

Causal effect vector and multiple correlation

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Abstract

In this article, we will describe the mechanism that links the notion of causality to correlations. This article answers yes to the following question: Can we deduce a causal relationship from correlations?

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1 Introduction

In this paper, we will understand from a proof how to relate the notion of causality to the correlation. For this, we will have to introduce the causal effect vector $\vec{\Delta}E$. This vector $\vec{\Delta}E$ will be related to the square root of a quadratic form containing the correlations $\sqrt{K_{X\Omega} \cdot K_{\Omega^2}^{-1} \cdot K_{\Omega X}}$.

2 Multiple correlation and causal effect vector

We will describe below the relationship relating the causal effect vector $\vec{\Delta}E$ and the correlations $\sqrt{K_{X\Omega} \cdot K_{\Omega^2}^{-1} \cdot K_{\Omega X}}$:

$$\frac{1}{2} \cdot \frac{\sqrt{\text{Var}(\vec{\Delta}E + \vec{X})}}{\sqrt{\text{Var}(\vec{X})}} = \sqrt{K_{X\Omega} \cdot K_{\Omega^2}^{-1} \cdot K_{\Omega X}}$$

Where:

1. $\text{Var}(\cdot)$ is the variance, Ω are the causes and \vec{X} is the vector on which Ω acts.
2. $\vec{\Delta}E$ is the causal effect vector of the causes Ω acting on the vector \vec{X} .
3. $\|\vec{\Delta}E\|$ is the causal effect magnitude of Ω on the vector X .
4. $\sqrt{K_{X\Omega} \cdot K_{\Omega^2}^{-1} \cdot K_{\Omega X}}$ is the multiple correlation and $0 \leq \sqrt{K_{X\Omega} \cdot K_{\Omega^2}^{-1} \cdot K_{\Omega X}} \leq 1$.

Proof:

From the conditional average $\vec{E}(\cdot|\cdot)$, we will define the following signals:

1. $\vec{E}(X|\Omega)$ is the causal signal obtained when the causes Ω **act** on the variable X .
This causal signal is produced by the intervention of causes Ω in the signal X .
2. $\vec{E}(X|\tilde{\Omega})$ is the signal obtained when the causes Ω **do not act** on the variable X .
This signal is produced without the intervention of causes Ω in the signal X .

The total signal \vec{X} is obtained by summing the two previous signals:

$$\vec{X} = \vec{E}(X|\Omega) + \vec{E}(X|\tilde{\Omega}) \quad (1)$$

Note that it is easy to show that the two signals $\vec{E}(X|\Omega)$ and $\vec{E}(X|\tilde{\Omega})$ are uncorrelated: $\text{cor}(\vec{E}(X|\Omega), \vec{E}(X|\tilde{\Omega})) = 0$, where $\text{cor}(\cdot)$ is the correlation.

We will now describe the causal effect vector $\vec{\Delta}E$:

$$\vec{\Delta}E = \vec{E}(X|\Omega) - \vec{E}(X|\tilde{\Omega}) \quad (2)$$

By combining relations (1) and (2):

$$\vec{E}(X|\Omega) = \frac{1}{2} \cdot (\vec{\Delta}E + \vec{X}) \quad (3)$$

As we know that (see appendix):

$$\frac{\sqrt{\text{Var}(\vec{E}(X|\Omega))}}{\sqrt{\text{Var}(\vec{X})}} = \sqrt{K_{X\Omega} \cdot K_{\Omega^2}^{-1} \cdot K_{\Omega X}} \quad (4)$$

We obtain by combining the relations (3) and (4):

$$\frac{1}{2} \cdot \frac{\sqrt{\text{Var}(\vec{\Delta}E + \vec{X})}}{\sqrt{\text{Var}(\vec{X})}} = \sqrt{K_{X\Omega} \cdot K_{\Omega^2}^{-1} \cdot K_{\Omega X}}$$

This relationship therefore links the causal effect vector $\vec{\Delta}E$ of the causes Ω acting on the variable X to the correlations $\sqrt{K_{X\Omega} \cdot K_{\Omega^2}^{-1} \cdot K_{\Omega X}}$.

3 Appendix

3.1 Conditional average vector and correlations

The relationship which links the conditional average vector to the correlations can be written as follows:

$$\boxed{\sqrt{K_{X,\Omega} \cdot K_{\Omega^2}^{-1} \cdot K_{\Omega,X}} = \sqrt{\frac{\text{Var}(E(X|\Omega))}{\text{Var}(X)}}$$

where $0 \leq \sqrt{K_{X,\Omega} \cdot K_{\Omega^2}^{-1} \cdot K_{\Omega,X}} \leq 1$ is the multiple correlation, $E(\cdot)$ is the conditional average and $\text{Var}(\cdot)$ is the variance.

Proof:

In what follows, we will factorize the variance Σ_{X^2} of the conditional variance $\Sigma_{X^2|\Omega}$ to show the correlations K :

$$\Sigma_{X^2|\Omega} = \Sigma_{X^2} - \Sigma_{X,\Omega} \cdot \Sigma_{\Omega^2}^{-1} \cdot \Sigma_{X,\Omega}$$

$$\Sigma_{X^2|\Omega} = \Sigma_{X^2} - \Sigma_{X,\Omega} \cdot (\text{diag}^{-1}(\Sigma_{\Omega^2}))^{\frac{1}{2}} \cdot K_{\Omega^2}^{-1} \cdot (\text{diag}^{-1}(\Sigma_{\Omega^2}))^{\frac{1}{2}} \cdot \Sigma_{\Omega,X}$$

$$\Sigma_{X^2|\Omega} = \Sigma_{X^2} - \Sigma_{X^2}^{\frac{1}{2}} \cdot K_{X,\Omega} \cdot K_{\Omega^2}^{-1} \cdot \Sigma_{X^2}^{\frac{1}{2}} \cdot K_{\Omega,X}$$

$$\Sigma_{X^2|\Omega} = \Sigma_{X^2} \cdot (1 - K_{X,\Omega} \cdot K_{\Omega^2}^{-1} \cdot K_{\Omega,X})$$

The relationship can also be written:

$$K_{X,\Omega} \cdot K_{\Omega^2}^{-1} \cdot K_{\Omega,X} = 1 - \frac{\Sigma_{X^2|\Omega}}{\Sigma_{X^2}} = 1 - \frac{\|X - E(X|\Omega)\|^2}{\|X - E(X)\|^2} = \frac{\|X - E(X)\|^2 - \|X - E(X|\Omega)\|^2}{\|X - E(X)\|^2}$$

Using the Pythagorean Theorem:

$$\|X - E(X)\|^2 = \|E(X|\Omega) - E(X)\|^2 + \|X - E(X|\Omega)\|^2$$

$$K_{X,\Omega} \cdot K_{\Omega^2}^{-1} \cdot K_{\Omega,X} = \frac{\|E(X|\Omega) - E(X)\|^2}{\|X - E(X)\|^2} = \frac{\frac{\|E(X|\Omega) - E(X)\|^2}{N}}{\frac{\|X - E(X)\|^2}{N}}$$

As we have: $E_{\Omega}(E(X|\Omega)) = \frac{1}{N} \sum_{\Omega} E(X|\Omega) = E(X)$, we obtain:

$$K_{X,\Omega} \cdot K_{\Omega^2}^{-1} \cdot K_{\Omega,X} = \frac{\text{Var}(E(X|\Omega))}{\text{Var}(X)}$$

By taking the square root we obtain the relationship.

4 Conclusion

In this paper, we have shown mathematically the steps to follow to obtain a relationship relating the notion of causality and correlation.

*[1]Optimal stastical decisions. Author: Morris H.DeGroot. Copyright 1970-2004
John Wiley and sons.*

*[2]Matrix Analysis. Author: Roger A.Horn and Charles R.Johnson. Copyright 2012,
Cambridge university press.*