ABOUT COUNTABILITY SETS

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Abstract
Dispute Cantor's theorem about power sets for infinite sets. Proof of the equivalence of sets of natural and real numbers. Theorem about countable of all sets.

NUMBER OF CANTOR

Georg Cantor argued that there are more elements in a set of real numbers than there are in a set of natural numbers.
Diagonalization proof looks like this:
Let's assume that all real numbers from the interval [0 , 1) can be set in an infinite sequence, i. e. assign a natural number to each real number. Cantor claimed that we could construct a number that's not in this sequence. The way we do this is we take the first digit after the decimal point from first number and add one, and from the second number we take a second digit and add one again. We do the same with the next digits, and if the digit will turn out to be nine, we enter zero. Let's call it Cantor's number.
By constructing such a number, we are sure that there is no same number among those that are at the beginning of the sequence, but what about the infinite number that are at the further places? Let's assume that we want to build a Cantor's number to the n number in the sequence that differs by the n decimal digit. The combination of all n digits is 10^n ,which is a finite number. It may be that all numbers containing combinations of the first n digits are already present in our sequence at further places. For example, for the third digit:

$$1 \leftrightarrow 0{,}294732 \ldots$$
$$2 \leftrightarrow 0{,}378820 \ldots$$
$$3 \leftrightarrow 0{,}515682 \ldots$$
$$\ldots \ldots \ldots$$
$$\ldots \ldots \ldots$$

What will be the Cantor's number ? 0,38? … if

$$\ldots \ldots \ldots$$
$$11 \leftrightarrow 0{,}381 \ldots$$
$$12 \leftrightarrow 0{,}382 \ldots$$
$$13 \leftrightarrow 0{,}383 \ldots$$
$$14 \leftrightarrow 0{,}384 \ldots$$
$$15 \leftrightarrow 0{,}385 \ldots$$
$$16 \leftrightarrow 0{,}386 \ldots$$
$$17 \leftrightarrow 0{,}387 \ldots$$
$$18 \leftrightarrow 0{,}388 \ldots$$
$$19 \leftrightarrow 0{,}389 \ldots$$
$$20 \leftrightarrow 0{,}380 \ldots$$
$$\ldots \ldots \ldots$$

In that case, we would have to go to the n+1 digit, but here the situation could be repeated. The construction of a Cantor number is possible only if the assumption that the sequence contains all $\mathbb{R}$ numbers in the range [0 , 1) is false. However, if the assumption is true, such a number cannot be constructed. The statement that you can always create a real number that does not exist in a given set is true only for finite sets.

Even if we create such a number, we can change the order on the basis of Hilbert's hotel to such that this number will have an assigned number from the set $\mathbb{N}$, i.e. we create a new sequence of numbers. Then we'd have to start creating a new Cantor number. Then we create a new order of numbers and so on. This conclusion has led us to a certain paradox. It's like a snake eating its own tail. Therefore, it cannot be considered proof $|\mathbb{N}| < |\mathbb{R}|$ .We can create more Cantor numbers and add them to the series until it becomes impossible. In this way, we create a series of all real numbers from this set, which shows that this is feasible. This is one way we can rank this set. A bit in spiteful of Cantor.

CANTOR'S THEOREM for power set *

Each set has less power than the family of its subsets, i.e. power set.

Proof:
Let $f: A \to P(A)$ be any function from a given set A into its power set $P(A)$. Let's define a set B of those elements of set A that do not belong to their images in the function f :

$$B = \{x \in A : x \notin f(x)\}$$

Set B , as a subset of set A, is an element of power set A:

$$B \subseteq A \implies B \in P(A)$$

Therefore, for any element m belonging to set A , there is:

$$m \notin f(m) \implies m \notin f(m) \land m \in B \implies f(m) \neq B$$

$$m \in f(m) \implies m \in f(m) \land m \notin B \implies f(m) \neq B$$

Thus, the set B is not an image of any element of the set A in the mapping f, hence the function f cannot be a surjection (the "onto" function), and in particular cannot be a bijection. This means that the sets A and $P(A)$ are not equal $|A| \neq |P(A)|$.

*Reference: Wikipedia

The proof of the theorem seems convincing, but only apparently.
It seems that the set B denies the existence of the bijection of the set into the power set. In fact, the opposite is true, it is bijection that prevents the formation of set B. It is the function f that creates the set B and does not vice versa. If f is bijection then

$$\exists m \in A \quad f(m) = B \Longrightarrow m \notin B \wedge m \in B$$

that means a contradiction. The set B does not exist, the bijection f does not define such a set. This is due to the fact that f, by assigning an element from the set A to the subset B, breaks the condition of the subset axiom which says that the predicate defining the subset B cannot contain B. Therefore, this set cannot be created within the Zermel-Fraenkel axiom. Illustratively, when the function f assigns an element m to a subset of B_n containing only elements satisfying the predicate $m \notin f(m)$, it simultaneously "breaks" this subset by adding another element to it. Therefore, subset B cannot be created. This is a version of Russell's paradox.

Can such a bijection exist? We can show that yes.

There are subsets B_n containing some (but not all) elements from the countable containing infinitely many elements of the set A satisfying the predicate $m \notin f(m)$.

Is there the largest set of B in which for every n $B_n \subset B$? Let's assume that for this function, yes. For function f there is no image of set B. Suppose this function contains all subsets of set A except set B. We can create a new function such that:

$$g(a_n) = \begin{cases} B, & n = 1 \\ f(a_{n-1}), & n > 1 \end{cases}$$

Such a function is a bijection of sets A and $P(A)$. For the function g there are different sets of B_n but for each there is an element from the set A for which B_n is an image. That is, there is no largest set of B_n for the function g. The predicate $m \notin f(m)$ determines a class of sets.

$$\forall i \exists j \;\; B_i \subset B_j$$

The series of sets of B_n determined by bijections is infinite and divergent, and there is no sum.

For finite sets it is impossible to construct the function g, i.e. the bijection sets A and $P(A)$ can only exist for sets having infinitely many elements. For finite sets, Cantor's theorem is true. Set B may exist for some functions , such as non-bijection and only such.

Let's consider the power set of natural numbers. The set $P(\mathbb{N})$ has two types of sets, finite sets and infinite sets. Let's designate them as $P`(\mathbb{N})$ and $P``(\mathbb{N})$ respectively. Finite sets are a countable quantity.

$$|P`(\mathbb{N})| = |\mathbb{N}^1| + |\mathbb{N}^2| + |\mathbb{N}^3| + \cdots = \aleph_0 + \aleph_0 + \aleph_0 + \cdots = \aleph_0$$

We can notice that:

$$\forall P`` \in P``(\mathbb{N}) \;\; \exists a_{n_1}, a_{n_2}, a_{n_3}, \ldots \in \mathbb{N} \;\; P`` = \{a_{n_1}, a_{n_2}, a_{n_3}, \ldots\} \Longrightarrow$$
$$\Longrightarrow \forall k \; \forall a_{n_k} \in P`` \;\; \exists P`_{n_k} \in P`(\mathbb{N}) \quad P`_{n_k} = \{a_{n_1}, a_{n_2}, \ldots, a_{n_k}\}$$

That is, for any finite number of elements of an infinite subset, there is a finite subset containing these elements and only these. This means that infinite sets cannot be more than finite sets. Intuitive is like as there are as many single-element subset as there are elements in a given set. Otherwise, there would have to be the largest finite set. Then the infinite subset would have to consist of several finite subsets, and thus there would be more combinations it means that more infinite subsets. There is no such set because there is no greatest natural number. We can rank subset $P`(\mathbb{N})$:

$$P`(\mathbb{N}) = \{P`_1, P`_2, P`_3, \dots\}$$

Let's define a function $h: \mathbb{N} \to P``(\mathbb{N})$

$$h(n) = \{P`` \in P``(\mathbb{N}), \quad P`_n \subset P`` \wedge \ P`` \neq h(1), h(2), \dots, h(n-1)$$

It is a function that assigns to every natural number a certain infinite subset of the power set $P(\mathbb{N})$. For every natural number n there is a finite subset $P`_n$ of a predefined series and there is an infinite subset $P``$ in which $P`_n$ is contained. The second condition guarantees that it is a differential function. Since each infinite subset differs from the other infinite subsets by at least one element, there is a corresponding finite subset which differs from the other finite subset by at least one element. This means that it's a function "onto" which assigns a natural number to every infinite subset. So, it's a bijection. It proves that in a power set there is a countable number of infinite subsets. So, there is a function f:

$$f(n) = \begin{cases} P`_n \in P(\mathbb{N}), & 2n-1 \\ P``_n \in P(\mathbb{N}), & 2n \end{cases}$$

This function can be written differently:

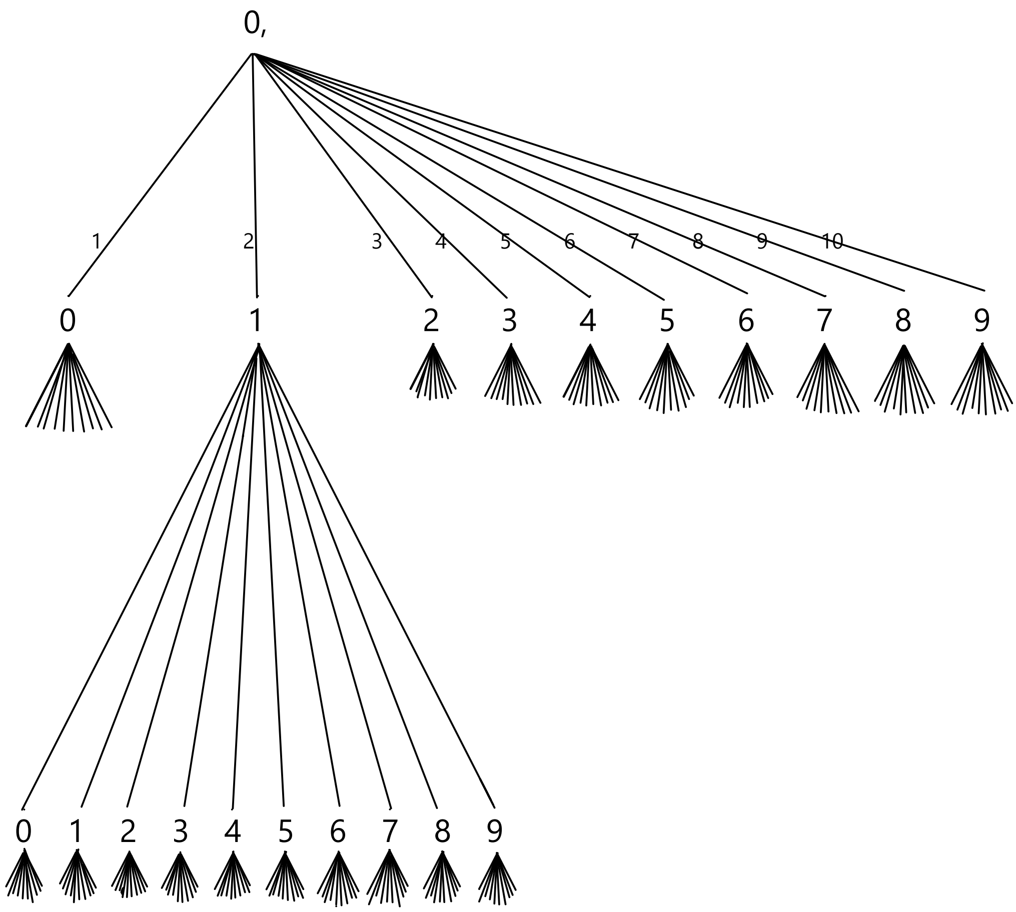
$$f(n) = \begin{cases} P`_n \in P(\mathbb{N}), & 2n-1 \\ \mathbb{N} \backslash P`_n \in P(\mathbb{N}), & 2n \end{cases}$$

In infinity $\mathbb{N} \backslash P`_n$ gives us all the infinite subsets of the set $\mathbb{N}$.

We can see an apparent paradox here. For finite sets there are only finite subsets and there are more of them than elements of the set, which is expressed by the formula $|P(A_n)| = 2^n$. On the other hand, for infinite sets, in addition to finite subsets, we have infinite subsets, and is there are as many of them as the elements of the set. This is because there is no greater quantity than infinitely many. Just as it may seem that there are more $\mathbb{Z}$ numbers than $\mathbb{N}$, but at infinity there are the same number of them.

REAL NUMBERS

Let's create a tree that represents the numbers of the range $[0, 1)$. There may be different ten digits in each place after the comma. We start with a zero to which we assign ten digits, and then to each of them another ten digits and so on.



Each of these branches of the tree is a certain real number. Two things are important to us here. One is that the tree contains all the numbers ℝ in the range $[0,1)$. The second is that each level has a finite number of branches, so each of them can be assigned a natural number. Thus, we obtained the order of all the numbers in this range. Numbers that are branches of this tree are numbered multiple times. To avoid this, we can perform a procedure. Note that a given branch whose natural number is assigned is clearly defined upwards but downwards it can be any part of the branch. Therefore, we can choose a given branch, i.e. a number $\pi - 3$ and going down do not assign its next natural number only numbering the next branch. By doing so we obtain the order of numbers from the set $[0,1)$. The set is equal to the set ℝ which was proven by Cantor. This proves that the set of real numbers is a countable set. We can therefore make a more general statement.

THEOREM

Any two sets of infinitely many elements are of the same power. Each set is countable.

$$\forall A_\infty \forall B_\infty \ \ |A| = |B| \qquad \forall A \ \ |A| \leq \aleph_0$$

THIS SOLVES THE SO-CALLED CONTINUUM HYPOTHESIS PROBLEM. There is no cardinal number continuum $\mathfrak{c}$ or anything greater than aleph zero. Quantitatively, there is nothing more than infinity. Set means that certain elements that we treat as a whole, but it does not determine where these elements are located or how they relate to each other. This means that we can “take” any of the element of the set and put it in the first place in the series. Then we can put any other element of the set in this series and so on. We don’t need any function for this, we can select elements completely chaotically. There are no mathematical rules to prevent us from doing so, so any set is countable.

We can show equinumerous of sets $\mathbb{N}$ and $\mathbb{R}$ in a different way. Let's imagine a countable number of concentric circles, each one larger than the last one. On the first, the smallest, we mark ten points and connect the center of the circles with these points by marking them with digits from 0 to 9. Each of these points is connected by successive with ten different points on the next circle. We do the same with the next circles, until infinity. In subsequent circles, the number of marked points and connecting sections increases by an order of magnitude. Thus, a kind of web is formed, and each of the paths leading from the centre of the circles determines some real number from the range $[0 , 1)$. The number of points and paths is countable and can be numbered with natural numbers. In infinity we will get all the numbers $\mathbb{R}$ from this range, and also all the points on the circle will be marked. This shows that the points on the circle are countable , and those in turn can be combined with points on the line that can be paired with real numbers.