A new cylindrical warp drive vector created using the methodology developed by Natario

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Abstract

The Natario warp drive appeared for the first time in 2001.([1])Natario defined a warp drive vector $nX = vs * (dx)$ where vs is the constant speed of the warp bubble and $*(dx)$ is the Hodge Star taken over the x-axis of motion in Polar Coordinates.We compute the Natario warp drive vector for variable velocities.Also we introduced a new warp drive vector $nX = vs*(dx)$ where vs is the constant speed of the warp bubble and $*(dx)$ is the Hodge Star taken over the x-axis of motion in Cylindrical Coordinates.We also compute the cylindrical warp drive vector for variable velocities.

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1 Introduction:

The Natario warp drive appeared for the first time in 2001.([1])

Natario (See pg 5 in [1]) defined a warp drive vector $nX = vs * (dx)$ where vs is the constant speed of the warp bubble and $*(dx)$ is the Hodge Star taken over the x-axis of motion in Polar Coordinates(See pg 4 in [1].(See also Appendices A and B for the detailed calculations)(see Appendix D about Polar Coordinates).The final form of the original Natario warp drive vector is given by:

$$nX = 2v_s f \cos\theta \mathbf{e}_r - v_s(2f + rf') \sin\theta \mathbf{e}_\theta \tag{1}$$

The Hodge Star actually must be taken over the product (xvs) giving the expression $nX = *(xvs) = vs * (dx) + x * (dvs)$ but due to a constant speed vs the term $x * d(vs) = 0$.Now we must examine what happens when the velocity is variable and then the term $x * d(vs)$ no longer vanishes.Remember that a real warp drive must accelerate or de-accelerate in order to be accepted as a physical valid model.The complete expression of the Hodge star that generates the Natario vector nX for a variable velocity vs is now given by(see Appendix C for detailed calculations)(The term $*(dx)$ is again taken in Polar Coordinates):

$$nX = 2f(r)rcos\theta ae_t + 2[2f(r)^2 + rf'(r)]atcos\theta e_r - 2f(r)at[2f(r) + rf'(r)] \sin\theta e_\theta \tag{2}$$

We defined a new warp drive vector $nX = vs * (dx)$ where vs is the constant speed of the warp bubble and $*(dx)$ is the Hodge Star taken over the x-axis of motion in Cylindrical Coordinates(See Appendices I and J for the detailed calculations)(see Appendix F about Cylindrical Coordinates).Our warp drive vector is given by:

$$nX = vs(t)f(r) \cos\theta e_r - vs(t) \sin\theta[f(r) + rf'(r)]e_\theta \tag{3}$$

Due to constant speed vs the term $x * d(vs) = 0$ but the Hodge Star must be taken over the product (xvs) giving the expression $nX = *(xvs) = vs * (dx) + x * (dvs)$.Now we must examine what happens when the velocity is variable and then the term $x * d(vs)$ no longer vanishes.Remember that a real warp drive must accelerate or de-accelerate in order to be accepted as a physical valid model.The complete expression of the Hodge star that generates the cylindrical warp drive vector nX for a variable velocity vs is now given by(see Appendix K for detailed calculations)(The term $*(dx)$ is again taken in Cylindrical Coordinates):

$$nX = f(r)rcos\theta ae_t + [f(r)^2 + rf'(r)]atcos\theta e_r - f(r)at[f(r) + rf'(r)] \sin\theta e_\theta \tag{4}$$

Natario used Polar Coordinates(See pg 4 in [1]) but for a real $3D$ Spherical Coordinates another warp drive vector must be calculated.This will appear in a future work.(see Appendix E for $3D$ Spherical Coordinates)

In order to fully understand the idea presented in this work(a new cylindrical warp drive vector) acquaintance with the Natario original warp drive paper is required but we provide all the mathematical demonstration QED(Quod Erad Demonstratum) in the Appendices.

2 The equation of the Natario warp drive vector with a constant speed vs

The equation of the Natario vector nX(pg 2 and 5 in [1]) is given by:

$$nX = X^{rs}drs + X^{\theta}rsd\theta \tag{5}$$

With the contravariant shift vector components X^{rs} and X^{θ} given by:(see pg 5 in [1])(see also Appendix A for details)

$$X^{rs} = 2v_s n(rs)\cos\theta \tag{6}$$

$$X^{\theta} = -v_s(2n(rs) + (rs)n'(rs))\sin\theta \tag{7}$$

Considering a valid $n(rs)$ as a Natario shape function being $n(rs) = \frac{1}{2}$ for large rs(outside the warp bubble) and $n(rs) = 0$ for small rs(inside the warp bubble) while being $0 < n(rs) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [1]):

We must demonstrate that the Natario warp drive vector given above satisfies the Natario requirements for a warp bubble defined by:

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nX = vs(t)dx$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [1])

Natario in its warp drive uses the spherical coordinates rs and θ.In order to simplify our analysis we consider motion in the $x - axis$ or the equatorial plane rs where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [1]).

In a $1 + 1$ spacetime the equatorial plane we get¿:

$$nX = X^{rs}drs \tag{8}$$

The contravariant shift vector component X^{rs} is then:

$$X^{rs} = 2v_s n(rs) \tag{9}$$

Remember that Natario(pg 4 in [1]) defines the x axis as the axis of motion.Inside the bubble $n(rs) = 0$ resulting in a $X^{rs} = 0$ and outside the bubble $n(rs) = \frac{1}{2}$ resulting in a $X^{rs} = vs$ and this illustrates the Natario definition for a warp drive spacetime.See Appendix D

3 The equation of the Natario warp drive vector with a variable speed vs due to a constant acceleration a

The equation of the Natario vector nX is given by:

$$nX = X^t dt + X^{rs} drs + X^{\theta} rs d\theta \tag{10}$$

The contravariant shift vector components X^t,X^{rs} and X^{θ} of the Natario vector are defined by(see Appendices B and C):

$$X^t = 2n(rs)rscos\theta a \tag{11}$$

$$X^{rs} = 2[2n(rs)^2 + rsn'(rs)]atcos\theta \tag{12}$$

$$X^{\theta} = -2n(rs)at[2n(rs) + rsn'(rs)]\sin\theta \tag{13}$$

Considering a valid $n(rs)$ as a Natario shape function being $n(rs) = \frac{1}{2}$ for large rs(outside the warp bubble) and $n(rs) = 0$ for small rs(inside the warp bubble) while being $0 < n(rs) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [1]):

We must demonstrate that the Natario warp drive vector given above satisfies the Natario requirements for a warp bubble defined by:

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nX = vs(t) * dx + x * dvs$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [1])

Natario in its warp drive uses the spherical coordinates rs and θ.In order to simplify our analysis we consider motion in the $x - axis$ or the equatorial plane rs where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [1]).

In a $1 + 1$ spacetime the equatorial plane we get¿:

$$nX = X^t dt + X^{rs} drs \tag{14}$$

$$X^t = 2n(rs)rsa \tag{15}$$

$$X^{rs} = 2[2n(rs)^2 + rsn'(rs)]at \tag{16}$$

The variable velocity vs due to a constant acceleration a is given by the following equation:

$$vs = 2n(rs)at \tag{17}$$

Remember that Natario(pg 4 in [1]) defines the x axis as the axis of motion.Inside the bubble $n(rs) = 0$ resulting in a $vs = 0$ and outside the bubble $n(rs) = \frac{1}{2}$ resulting in a $vs = at$ as expected from a variable velocity vs in time t due to a constant acceleration a.Since inside and outside the bubble $n(rs)$ always possesses the same values of 0 or $\frac{1}{2}$ then the derivative $n'(rs)$ of the Natario shape function $n(rs)$ is zero and the shift vector $X^{rs} = 2[2n(rs)^2]at$ with $X^{rs} = 0$ inside the bubble and $X^{rs} = 2[2n(rs)^2]at = 2[2\frac{1}{4}]at = at = vs$ outside the bubble and this illustrates the Natario definition for a warp drive spacetime.See Appendix D

4 The equation of the cylindrical warp drive vector with a constant speed vs

The equation of the cylindrical warp drive vector nX is given by:

$$nX = X^{rs}drs + X^{\theta}rsd\theta \tag{18}$$

With the contravariant shift vector components X^{rs} and X^{θ} given by:(see Appendix I for details)

$$X^{rs} = v_s n(rs)\cos\theta \tag{19}$$

$$X^{\theta} = -v_s(n(rs) + (rs)n'(rs))\sin\theta \tag{20}$$

Considering a valid $n(rs)$ as a shape function being $n(rs) = 1$ for large rs(outside the warp bubble) and $n(rs) = 0$ for small rs(inside the warp bubble) while being $0 < n(rs) < 1$ in the walls of the warp bubble the warped region:

We must demonstrate that the cylindrical warp drive vector given above satisfies the Natario requirements for a warp bubble defined by:

any cylindrical vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nX = vs(t)dx$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [1])

In order to simplify our analysis we consider motion in the $x - axis$ or the equatorial plane rs where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.

In a $1 + 1$ spacetime the equatorial plane we get¿:

$$nX = X^{rs}drs \tag{21}$$

The contravariant shift vector component X^{rs} is then:

$$X^{rs} = v_s n(rs) \tag{22}$$

Remember that Natario(pg 4 in [1]) defines the x axis as the axis of motion.Inside the bubble $n(rs) = 0$ resulting in a $X^{rs} = 0$ and outside the bubble $n(rs) = 1$ resulting in a $X^{rs} = vs$ and this illustrates the Natario definition for a warp drive spacetime.See Appendix F

5 The equation of the cylindrical warp drive vector with a variable speed vs due to a constant acceleration a

The equation of the cylindrical warp drive vector nX is given by:

$$nX = X^t dt + X^{rs} drs + X^{\theta} rs d\theta \tag{23}$$

The contravariant shift vector components X^t,X^{rs} and X^{θ} of the cylindrical warp drive vector are defined by(see Appendices J and K):

$$X^t = n(rs) rs cos\theta a \tag{24}$$

$$X^{rs} = [n(rs)^2 + rsn'(rs)] atcos\theta \tag{25}$$

$$X^{\theta} = -n(rs)at[n(rs) + rsn'(rs)] \sin\theta \tag{26}$$

Considering a valid $n(rs)$ as a shape function being $n(rs) = 1$ for large rs(outside the warp bubble) and $n(rs) = 0$ for small rs(inside the warp bubble) while being $0 < n(rs) < 1$ in the walls of the warp bubble also known as the warped region:

We must demonstrate that the cylindrical warp drive vector given above satisfies the Natario requirements for a warp bubble defined by:

any cylindrical vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nX = vs(t) * dx + x * dvs$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [1])

Natario in its warp drive uses the spherical coordinates rs and θ.In order to simplify our analysis we consider motion in the $x - axis$ or the equatorial plane rs where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.

In a $1 + 1$ spacetime the equatorial plane we get¿:

$$nX = X^t dt + X^{rs} drs \tag{27}$$

$$X^t = n(rs) rsa \tag{28}$$

$$X^{rs} = [n(rs)^2 + rsn'(rs)] at \tag{29}$$

The variable velocity vs due to a constant acceleration a is given by the following equation:

$$vs = n(rs) at \tag{30}$$

Remember that Natario(pg 4 in [1]) defines the x axis as the axis of motion.Inside the bubble $n(rs) = 0$ resulting in a $vs = 0$ and outside the bubble $n(rs) = 1$ resulting in a $vs = at$ as expected from a variable velocity vs in time t due to a constant acceleration a.Since inside and outside the bubble $n(rs)$ always possesses the same values of 0 or 1 then the derivative $n'(rs)$ of the shape function $n(rs)$ is zero and the shift vector $X^{rs} = [n(rs)^2]at$ with $X^{rs} = 0$ inside the bubble and $X^{rs} = [n(rs)^2]at = vs$ outside the bubble and this illustrates the Natario definition for a warp drive spacetime.See Appendix F

6 Conclusion

In this work we introduced a new cylindrical warp drive vector using the Natario mathematical techniques.We focused ourselves in the application of the Hodge Star in cylindrical coordinates for both constant and variable speeds.

Our focus was concentrated in the Natario methods to obtain a warp drive vector.

The application of the cylindrical warp drive vector to the ADM equation in General Relativit5y will appear in a future work.

Natario used Polar Coordinates(See pg 4 in [1]) but for a real $3D$ Spherical Coordinates another warp drive vector must be calculated.This will appear in a future work.(see Appendix E for $3D$ Spherical Coordinates)

7 Appendix A:differential forms,Hodge star and the mathematical demonstration of the Natario vectors $nX = -vsdx$ and $nX = vsdx$ for a constant speed vs in a R^3 space basis

This appendix is being written for novice or newcomer students on Warp Drive theory still not acquainted with the methods Natario used to arrive at the final expression of the Natario Vector nX

The Canonical Basis of the Hodge Star in spherical coordinates can be defined as follows(see pg 4 in [1],eq 3.72 pg 69$(a)(b)$ in [2]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2 \sin\theta (d\theta \wedge d\varphi) \tag{31}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial \theta} \sim rd\theta \sim (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta (d\varphi \wedge dr) \tag{32}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi} \sim r\sin\theta d\varphi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{33}$$

From above we get the following results

$$dr \sim r^2 \sin\theta (d\theta \wedge d\varphi) \tag{34}$$

$$rd\theta \sim r\sin\theta (d\varphi \wedge dr) \tag{35}$$

$$r\sin\theta d\varphi \sim r(dr \wedge d\theta) \tag{36}$$

Note that this expression matches the common definition of the Hodge Star operator * applied to the spherical coordinates as given by(see eq 3.72 pg 69$(a)(b)$ in [2]):

$$*dr = r^2 \sin\theta (d\theta \wedge d\varphi) \tag{37}$$

$$*rd\theta = r\sin\theta (d\varphi \wedge dr) \tag{38}$$

$$*r\sin\theta d\varphi = r(dr \wedge d\theta) \tag{39}$$

Back again to the Natario equivalence between spherical and cartezian coordinates(pg 5 in [1]):

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \sim r^2 \sin\theta\cos\theta d\theta \wedge d\varphi + r\sin^2\theta dr \wedge d\varphi = d\left(\frac{1}{2}r^2 \sin^2\theta d\varphi\right) \tag{40}$$

Look that

$$dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \tag{41}$$

Or

$$dx = d(r\cos\theta) = \cos\theta dr - \sin\theta r d\theta \tag{42}$$

Applying the Hodge Star operator * to the above expression:

$$*dx = *d(r\cos\theta) = \cos\theta(*dr) - \sin\theta(*rd\theta) \tag{43}$$

$$*dx = *d(r\cos\theta) = \cos\theta[r^2\sin\theta(d\theta\wedge d\varphi)] - \sin\theta[r\sin\theta(d\varphi\wedge dr)] \tag{44}$$

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(d\theta\wedge d\varphi)] - [r\sin^2\theta(d\varphi\wedge dr)] \tag{45}$$

We know that the following expression holds true(see eq 3.79 pg 70$(a)(b)$ in [2]):

$$d\varphi\wedge dr = -dr\wedge d\varphi \tag{46}$$

Then we have

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(d\theta\wedge d\varphi)] + [r\sin^2\theta(dr\wedge d\varphi)] \tag{47}$$

And the above expression matches exactly the term obtained by Natario using the Hodge Star operator applied to the equivalence between cartezian and spherical coordinates(pg 5 in [1]).

Now examining the expression:

$$d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{48}$$

We must also apply the Hodge Star operator to the expression above

And then we have:

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{49}$$

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \sim \frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] + \frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] \tag{50}$$

According to eq 3.90 pg 74$(a)(b)$ in [2] the term $\frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] = 0$

This leaves us with:

$$\frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] \sim \frac{1}{2}r^2 2\sin\theta\cos\theta(d\theta\wedge d\varphi) + \frac{1}{2}\sin^2\theta 2r(dr\wedge d\varphi) \tag{51}$$

$$\frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] \sim \frac{1}{2}r^2 2\sin\theta\cos\theta(d\theta\wedge d\varphi) + \frac{1}{2}\sin^2\theta 2r(dr\wedge d\varphi) \qquad (52)$$

Because and according to eqs 3.90 and 3.91 pg 74$(a)(b)$ in [2],tb 3.2 pg 68$(a)(b)$ in [2]:

$$*d(\alpha+\beta) = d\alpha + d\beta \qquad (53)$$

$$*d(f\alpha) = df\wedge\alpha + (-1)^p f\wedge d\alpha \dashrightarrow p = 2 \dashrightarrow *d(f\alpha) = df\wedge\alpha + f\wedge d\alpha \qquad (54)$$

$$*d(dx) = d(dy) = d(dz) = 0 \qquad (55)$$

From above we can see for example that

$$*d[(\sin^2\theta)d\varphi] = d(\sin^2\theta)\wedge d\varphi + \sin^2\theta\wedge dd\varphi = 2sin\theta\cos\theta(d\theta\wedge d\varphi) \qquad (56)$$

$$*[d(r^2)d\varphi] = 2rdr\wedge d\varphi + r^2\wedge dd\varphi = 2r(dr\wedge d\varphi) \qquad (57)$$

And then we derived again the Natario result of pg 5 in [1]

$$r^2\sin\theta\cos\theta(d\theta\wedge d\varphi) + r\sin^2\theta(dr\wedge d\varphi) \qquad (58)$$

Now we will examine the following expression equivalent to the one of Natario pg 5 in [1] except that we replaced $\frac{1}{2}$ by the function $f(r)$:

$$*d[f(r)r^2\sin^2\theta d\varphi] \qquad (59)$$

From above we can obtain the next expressions

$$f(r)r^2 * d[(\sin^2\theta)d\varphi] + f(r)\sin^2\theta * [d(r^2)d\varphi] + r^2\sin^2\theta * d[f(r)d\varphi] \qquad (60)$$

$$f(r)r^2 2sin\theta\cos\theta(d\theta\wedge d\varphi) + f(r)\sin^2\theta 2r(dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dr\wedge d\varphi) \qquad (61)$$

$$2f(r)r^2 sin\theta\cos\theta(d\theta\wedge d\varphi) + 2f(r)r\sin^2\theta(dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dr\wedge d\varphi) \qquad (62)$$

$$2f(r)r^2 sin\theta\cos\theta(d\theta\wedge d\varphi)+2f(r)r\sin^2\theta(dr\wedge d\varphi)+r^2\sin^2\theta f'(r)(dr\wedge d\varphi) \tag{63}$$

Comparing the above expressions with the Natario definitions of pg 4 in [1]):

$$e_r\equiv\frac{\partial}{\partial r}\sim dr\sim(rd\theta)\wedge(r\sin\theta d\varphi)\sim r^2\sin\theta(d\theta\wedge d\varphi) \tag{64}$$

$$e_\theta\equiv\frac{1}{r}\frac{\partial}{\partial\theta}\sim rd\theta\sim(r\sin\theta d\varphi)\wedge dr\sim r\sin\theta(d\varphi\wedge dr)\sim -r\sin\theta(dr\wedge d\varphi) \tag{65}$$

$$e_\varphi\equiv\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\sim r\sin\theta d\varphi\sim dr\wedge(rd\theta)\sim r(dr\wedge d\theta) \tag{66}$$

We can obtain the following result:

$$2f(r)\ cos\theta[r^2 sin\theta(d\theta\wedge d\varphi)]+2f(r)\ sin\theta[r\sin\theta(dr\wedge d\varphi)]+f'(r)r\sin\theta[r\sin\theta(dr\wedge d\varphi)] \tag{67}$$

$$2f(r)\ cos\theta e_r-2f(r)\ sin\theta e_\theta-rf'(r)\sin\theta e_\theta \tag{68}$$

$$*d[f(r)r^2\sin^2\theta d\varphi]=2f(r)\ cos\theta e_r-[2f(r)+rf'(r)]\sin\theta e_\theta \tag{69}$$

Defining the Natario Vector as in pg 5 in [1] with the Hodge Star operator * explicitly written :

$$nX=vs(t)*d\left(f(r)r^2\sin^2\theta d\varphi\right) \tag{70}$$

$$nX=-vs(t)*d\left(f(r)r^2\sin^2\theta d\varphi\right) \tag{71}$$

We can get finally the latest expressions for the Natario Vector nX also shown in pg 5 in [1]

$$nX=2vs(t)f(r)\ cos\theta e_r-vs(t)[2f(r)+rf'(r)]\sin\theta e_\theta \tag{72}$$

$$nX=-2vs(t)f(r)\ cos\theta e_r+vs(t)[2f(r)+rf'(r)]\sin\theta e_\theta \tag{73}$$

With our pedagogical approaches

$$nX=2vs(t)f(r)\ cos\theta dr-vs(t)[2f(r)+rf'(r)]r\sin\theta d\theta \tag{74}$$

$$nX=-2vs(t)f(r)\ cos\theta dr+vs(t)[2f(r)+rf'(r)]r\sin\theta d\theta \tag{75}$$

8 Appendix B:differential forms,Hodge star and the mathematical demonstration of the Natario vectors $nX = -vsdx$ and $nX = vsdx$ for a constant speed vs or for the first term $vsdx$ from the Natario vector $nX = vsdx + xdvs$(a variable speed) in a R^4 space basis

This appendix is being written for novice or newcomer students on Warp Drive theory still not acquainted with the methods Natario used to arrive at the final expression of the Natario Vector nX

The Canonical Basis of the Hodge Star in spherical coordinates can be defined as follows(see pg 4 in [1],eqs 3.135 and 3.137 pg 82$(a)(b)$ in [2],eq 3.74 pg 69$(a)(b)$ in [2])(see pg 47 eqs 2.67 to 2.70 and pg 92 in [3]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim dt \wedge (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2 \sin\theta (dt \wedge d\theta \wedge d\varphi) \tag{76}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim dt \wedge (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta (dt \wedge d\varphi \wedge dr) \tag{77}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dt \wedge dr \wedge (rd\theta) \sim r(dt \wedge dr \wedge d\theta) \tag{78}$$

From above we get the following results

$$dr \sim r^2 \sin\theta (dt \wedge d\theta \wedge d\varphi) \tag{79}$$

$$rd\theta \sim r\sin\theta (dt \wedge d\varphi \wedge dr) \tag{80}$$

$$r\sin\theta d\varphi \sim r(dt \wedge dr \wedge d\theta) \tag{81}$$

Note that this expression matches the common definition of the Hodge Star operator * applied to the spherical coordinates as given by(see eq 3.74 pg 69$(a)(b)$ in [2]):

$$*dr = r^2 \sin\theta (dt \wedge d\theta \wedge d\varphi) \tag{82}$$

$$*rd\theta = r\sin\theta (dt \wedge d\varphi \wedge dr) \tag{83}$$

$$*r\sin\theta d\varphi = r(dt \wedge dr \wedge d\theta) \tag{84}$$

Back again to the Natario equivalence between spherical and cartezian coordinates(pg 5 in [1]):

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \sim r^2 \sin\theta\cos\theta dt \wedge d\theta \wedge d\varphi + r\sin^2\theta dt \wedge dr \wedge d\varphi = d\left(\frac{1}{2}r^2 \sin^2\theta d\varphi\right) \tag{85}$$

Look that

$$dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \tag{86}$$

Or

$$dx = d(r\cos\theta) = \cos\theta dr - \sin\theta r d\theta \tag{87}$$

Applying the Hodge Star operator * to the above expression:

$$*dx = *d(r\cos\theta) = \cos\theta(*dr) - \sin\theta(*rd\theta) \tag{88}$$

$$*dx = *d(r\cos\theta) = \cos\theta[r^2\sin\theta(dt\wedge d\theta\wedge d\varphi)] - \sin\theta[r\sin\theta(dt\wedge d\varphi\wedge dr)] \tag{89}$$

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi)] - [r\sin^2\theta(dt\wedge d\varphi\wedge dr)] \tag{90}$$

We know that the following expression holds true(see eq 3.79 pg 70$(a)(b)$ in [2]):

$$d\varphi\wedge dr = -dr\wedge d\varphi \tag{91}$$

Then we have

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi)] + [r\sin^2\theta(dt\wedge dr\wedge d\varphi)] \tag{92}$$

And the above expression matches exactly the term obtained by Natario using the Hodge Star operator applied to the equivalence between cartezian and spherical coordinates(pg 5 in [1]).

Now examining the expression:

$$d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{93}$$

We must also apply the Hodge Star operator to the expression above

And then we have:

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{94}$$

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \sim \frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] + \frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] \tag{95}$$

According to eq 3.90 pg 74$(a)(b)$ in [2] the term $\frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] = 0$

This leaves us with:

$$\frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] \sim \frac{1}{2}r^2 2\sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi) + \frac{1}{2}\sin^2\theta 2r(dt\wedge dr\wedge d\varphi) \tag{96}$$

$$\frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] \sim \frac{1}{2}r^2 2\sin\theta\cos\theta(dt \wedge d\theta \wedge d\varphi) + \frac{1}{2}\sin^2\theta 2r(dt \wedge dr \wedge d\varphi) \quad (97)$$

Because and according to eqs 3.90 and 3.91 pg 74$(a)(b)$ in [2],tb 3.3 pg 68$(a)(b)$ in [2]::

$$*d(\alpha+\beta) = d\alpha + d\beta \quad (98)$$

$$*d(f\alpha) = df \wedge \alpha + (-1)^p f \wedge d\alpha \dashrightarrow p = 3 \dashrightarrow *d(f\alpha) = df \wedge \alpha - f \wedge d\alpha \quad (99)$$

$$*d(dx) = d(dy) = d(dz) = 0 \quad (100)$$

From above we can see for example that

$$*d[(\sin^2\theta)d\varphi] = dt \wedge d(\sin^2\theta) \wedge d\varphi - dt \wedge \sin^2\theta \wedge dd\varphi = 2sin\theta\cos\theta(dt \wedge d\theta \wedge d\varphi) \quad (101)$$

$$*[d(r^2)d\varphi] = 2rdt \wedge dr \wedge d\varphi - dt \wedge r^2 \wedge dd\varphi = 2r(dt \wedge dr \wedge d\varphi) \quad (102)$$

And then we derived again the Natario result of pg 5 in [1]

$$r^2\sin\theta\cos\theta(dt \wedge d\theta \wedge d\varphi) + r\sin^2\theta(dt \wedge dr \wedge d\varphi) \quad (103)$$

Now we will examine the following expression equivalent to the one of Natario pg 5 in [1] except that we replaced $\frac{1}{2}$ by the function $f(r)$:

$$*d[f(r)r^2\sin^2\theta d\varphi] \quad (104)$$

From above we can obtain the next expressions

$$f(r)r^2 * d[(\sin^2\theta)d\varphi] + f(r)\sin^2\theta * [d(r^2)d\varphi] + r^2\sin^2\theta * d[f(r)d\varphi] \quad (105)$$

$$f(r)r^2 2sin\theta\cos\theta(dt \wedge d\theta \wedge d\varphi) + f(r)\sin^2\theta 2r(dt \wedge dr \wedge d\varphi) + r^2\sin^2\theta f'(r)(dt \wedge dr \wedge d\varphi) \quad (106)$$

$$2f(r)r^2 sin\theta\cos\theta(dt \wedge d\theta \wedge d\varphi) + 2f(r)r\sin^2\theta(dt \wedge dr \wedge d\varphi) + r^2\sin^2\theta f'(r)(dt \wedge dr \wedge d\varphi) \quad (107)$$

$$2f(r)r^2 sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi)+2f(r)r\sin^2\theta(dt\wedge dr\wedge d\varphi)+r^2\sin^2\theta f'(r)(dt\wedge dr\wedge d\varphi) \tag{108}$$

Comparing the above expressions with the Natario definitions of pg 4 in [1]):

$$e_r\equiv\frac{\partial}{\partial r}\sim dr\sim dt\wedge(rd\theta)\wedge(r\sin\theta d\varphi)\sim r^2\sin\theta(dt\wedge d\theta\wedge d\varphi) \tag{109}$$

$$e_\theta\equiv\frac{1}{r}\frac{\partial}{\partial\theta}\sim rd\theta\sim dt\wedge(r\sin\theta d\varphi)\wedge dr\sim r\sin\theta(dt\wedge d\varphi\wedge dr)\sim -r\sin\theta(dt\wedge dr\wedge d\varphi) \tag{110}$$

$$e_\varphi\equiv\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\sim r\sin\theta d\varphi\sim dt\wedge dr\wedge(rd\theta)\sim r(dt\wedge dr\wedge d\theta) \tag{111}$$

We can obtain the following result:

$$2f(r)\ cos\theta[r^2 sin\theta(dt\wedge d\theta\wedge d\varphi)]+2f(r)\ sin\theta[r\sin\theta(dt\wedge dr\wedge d\varphi)]+f'(r)r\sin\theta[r\sin\theta(dt\wedge dr\wedge d\varphi)] \tag{112}$$

$$2f(r)\ cos\theta e_r-2f(r)\ sin\theta e_\theta-rf'(r)\sin\theta e_\theta \tag{113}$$

$$*d[f(r)r^2\sin^2\theta d\varphi]=2f(r)\ cos\theta e_r-[2f(r)+rf'(r)]\sin\theta e_\theta \tag{114}$$

Defining the Natario Vector as in pg 5 in [1] with the Hodge Star operator * explicitly written :

$$nX=vs(t)*d\left(f(r)r^2\sin^2\theta d\varphi\right) \tag{115}$$

$$nX=-vs(t)*d\left(f(r)r^2\sin^2\theta d\varphi\right) \tag{116}$$

We can get finally the latest expressions for the Natario Vector nX also shown in pg 5 in [1]

$$nX=2vs(t)f(r)\ cos\theta e_r-vs(t)[2f(r)+rf'(r)]\sin\theta e_\theta \tag{117}$$

$$nX=-2vs(t)f(r)\ cos\theta e_r+vs(t)[2f(r)+rf'(r)]\sin\theta e_\theta \tag{118}$$

With our pedagogical approaches

$$nX=2vs(t)f(r)\ cos\theta dr-vs(t)[2f(r)+rf'(r)]r\sin\theta d\theta \tag{119}$$

$$nX=-2vs(t)f(r)\ cos\theta dr+vs(t)[2f(r)+rf'(r)]r\sin\theta d\theta \tag{120}$$

9 Appendix C:differential forms,Hodge star and the mathematical demonstration of the Natario vector $nX = *(vsx)$ for a variable speed vs and a constant acceleration a

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of r defined by Natario as the interior of the warp bubble and $nX = vs(t)dx$ with $X = vs$ for a large value of r defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [1])

In the Appendices A and B we gave the mathematical demonstration of the Natario vector nX in the R^3 and R^4 space basis when the velocity vs is constant.Hence the complete expression of the Hodge star that generates the Natario vector nX for a constant velocity vs is given by:

$$nX = *(vsx) = vs * (dx) \tag{121}$$

$$*dx = *d(rcos\theta) = *d\left(\frac{1}{2}r^2 \sin^2\theta d\varphi\right) = *d[f(r)r^2 \sin^2\theta d\varphi] \tag{122}$$

The equation of the Natario vector nX(pg 2 and 5 in [1]) is given by:

$$nX = X^r e_r + X^\theta e_\theta \tag{123}$$

$$nX = X^r dr + X^\theta r d\theta \tag{124}$$

$$nX = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{125}$$

$$nX = 2vs(t)f(r)\ cos\theta dr - vs(t)[2f(r) + rf'(r)]r\sin\theta d\theta \tag{126}$$

With the contravariant shift vector components explicitly given by:

$$X^r = 2v_s f(r)\cos\theta \tag{127}$$

$$X^\theta = -v_s(2f(r) + (r)f'(r))\sin\theta \tag{128}$$

Because due to a constant speed vs the term $x * d(vs) = 0$.Now we must examine what happens when the velocity is variable and then the term $x * d(vs)$ no longer vanishes.Remember that a real warp drive must accelerate or de-accelerate in order to be accepted as a physical valid model.The complete expression of the Hodge star that generates the Natario vector nX for a variable velocity vs is now given by:

$$nX = *(vsx) = vs * (dx) + x * (dvs) \tag{129}$$

In order to study the term $x * d(vs)$ we must introduce a new Canonical Basis for the coordinate time in the R^4 space basis defined as follows:(see eqs 10.102 and 10.103 pgs 363$(a)(b)$ and 364$(a)(b)$ in [2] with the terms $S = u = 1$[1],eq 3.74 pg 69$(a)(b)$ in [2],eqs 11.131 and 11.133 with the term $m = 0$[2] pg 417$(a)(b)$ in [2].)(see pg 47 eqs 2.67 to 2.70 and pg 92 in [3]):

$$e_t \equiv \frac{\partial}{\partial t} \sim dt \sim dr \wedge (rd\theta) \wedge (r \sin\theta d\varphi) \sim r^2 \sin\theta(dr \wedge d\theta \wedge d\varphi) \tag{130}$$

$$dt \sim r^2 \sin\theta(dr \wedge d\theta \wedge d\varphi) \tag{131}$$

The Hodge star operator defined for the coordinate time is given by:(see eq 3.74 pg 69$(a)(b)$ in [2]):

$$*dt = r^2 \sin\theta(dr \wedge d\theta \wedge d\varphi) \tag{132}$$

The valid expression for a variable velocity $vs(t)$ in the Natario warp drive spacetime due to a constant acceleration a must be given by:

$$vs = 2f(r)at \tag{133}$$

Because and considering a valid $f(r)$ as a Natario shape function being $f(r) = \frac{1}{2}$ for large r(outside the warp bubble where $X = vs(t)$ and $nX = vs(t) * dx + x * d(vs(t)))$ and $f(r) = 0$ for small r(inside the warp bubble where $X = 0$ and $nX = 0$) while being $0 < f(r) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pgs 4 and 5 in [1]) and considering also that the Natario warp drive is a ship-frame based coordinates system(a reference frame placed in the center of the warp bubble where the ship resides-or must reside!!) then an observer in the ship inside the bubble sees every point inside the bubble at the rest with respect to him because inside the bubble $vs(t) = 0$ because $f(r) = 0$.

To illustrate the statement pointed above imagine a fish inside an aquarium and the aquarium is floating in the surface of a river but carried out by the river stream.The stream varies its velocity with time.The warp bubble in this case is the aquarium and the walls of the aquarium are the walls of the warp bubble-Natario warped region.An observer in the margin of the river would see the aquarium passing by him at a large speed considering a coordinates system(a reference frame) placed in the margin of the river but inside the aquarium the fish is at the rest with respect to his local neighborhoods.Then for the fish any point inside the aquarium is at the rest with respect to him because inside the aquarium $vs = 2f(r)at$ with $f(r) = 0$ and consequently giving a $vs(t) = 0$.Again with respect to the fish the fish "sees" the margin passing by him with a large relative velocity.The margin in this case is the region outside the bubble "seen" by the fish with a variable velocity $vs(t) = v1$ in the time $t1$ and $vs(t) = v2$ in the time $t2$ because outside the bubble the generic expression for a variable velocity vs is given by $vs = 2f(r)at$ and outside the bubble $f(r) = \frac{1}{2}$ giving a generic expression for a variable velocity vs as $vs(t) = at$ and consequently a $v1 = at1$ in the time $t1$ and a $v2 = at2$ in the time $t2$.Then the variable velocity in not only a function of time alone but must consider also the position of the bubble where the measure is being taken wether inside or outside the bubble.So the velocity must also be a function of r.Its total differential is then given by:

$$dvs = 2[atf'(r)dr + f(r)tda + f(r)adt] \tag{134}$$

[1]These terms are needed to deal with the Robertson-Walker equation in Cosmology using differential forms.We dont need these terms here and we can make $S = u = 1$

[2]This term is needed to describe the Dirac equation in the Schwarzschild spacetime we dont need the term here so we can make $m = 1$.Remember also that here we consider geometrized units in which $c = 1$

Applying the Hodge star to the total differential dvs we get:

$$*dvs = 2[atf'(r) * dr + f(r)t * da + f(r)a * dt] \quad (135)$$

But we consider here the acceleration a a constant.Then the term $f(r)tda = 0$ and in consequence $f(r)t * da = 0$.This leaves us with:

$$*dvs = 2[atf'(r) * dr + f(r)a * dt] \quad (136)$$

$$*dvs = 2[atf'(r) * dr + f(r)a * dt] = 2[atf'(r)r^2 \sin\theta(dt \wedge d\theta \wedge d\varphi) + f(r)ar^2 \sin\theta(dr \wedge d\theta \wedge d\varphi)] \quad (137)$$

$$*dvs = 2[atf'(r) * dr + f(r)a * dt] = 2[atf'(r)e_r + f(r)ae_t] \quad (138)$$

The complete expression of the Hodge star that generates the Natario vector nX for a variable velocity vs is given by:

$$nX = *(vsx) = vs * (dx) + x * d(vs) \quad (139)$$

The term $*dx$ was obtained in the Appendices A and B as follows:(see pg 5 in [1])

$$*dx = 2f(r)\ cos\theta e_r - [2f(r) + rf'(r)] \sin\theta e_\theta \quad (140)$$

The complete expression of the Hodge star that generates the Natario vector nX for a variable velocity vs is now given by:

$$nX = *(vsx) = vs(2f(r)\ cos\theta e_r - [2f(r) + rf'(r)] \sin\theta e_\theta) + x(2[atf'(r)e_r + f(r)ae_t]) \quad (141)$$

But remember that $x = rcos\theta$(see pg 5 in [1]) and this leaves us with:

$$nX = *(vsx) = vs(2f(r)\ cos\theta e_r - [2f(r) + rf'(r)] \sin\theta e_\theta) + rcos\theta(2[atf'(r)e_r + f(r)ae_t]) \quad (142)$$

But we know that $vs = 2f(r)at$.Hence we get:

$$nX = *(vsx) = 2f(r)at(2f(r)\ cos\theta e_r - [2f(r) + rf'(r)] \sin\theta e_\theta) + rcos\theta(2[atf'(r)e_r + f(r)ae_t]) \quad (143)$$

Then we can start with a warp bubble initially at the rest using the Natario vector shown above and accelerate the bubble to a desired speed of 200 times faster than light.When we achieve the desired speed we turn off the acceleration and keep the speed constant.The terms due to the acceleration now disappears and we are left again with the Natario vector for constant speeds shown below:

$$nX = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r) + rf'(r)] \sin\theta e_\theta \quad (144)$$

Working some algebra with the Natario vector for variable velocities we get:

$$nX = *(vsx) = 2f(r)at(2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta) + rcos\theta(2[atf'(r)e_r + f(r)ae_t]) \quad (145)$$

$$nX = 4f(r)^2at\ cos\theta e_r - 2f(r)at[2f(r) + rf'(r)]\sin\theta e_\theta + 2atf'(r)rcos\theta e_r + 2f(r)rcos\theta ae_t \quad (146)$$

$$nX = 2f(r)rcos\theta ae_t + 4f(r)^2at\ cos\theta e_r + 2atf'(r)rcos\theta e_r - 2f(r)at[2f(r) + rf'(r)]\sin\theta e_\theta \quad (147)$$

$$nX = 2f(r)rcos\theta ae_t + 2[2f(r)^2 + rf'(r)]atcos\theta e_r - 2f(r)at[2f(r) + rf'(r)]\sin\theta e_\theta \quad (148)$$

Then the Natario vector for variable velocities defined using contravariant shift vector components is given by the following expressions:

$$nX = X^t e_t + X^r e_r + X^\theta e_\theta \quad (149)$$

$$nX = X^t dt + X^r dr + X^\theta r d\theta \quad (150)$$

Or being:

$$nX = 2f(r)rcos\theta ae_t + 2[2f(r)^2 + rf'(r)]atcos\theta e_r - 2f(r)at[2f(r) + rf'(r)]\sin\theta e_\theta \quad (151)$$

$$nX = 2f(r)rcos\theta adt + 2[2f(r)^2 + rf'(r)]atcos\theta dr - 2f(r)at[2f(r) + rf'(r)]r\sin\theta d\theta \quad (152)$$

The contravariant shift vector components are respectively given by the following expressions:

$$X^t = 2f(r)rcos\theta a \quad (153)$$

$$X^r = 2[2f(r)^2 + rf'(r)]atcos\theta \quad (154)$$

$$X^\theta = -2f(r)at[2f(r) + rf'(r)]\sin\theta \quad (155)$$

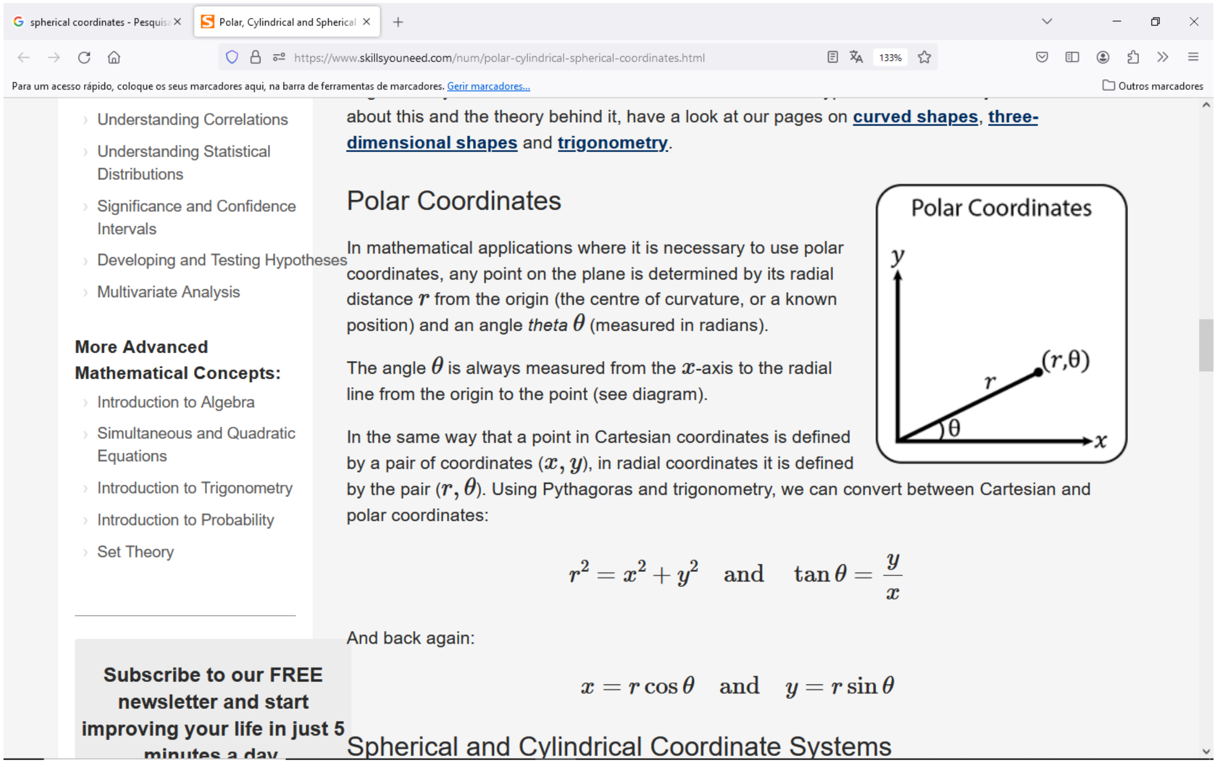

Figure 1: Polar Coordinates.(Source:Internet)

10 Appendix D:Polar Coordinates

Natario (See pg 5 in [1]) defined a warp drive vector $nX = vs * (dx)$ where vs is the **constant** speed of the warp bubble and $*(dx)$ is the Hodge Star taken over the x-axis of motion in **Polar Coordinates**(See pg 4 in [1].(See also Appendices A and B for the detailed calculations).

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \sim \ \sim r^2\sin\theta\cos\theta d\theta \wedge d\varphi + r\sin^2\theta dr \wedge d\varphi = d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right). \tag{156}$$

Consequently if we set exactly what Natario did in pg 5 in [1]:

$$\mathbf{X} \sim -v_s(t)d\left[f(r)r^2\sin^2\theta d\varphi\right] \sim -2v_s f\cos\theta\mathbf{e}_r + v_s(2f + rf')\sin\theta\mathbf{e}_\theta \tag{157}$$

$$\mathbf{X} \sim v_s(t)d\left[f(r)r^2\sin^2\theta d\varphi\right] \sim 2v_s f\cos\theta\mathbf{e}_r - v_s(2f + rf')\sin\theta\mathbf{e}_\theta \tag{158}$$

$$nX = X^r e_r + X^\theta e_\theta \tag{159}$$

$$X^{rs} = 2v_s f\cos\theta \tag{160}$$

$$X^\theta = -v_s(2f + rf')\sin\theta \tag{161}$$

Considering a valid f as a Natario shape function being $f = \frac{1}{2}$ for large r(outside the warp bubble) and $f = 0$ for small r(inside the warp bubble) while being $0 < f < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [1]):

We must demonstrate that the Natario warp drive vector given above satisfies the Natario requirements for a warp bubble defined by:

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nX = vs(t)dx$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [1])

Inside the bubble $f = 0$ and the Natario vector components are zero too.Outside the bubble $f = \frac{1}{2}$,$X^{rs} = v_s \cos\theta$ and $X^{\theta} = -v_s \sin\theta$.In motion over the x-axis only in the equatorial plane $X^{rs} = v_s$ because $\cos\theta = 1$

Due to a **constant** speed vs the term $x * d(vs) = 0$.Now we must examine what happens when the velocity is **variable** and then the term $x * d(vs)$ no longer vanishes.Remember that a real warp drive must accelerate or de-accelerate in order to be accepted as a physical valid model.The complete expression of the Hodge star that generates the Natario vector nX for a variable velocity vs is now given by(see Appendix C for detailed calculations):

$$nX = *(vsx) = vs * (dx) + x * (dvs) \tag{162}$$

The term $*(dx)$ is **again** taken in **Polar Coordinates**

$$nX = X^t e_t + X^r e_r + X^{\theta} e_{\theta} \tag{163}$$

$$nX = 2f(r)rcos\theta ae_t + 2[2f(r)^2 + rf'(r)]atcos\theta e_r - 2f(r)at[2f(r) + rf'(r)]\sin\theta e_{\theta} \tag{164}$$

$$X^t = 2f(r)rcos\theta a \tag{165}$$

$$X^r = 2[2f(r)^2 + rf'(r)]atcos\theta \tag{166}$$

$$X^{\theta} = -2f(r)at[2f(r) + rf'(r)]\sin\theta \tag{167}$$

The variable velocity vs due to a constant acceleration a is given by the following equation:

$$vs = 2fat \tag{168}$$

Remember that Natario(pg 4 in [1]) defines the x axis as the axis of motion.Inside the bubble $f = 0$ resulting in a $vs = 0$ and outside the bubble $f = \frac{1}{2}$ resulting in a $vs = at$ as expected from a variable velocity vs in time t due to a constant acceleration a.Since inside and outside the bubble f always possesses the same values of 0 or $\frac{1}{2}$ then the derivative f' of the Natario shape function f is zero and the shift vector $X^{rs} = 2[2f^2]at$ with $X^{rs} = 0$ inside the bubble and $X^{rs} = 2[2f^2]at = 2[2\frac{1}{4}]at = at = vs$ outside the bubble and this illustrates the Natario definition for a warp drive spacetime.See Appendix G

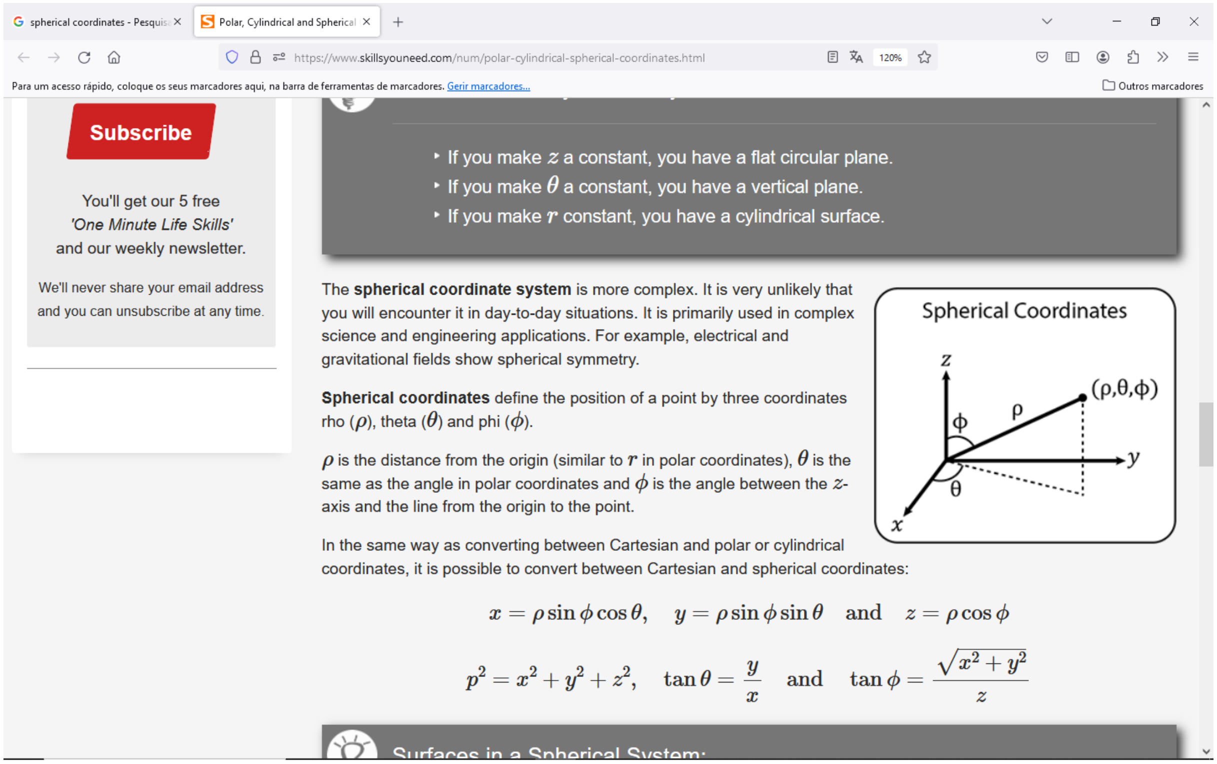

Figure 2: Tridimensional Spherical Coordinates.(Source:Internet)

11 Appendix E:Tridimensional Spherical Coordinates

Natario (See pg 5 in [1]) defined a warp drive vector $nX = vs * (dx)$ where vs is the **constant** speed of the warp bubble and $*(dx)$ is the Hodge Star taken over the x-axis of motion in **Polar Coordinates**(See pg 4 in [1].(See also Appendix D).

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \sim \ \sim r^2\sin\theta\cos\theta d\theta\wedge d\varphi + r\sin^2\theta dr\wedge d\varphi = d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right). \tag{169}$$

Note that in this case the Hodge Star must be taken no longer over $d(r\cos\theta)$ but instead over $d(\rho\sin\phi\cos\theta)$ and this demands more calculations that will appear in a future work.

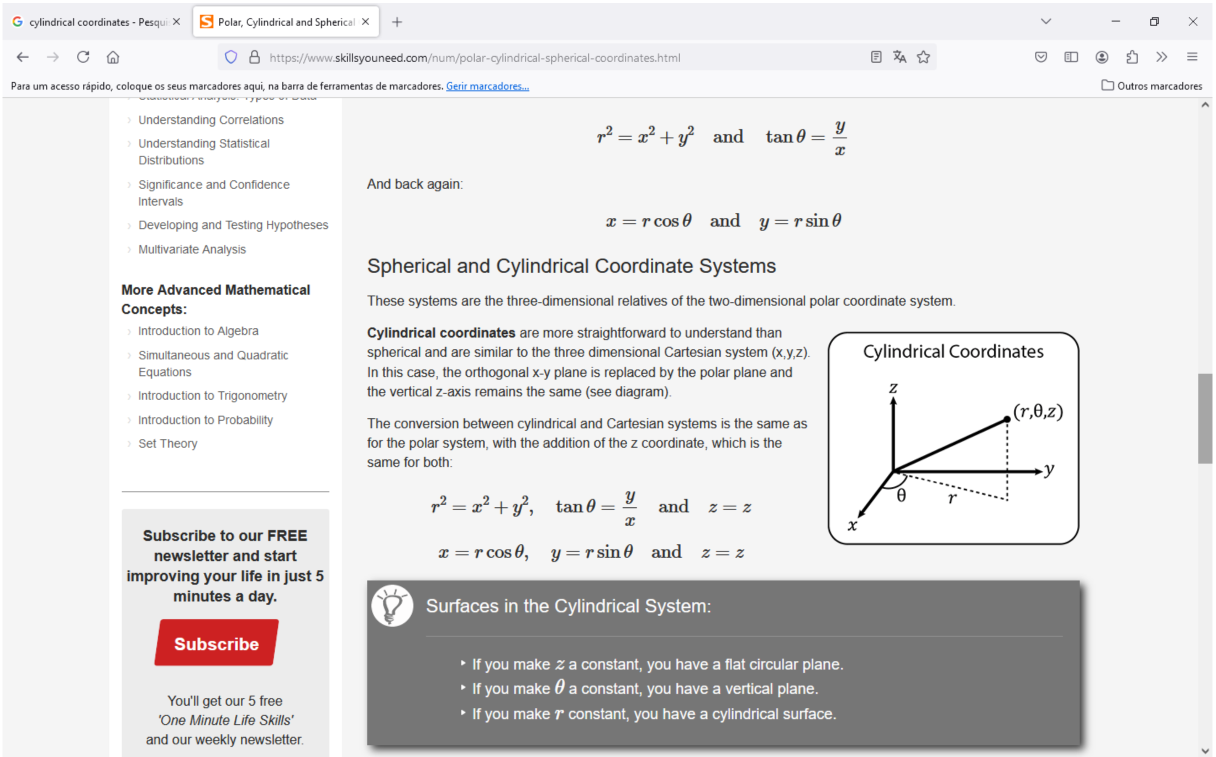

Figure 3: Tridimensional Cylindrical Coordinates.(Source:Internet)

12 Appendix F:Tridimensional Cylindrical Coordinates

We defined a warp drive vector $nX = vs * (dx)$ where vs is the **constant** speed of the warp bubble and $*(dx)$ is the Hodge Star taken over the x-axis of motion in **Cylindrical Coordinates**(See Appendices I and J for the detailed calculations).

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \sim \cos\theta r(d\theta \wedge dz) - \sin\theta(dz \wedge dr) = d\left(r\sin\theta dz\right) \tag{170}$$

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - \sin\theta r d\theta \sim \cos\theta r(d\theta \wedge dz) - \sin\theta(dz \wedge dr) = d\left(r\sin\theta dz\right) \tag{171}$$

$$nX = vs(t)f(r)\cos\theta e_r - vs(t)\sin\theta[f(r) + rf'(r)]e_\theta \tag{172}$$

$$nX = X^r e_r + X^\theta e_\theta \tag{173}$$

$$X^{rs} = v_s f\cos\theta \tag{174}$$

$$X^\theta = -v_s(f + rf')\sin\theta \tag{175}$$

Considering a valid f as a Natario shape function being $f = 1$ for large r(outside the warp bubble) and $f = 0$ for small r(inside the warp bubble) while being $0 < f < 1$ in the walls of the warp bubble also known as the warped region:

We must demonstrate that the cylindrical warp drive vector given above satisfies the Natario requirements for a warp bubble defined by:

any vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nX = vs(t)dx$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [1])

Inside the bubble $f = 0$ and the cylindrical warp drive vector components are zero too.Outside the bubble $f = 1$,$X^{rs} = v_s \cos\theta$ and $X^{\theta} = -v_s \sin\theta$ because f is constant.In motion over the x-axis only in the equatorial plane $X^{rs} = v_s$ because $\cos\theta = 1$

Due to a **constant** speed vs the term $x * d(vs) = 0$.Now we must examine what happens when the velocity is **variable** and then the term $x * d(vs)$ no longer vanishes.Remember that a real warp drive must accelerate or de-accelerate in order to be accepted as a physical valid model.The complete expression of the Hodge star that generates the cylindrical warp drive vector nX for a variable velocity vs is now given by(see Appendix K for detailed calculations):

$$nX = *(vsx) = vs * (dx) + x * (dvs) \tag{176}$$

The term $*(dx)$ is **again** taken in **Cylindrical Coordinates**

$$nX = X^t e_t + X^r e_r + X^{\theta} e_{\theta} \tag{177}$$

$$nX = f(r)rcos\theta ae_t + [f(r)^2 + rf'(r)]atcos\theta e_r - f(r)at[f(r) + rf'(r)]\sin\theta e_{\theta} \tag{178}$$

$$X^t = f(r)rcos\theta a \tag{179}$$

$$X^r = [f(r)^2 + rf'(r)]atcos\theta \tag{180}$$

$$X^{\theta} = -f(r)at[f(r) + rf'(r)]\sin\theta \tag{181}$$

The variable velocity vs due to a constant acceleration a is given by the following equation:

$$vs = fat \tag{182}$$

Remember that Natario(pg 4 in [1]) defines the x axis as the axis of motion.Inside the bubble $f = 0$ resulting in a $vs = 0$ and outside the bubble $f = 1$ resulting in a $vs = at$ as expected from a variable velocity vs in time t due to a constant acceleration a.Since inside and outside the bubble f always possesses the same values of 0 or 1 then the derivative f' of the shape function f is zero and the shift vector $X^{rs} = [f^2]at$ with $X^{rs} = 0$ inside the bubble and $X^{rs} = [f^2]at = vs$ outside the bubble and this illustrates the Natario definition for a warp drive spacetime.See Appendix G

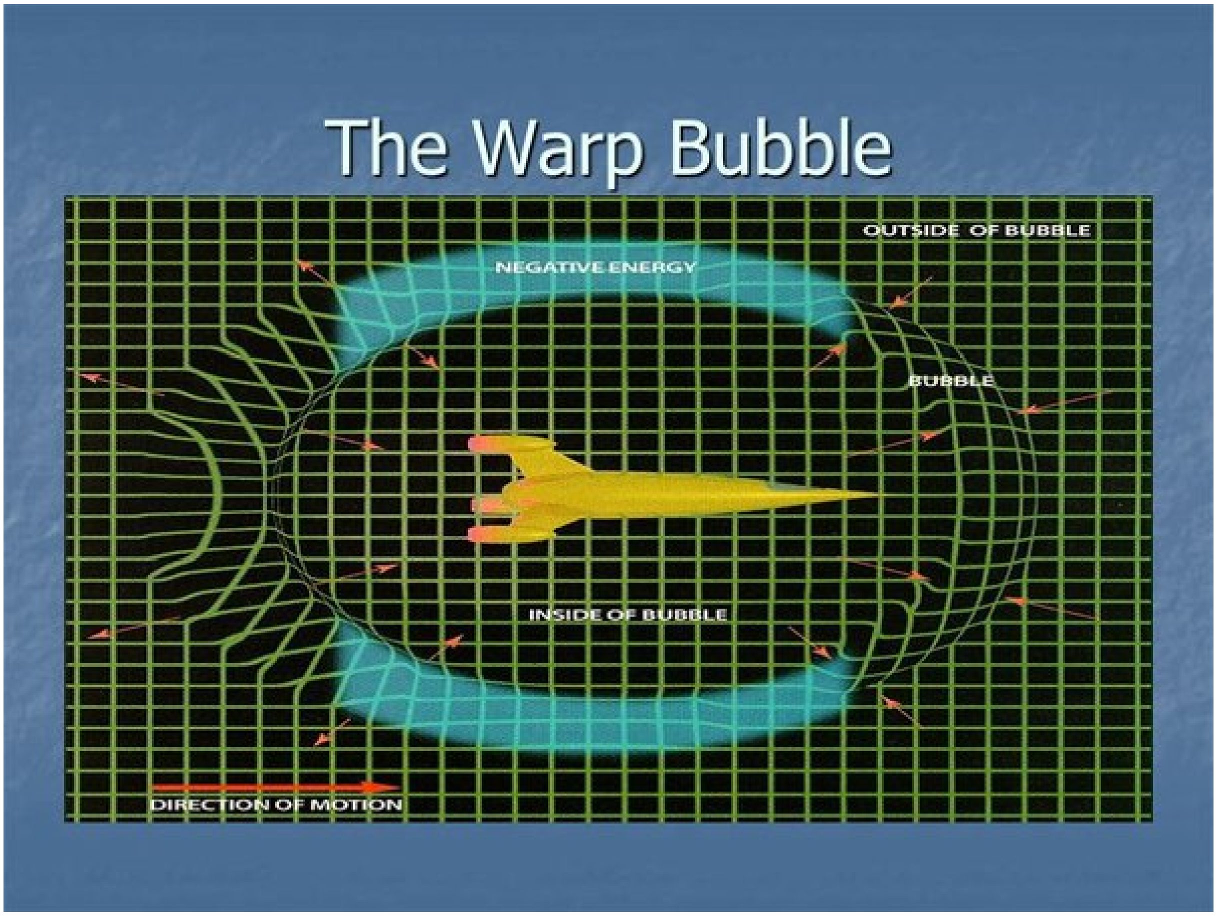

Figure 4: Artistic Presentation of a Warp Bubble.(Source:Internet)

13 Appendix G:Artistic Presentation of a Warp Bubble

In 2001 the Natario warp drive appeared.([1]).This warp drive deals with the spacetime as a "strain" tensor of Fluid Mechanics(pg 5 in [1]). Imagine a fish inside an aquarium and the aquarium is floating in the surface of a river but carried out by the river stream.The warp bubble in this case is the aquarium.An observer at the rest in the margin of the river would see the aquarium passing by him at a large speed but inside the aquarium the fish is at the rest with respect to his local neighborhoods.Since the fish is at the rest inside the aquarium the fish would see the observer in the margin passing by him with a large relative speed since for the fish is the margin that moves with a large relative velocity

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nX = vs(t)dx$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [1])

Lets explain better this statement:Natario considered in this case a coordinates reference frame placed inside the bubble where the fish inside the aquarium or the astronaut in a spaceship inside the bubble depicted above are at the rest with respect to their local neighborhoods.Then any Natario vector must be zero inside the bubble or the aquarium or the spaceship.

On the other hand since the fish sees the margin passing by him with a large relative velocity or the astronaut would see a stationary observer in outer space outside the bubble passing by him with a large relative velocity then any Natario vector outside the bubble must have a value equal to the relative velocity seen by both the fish and the astronaut.

Considering a valid f as a Natario shape function being $f = \frac{1}{2}$ for large r(outside the warp bubble) and $f = 0$ for small r(inside the warp bubble) while being $0 < f < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [1]):The walls of the bubble the Natario warped region corresponds to the distorted region in the picture depicted in this Appendix.

The cylindrical warp drive vector is an identical case:the only difference is the value of the shape function outside the bubble which is 1.

See also Appendix H.

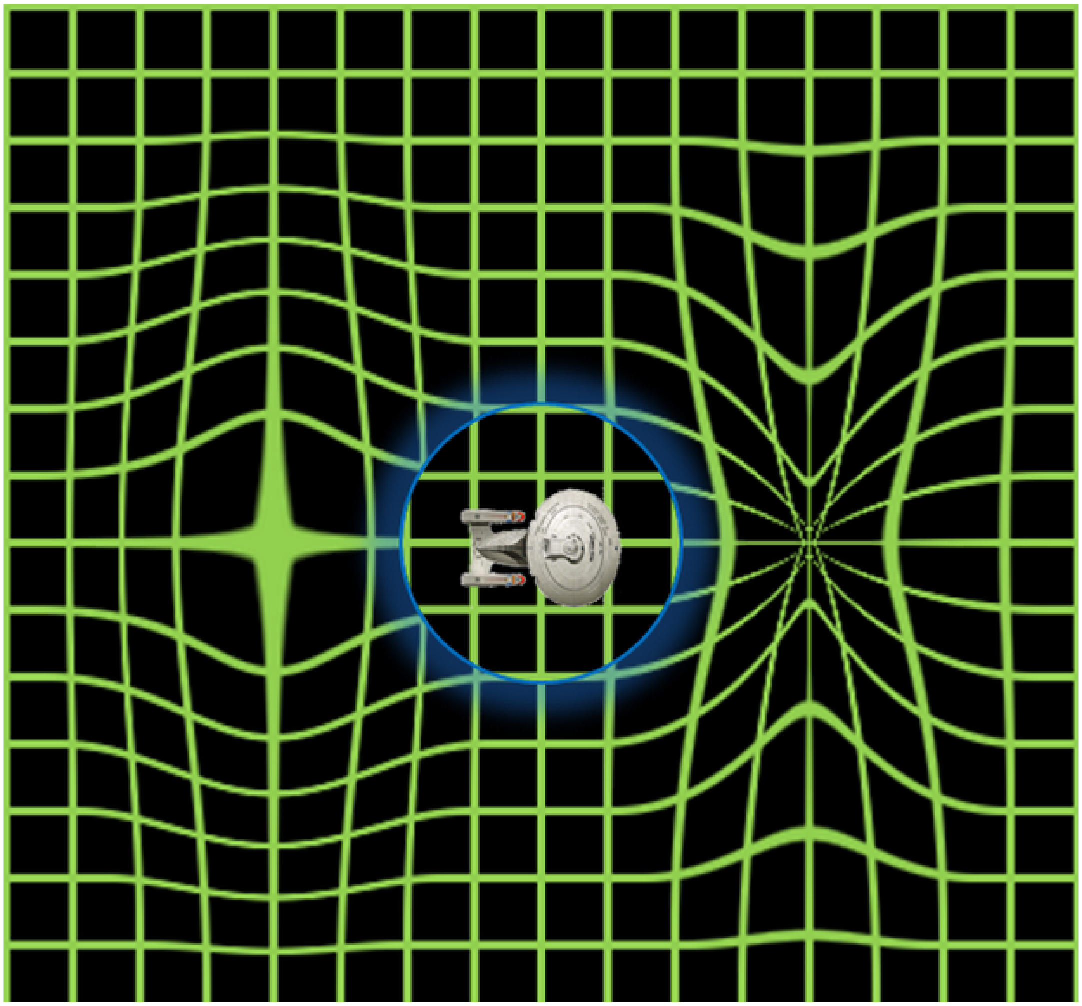

Figure 5: Another Artistic Presentation of a Warp Bubble.(Source:Internet)

14 Appendix H:Another Artistic Presentation of a Warp Bubble

Natario considered a coordinates reference frame placed inside the bubble.Now we must consider a coordinates reference frame placed outside the bubble:In this case the observer at the rest in the margin of the river would see the aquarium passing by him with a large velocity with the fish inside.Also a stationary observer at the rest in outer space would see the spaceship depicted in the picture above passing by him with a large velocity with the astronaut inside.

Now the rules originally defined by Natario are interchanged:

Since the observer in the margin and the observer in outer space are at the rest any Natario vector in this case must be zero outside the bubble.

But since the fish and the spaceship are being seen by the observer at the rest in the margin and the observer at the rest in outer space both fish and spaceship with a large velocity then the Natario vector inside the bubble must have a value equal to the velocity seen by both observers.

Considering a valid f as a Natario shape function being $f = 0$ for large r(outside the warp bubble) and $f = \frac{1}{2}$ for small r(inside the warp bubble) while being $0 < f < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region:The walls of the bubble the Natario warped region corresponds to the distorted region the "blue circle" in the picture depicted in this Appendix.

The cylindrical warp drive vector is an identical case:the only difference is the value of the shape function inside the bubble which is 1.

15 Appendix I:differential forms,Hodge star and the mathematical demonstration of the cylindrical warp drive vectors $nX = -vsdx$ and $nX = vsdx$ for a constant speed vs in a R^3 space basis

This appendix is being written for novice or newcomer students on Warp Drive theory still not acquainted with the methods used to obtain the final expression of the cylindrical warp drive vectors.

The Canonical Basis of the Hodge Star in cylindrical coordinates can be defined as follows(see eq 3.72 pg 69$(a)(b)$ in [2]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge dz \sim r(d\theta \wedge dz) \tag{183}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial \theta} \sim rd\theta \sim (dz) \wedge dr \sim (dz \wedge dr) \tag{184}$$

$$e_z \equiv \frac{\partial}{\partial z} \sim (dz) \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{185}$$

From above we get the following results

$$dr \sim r(d\theta \wedge dz) \tag{186}$$

$$rd\theta \sim (dz \wedge dr) \tag{187}$$

$$dz \sim r(dr \wedge d\theta) \tag{188}$$

Note that this expression matches the common definition of the Hodge Star operator * applied to the cylindrical coordinates as given by(see eq 3.72 pg 69$(a)(b)$ in [2]):

$$*dr = r(d\theta \wedge dz) \tag{189}$$

$$*rd\theta = (dz \wedge dr) \tag{190}$$

$$*dz = r(dr \wedge d\theta) \tag{191}$$

Applying the Hodge Star to the x-axis as the motion axis we get:

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \sim \cos\theta r(d\theta \wedge dz) - \sin\theta(dz \wedge dr) = d\left(r\sin\theta dz\right) \tag{192}$$

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - \sin\theta rd\theta \sim \cos\theta r(d\theta \wedge dz) - \sin\theta(dz \wedge dr) = d\left(r\sin\theta dz\right) \tag{193}$$

Look that

$$dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \tag{194}$$

Or

$$dx = d(r\cos\theta) = \cos\theta dr - \sin\theta r d\theta \tag{195}$$

Applying the Hodge Star operator * to the above expression:

$$*dx = *d(r\cos\theta) = \cos\theta(*dr) - \sin\theta(*rd\theta) \tag{196}$$

$$*dx = *d(r\cos\theta) = \cos\theta r(d\theta \wedge dz) - \sin\theta(dz \wedge dr) \tag{197}$$

$$*dx = *d(r\cos\theta) = \cos\theta e_r - \sin\theta e_\theta \tag{198}$$

Now examining the expression:

$$d\,(r\sin\theta dz) \tag{199}$$

We must also apply the Hodge Star operator to the expression above

And then we have:

$$*d\,(r\sin\theta dz) \tag{200}$$

$$*d\,(r\sin\theta dz) \sim r * d[(\sin\theta)dz] + \sin\theta * d[rdz] + r\sin\theta * d[(dz)] \tag{201}$$

According to eq 3.90 pg $74(a)(b)$ in [2] the term $r\sin\theta * d[(dz)] = 0$

This leaves us with:

$$*d\,(r\sin\theta dz) \sim r * d[(\sin\theta)dz] + \sin\theta * d[rdz] \sim \cos\theta r(d\theta \wedge dz) - \sin\theta(dz \wedge dr) \tag{202}$$

Because and according to eqs 3.90 and 3.91 pg $74(a)(b)$ in [2],tb 3.2 pg $68(a)(b)$ in [2]:

$$*d(\alpha + \beta) = d\alpha + d\beta \tag{203}$$

$$*d(f\alpha) = df \wedge \alpha + (-1)^p f \wedge d\alpha \dashrightarrow p = 2 \dashrightarrow *d(f\alpha) = df \wedge \alpha + f \wedge d\alpha \tag{204}$$

$$*d(dx) = d(dy) = d(dz) = 0 \tag{205}$$

From above we can see for example that

$$*d[(\sin\theta)dz] = d(\sin\theta) \wedge dz + \sin\theta \wedge ddz = \cos\theta(d\theta \wedge dz) \tag{206}$$

$$*d[rdz] = dr \wedge dz + r \wedge ddz = (dr \wedge dz) = -(dz \wedge dr) \tag{207}$$

We know that the following expression holds true(see eq 3.79 pg 70$(a)(b)$ in [2])):

$$dz \wedge dr = -dr \wedge dz \tag{208}$$

And then we have:

$$*d\left(r \sin\theta dz\right) \sim r * d[(\sin\theta) dz] + \sin\theta * d[rdz] \tag{209}$$

With:

$$*d[(\sin\theta) dz] = d(\sin\theta) \wedge dz + \sin\theta \wedge ddz = \cos\theta(d\theta \wedge dz) \tag{210}$$

$$*d[rdz] = dr \wedge dz + r \wedge ddz = (dr \wedge dz) = -(dz \wedge dr) \tag{211}$$

$$*d\left(r \sin\theta dz\right) \sim r * d[(\sin\theta) dz] + \sin\theta * d[rdz] \sim rcos\theta(d\theta \wedge dz) + \sin\theta(dr \wedge dz) \tag{212}$$

$$*d\left(r \sin\theta dz\right) \sim r * d[(\sin\theta) dz] + \sin\theta * d[rdz] \sim cos\theta r(d\theta \wedge dz) - \sin\theta(dz \wedge dr) \tag{213}$$

$$*d\left(r \sin\theta dz\right) \sim r * d[(\sin\theta) dz] + \sin\theta * d[rdz] \sim cos\theta e_r - \sin\theta e_\theta \tag{214}$$

Now we will examine the following expression:

$$*d[f(r) r \sin\theta dz] \tag{215}$$

From above we can obtain the next expressions

$$f(r) r * d[(\sin\theta) dz] + f(r) \sin\theta * d[rdz] + f(r) r \sin\theta * d(dz) + r \sin\theta * d[f(r) dz] \tag{216}$$

$$f(r) r * d[(\sin\theta) dz] + f(r) \sin\theta * d[rdz] + r \sin\theta * d[f(r) dz] \tag{217}$$

$$*d[f(r) dz] = df(r) \wedge dz + f(r) \wedge ddz = f'(r)(dr \wedge dz) = -f'(r)(dz \wedge dr) \tag{218}$$

$$f(r) r \cos\theta(d\theta \wedge dz) + f(r) \sin\theta(-(dz \wedge dr)) + r \sin\theta(-f'(r)(dz \wedge dr)) \tag{219}$$

$$f(r) r \cos\theta(d\theta \wedge dz) - f(r) \sin\theta(dz \wedge dr) - r \sin\theta f'(r)(dz \wedge dr) \tag{220}$$

$$f(r) r \cos\theta(d\theta \wedge dz) - f(r) \sin\theta(dz \wedge dr) - \sin\theta r f'(r)(dz \wedge dr) \tag{221}$$

$$f(r) r \cos\theta(d\theta \wedge dz) - \sin\theta[f(r) + r f'(r)](dz \wedge dr) \tag{222}$$

$$f(r) \cos\theta r(d\theta \wedge dz) - \sin\theta[f(r) + r f'(r)](dz \wedge dr) \tag{223}$$

$$*d[f(r) r \sin\theta dz] \sim f(r) \cos\theta r(d\theta \wedge dz) - \sin\theta[f(r) + r f'(r)](dz \wedge dr) \tag{224}$$

$$*d[f(r)r\sin\theta dz] \sim f(r)\cos\theta r(d\theta\wedge dz) - \sin\theta[f(r)+rf'(r)](dz\wedge dr) \tag{225}$$

Comparing the above expression with the Canonical Basis of the Hodge Star in cylindrical coordinates:

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta)\wedge dz \sim r(d\theta\wedge dz) \tag{226}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim (dz)\wedge dr \sim (dz\wedge dr) \tag{227}$$

$$e_z \equiv \frac{\partial}{\partial z} \sim (dz) \sim dr\wedge(rd\theta) \sim r(dr\wedge d\theta) \tag{228}$$

We can obtain the following result:

$$*d[f(r)r\sin\theta dz] \sim f(r)\cos\theta r(d\theta\wedge dz) - \sin\theta[f(r)+rf'(r)](dz\wedge dr) \tag{229}$$

$$*d[f(r)r\sin\theta dz] \sim f(r)\cos\theta e_r - \sin\theta[f(r)+rf'(r)]e_\theta \tag{230}$$

Defining the cylindrical warp drive vectors with the Hodge Star operator * explicitly written :

$$nX = vs(t) * d[f(r)r\sin\theta dz] \tag{231}$$

$$nX = -vs(t) * d[f(r)r\sin\theta dz] \tag{232}$$

We can get finally the latest expressions for the cylindrical warp drive vectors nX

$$nX = vs(t)f(r)\cos\theta e_r - vs(t)\sin\theta[f(r)+rf'(r)]e_\theta \tag{233}$$

$$nX = -vs(t)f(r)\cos\theta e_r + vs(t)\sin\theta[f(r)+rf'(r)]e_\theta \tag{234}$$

16 Appendix J:differential forms,Hodge star and the mathematical demonstration of the cylindrical warp drive vectors $nX = -vsdx$ and $nX = vsdx$ for a constant speed vs or for the first term $vsdx$ from the cylindrical warp drive vector $nX = vsdx + xdvs$(a variable speed) in a R^4 space basis

This appendix is being written for novice or newcomer students on Warp Drive theory still not acquainted with the methods used to arrive at the final expression of the cylindrical warp drive vector nX

The Canonical Basis of the Hodge Star in cylindrical coordinates can be defined as follows(see eq 3.74 pg 69$(a)(b)$ in [2])(see pg 47 eqs 2.67 to 2.70 and pg 92 in [3]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim dt \wedge (rd\theta) \wedge dz \sim r(dt \wedge d\theta \wedge dz) \tag{235}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial \theta} \sim rd\theta \sim dt \wedge (dz) \wedge dr \sim (dt \wedge dz \wedge dr) \tag{236}$$

$$e_z \equiv \frac{\partial}{\partial z} \sim (dz) \sim dt \wedge dr \wedge (rd\theta) \sim r(dt \wedge dr \wedge d\theta) \tag{237}$$

From above we get the following results

$$dr \sim r(dt \wedge d\theta \wedge dz) \tag{238}$$

$$rd\theta \sim (dt \wedge dz \wedge dr) \tag{239}$$

$$dz \sim r(dt \wedge dr \wedge d\theta) \tag{240}$$

Note that this expression matches the common definition of the Hodge Star operator * applied to the cylindrical coordinates as given by(see eq 3.74 pg 69$(a)(b)$ in [2])(see pg 47 eqs 2.67 to 2.70 and pg 92 in [3]):

$$*dr = r(dt \wedge d\theta \wedge dz) \tag{241}$$

$$*rd\theta = (dt \wedge dz \wedge dr) \tag{242}$$

$$*dz = r(dt \wedge dr \wedge d\theta) \tag{243}$$

Applying the Hodge Star to the x-axis as the motion axis we get:

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \sim \cos\theta r(dt \wedge d\theta \wedge dz) - \sin\theta(dt \wedge dz \wedge dr) = d\left(r\sin\theta dz\right) \tag{244}$$

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - \sin\theta rd\theta \sim \cos\theta r(dt \wedge d\theta \wedge dz) - \sin\theta(dt \wedge dz \wedge dr) = d\left(r\sin\theta dz\right) \tag{245}$$

Look that

$$dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \tag{246}$$

Or

$$dx = d(r\cos\theta) = \cos\theta dr - \sin\theta r d\theta \tag{247}$$

Applying the Hodge Star operator * to the above expression:

$$*dx = *d(r\cos\theta) = \cos\theta(*dr) - \sin\theta(*rd\theta) \tag{248}$$

$$*dx = *d(r\cos\theta) = \cos\theta r(dt\wedge d\theta\wedge dz) - \sin\theta(dt\wedge dz\wedge dr) \tag{249}$$

$$*dx = *d(r\cos\theta) = \cos\theta e_r - \sin\theta e_\theta \tag{250}$$

Now examining the expression:

$$d\left(r\sin\theta dz\right) \tag{251}$$

We must also apply the Hodge Star operator to the expression above

And then we have:

$$*d\left(r\sin\theta dz\right) \tag{252}$$

$$*d\left(r\sin\theta dz\right) \sim r * d[(\sin\theta)dz] + \sin\theta * d[rdz] + r\sin\theta * d[(dz)] \tag{253}$$

According to eq 3.90 pg $74(a)(b)$ in [2] the term $r\sin\theta * d[(dz)] = 0$

This leaves us with:

$$*d\left(r\sin\theta dz\right) \sim r * d[(\sin\theta)dz] + \sin\theta * d[rdz] \sim \cos\theta r(dt\wedge d\theta\wedge dz) - \sin\theta(dt\wedge dz\wedge dr) \tag{254}$$

Because and according to eqs 3.90 and 3.91 pg $74(a)(b)$ in [2],tb 3.2 pg $68(a)(b)$ in [2]:

$$*d(\alpha+\beta) = d\alpha + d\beta \tag{255}$$

$$*d(f\alpha) = df\wedge\alpha + (-1)^p f\wedge d\alpha \dashrightarrow p = 2 \dashrightarrow *d(f\alpha) = df\wedge\alpha + f\wedge d\alpha \tag{256}$$

$$*d(dx) = d(dy) = d(dz) = 0 \tag{257}$$

From above we can see for example that

$$*d[(\sin\theta)dz] = dt\wedge d(\sin\theta)\wedge dz + dt\wedge\sin\theta\wedge ddz = \cos\theta(dt\wedge d\theta\wedge dz) \tag{258}$$

$$*d[rdz] = dt \wedge dr \wedge dz + dt \wedge r \wedge ddz = (dr \wedge dz) = -(dz \wedge dr) \tag{259}$$

We know that the following expression holds true(see eq 3.79 pg 70$(a)(b)$ in [2])):

$$dz \wedge dr = -dr \wedge dz \tag{260}$$

And then we have:

$$*d\,(r \sin\theta dz) \sim r * d[(\sin\theta)dz] + \sin\theta * d[rdz] \tag{261}$$

With:

$$*d[(\sin\theta)dz] = dt \wedge d(\sin\theta) \wedge dz + dt \wedge \sin\theta \wedge ddz = \cos\theta(dt \wedge d\theta \wedge dz) \tag{262}$$

$$*d[rdz] = dt \wedge dr \wedge dz + dt \wedge r \wedge ddz = (dt \wedge dr \wedge dz) = -(dt \wedge dz \wedge dr) \tag{263}$$

$$*d\,(r \sin\theta dz) \sim r * d[(\sin\theta)dz] + \sin\theta * d[rdz] \sim rcos\theta(dt \wedge d\theta \wedge dz) + \sin\theta(dt \wedge dr \wedge dz) \tag{264}$$

$$*d\,(r \sin\theta dz) \sim r * d[(\sin\theta)dz] + \sin\theta * d[rdz] \sim cos\theta r(dt \wedge d\theta \wedge dz) - \sin\theta(dt \wedge dz \wedge dr) \tag{265}$$

$$*d\,(r \sin\theta dz) \sim r * d[(\sin\theta)dz] + \sin\theta * d[rdz] \sim cos\theta e_r - \sin\theta e_\theta \tag{266}$$

Now we will examine the following expression:

$$*d[f(r)r \sin\theta dz] \tag{267}$$

From above we can obtain the next expressions

$$f(r)r * d[(\sin\theta)dz] + f(r)\sin\theta * d[rdz] + f(r)r\sin\theta * d(dz) + r\sin\theta * d[f(r)dz] \tag{268}$$

$$f(r)r * d[(\sin\theta)dz] + f(r)\sin\theta * d[rdz] + r\sin\theta * d[f(r)dz] \tag{269}$$

$$*d[f(r)dz] = dt \wedge df(r) \wedge dz + dt \wedge f(r) \wedge ddz = f'(r)(dt \wedge dr \wedge dz) = -f'(r)(dt \wedge dz \wedge dr) \tag{270}$$

$$f(r)r\cos\theta(dt \wedge d\theta \wedge dz) + f(r)\sin\theta(-(dt \wedge dz \wedge dr)) + r\sin\theta(-f'(r)(dt \wedge dz \wedge dr)) \tag{271}$$

$$f(r)r\cos\theta(dt \wedge d\theta \wedge dz) - f(r)\sin\theta(dt \wedge dz \wedge dr) - r\sin\theta f'(r)(dt \wedge dz \wedge dr) \tag{272}$$

$$f(r)r\cos\theta(dt \wedge d\theta \wedge dz) - f(r)\sin\theta(dt \wedge dz \wedge dr) - \sin\theta r f'(r)(dt \wedge dz \wedge dr) \tag{273}$$

$$f(r)r\cos\theta(dt\wedge d\theta\wedge dz)-\sin\theta[f(r)+rf'(r)](dt\wedge dz\wedge dr) \tag{274}$$

$$f(r)\cos\theta r(dt\wedge d\theta\wedge dz)-\sin\theta[f(r)+rf'(r)](dt\wedge dz\wedge dr) \tag{275}$$

$$*d[f(r)r\sin\theta dz]\sim f(r)\cos\theta r(dt\wedge d\theta\wedge dz)-\sin\theta[f(r)+rf'(r)](dt\wedge dz\wedge dr) \tag{276}$$

$$*d[f(r)r\sin\theta dz]\sim f(r)\cos\theta r(dt\wedge d\theta\wedge dz)-\sin\theta[f(r)+rf'(r)](dt\wedge dz\wedge dr) \tag{277}$$

Comparing the above expression with the Canonical Basis of the Hodge Star in cylindrical coordinates:

$$e_r\equiv\frac{\partial}{\partial r}\sim dr\sim dt\wedge(rd\theta)\wedge dz\sim r(dt\wedge d\theta\wedge dz) \tag{278}$$

$$e_\theta\equiv\frac{1}{r}\frac{\partial}{\partial\theta}\sim rd\theta\sim dt\wedge(dz)\wedge dr\sim(dt\wedge dz\wedge dr) \tag{279}$$

$$e_z\equiv\frac{\partial}{\partial z}\sim(dz)\sim dt\wedge dr\wedge(rd\theta)\sim r(dt\wedge dr\wedge d\theta) \tag{280}$$

We can obtain the following result:

$$*d[f(r)r\sin\theta dz]\sim f(r)\cos\theta r(dt\wedge d\theta\wedge dz)-\sin\theta[f(r)+rf'(r)](dt\wedge dz\wedge dr) \tag{281}$$

$$*d[f(r)r\sin\theta dz]\sim f(r)\cos\theta e_r-\sin\theta[f(r)+rf'(r)]e_\theta \tag{282}$$

Defining the cylindrical warp drive vectors with the Hodge Star operator * explicitly written :

$$nX=vs(t)*d[f(r)r\sin\theta dz] \tag{283}$$

$$nX=-vs(t)*d[f(r)r\sin\theta dz] \tag{284}$$

We can get finally the latest expressions for the cylindrical warp drive vectors nX

$$nX=vs(t)f(r)\cos\theta e_r-vs(t)\sin\theta[f(r)+rf'(r)]e_\theta \tag{285}$$

$$nX=-vs(t)f(r)\cos\theta e_r+vs(t)\sin\theta[f(r)+rf'(r)]e_\theta \tag{286}$$

17 Appendix K:differential forms,Hodge star and the mathematical demonstration of the cylindrical warp drive vector $nX = *(vsx)$ for a variable speed vs and a constant acceleration a

any vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of r defined by Natario as the interior of the warp bubble and $nX = vs(t)dx$ with $X = vs$ for a large value of r defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [1])

In the Appendices I and J we gave the mathematical demonstration of the cylindrical warp drive vector nX in the R^3 and R^4 space basis when the velocity vs is constant.Hence the complete expression of the Hodge star that generates the cylindrical warp drive vector nX for a constant velocity vs is given by:

$$nX = *(vsx) = vs * (dx) \tag{287}$$

$$*dx = *d(rcos\theta) = *d\,(r \sin\theta dz) = *d[f(r)r \sin\theta dz] \tag{288}$$

The equation of the cylindrical warp drive vector nX is given by:

$$nX = X^r e_r + X^\theta e_\theta \tag{289}$$

$$nX = X^r dr + X^\theta r d\theta \tag{290}$$

$$nX = vs(t)f(r)\cos\theta e_r - vs(t)\sin\theta[f(r) + rf'(r)]e_\theta \tag{291}$$

$$nX = -vs(t)f(r)\cos\theta e_r + vs(t)\sin\theta[f(r) + rf'(r)]e_\theta \tag{292}$$

With the contravariant shift vector components explicitly given by:

$$X^r = v_s f(r)\cos\theta \tag{293}$$

$$X^\theta = -v_s(f(r) + (r)f'(r))\sin\theta \tag{294}$$

Because due to a constant speed vs the term $x * d(vs) = 0$.Now we must examine what happens when the velocity is variable and then the term $x * d(vs)$ no longer vanishes.Remember that a real warp drive must accelerate or de-accelerate in order to be accepted as a physical valid model.The complete expression of the Hodge star that generates the Natario vector nX for a variable velocity vs is now given by:

$$nX = *(vsx) = vs * (dx) + x * (dvs) \tag{295}$$

In order to study the term $x * d(vs)$ we must introduce a new Canonical Basis for the coordinate time in the R^4 space basis defined as follows:(see eq 3.74 pg 69$(a)(b)$ in [2])(see pg 47 eqs 2.67 to 2.70 and pg 92 in [3]):

$$e_t \equiv \frac{\partial}{\partial t} \sim dt \sim dr \wedge (rd\theta) \wedge (dz) \sim r(dr \wedge d\theta \wedge dz) \tag{296}$$

$$dt \sim r(dr \wedge d\theta \wedge dz) \tag{297}$$

The Hodge star operator defined for the coordinate time is given by:(see eq 3.74 pg 69$(a)(b)$ in [2]):

$$*dt = r(dr \wedge d\theta \wedge dz) \tag{298}$$

The valid expression for a variable velocity $vs(t)$ in the cylindrical warp drive spacetime due to a constant acceleration a must be given by:

$$vs = f(r)at \tag{299}$$

Because and considering a valid $f(r)$ as a shape function being $f(r) = 1$ for large r(outside the warp bubble where $X = vs(t)$ and $nX = vs(t) * dx + x * d(vs(t)))$ and $f(r) = 0$ for small r(inside the warp bubble where $X = 0$ and $nX = 0$) while being $0 < f(r) < 1$ in the walls of the warp bubble also known as the warped region and considering also that the cylindrical warp drive is a ship-frame based coordinates system(a reference frame placed in the center of the warp bubble where the ship resides-or must reside!!) then an observer in the ship inside the bubble sees every point inside the bubble at the rest with respect to him because inside the bubble $vs(t) = 0$ because $f(r) = 0$.

To illustrate the statement pointed above imagine a fish inside an aquarium and the aquarium is floating in the surface of a river but carried out by the river stream.The stream varies its velocity with time.The warp bubble in this case is the aquarium and the walls of the aquarium are the walls of the warp bubble-warped region.An observer in the margin of the river would see the aquarium passing by him at a large speed considering a coordinates system(a reference frame) placed in the margin of the river but inside the aquarium the fish is at the rest with respect to his local neighborhoods.Then for the fish any point inside the aquarium is at the rest with respect to him because inside the aquarium $vs = f(r)at$ with $f(r) = 0$ and consequently giving a $vs(t) = 0$.Again with respect to the fish the fish "sees" the margin passing by him with a large relative velocity.The margin in this case is the region outside the bubble "seen" by the fish with a variable velocity $vs(t) = v1$ in the time $t1$ and $vs(t) = v2$ in the time $t2$ because outside the bubble the generic expression for a variable velocity vs is given by $vs = f(r)at$ and outside the bubble $f(r) = 1$ giving a generic expression for a variable velocity vs as $vs(t) = at$ and consequently a $v1 = at1$ in the time $t1$ and a $v2 = at2$ in the time $t2$.Then the variable velocity in not only a function of time alone but must consider also the position of the bubble where the measure is being taken wether inside or outside the bubble.So the velocity must also be a function of r.Its total differential is then given by:

$$dvs = [atf'(r)dr + f(r)tda + f(r)adt] \tag{300}$$

Applying the Hodge star to the total differential dvs we get:

$$*dvs = [atf'(r) * dr + f(r)t * da + f(r)a * dt] \tag{301}$$

But we consider here the acceleration a a constant.Then the term $f(r)tda = 0$ and in consequence $f(r)t * da = 0$.This leaves us with:

$$*dvs = [atf'(r) * dr + f(r)a * dt] \tag{302}$$

$$*dvs = [atf'(r) * dr + f(r)a * dt] = [atf'(r)r(dt \wedge d\theta \wedge dz) + f(r)ar(dr \wedge d\theta \wedge dz)] \tag{303}$$

$$*dvs = [atf'(r) * dr + f(r)a * dt] = [atf'(r)e_r + f(r)ae_t] \tag{304}$$

The complete expression of the Hodge star that generates the cylindrical warp drive vector nX for a variable velocity vs is given by:

$$nX = *(vsx) = vs * (dx) + x * d(vs) \tag{305}$$

The term $*dx$ was obtained in the Appendices I and J as follows:

$$*dx = f(r)\cos\theta e_r - \sin\theta[f(r) + rf'(r)]e_\theta \tag{306}$$

The complete expression of the Hodge star that generates the cylindrical warp drive vector nX for a variable velocity vs is now given by:

$$nX = *(vsx) = vs(f(r)\ cos\theta e_r - [f(r) + rf'(r)]\sin\theta e_\theta) + x([atf'(r)e_r + f(r)ae_t]) \tag{307}$$

But remember that $x = rcos\theta$(see Appendix F) and this leaves us with:

$$nX = *(vsx) = vs(f(r)\ cos\theta e_r - [f(r) + rf'(r)]\sin\theta e_\theta) + rcos\theta([atf'(r)e_r + f(r)ae_t]) \tag{308}$$

But we know that $vs = f(r)at$.Hence we get:

$$nX = *(vsx) = f(r)at(f(r)\ cos\theta e_r - [f(r) + rf'(r)]\sin\theta e_\theta) + rcos\theta([atf'(r)e_r + f(r)ae_t]) \tag{309}$$

Then we can start with a warp bubble initially at the rest using the cylindrical warp drive vector shown above and accelerate the bubble to a desired speed of 200 times faster than light.When we achieve the desired speed we turn off the acceleration and keep the speed constant.The terms due to the acceleration now disappears and we are left again with the cylindrical warp drive vector for constant speeds shown below:

$$nX = vs(t)f(r)\ cos\theta e_r - vs(t)[f(r) + rf'(r)]\sin\theta e_\theta \tag{310}$$

Working some algebra with the cylindrical warp drive vector for variable velocities we get:

$$nX = *(vsx) = f(r)at(f(r)\ cos\theta e_r - [f(r) + rf'(r)]\sin\theta e_\theta) + rcos\theta([atf'(r)e_r + f(r)ae_t]) \tag{311}$$

$$nX = f(r)^2 at\ cos\theta e_r - f(r)at[f(r) + rf'(r)]\sin\theta e_\theta + atf'(r)rcos\theta e_r + f(r)rcos\theta ae_t \tag{312}$$

$$nX = f(r)rcos\theta ae_t + f(r)^2 at\ cos\theta e_r + atf'(r)rcos\theta e_r - f(r)at[f(r) + rf'(r)]\sin\theta e_\theta \tag{313}$$

$$nX = f(r)rcos\theta ae_t + [f(r)^2 + rf'(r)]atcos\theta e_r - f(r)at[2f(r) + rf'(r)]\sin\theta e_\theta \tag{314}$$

Then the cylindrical warp drive vector for variable velocities defined using contravariant shift vector components is given by the following expressions:

$$nX = X^t e_t + X^r e_r + X^\theta e_\theta \tag{315}$$

$$nX = X^t dt + X^r dr + X^\theta r d\theta \tag{316}$$

Or being:

$$nX = f(r)rcos\theta ae_t + [f(r)^2 + rf'(r)]atcos\theta e_r - f(r)at[f(r) + rf'(r)]\sin\theta e_\theta \tag{317}$$

$$nX = f(r)rcos\theta adt + [f(r)^2 + rf'(r)]atcos\theta dr - f(r)at[f(r) + rf'(r)]r\sin\theta d\theta \tag{318}$$

The contravariant shift vector components are respectively given by the following expressions:

$$X^t = f(r)rcos\theta a \tag{319}$$

$$X^r = [f(r)^2 + rf'(r)]atcos\theta \tag{320}$$

$$X^\theta = -f(r)at[f(r) + rf'(r)]\sin\theta \tag{321}$$

References

[1] *Natario J.,(*2002*). Classical and Quantum Gravity. 19 1157-1166,arXiv gr-qc/0110086*

[2] *Ryder L.,(Introduction to General Relativity)*
(Cambridge University Press 2009*)*

[3] *Renteln P.,(Manyfolds Tensors and Forms:An Introduction for mathematicians and physicists)*
(Cambridge University Press 2014*)*

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