only by linear transformations".

If the coordinate axes are so chosen that one frame moves with respect to the other only in the x-direction, then the problem becomes a 2-dimension problem. This note shows that if $\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}x^2$ is zero in one frame if and only if in another frame, $\mathrm{d}s'^2 = c^2\mathrm{d}t'^2 - \mathrm{d}x'^2$ is also zero, and if we are allowed only linear transformations, then $\mathrm{d}s^2$ and $\mathrm{d}s'^2$ are proportional to each other. Two proofs of this result are given. These proofs are "elementary" and should be accessible to school students.

Several proofs are available on net[3, 4, 5], but these are not elementary.

2 First Proof

If in one frame a beam of light travels distance Δx in time Δt, then $c = \Delta x/\Delta t$, as the speed of light (not velocity) of light is constant, $\Delta s^2 = \Delta x^2 - c^2\Delta t^2 = 0$.

As the speed of light is constant, the same beam of light may travel $\Delta x'$ distance in $\Delta t'$ time when

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observed from another frame. Thus, again $\Delta s'^2 = \Delta x'^2 - c^2\Delta t'^2 = 0$.

Thus, $\Delta s^2 = 0$ implies (and is implied by) $\Delta s'^2 = 0$.

Assume that (the simplest possible) linear transforms between primed and non-primed frames:[1] $\Delta x' = \alpha\Delta x + c\beta\Delta t$ and $c\Delta t' = \gamma\Delta x + c\delta\Delta t$

Let us try to determine how Δs^2 changes from a primed to a non-primed frame. We will (only) use the fact that when it is zero in one frame, it is also zero in the other.

When $\Delta s^2 = 0$ then $\Delta x^2 = c^2\Delta t^2$. Then $\Delta x = \pm c\Delta t$.

First, we choose $\Delta x = c\Delta t$. For this choice, $\Delta s^2 = 0$, hence $\Delta s'^2$ must also be zero, or $c^2\Delta t'^2 = \Delta x'^2$, thus

$$\begin{aligned}
(\gamma\Delta x + c\delta\Delta t)^2 &= (\alpha\Delta x + c\beta\Delta t)^2 \text{ or} \\
(\gamma c\Delta t + c\delta\Delta t)^2 &= (\alpha c\Delta t + c\beta\Delta t)^2 \text{ or} \\
(\gamma + \delta)^2 &= (\alpha + \beta)^2 \text{ or} \\
\gamma^2 + \delta^2 + 2\gamma\delta &= \alpha^2 + \beta^2 + 2\alpha\beta
\end{aligned}$$

This is the first condition, which $\alpha, \beta, \gamma, \delta$ must satisfy.

Next, we choose $\Delta x = -c\Delta t$. For this choice, $\Delta s^2 = 0$, hence $\Delta s'^2$ must also be zero, or $c^2\Delta t'^2 = \Delta x'^2$, thus

$$\begin{aligned}
(\gamma\Delta x + c\delta\Delta t)^2 &= (\alpha\Delta x + c\beta\Delta t)^2 \text{ or} \\
(-\gamma c\Delta t + c\delta\Delta t)^2 &= (alphac\Delta t + c\beta\Delta t)^2 \text{ or} \\
(-\gamma + \delta)^2 &= (-\alpha + \beta)^2 \text{ or} \\
\gamma^2 + \delta^2 - 2\gamma\delta &= \alpha^2 + \beta^2 - 2\alpha\beta
\end{aligned}$$

This is the second condition, which $\alpha, \beta, \gamma, \delta$ must satisfy.

Adding the two conditions, we find $\gamma^2 + \delta^2 = \alpha^2 + \beta^2$ and subtracting one from the other gives $\gamma\delta = \alpha\beta$.

Now,

$$\begin{aligned}
\Delta s'^2 &= \Delta x'^2 - c^2\Delta t'^2 \\
&= (\alpha\Delta x + c\beta\Delta t)^2 - (\gamma\Delta x + c\delta\Delta t)^2 \\
&= \Delta x^2(\alpha^2 - \gamma^2) - c^2\Delta t^2(\delta^2 - \beta^2) + 2c\Delta x\Delta t(\cancel{\alpha\beta} - \cancel{\gamma\delta}) \\
&= \Delta x^2(\alpha^2 - \gamma^2) - c^2\Delta t^2(\delta^2 - \beta^2) \\
&= \Delta x^2(\alpha^2 - \gamma^2) - c^2\Delta t^2(\alpha^2 - \gamma^2) \\
&= (\alpha^2 - \gamma^2)(\Delta x^2 - c^2\Delta t^2) \\
&= (\alpha^2 - \gamma^2)\Delta s^2
\end{aligned}$$

Thus, $\Delta s'^2$ and Δs^2 differ by only a constant.

[1] Actually, from dimensional analysis, since everything is of the dimension of $[L^2]$ only second-order terms can be there, hence transformation should be linear.

3 Second Proof

Let $\Delta s^2 = \Delta x^2 - c^2 \Delta t^2$, and let (x', t') be linear combinations of x and t and let $\Delta s'^2 = \Delta x'^2 - c^2 \Delta t'^2$.

Δs^2, under linear transform becomes, a quadratic form (say) $\alpha x^2 + \beta t^2 + 2\gamma xt$.

In general, if we have two quadratic forms $Q = ax^2 + by^2 + 2cxy$ and $R = fx^2 + gy^2 + 2hxy$, then it is sufficient to show that $Q = 0 \Leftrightarrow R = 0$ imply Q is a constant times R.

$Q/y^2 = a(x/y)^2 + 2c(x/y) + b$ and $R/y^2 = f(x/y)^2 + 2h(x/y) + g$. If $Q/y^2 = 0$, the quadratic equation in (x/y) will be zero. Let the two roots be α_1 and α_2 then $Q/y^2 = \lambda((x/y) - \alpha_1)((x/y) - \alpha_2)$. Similarly let β_1 and β_2 be two roots of R/y^2 then $R/y^2 = \mu((x/y) - \beta_1)((x/y) - \beta_2)$. If for some x/y both Q and R are equal then that $x/y = \alpha_1 = \beta_1$ or $x/y = \alpha_1 = \beta_2$. And for other root $x/y = \alpha_2 = \beta_2$ or $x/y = \alpha_2 = \beta_1$ (respectively).

Thus, Q and R differ by only a multiplicative constant.

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References

[1] Albert Einstein, The Meaning of Relativity, 1922
[2] L.D.Landau and E.M.Lifshitz, The classical Theory of Fields, 3rd Ed, Pergamon Press, 1971.
[3] Quadratic Forms on a (finite dimensional real) vector space with same zero set are scalar multiples?, Maths Stack Exchange, question 3643404.
[4] Proving invariance of $\mathrm{d}s^2$ from the invariance of the speed of light, Physics Stack Exchange, question 89603.
[5] Einstein's postulates $\leftrightarrow$ Minkowski space for a Layman, Physics Stack Exchange, question 12435.