Spin Density Plane Waves in an Elastic Solid Model of the Vacuum

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(Dated: July 6, 2024)

Plane waves of spin angular momentum density in an ideal elastic solid are analyzed using vector and bispinor descriptions. In both classical and quantum physics, spin density is the axial vector field whose curl is equal to twice the incompressible intrinsic momentum density. The second-order vector wave equation assumes that temporal changes of spin density in an ideal elastic solid are attributable to convection, rotation, and torque density. The corresponding first-order wave equation for Dirac bispinors incorporates terms describing wave propagation, convection, rotations of the medium and rotations of wave velocity relative to the medium. The two rotation terms are also operators for rotational kinetic energy and conventional potential energy, respectively. The potential energy corresponds to half the mass term of the free electron Dirac equation. Bispinor plane wave solutions are constructed consistent with the usual dynamical operators of relativistic quantum mechanics. Lagrangian and Hamiltonian densities are also constructed with each term having a clear classical physics interpretation. The intrinsic momentum associated with the Belinfante-Rosenfeld stress tensor is explained. Application to elementary particles is discussed, including classical physics analogues of the Pauli exclusion principle, interaction potentials, fermions, bosons, and antimatter.

Keywords: angular momentum, Dirac equation, elastic solid, intrinsic momentum, quantum mechanics, spin, wave mechanics

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1. INTRODUCTION

Recent experimental and theoretical work has demonstrated that many phenomena previously thought to be in the exclusive realm of quantum mechanics can actually be realized via classical physics. Most notably, quantum statistics such as single-particle diffraction and interference, wave-like probability distributions, tunneling, quantized orbits, and orbital level splitting have been experimentally demonstrated using silicone oil droplets bouncing on a vibrating tank of fluid. [1–7] These experiments are classical realizations of pilot-wave theory, or Bohmian mechanics, which was an early attempt to reconcile the deterministic nature of quantum mechanical equations with the probabilistic nature of measurements. [8–11]

The discovery of a classical interpretation of spin angular momentum evolving according to a Dirac-type equation further lessens the distinction between classical and quantum physics. [12–14] The Dirac formalism has been used in a variety of contexts to describe classical wave dynamics. [15–22] Despite the probabilistic nature of measurements, the quantum mechanical Dirac equation is fundamentally a deterministic equation describing the evolution of physical quantities such as spin density, momentum density, and energy density. These quantities are independent of any interpretation of the wave function as representing a "particle".

While it is clear that the Dirac equation has application to classical physics, it is unclear to what extent classical physics can describe elementary particles and their interactions. To make progress in this area requires a thorough understanding of the equations describing spin angular momentum. Given that angular momentum is naturally interpreted as rotational motion of a substance with inertia, a thorough analysis of the Dirac equation with this interpretation of spin angular momentum is long overdue.

A fundamental principle of analysis is that one should strive to understand simple systems before attempting to analyze more complex systems. Rather than attempting to derive results via mathematical proofs or fit mathematical parameters to experimental data, we instead use simple examples of plane waves to demonstrate how terms in the Dirac equation relate to a specific physical model.

We start by modeling an ideal elastic solid, and assume a simple vector wave equation for the evolution of spin density. We then factor the vector wave equation to obtain a first-order Dirac equation for bispinor fields, and construct plane wave solutions. Calculations of physical quantities utilize operators that are compared with those of relativistic quantum mechanics. We construct an appropriate Lagrangian and Hamiltonian, including operators for potential and kinetic energy. Finally, we discuss possibilities for applying these results to the study of elementary particles and their interactions.

2. AN EQUATION FOR SPIN DENSITY

2.1. Ideal elastic solid

We consider the case of an isotropic, homogeneous solid with a linear relationship between infinitesimal stress and strain. The usual expression for potential energy is (e.g. Ref. 23):

$$\int U\, d^3\mathbf{r} = \int \left(\frac{1}{2}\lambda(\nabla \cdot \boldsymbol{\xi})^2 + \mu e_{ij} e_{ij} \right) d^3\mathbf{r} \tag{1}$$

where $\boldsymbol{\xi}(\mathbf{r}, t)$ represents displacement, $e_{ij} = (\partial_i \xi_j + \partial_j \xi_i)/2$ is the symmetric strain tensor, and λ and μ are the Lamé parameters. This expression has the drawback that it does not cleanly separate compressible and rotational motion. We can remedy this as follows:

Expanding the square of the symmetrical strain tensor yields:

$$\begin{aligned} e_{ij} e_{ij} &= \left[(\partial_x \xi_x)^2 + (\partial_y \xi_y)^2 + (\partial_z \xi_z)^2 \right] \\ &+ \frac{1}{2} \left[(\partial_x \xi_y + \partial_y \xi_x)^2 + (\partial_y \xi_z + \partial_z \xi_y)^2 + (\partial_z \xi_x + \partial_x \xi_z)^2 \right]. \end{aligned} \tag{2}$$

Add $2(\partial_x \xi_x \partial_y \xi_y + \partial_y \xi_y \partial_z \xi_z + \partial_z \xi_z \partial_x \xi_x)$ to the first term in square brackets and subtract it from the second term to obtain:

$$\begin{aligned} e_{ij} e_{ij} &= (\nabla \cdot \boldsymbol{\xi})^2 \\ &\quad + \frac{1}{2} \left[(\partial_x \xi_y + \partial_y \xi_x)^2 + (\partial_y \xi_z + \partial_z \xi_y)^2 + (\partial_z \xi_x + \partial_x \xi_z)^2 \right] \\ &\quad - 2(\partial_x \xi_x \partial_y \xi_y + \partial_y \xi_y \partial_z \xi_z + \partial_z \xi_z \partial_x \xi_x). \end{aligned} \tag{3}$$

Since this term occurs as an integrand for the potential energy, we can integrate the extra terms by parts on each of the two derivatives (neglecting contributions from total derivatives, which are assumed to integrate to zero) to obtain:

$$\begin{aligned} e_{ij}e_{ij} \rightarrow & (\nabla \cdot \boldsymbol{\xi})^2 \\ & + \frac{1}{2}\left((\partial_x \xi_y + \partial_y \xi_x)^2 + (\partial_y \xi_z + \partial_z \xi_y)^2 + (\partial_z \xi_x + \partial_x \xi_z)^2\right) \\ & - 2(\partial_x \xi_y \partial_y \xi_x + \partial_y \xi_z \partial_z \xi_y + \partial_z \xi_x \partial_x \xi_z). \end{aligned} \tag{4}$$

This is equivalent to:

$$e_{ij}e_{ij} \rightarrow (\nabla \cdot \boldsymbol{\xi})^2 + \frac{1}{2}(\nabla \times \boldsymbol{\xi})^2 \,. \tag{5}$$

The potential energy density may therefore be expressed as:

$$U = \frac{1}{2}(\lambda + 2\mu)(\nabla \cdot \xi)^2 + \frac{1}{2}\mu(\nabla \times \boldsymbol{\xi})^2. \tag{6}$$

This form of the potential energy density separates infinitesimal irrotational and incompressible motion. It is a quadratic function of the first derivatives of displacement. The Lagrangian for infinitesimal incompressible motion is the difference between kinetic and potential energies:

$$\mathcal{L} = \int \left(\frac{1}{2}\rho(\partial_t \boldsymbol{\xi})^2 - \frac{1}{2}\mu(\nabla \times \boldsymbol{\xi})^2\right) d^3r. \tag{7}$$

The Euler-Lagrange equation is the usual equation for infinitesimal shear waves:

$$\partial_t^2 \boldsymbol{\xi} = -\frac{\mu}{\rho}\nabla \times \nabla \times \boldsymbol{\xi} \tag{8}$$

for which the wave speed is $c = \sqrt{\mu/\rho}$. The incompressible potential energy in equation (7) was used by MacCullagh in 1837 to derive equation (8) as a description of light waves. [24]

The wave momentum is:

$$P_i = -\frac{\partial \mathcal{L}}{\partial_t \xi_j}\partial_i \xi_j = -\rho \partial_t \xi_j \partial_i \xi_j \,. \tag{9}$$

We are interested in incompressible plane wave solutions. Multiplying by the wave velocity component $v_i = ce_i$ (where e_i is the direction cosine), and applying the continuity equation $\partial_t \xi_j = -v_k \partial_k \xi_j$ yields:

$$v_i P_i = \rho(ce_i \partial_i \xi_j)(ce_k \partial_k \xi_j) = \mu(\partial_w \xi_j)(\partial_w \xi_j) \tag{10}$$

where ∂_w is the spatial derivative in the direction of wave propagation. Since shear waves propagate perpendicular to $\boldsymbol{\xi}$, this is equivalent to:

$$v_i P_i = \mu(\nabla \times \boldsymbol{\xi})^2 \tag{11}$$

which is twice the potential energy density. This result will later be compared with its Dirac equivalent.

2.2. Spin angular momentum

It is well known that elastic waves in solids have two types of momentum: that of the medium ($\rho\partial_t \boldsymbol{\xi}$) and that of the wave: $\rho(\nabla \xi_j)\partial_t \xi_j$ (see e.g. Ref. 25). Clearly there must also be two types of angular momentum in an elastic solid: "spin" associated with rotation of the medium, and "orbital" associated with rotation of the wave. However, spin angular momentum has not been considered to be a classical physics concept until recently. A brief review is presented here.

Considering only incompressible motion, the Helmholtz decomposition of momentum density $\mathbf{p}$ yields the curl of a vector field, e.g. $\mathbf{p} = \frac{1}{2}\nabla \times \mathbf{s}$. The vector field $\mathbf{s}$ has been shown to represent angular momentum density corresponding to spin in relativistic quantum mechanics. [12–14] Hence we refer to $\mathbf{s}$ as "spin density".

This relationship between spin and intrinsic momentum densities is quite general. Belinfante and Rosenfeld showed that it must be true quantum mechanically. [26, 27] More recently, Bliokh *et. al.* showed that this relationship holds for water gravity waves. [28]

Assuming sufficiently rapid fall-off at large distances, the volume integral of spin density is equal to the volume integral of the first moment of momentum $\mathbf{r} \times \mathbf{p}$. The two representations of angular momentum density are related by integration by parts: [14]

$$\begin{aligned} &\int \mathbf{r} \times \frac{1}{2}(\nabla \times \mathbf{s}) d^3r = \frac{1}{2}\int \left(\nabla(\mathbf{r}\cdot\mathbf{s}) - \mathbf{r}\cdot\nabla\mathbf{s} - \mathbf{s}\cdot\nabla\mathbf{r}\right) d^3r \\ &= \frac{1}{2}\int \left(\nabla(\mathbf{r}\cdot\mathbf{s}) - \partial_i(r_i\mathbf{s}) + \mathbf{s}(\nabla\cdot\mathbf{r}) - \mathbf{s}\cdot\nabla\mathbf{r}\right) d^3r \\ &= \int \mathbf{s}\, d^3r\,. \end{aligned} \tag{12}$$

The total derivatives do not contribute to the last line because they can be converted into surface integrals that are assumed to vanish.

Unlike the "moment of momentum" definition of angular momentum, spin density is an intrinsic property defined at each point in space. Coordinate-independent descriptions of rotational dynamics can actually be traced back to the nineteenth century.[29] In 1891 Oliver Heaviside recognized MacCullagh's force density in equation (8) as being the curl of a torque density that is proportional to an infinitesimal rotation angle. [30] However, this idea seems to have been largely forgotten.

The rotational kinetic energy is: [13]

$$\begin{aligned} K_R &= \frac{1}{2\rho}\int \tilde{p}^2 d^3r = \frac{1}{2\rho}\int \left[\frac{1}{2}\nabla\times\mathbf{s}\right]^2 d^3r \\ &= \frac{1}{8\rho}\int \left[\mathbf{s}\cdot[\nabla\times(\nabla\times\mathbf{s})] + \nabla\cdot(\mathbf{s}\times(\nabla\times\mathbf{s}))\right] d^3r \\ &= \frac{1}{2}\int \mathbf{w}\cdot\mathbf{s}\, d^3r, \end{aligned} \tag{13}$$

where $\mathbf{w} = \nabla\times\mathbf{u}/2$ is the instantaneous angular velocity (sometimes confusingly referred to as "spin" in the literature). In this case the divergence term does not to contribute to the volume integral because it can be converted into a surface integral at infinity (and assumed to vanish).

For a Lagrangian density dependent on motion only through kinetic energy, the spin density ($\mathbf{s}$) is the momentum conjugate to angular velocity:

$$\frac{\delta}{\delta w_i}\int \frac{1}{2} w_j s_j\, d^3r = \frac{1}{2}\int \left(\frac{\delta w_j}{\delta w_i} s_j + w_j \frac{\delta s_j}{\delta w_i}\right) d^3r = \frac{1}{2}s_i + \frac{1}{2}s_i = s_i\,, \tag{14}$$

where integration by parts was used twice to evaluate the second term in the integral.

Spin density can be used to describe rigid rotations as well. See Ref. 14 for an example.

A popular introductory text on quantum mechanics states that "these phenomena involve a quantum degree of freedom called spin, which has no classical counterpart". [31] This common claim that spin angular momentum has no classical physics analogue is incorrect. Spin angular momentum is simply the coordinate-independent form of classical angular momentum.

2.3. Equation of evolution

Assuming incompressible motion with velocity $\mathbf{u} = (1/(2\rho))\nabla\times\mathbf{s} = \partial_t\boldsymbol{\xi}$, equation (8) becomes:

$$\frac{1}{2}\partial_t(\nabla\times\mathbf{s}) + \mu\nabla\times(\nabla\times\boldsymbol{\xi}) = 0. \tag{15}$$

Assuming $\nabla\cdot\mathbf{s} = 0$, the Helmholtz decomposition yields:

$$\partial_t\mathbf{s} + 2\mu\nabla\times\boldsymbol{\xi} = 0. \tag{16}$$

This equation states that the rate of change of spin density is equal to torque density, which is proportional to rotation angle $(1/2)\nabla\times\boldsymbol{\xi}$ for infinitesimal displacements.

The next step is to relate the displacement $\boldsymbol{\xi}$ to the spin density $\mathbf{s}$. Define a vector potential $\mathbf{Q}$ such that $\partial_t \mathbf{Q} = \mathbf{s}$. Since the curl of $\mathbf{s}$ is proportional to velocity, the curl of $\mathbf{Q}$ must be proportional to displacement:

$$\frac{1}{2\rho}\nabla \times \mathbf{Q} = \boldsymbol{\xi}\,. \tag{17}$$

Therefore the linear equation for $\mathbf{s}$ is equivalent to:

$$\partial_t^2 \mathbf{Q} + c^2 \nabla \times \nabla \times \mathbf{Q} = 0\,, \tag{18}$$

where $c^2 = \mu/\rho$. The curl of this equation yields equation (8). The torque density is $\boldsymbol{\tau} = -c^2 \nabla \times \nabla \times \mathbf{Q}$.

Thus far we have assumed infinitesimal motion. We could instead start from the nonlinear equation for momentum density:

$$\partial_t \mathbf{p} + \mathbf{u} \cdot \nabla \mathbf{p} = \mathbf{f}, \tag{19}$$

where $\mathbf{f}$ is the force density. This equation implies that changes to momentum density can only result from translation or force. It is the consequence of translational symmetry of the physical system. Newton's third law implies that the force may be regarded as an equal and opposite change of momentum of its source. In an elastic solid, this means that the change in canonical momentum is equal and opposite to the change in dynamical momentum. One drawback of equation (19) is that it combines both incompressible and irrotational contributions to momentum density.

In addition to translational symmetry, the physical system also has rotational symmetry, implying conservation of angular momentum. This constraint is expressed by the equation:

$$\partial_t \mathbf{s} + \mathbf{u} \cdot \nabla \mathbf{s} - \mathbf{w} \times \mathbf{s} = -c^2 \nabla \times \nabla \times \mathbf{Q}. \tag{20}$$

The logic of this equation is that changes of spin density can only result from translation, rotation, or torque. Since total angular momentum is conserved, torque density is equivalent to minus the rate of change of orbital angular momentum density.

Since spin density is a fundamental physical quantity, it is reasonable to assume that it satisfies a single equation of evolution everywhere in space and time. Eq. 20 is a sensible candidate for such an "equation of everything."

Eq. 20 can be put in Lorentz-covariant form using the four-position $x^\alpha = (ct, x, y, z)$ and metric $g_{\mu\nu} = g^{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. We assume that $Q^\alpha = (0, Q_x, Q_y, Q_z)$ and $\nabla \cdot \mathbf{Q} = 0$ in the "rest" or "lab" frame of reference. Define the Lorentz four-velocity as $U^\alpha = (c, 0, 0, 0)$ in the "rest" frame. The four-displacement is $\xi^\alpha = \epsilon^{\alpha\beta\chi\delta}\partial_\beta Q_\chi U_\delta$. This can be combined with the four-vector Q^α to form an antisymmetric tensor $\tilde{Q}^{\mu\nu} = (1/c)(U^\mu Q^\nu - Q^\mu U^\nu)$. Then the four-spin density is $s^\alpha = \partial_\mu \tilde{Q}^{\mu\alpha}$, the four-momentum density is $p^\alpha = \rho u^\alpha = (1/2)\epsilon^{\alpha\beta\chi\delta}\partial_\beta s_\chi U_\delta$, and the four-angular velocity is $w^\alpha = (1/4\rho)\epsilon^{\alpha\beta\chi\delta}\partial_\beta p_\chi U_\delta$. The rotation rate matrix is $w^\alpha_{\ \mu} = \epsilon^{\alpha\beta\chi\delta} g_{\beta\mu} U_\chi w_\delta$. The Lorentz-covariant equation is then:

$$c^2 \partial^\mu \partial_\mu Q^\alpha + u^\mu \partial_\mu s^\alpha - w^\alpha_{\ \mu} s^\mu = 0 \tag{21}$$

The velocity of the medium u^α should not be confused with the Lorentz four-velocity (U^α), which only depends on the relative motion of reference frames, or with wave velocity $v^\alpha = (c, v_x, v_y, v_z)$, which quantifies wave propagation rather than motion of the medium.

The Lorentz transformations relate measurements in different reference frames, and are applicable whenever measurements are made exclusively with waves having a single characteristic speed. [32] Since absolute motion cannot be measured in this way, each inertial observer naturally treats their own reference frame as the"rest" frame. Although the waves propagate in Galilean space-time, the measurements made with these waves form a Minkowski space. Lorentz transformations are applicable to light and matter because both are described by Lorentz-covariant wave equations with the same characteristic speed (c), even though matter waves have group velocities with magnitudes less than c. MacCullagh [24] and Maxwell [33] similarly assumed a Galilean physical space-time in deriving relativistic equations for light and electromagnetism, respectively.

Although equation (20) may be sensible, an alternative would be:

$$\partial_t \mathbf{s} + \frac{1}{2}\nabla \times (\mathbf{s} \times \mathbf{u}) = -c^2 \nabla \times \nabla \times \mathbf{Q}. \tag{22}$$

This equation differs from equation (20) only by factors proportional to $\nabla \cdot \mathbf{u}$, $\nabla \cdot \mathbf{s}$, and $\nabla(\mathbf{u} \cdot \mathbf{s})$. Incompressibility requires $\nabla \cdot \mathbf{u} = 0$. We can choose to make $\nabla \cdot \mathbf{s} = 0$ everywhere since only the curl of spin density has physical significance. The equation of evolution (22) then guarantees that $\nabla \cdot \mathbf{s}$ does not change over time.

For simple plane waves, there is no difference between equations (20) and (22). The rest of this paper only deals with equation (20).

2.4. Dirac equation

To understand the Dirac equation, consider equation (18), which is a second-order differential equation for the vector field $\mathbf{Q}(\mathbf{r}, t)$. There are often benefits in converting a second-order equation to a set of first-order equations. We will do this by following Refs. [12] and [14], starting with one-dimensional waves and then generalizing to three dimensions.

2.4.1. One-dimensional waves

Consider a one-component wave propagating in one-dimension with amplitude of $Q(z, t)$. If the wave equation is

$$\partial_t^2 Q - c^2 \partial_z^2 Q = 0\,, \tag{23}$$

the derivative operators can be factored to yield:

$$(\partial_t + c\partial_z)(\partial_t - c\partial_z)Q = 0\,. \tag{24}$$

The general solution consists of backward (B) and forward (F) propagating waves:

$$Q = Q_B(ct + z) + Q_F(ct - z)\,. \tag{25}$$

The two directions of wave propagation are clearly independent states, and they are separated in space by a 180° rotation. This property is the fundamental characteristic of spin one-half states. Generalization to three dimensional space therefore involves spinor or bispinor wave functions.

The forward and backward waves satisfy the equations:

$$\begin{aligned} \partial_t Q_B &= \partial_z Q_B\,, \\ \partial_t Q_F &= -\partial_z Q_F\,. \end{aligned} \tag{26}$$

Defining $\dot{Q} = \partial_t Q$, we can write the wave equation as a first-order matrix equation:

$$\partial_t \begin{bmatrix} \dot{Q}_B \\ \dot{Q}_F \end{bmatrix} - c\partial_z \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{bmatrix} \dot{Q}_B \\ \dot{Q}_F \end{bmatrix} = 0\,. \tag{27}$$

The matrix simply transforms temporal derivatives to spatial derivatives as in equation (26). Applying this transformation and summing the equations for $\dot{Q}_B$ and $\dot{Q}_F$ then recovers the original wave equation.

We have thus achieved the goal of converting a one-dimensional second-order wave equation into a first-order matrix equation. Although generalization to three dimensional vector waves involves some mathematical complexity, it does not involve any fundamentally new concepts. A clue can be found in the fact that the matrix for spatial derivatives is the Pauli matrix σ_z.

First, note that the procedure above specifies independent components with positive and negative wave velocity, and uses a diagonal matrix to relate spatial and temporal derivatives. We can apply a similar technique to separate positive and negative values of the wave time derivatives. Letting Q_B and Q_F represent the z-components of vectors, separate each component of the wave into positive and negative contributions ($\dot{Q}_B = \dot{Q}_{B+} - \dot{Q}_{B-}$ and $\dot{Q}_F = \dot{Q}_{F+} - \dot{Q}_{F-}$) so that each of the four wave components ($\dot{Q}_{B+}, \dot{Q}_{B-}, \dot{Q}_{F+}, \dot{Q}_{F-}$) is positive-definite. With these definitions, we can use a matrix expression for $\dot{Q}$:

$$\dot{Q} = \partial_t Q = \frac{1}{2} \begin{bmatrix} \dot{Q}_{B+}^{1/2} \\ \dot{Q}_{F-}^{1/2} \\ \dot{Q}_{F+}^{1/2} \\ \dot{Q}_{B-}^{1/2} \end{bmatrix}^T \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{bmatrix} \dot{Q}_{B+}^{1/2} \\ \dot{Q}_{F-}^{1/2} \\ \dot{Q}_{F+}^{1/2} \\ \dot{Q}_{B-}^{1/2} \end{bmatrix} = \frac{1}{2}\psi^T \sigma_z \psi \tag{28}$$

where σ_z is the 4×4 Dirac matrix for the z-component of spin density, and the four-component column vector is called a (one-dimensional) Dirac bispinor. In one dimension, the significance of simultaneous positive and negative components is unclear. We will see that in three dimensions, simultaneous positive and negative components for one direction can (but doesn't necessarily) describe polarization in a different direction.

The spatial derivative is now given by:

$$c\partial_z Q = \frac{1}{2}\begin{bmatrix}\dot{Q}_{B+}^{1/2}\\ \dot{Q}_{F-}^{1/2}\\ \dot{Q}_{F+}^{1/2}\\ \dot{Q}_{B-}^{1/2}\end{bmatrix}^T \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}\begin{bmatrix}\dot{Q}_{B+}^{1/2}\\ \dot{Q}_{F-}^{1/2}\\ \dot{Q}_{F+}^{1/2}\\ \dot{Q}_{B-}^{1/2}\end{bmatrix} = \frac{1}{2}\psi^T\beta^3\psi\,. \tag{29}$$

The matrix $-\beta^3$ is the Dirac matrix for chirality (equal to the matrix γ^5 in the standard chiral representation). If the amplitude (Q) represents rotation angle, then positive and negative chirality ($-\partial_z a$) are analogous to right- and left-handed threads on a screw (denoted by R and L, respectively). The chirality projection operators are:

$$\begin{aligned}\frac{1}{2}(I+\beta^3)\psi &\equiv \psi_L\\ \frac{1}{2}(I-\beta^3)\psi &\equiv \psi_R\end{aligned} \tag{30}$$

2.4.2. *Wave velocity and Lorentz boosts*

Wave velocity (v) is obtained by combining the two matrices used above:

$$v\psi = c\begin{pmatrix}-1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}\begin{bmatrix}\dot{Q}_{B+}^{1/2}\\ \dot{Q}_{F-}^{1/2}\\ \dot{Q}_{F+}^{1/2}\\ \dot{Q}_{B-}^{1/2}\end{bmatrix} = -c\beta^3\sigma_z\psi\,. \tag{31}$$

We can define a "weighted wave velocity" from the difference between forward and backward amplitudes divided by the sum of forward and backward amplitudes:

$$v = c\,\frac{|\dot{Q}_{F+}|+|\dot{Q}_{F-}|-|\dot{Q}_{B+}|-|\dot{Q}_{B-}|}{|\dot{Q}_{F+}|+|\dot{Q}_{F-}|+|\dot{Q}_{B+}|+|\dot{Q}_{B-}|} = -\frac{\psi^\dagger c\beta^3\sigma_z\psi}{\psi^\dagger\psi} \tag{32}$$

The magnitude $|\dot{Q}|$ and rapidity α can be suitable chosen to satisfy:

$$\begin{aligned}|\dot{Q}_{F+}|+|\dot{Q}_{F-}| &= |\dot{Q}|\exp(\alpha)\\ |\dot{Q}_{B+}|+|\dot{Q}_{B-}| &= |\dot{Q}|\exp(-\alpha),\end{aligned} \tag{33}$$

so that the weighted wave velocity becomes:

$$v = c\,\frac{|\dot{Q}|\exp(\alpha)-|\dot{Q}|\exp(-\alpha)}{|\dot{Q}|\exp(\alpha)+|\dot{Q}|\exp(-\alpha)} = c\,\tanh(\alpha)\,. \tag{34}$$

A Lorentz boost $\psi' = \exp(-\beta^3\sigma_z\alpha_1)\psi$ changes the weighted wave velocity ($v \to v'$) by altering the relative strength of forward and backward waves:

$$\begin{aligned}v' &= c\,\frac{(\exp(-\beta^3\sigma_z\alpha_1/2)\psi)^\dagger(-\beta^3\sigma_z)(\exp(-\beta^3\sigma_z\alpha_1/2)\psi)}{(\exp(-\beta^3\sigma_z\alpha_1/2)\psi)^\dagger(\exp(-\beta^3\sigma_z\alpha_1/2)\psi)}\\ &= c\,\frac{|\dot{Q}|\exp(\alpha+\alpha_1)-|\dot{Q}|\exp(-\alpha-\alpha_1)}{|\dot{Q}|\exp(\alpha+\alpha_1)+|\dot{Q}|\exp(-\alpha-\alpha_1)} = c\,\tanh(\alpha+\alpha_1)\,.\end{aligned} \tag{35}$$

Thus, the concept of rapidity emerges naturally from the separation of forward and backward waves propagating in Galilean space-time.

2.4.3. Three-dimensional vector waves

Combining equations (28) and (29), the one-dimensional linear wave equation may be written in the form:

$$\partial_t[\psi^T \sigma_z \psi] - c\partial_z[\psi^T \beta^3 \psi] = 2(\partial_t^2 Q - c^2 \partial_z^2 Q) = 0\,. \tag{36}$$

Expanding the derivatives yields:

$$\psi^T \sigma_z \partial_t \psi - \psi^T c\beta^3 \partial_z \psi + \text{adjoint} = 0\,. \tag{37}$$

Factoring $\psi^T \sigma_z$ then yields:

$$\psi^T \sigma_z (\partial_t \psi - c\beta^3 \partial_z \sigma_z \psi) + \text{adjoint} = 0\,. \tag{38}$$

This one-dimensional Dirac equation is itself useful for teaching purposes. [34, 35] However, its equivalence with the one-dimensional second-order wave equation has not been widely recognized. Next we will show how to generalize the first-order equation to three spatial dimensions.

Generalization to three dimensions is based on geometric algebra. This algebra derives from the fact that there are two independent ways to construct a product of 3-vectors: scalar product and cross product. These two products measure the degree to which two vectors are parallel (scalar product) or perpendicular (cross product). The cross product additionally defines the plane of the two vectors, and is therefore sometimes called the "directed area product". These two products can be combined into a single product by making the cross product imaginary: [36]

$$\mathbf{ab} = \mathbf{a} \cdot \mathbf{b} + \mathrm{i}(\mathbf{a} \times \mathbf{b})\,. \tag{39}$$

The unit imaginary defines an oriented volume:

$$\begin{aligned} \hat{\mathbf{x}}\hat{\mathbf{y}}\hat{\mathbf{z}} &= (\mathrm{i}\hat{\mathbf{x}} \times \hat{\mathbf{y}}) \cdot \hat{\mathbf{z}} = \mathrm{i}\,, \\ \hat{\mathbf{z}}\hat{\mathbf{y}}\hat{\mathbf{x}} &= (\mathrm{i}\hat{\mathbf{z}} \times \hat{\mathbf{y}}) \cdot \hat{\mathbf{x}} = -\mathrm{i}\,. \end{aligned} \tag{40}$$

Generalization of the Dirac equation to three dimensions consists of finding spin and velocity matrices with the same algebra as unit vectors:

$$\hat{x}_i \hat{x}_j = \delta_{ij} + \mathrm{i}\epsilon_{ijk}\hat{x}_k \tag{41}$$

The Pauli spin matrices $\boldsymbol{\sigma}^P = (\sigma_x^P, \sigma_y^P, \sigma_z^P)$ have this property. Arbitrary vector components a_i can be computed from a 2-component complex wave function η as follows:

$$\begin{aligned} a_x &= \eta^\dagger \sigma_x^P \eta = \eta^\dagger \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \eta\,, \\ a_y &= \eta^\dagger \sigma_y^P \eta = \eta^\dagger \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \eta\,, \\ a_z &= \eta^\dagger \sigma_z^P \eta = \eta^\dagger \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \eta\,. \end{aligned} \tag{42}$$

The Pauli matrices may in general represent axial or polar vectors, but they are most commonly associated with spin density, which is an axial vector. The fourth independent matrix in this algebra is the identity matrix (I). At each point, the direction of the vector $\eta^\dagger \boldsymbol{\sigma}^P \eta$ can be rotated by an arbitrary angle φ about an axis $\hat{\mathbf{e}}_\varphi$ using operations of the form (with $\boldsymbol{\varphi} = \varphi \hat{\mathbf{e}}_\varphi$):

$$R_{\boldsymbol{\varphi}}(\eta^\dagger \boldsymbol{\sigma}^P \eta) = \eta^\dagger \exp\left(\mathrm{i}\boldsymbol{\sigma}^P \cdot \boldsymbol{\varphi}/2\right) \boldsymbol{\sigma}^P \exp\left(-\mathrm{i}\boldsymbol{\sigma}^P \cdot \boldsymbol{\varphi}/2\right) \eta. \tag{43}$$

For example, $\exp\left(-\mathrm{i}\sigma_z^P \pi/4\right) \sigma_x^P \exp\left(\mathrm{i}\sigma_z^P \pi/4\right) = \sigma_y^P$. So to find η' such that $\eta'^\dagger \sigma_y^P \eta' = \eta^\dagger \sigma_x^P \eta$, the rotated wave function must be $\eta' = \exp\left(-\mathrm{i}\sigma_z^P \pi/4\right)\eta$. This transformation rotates the wave polarization direction from $\hat{\mathbf{x}}$ to $\hat{\mathbf{y}}$.

Rotations of the field (as opposed to a single point) would also require $\mathbf{r} \to R_{\boldsymbol{\varphi}}^{-1}\mathbf{r}$. Thus

$$R_\varphi(\eta(\mathbf{r}, t)) = \exp\left(-\mathrm{i}\boldsymbol{\sigma}^P \cdot \boldsymbol{\varphi}/2\right) \eta(R_\varphi^{-1}\mathbf{r}, t)\,. \tag{44}$$

The Dirac wave functions specify not a single vector, but spatial and temporal derivatives of a vector field. Forward and backward waves along each axis are combined by replacing the Pauli matrices with the corresponding 4×4

Dirac spin matrices and replacing the two-component spinor η with a 4-component bispinor ψ. In terms of the Pauli matrices, the Dirac spin matrices $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ are:

$$\sigma_x = \begin{pmatrix} \sigma_x^P & \mathbf{0} \\ \mathbf{0} & \sigma_x^P \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} \sigma_y^P & \mathbf{0} \\ \mathbf{0} & \sigma_y^P \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} \sigma_z^P & \mathbf{0} \\ \mathbf{0} & \sigma_z^P \end{pmatrix}, \tag{45}$$

where $\mathbf{0}$ is the 2×2 null matrix.

Just as there are three Pauli matrices indicating different directions of wave polarization, there are also three orthogonal matrices associated with spatial derivatives (and also related to wave velocity). We will denote these as:

$$\beta^1 = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix}, \quad \beta^2 = \begin{pmatrix} \mathbf{0} & -\mathrm{i}\mathbf{I} \\ \mathrm{i}\mathbf{I} & \mathbf{0} \end{pmatrix}, \quad \beta^3 = \begin{pmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{I} \end{pmatrix}, \tag{46}$$

where $\mathbf{I}$ is the 2×2 identity matrix. Compared with the chiral notation of relativistic quantum mechanics, $\beta^3 = -\gamma^5$ and $\beta^1 = \gamma^0$. Equation (31) implies that the matrix $-\beta^3\boldsymbol{\sigma}$ tabulates wave velocity. Since $\beta^1\beta^2 = \mathrm{i}\beta^3$, rotations in β-space are performed similarly to rotations in σ-space. Although the β matrices are clearly associated with spatial derivatives, they are not explicitly associated with the directional unit vectors that define the spin direction.

The one-dimensional wave equation (36) has the bispinor form:

$$\left\{\psi^T\sigma_z\partial_t\psi - c\psi^T\beta^3\partial_z\psi\right\} + \mathrm{Transpose} = 0\,. \tag{47}$$

We can separate a common factor of $\psi^T\sigma_z$:

$$\psi^T\sigma_z\left\{\partial_t\psi - c\beta^3\sigma_z\partial_z\psi\right\} + \mathrm{Transpose} = 0\,. \tag{48}$$

For arbitrary vector components and derivatives, the matrices and spatial derivatives are generalized to arbitrary directions by allowing for three indices ($i = (x, y, z)$ and $j = (x, y, z)$), and the bispinor wave functions are allowed to be complex:

$$\psi^\dagger\sigma_i\left\{\partial_t\psi - c\beta^3\sigma_j\partial_j\psi\right\} + \mathrm{adjoint} = 0\,. \tag{49}$$

This is the first-order wave equation for vector waves in three dimensions. The wave function of a free electron satisfies the same equation. Start with the Dirac equation for a free electron:

$$\partial_t\psi - c\beta^3\sigma_j\partial_j\psi + \mathrm{i}\Omega\beta^1\psi = 0 \tag{50}$$

with $\Omega = m_ec^2/\hbar$. Multiplication by $\psi^\dagger\sigma_i$ and addition of the adjoint yields equation (49).

Expanding the spatial derivative term in equation (49) yields the 3-D generalization of the wave equation (36):

$$\partial_t\left[\psi^\dagger\boldsymbol{\sigma}\psi\right] - c\nabla\left[\psi^\dagger\beta^3\psi\right] + \mathrm{i}c\left\{\left[\nabla\psi^\dagger\right]\times\beta^3\boldsymbol{\sigma}\psi + \psi^\dagger\beta^3\boldsymbol{\sigma}\times\nabla\psi\right\} = 0\,. \tag{51}$$

The terms correspond, in order, to twice those in the vector wave equation:

$$\partial_t^2\mathbf{Q} - c^2\nabla(\nabla\cdot\mathbf{Q}) + c^2\nabla\times(\nabla\times\mathbf{Q}) = 0\,. \tag{52}$$

Thus equation (51) is the result we have been seeking. We have rewritten the second-order vector wave equation as a first order equation involving Dirac bispinors. The validity of this correspondence, which we will confirm with examples, demonstrates that the Dirac equation of relativistic quantum mechanics may be regarded as a first-order representation of an ordinary second-order vector wave equation.

Furthermore, the evolution of the spin density vector field of a free electron is identical to the linear evolution of spin density in an elastic solid. This simple fact justifies the study of an elastic solid as a model of the vacuum.

Equation (52) yields the following physical correspondences: [12]

$$\mathbf{s} = \partial_t\mathbf{Q} \equiv \frac{1}{2}\left[\psi^\dagger\boldsymbol{\sigma}\psi\right]; \tag{53a}$$

$$c\nabla\cdot\mathbf{Q} \equiv \frac{1}{2}\left[\psi^\dagger\beta^3\psi\right]; \tag{53b}$$

$$c^2\left\{\nabla\times\nabla\times\mathbf{Q}\right\} \equiv \frac{\mathrm{i}c}{2}\left\{\left[\nabla\psi^\dagger\right]\times\beta^3\boldsymbol{\sigma}\psi + \psi^\dagger\beta^3\boldsymbol{\sigma}\times\nabla\psi\right\}; \tag{53c}$$

$$0 = \frac{\mathrm{i}c}{2}\nabla\cdot\left\{\left[\nabla\psi^\dagger\right]\times\beta^3\boldsymbol{\sigma}\psi + \psi^\dagger\beta^3\boldsymbol{\sigma}\times\nabla\psi\right\}\ . \tag{53d}$$

These identifications provide seven independent constraints on the eight free parameters of the complex Dirac bispinor: three for the first, one for the second, two for the third (since a curl has only two independent components), and one for the fourth. There is also an arbitrary overall phase factor. The last equation simply states that the divergence of a curl is zero. This condition is necessary for consistency. Velocity and angular velocity are:

$$\mathbf{u} = \tfrac{1}{2\rho}\nabla \times \mathbf{s} = \tfrac{1}{2\rho}\nabla \times \partial_t \mathbf{Q} = \tfrac{1}{4\rho}\nabla \times \left[\psi^\dagger \boldsymbol{\sigma}\psi\right] ; \tag{54a}$$

$$\mathbf{w} = \tfrac{1}{2}\nabla \times \mathbf{u} = \tfrac{1}{4\rho}\nabla \times \nabla \times \partial_t \mathbf{Q} = \tfrac{1}{8\rho}\nabla \times \nabla \times \left[\psi^\dagger \boldsymbol{\sigma}\psi\right] . \tag{54b}$$

The classical and quantum mechanical expressions for spin angular momentum differ by a factor of $\hbar$. This is of course a mere convention. There is no question that the quantum mechanical single-particle Dirac equation describes the deterministic evolution of spin density.

According to the above analysis, the first-order Dirac equation is a kind of factorization (or square root) of a second-order vector wave equation. Others have made different factorizations of wave equations using multivariate 4-vectors, quaternions, or octonions. [37–40]

The first-order wave equation (49) can be reduced to:

$$\partial_t \psi - c\beta^3 \boldsymbol{\sigma} \cdot \nabla \psi + \mathrm{i}\chi\psi = 0 \,, \tag{55}$$

where χ is any operator with the property

$$\mathrm{Re}\left\{\psi^\dagger \sigma_j \mathrm{i}\chi\psi\right\} = 0 \,. \tag{56}$$

The equation for a free electron is obtained by the choosing $\chi = \Omega\beta^1 = \Omega\gamma^0$ with $\Omega = m_e c^2/\hbar$. This term represents rotation of wave velocity,[18] and has also been interpreted as describing circular particle motion. [41]

Multiplying equation (55) by $\psi^\dagger$ and adding the adjoint yields a conservation law with density $\psi^\dagger\psi$ and current $-\psi^\dagger c\beta^3\boldsymbol{\sigma}\psi$:

$$\partial_t(\psi^\dagger\psi) - \nabla \cdot (\psi^\dagger c\beta^3 \boldsymbol{\sigma}\psi) = 0 \,. \tag{57}$$

In quantum mechanics this equation is regarded as a conservation law for probability density, but in both classical and quantum mechanics it is part of the description of the evolution of spin density.

The four-vector for spin density is $(-\psi^\dagger\beta^3\psi, \psi^\dagger\sigma^i\psi)$. Since the time component represents a divergence in equation (53b), its volume integral can be converted to a surface integral at infinity. Assuming that the wave amplitude falls to zero sufficiently rapidly, this surface integral is zero. Thus, the time component of the total spin of elementary particles can be taken as zero in the rest frame. The stronger assumption that $\nabla \cdot \mathbf{Q} = 0$ everywhere in the rest frame may also be valid (as assumed earlier when constructing a Lorentz-covariant equation).

3. SPIN DENSITY PLANE WAVES

We present bispinor descriptions of plane wave solutions to the vector wave equation. These represent physical plane waves with oscillating spin density, unlike quantum mechanical so-called "plane waves" that merely have an oscillating phase factor. The nonlinear vector terms are zero for plane waves. However, we can use these solutions to determine appropriate nonlinear terms in the bispinor wave equation.

3.1. Linear plane wave solutions

We start with a description of a longitudinal wave:

$$\psi'_{s_z,v_z} = \sqrt{\frac{\omega Q_0}{2}} \begin{bmatrix} 0 \\ -1 + \cos(\omega t - kz) \\ 1 + \cos(\omega t - kz) \\ 0 \end{bmatrix} . \tag{58}$$

for which the only nonzero spin density component is $s_z = (1/2)\psi^\dagger\sigma_z\psi = \omega Q_0 \cos(\omega t - kz)$. The wave velocity operator is $-c\beta^3\sigma_z\hat{\mathbf{z}}$ as in equation (32). The divergence is given by $(1/2c)\psi^\dagger\beta^3\psi = -kQ_0\cos(\omega t - kz)$, where $k = \omega/c$. This wave function cannot describe spin density because it has zero curl. Also, the Dirac representation is not unique. This wave function has both positive and negative contributions to the scalar wave amplitude at each point, but it does not have discontinuities that would result from strict separation of positive and negative values. This wave function also has all real-valued Dirac components, and these remain real-valued under velocity rotation in the x-z plane.

We can rotate wave velocity by using the β matrices, with β^1 initially aligned with $\hat{\mathbf{x}}$ and β^2 initially aligned with $\hat{\mathbf{y}}$. Since, according to equation (29), the matrix β^3 is aligned with the z-axis, the β matrices form a right-handed coordinate system. Thus, spin-independent wave velocity rotation of $-\pi/2$ about the x-axis is accomplished by:

$$\psi'_{s_z,v_y} = \exp\left(\mathrm{i}\beta^1\pi/4\right)\psi'_{s_z,v_z}(z \to y, t) = \frac{\sqrt{\omega Q_0}}{2}\begin{bmatrix} \mathrm{i}(1+\cos(\omega t - ky)) \\ -1+\cos(\omega t - ky) \\ 1+\cos(\omega t - ky) \\ \mathrm{i}(-1+\cos(\omega t - ky)) \end{bmatrix}. \tag{59}$$

where the argument $z \to y$ indicates the effect of rotating the coordinates about the negative x-axis. The spin density for this wave function is $s_z = \omega Q_0 \cos(\omega t - ky)$. Interestingly, the quantity $-c\psi^\dagger\beta^3\sigma_y\psi = -c\omega Q_0 \sin(\omega t - ky)^2$ is not equal to $c|\psi|^2$ as expected for a velocity operator (it even has the wrong sign). As we will see more clearly below, this behavior arises from the fact that some of the terms in the wave function have zero gradient. An alternative wave velocity (or wave flow) calculation, $-c\psi^\dagger\beta^2\sigma_z\psi\hat{\mathbf{y}}$, does have magnitude of $c|\psi|^2$. This is expected since the $-\pi/2$ rotation about β^1 moves β^3 to β^2. The spatial derivative of Q_z is given by $\partial_y Q_z = (1/2c)\psi^\dagger\beta^2\psi = -kQ_0\cos(\omega t - ky)$. This is proportional to the displacement: $\boldsymbol{\xi} = (1/2\rho)\partial_y Q_z\hat{\mathbf{x}}$.

Similarly, spin-independent rotation of wave velocity by $\pi/2$ about the y-axis is given by:

$$\psi'_{s_z,v_x} = \exp\left(-\mathrm{i}\beta^2\pi/4\right)\psi'_{s_z,v_z}(z \to x, t) = \frac{\sqrt{\omega Q_0}}{2}\begin{bmatrix} -1-\cos(\omega t - kx) \\ -1+\cos(\omega t - kx) \\ 1+\cos(\omega t - kx) \\ -1+\cos(\omega t - kx) \end{bmatrix}, \tag{60}$$

which yields spin density of $s_z = \omega Q_0\cos(\omega t - kx)$. The wave velocity operator $-c\beta^3\sigma_x\hat{\mathbf{x}}$ again does not evaluate to $c|\psi|^2$, but the alternative wave velocity operator $-c\beta^1\sigma_z\hat{\mathbf{x}}$ does (the rotation about β^2 moves β^3 to β^1). The spatial derivative of Q_z is given by $\partial_x Q_z = (1/2c)\psi^\dagger\beta^1\psi = -kQ_0\cos(\omega t - kx)$. This is proportional to the displacement: $\boldsymbol{\xi} = -(1/2\rho)\partial_x Q_z\hat{\mathbf{y}}$.

Each of the above wave functions satisfies the linear Dirac equation:

$$\partial_t\psi - c\beta^3\sigma_j\partial_j\psi = 0\,. \tag{61}$$

To obtain a spin density aligned with the x-axis and propagating in the z-direction, we start with the longitudinal wave ψ'_{s_z,v_z}, rotate the velocity by $-\pi/2$ around the y-axis (from $\hat{\mathbf{z}}$ to $-\hat{\mathbf{x}}$) using β^2, then rotate the entire wave function by $\pi/2$ around the y-axis using σ_y:

$$\psi'_{s_x,v_z} = \exp\left(-\mathrm{i}\sigma_y\pi/4\right)\exp\left(\mathrm{i}\beta^2\pi/4\right)\psi'_{s_z,v_z}(z,t) = \sqrt{\frac{\omega Q_0}{2}}\begin{bmatrix} 1 \\ \cos(\omega t - kz) \\ \cos(\omega t - kz) \\ 1 \end{bmatrix}. \tag{62}$$

which yields spin density of $s_x = \omega Q_0\cos(\omega t - kz)$. With velocity aligned with the z-axis so that the wave velocity matrix is diagonal, we now see why the wave velocity operator $-c\beta^3\sigma_z$ does not evaluate to $c|\psi|^2$: the first and fourth wave function components contribute to $-c\psi^\dagger\beta^3\sigma_z\psi$ but not to the wave propagation term $-\psi^\dagger\beta^3\sigma_z\partial_z\psi$. Thus, $-c\beta^3\boldsymbol{\sigma}$ is a valid velocity operator when operating on the gradient of the wave function, but not when operating independent of the gradient operator. The alternative wave velocity operator $c\beta^1\sigma_x\hat{\mathbf{z}}$ does evaluate to $c|\psi|^2\hat{\mathbf{z}}$. The σ matrix associated with the alternative wave velocity operator is the matrix of the spin direction (in this case σ_x for spin density polarization along the x-axis).

This wave function ($\psi = \psi'_{s_x,v_z}$) yields the following terms in the second-order wave equation:

$$\mathbf{s} = \partial_t\mathbf{Q} = \frac{1}{2}\left[\psi^\dagger\boldsymbol{\sigma}\psi\right] = (\omega Q_0\cos(\omega t - kz), 0, 0)\,; \tag{63a}$$

$$c\nabla\cdot\mathbf{Q} = \frac{1}{2}\left[\psi^\dagger\beta^3\psi\right] = 0\,; \tag{63b}$$

$$(\nabla\times\nabla\times\mathbf{Q}) = \frac{\mathrm{i}}{2c}\left\{\left[\nabla\psi^\dagger\right]\times\beta^3\boldsymbol{\sigma}\psi - \psi^\dagger\beta^3\boldsymbol{\sigma}\times\nabla\psi\right\} = (k^2Q_0\sin(\omega t - kz), 0, 0). \tag{63c}$$

Since the wave velocity was rotated about β^2, the displacement $\boldsymbol{\xi}$ is now computed from the spatial derivative operator using the matrix $-\beta^1$ instead of β^3:

$$\xi_y = (1/2\rho)\partial_z Q_x = -(1/4c\rho)\psi^\dagger\beta^1\psi = -(kQ_0/2\rho)\cos(\omega t - kz)\,. \tag{64}$$

Given the transverse wave function ψ'_{s_x,v_z}, we can rotate the transverse wave velocity direction using the β^1 matrix. Keeping spin density along the x-axis, wave velocity in the y-direction is obtained by:

$$\psi'_{s_x,v_y}(y,t) = \exp(\mathrm{i}\beta^1\pi/4)\psi_{s_x,v_z}(z\to y,t) = \frac{\sqrt{\omega Q_0}}{2}\begin{bmatrix}1+\mathrm{i}\cos(\omega t-ky)\\ \mathrm{i}+\cos(\omega t-ky)\\ \mathrm{i}+\cos(\omega t-ky)\\ 1+\mathrm{i}\cos(\omega t-ky)\end{bmatrix}, \tag{65}$$

for which the spin density is $s_x = \omega Q_0\cos(\omega t - ky)$. Thus, the β matrix associated with displacements is also used for velocity rotations about the spin axis. Positive values of $-(1/4c\rho)\psi^\dagger\beta^1\psi = -(kq_0/2\rho)\cos(\omega t-ky)$ now represents displacement along the $-z$-axis since the rotation $z\to y$ moved the β^1 direction from $\hat{\mathbf{y}}$ to $-\hat{\mathbf{z}}$. Thus $\xi_z = -(1/2\rho)\partial_y Q_x = (kq_0/2\rho)\cos(\omega t-ky)$.

As shown in the sample wave functions above, we can always choose a representation in which β^1 is the operator for the transverse spatial derivative of $\mathbf{Q}\cdot\hat{\mathbf{s}}$, and the same matrix is used for rotations of wave velocity about the spin axis. Just as rotation about the spin axis preserves spin but rotates the perpendicular axes, rotation about β^1 preserves the value of $\psi^\dagger\beta^1\psi$, while $\psi^\dagger\beta^2\psi$ and $\psi^\dagger\beta^3\psi$ both remain zero. Rotation of β^3 about β^1 changes the direction of wave velocity, as represented by $-c\beta^3\boldsymbol{\sigma}$.

3.2. Nonlinear plane wave solutions

The preceding analysis is incomplete because the wave functions described above do not include any effect of the motion of the medium. To see what is missing, rewrite equation (20) in terms of the bispinor wave function:

$$0 = \psi^\dagger\sigma_i\left(\partial_t\psi - c\beta^3\sigma_j\partial_j\psi + u_j\partial_j\psi + \frac{\mathrm{i}}{2}w_j\sigma_j\psi\right) + c.c. \tag{66}$$

where "*c.c.*" stands for "complex conjugate." The third term in parentheses is zero, but the last term describes the effect of rotation of the solid medium on the wave function. If we were to describe the wave function evolution independent of the multiplier $\psi^\dagger\sigma_j$, we would set the expression in parentheses equal to zero. The simple wave function in equation (62) would not satisfy that equation because it omits the rotation effect. Furthermore, the expression in parentheses is also incomplete because the wave function does not completely rotate with the medium. Instead, as the medium rotates, the wave velocity remains constant. In other words, as the medium rotates about the spin axis, the wave velocity rotates back relative to the medium in order to remain unchanged. For plane waves, this rotation is about the spin axis and utilizes the matrix β^1:

$$0 = \partial_t\psi - c\beta^3\sigma_j\partial_j\psi + u_j\partial_j\psi + \frac{\mathrm{i}}{2}\breve{w}_1\beta^1\psi + \frac{\mathrm{i}}{2}w_j\sigma_j\psi \tag{67}$$

where $\breve{w}_1$ represents the rotation rate of wave velocity about the spin axis. The relative alignment of wave variables is shown in figure 1.

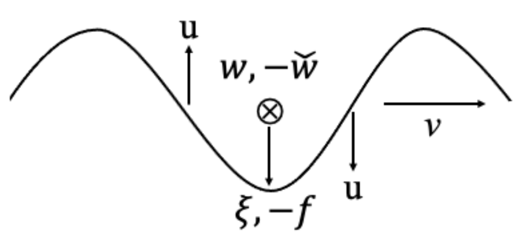

Figure 1. Wave variables at their maximal positions for a plane wave propagating toward the right with speed v. When displacement ξ is downward, the force density f is upward, the angular velocity w of the medium is into the page, and the wave velocity rotation rate $\breve{w}$ relative to the medium is opposite to w.

Equation (67) attributes temporal changes in the wave function to propagation, convection, rotation of wave velocity, and rotation of the medium. Additional terms may be necessary in some circumstances (e.g. interactions with other waves), but equation (67) is sufficient for plane waves.

The conservation law of equation (57) is now modified to include convection in addition to wave propagation:

$$\partial_t(\psi^\dagger\psi) + \mathbf{u}\cdot\nabla(\psi^\dagger\psi) - \nabla\cdot(\psi^\dagger c\beta^3\boldsymbol{\sigma}\psi) = 0\,. \tag{68}$$

For plane waves the additional convection term is zero.

Although the two rotation terms in equation (67) cancel for plane waves, we modify the wave function in equation (62) so that each term is consistent with its interpretation in the vector wave equation:

$$\psi_{s_x,v_z} = \sqrt{\frac{\omega Q_0}{2}}\begin{bmatrix} 1 \\ \cos(\omega t - kz) - \mathrm{i}\frac{k^2Q_0}{4\rho}\sin(\omega t - kz) \\ \cos(\omega t - kz) - \mathrm{i}\frac{k^2Q_0}{4\rho}\sin(\omega t - kz) \\ 1 \end{bmatrix}. \tag{69}$$

This wave function still yields $s_x = \omega Q_0\cos(\omega t - kz)$ and satisfies the full nonlinear Dirac equation in equation (67).

3.2.1. *Energy operators*

Now consider the physical interpretation of the terms in equation (67). The nonlinear terms represent rotations of wave velocity and of the medium as a whole. But they are also related to energy. Multiplying equation (67) by $\mathrm{i}\psi^\dagger/2$ and adding the complex conjugate yields, with some rearranging:

$$\mathrm{Re}(\psi^\dagger\mathrm{i}\partial_t\psi) = \mathrm{Re}(c\psi^\dagger\beta^3\sigma_j\mathrm{i}\partial_j\psi) - u_j\mathrm{Re}(\psi^\dagger\mathrm{i}\partial_j\psi) + \frac{1}{2}\breve{w}_1\psi^\dagger\beta^1\psi + \frac{1}{2}w_j\psi^\dagger\sigma_j\psi\,. \tag{70}$$

The last term in this equation is $w_j s_j$, which is twice the rotational kinetic energy. The next to last term is proportional to the displacement $\xi_1 = -(1/4c\rho)\psi^\dagger\beta^1\psi$ as in equation (64). Using $\nabla\cdot\boldsymbol{\xi} = 0$, the corresponding component of force density is equal to:

$$f_1 = \mu\nabla^2\xi_1 = -\frac{c}{4}\nabla^2\left[\psi^\dagger\beta^1\psi\right]. \tag{71}$$

For plane waves, the force density is proportional to displacement, so that as displacement increases from zero, the average force is half of the final force. Therefore, the conventional potential energy is

$$U = -\int\mathbf{f}\cdot d\ell = -\frac{1}{2}\mathbf{f}\cdot\boldsymbol{\xi} = -\frac{1}{32\rho}\nabla^2(\psi^\dagger\beta^1\psi)\cdot\psi^\dagger\beta^1\psi\,. \tag{72}$$

Comparing with equation (70), it appears that $\breve{w}_1$ is proportional to force and the next-to-last term in equation (70) is proportional to conventional potential energy density. To keep the wave velocity constant, its spin-independent rotation rate must be the opposite of the medium rotation rate:

$$\breve{w}_1 = -w_x = -\frac{\omega k^2 Q_0}{4\rho}\cos(\omega t - kz) = \frac{1}{8\rho}\nabla^2(\psi^\dagger\beta^1\psi)\,. \tag{73}$$

This is similar to the expression for angular velocity in equation (54b), with $-\beta^1$ replacing σ matrices. With this value of $\breve{w}_1$, the next-to-last term in equation (70) is equal to minus two times the conventional potential energy density, which cancels the the last term in equation (70) (since the rotational kinetic energy is in quadrature to the conventional kinetic energy: i.e. $\sin^2\leftrightarrow\cos^2$).

The terms in equation (70) correspond to different energies as follows:

$$\mathrm{Re}(\psi^\dagger\mathrm{i}\partial_t\psi) = \mathrm{Re}(c\psi^\dagger\beta^3\sigma_j\mathrm{i}\partial_j\psi) - u_j\mathrm{Re}(\psi^\dagger\mathrm{i}\partial_j\psi) + \frac{1}{2}\breve{w}_1\psi^\dagger\beta^1\psi + \frac{1}{2}w_j\psi^\dagger\sigma_j\psi\,, \tag{74a}$$

$$\mathcal{E} \quad = \quad c\hat{\mathbf{v}}\cdot\mathbf{P} \quad + \quad 0 \quad + \quad \mathbf{f}\cdot\boldsymbol{\xi} \quad + \quad \mathbf{w}\cdot\mathbf{s}\,, \tag{74b}$$

where $c\hat{\mathbf{v}}\cdot\mathbf{P}$ is shorthand for $\psi^\dagger\mathbf{v}_{op}\mathbf{P}_{op}\psi$ with wave velocity operator $\mathbf{v}_{op} = -c\beta^3\boldsymbol{\sigma}$ and wave momentum operator $\mathbf{P}_{op} = -\mathrm{i}\nabla$. The total energy density is $\mathcal{E} = \omega^2k^2Q_0^2/(8\rho)$, which is also the value of $c\hat{\mathbf{v}}\cdot\mathbf{P}$.

Rotational potential energy density can be defined as $U_R = \mathbf{P}\cdot c\hat{\mathbf{v}} - U$. The rotational potential energy density (U_R) and rotational kinetic energy density (K_R) are in quadrature to their usual expressions. The term $c\hat{\mathbf{v}}\cdot\mathbf{P}$ represents the product of wave momentum density and wave velocity, which was shown in equation (11) to be twice the potential energy for the vector representation. For the bispinor representation of plane waves, $c\hat{\mathbf{v}}\cdot\mathbf{P}$ is equal to the total energy, which has the same integrated value as twice the potential energy.

The different energy expressions are therefore:

$$K_R = \frac{1}{2}\mathbf{w}\cdot\mathbf{s} = \frac{1}{2}(\frac{1}{8\rho}\nabla\times\nabla\times\psi^\dagger\boldsymbol{\sigma}\psi)\cdot\frac{1}{2}\psi^\dagger\boldsymbol{\sigma\psi}\,, \tag{75a}$$

$$U_R = c\hat{\mathbf{v}}\cdot\mathbf{P} + \frac{1}{2}\mathbf{f}\cdot\boldsymbol{\xi} = \mathrm{Re}(\psi^\dagger c\beta^3\sigma_j \mathrm{i}\partial_j\psi) + \frac{1}{2}\nabla^2(\frac{1}{8\rho}\psi^\dagger\beta^1\psi)\cdot\frac{1}{2}\psi^\dagger\beta^1\psi\,. \tag{75b}$$

For plane waves we also have $\mathcal{E} = \mathrm{Re}(\psi^\dagger \mathrm{i}\partial_t\psi)$, but that result is only valid due to cancelation of the medium rotation and wave rotation terms.

In comparison, the equation of evolution for a free electron corresponding to equation (74) is:

$$\hbar\,\mathrm{Re}(\psi^\dagger \mathrm{i}\partial_t\psi) = \hbar\,\mathrm{Re}(c\psi^\dagger\beta^3\sigma_j\mathrm{i}\partial_j\psi) + m_e c^2\psi^\dagger\beta^1\psi\,, \tag{76}$$

$$\mathcal{E} \quad = \quad \hbar c\hat{\mathbf{v}}\cdot\mathbf{P} \quad + \quad m_e c^2\,. \tag{77}$$

An electron at rest is commonly presumed to have no internal wave structure, resulting in $c\hat{\mathbf{v}}\cdot\mathbf{P} = 0$. The mass term corresponds in equation (74) to $\pm 2U$. With equipartition between potential and kinetic energy, the potential energy density integrates to half of the total energy. Thus, the electron equation approximates the elastic solid equation by ignoring kinetic energy and internal wave structure, and by substituting mass for twice the potential energy.

Even for an electron, the mass term clearly describes rotation of wave velocity. However, standard theories of the electron offer no insight as to why such velocity rotation should be associated with energy. Hestenes interpreted the wave velocity as particle velocity and proposed that the rest energy is kinetic in origin. [41] The analysis of spin density plane waves instead provides a clear physical process by which quantum mechanical rest mass is associated with potential energy. However, a correct description of particle-like waves in an elastic solid would require an internal wave structure with both kinetic and potential energies.

3.2.2. Lagrangian and Hamiltonian

Having a first-order equation of evolution enables the use of variational methods. Interpreting equation (67) as an Euler-Lagrange equation requires distinction between terms containing one factor each of ψ and $\psi^\dagger$ or their derivatives, and terms containing two such factors. Just as spin density had to be regarded as functionally dependent on angular velocity in equation (14), angular velocity ($\mathbf{w}$ or $\breve{w}$) should be regarded as functionally dependent on the wave function.

Treating ψ and $\psi^\dagger$ as independent variables, we construct a Lagrangian density $\mathcal{L} = 0$ so that terms linear in ψ and its derivatives have coefficient of one as in equation (67), and the two rotation terms are cut in half:

$$\mathcal{L} = \mathrm{Re}(\psi^\dagger\mathrm{i}\partial_t\psi - c\psi^\dagger\beta^3\sigma_j\mathrm{i}\partial_j\psi + \psi^\dagger u_j\mathrm{i}\partial_j\psi) - \frac{1}{4}\breve{w}_1\psi^\dagger\beta^1\psi - \frac{1}{4}w_j\psi^\dagger\sigma_j\psi\,. \tag{78}$$

The Euler-Lagrange equation is:

$$\partial_t\frac{\partial\mathcal{L}}{\partial(\partial_t\psi^\dagger)} + \partial_i\frac{\partial\mathcal{L}}{\partial(\partial_i\psi^\dagger)} - \frac{\partial\mathcal{L}}{\partial\psi^\dagger} = 0 \tag{79}$$

Application to equation (78) yields equation (67). The rotation terms are evaluated using integration by parts. For example:

$$-\int\nabla^2(\psi^\dagger\sigma_j\psi)(\psi^\dagger\sigma_j\psi)d^3r = \int\partial_i(\psi^\dagger\sigma_j\psi)\partial_i(\psi^\dagger\sigma_j\psi)d^3r\,. \tag{80}$$

The conjugate momentum to the field ψ is p_ψ:

$$p_\psi = \frac{\partial\mathcal{L}}{\partial\left[\partial_t\psi\right]} = \frac{\mathrm{i}}{2}\psi^\dagger\,, \tag{81}$$

and similarly for $p_{\psi^\dagger}$. The real-valued Hamiltonian is

$$\mathcal{H} = \frac{1}{2}\left\{c\psi^\dagger\beta^3\sigma_j\mathrm{i}\partial_j\psi - c\mathrm{i}\partial_j\psi^\dagger\beta^3\sigma_j\psi\right\} + \frac{1}{4}\breve{w}_1\psi^\dagger\beta^1\psi + \frac{1}{4}w_j\psi^\dagger\sigma_j\psi\,. \tag{82}$$

This is equal to the total energy as demonstrated in equation (75). With the nonlinear rotational terms in the Hamiltonian, we must consider the possibility that $\mathrm{i}\partial_t\psi \neq H\psi$. However, the equality holds for plane waves due to cancellation of rotational terms.

3.2.3. Dynamical quantities

The Hamiltonian is a special case (T^0_0) of the stress-energy (or energy-momentum) tensor: [42]

$$T^\mu_\nu = \partial_\nu \psi^\dagger \frac{\partial \mathcal{L}}{\partial\left[\partial_\mu \psi^\dagger\right]} + \frac{\partial \mathcal{L}}{\partial\left[\partial_\mu \psi\right]}\partial_\nu \psi - \mathcal{L}\delta^\mu_\nu \,. \tag{83}$$

In the Lagrangian, the kinetic energy term is negative. Therefore, the conjugate momenta computed from the Lagrangian should include a minus sign. The dynamical (or wave) momentum density P_i is

$$P_i = -T^0_i = -\frac{\partial \mathcal{L}}{\partial\left[\partial_t \psi^\dagger\right]}\partial_i \psi^\dagger - \frac{\partial \mathcal{L}}{\partial\left[\partial_t \psi\right]}\partial_i \psi = -\mathrm{Re}\left\{\psi^\dagger \mathrm{i}\partial_i \psi\right\} \,. \tag{84}$$

The wave angular momentum density is likewise

$$\mathbf{L} = -\partial_{\boldsymbol{\varphi}} \psi^\dagger \frac{\partial \mathcal{L}}{\partial\left[\partial_t \psi^\dagger\right]} - \frac{\partial \mathcal{L}}{\partial\left[\partial_t \psi\right]}\partial_{\boldsymbol{\varphi}} \psi = -\mathrm{Re}(\mathrm{i}\psi^\dagger \partial_{\boldsymbol{\varphi}} \psi) = -\mathrm{Re}(\frac{\mathrm{i}}{2}\psi^\dagger \frac{\partial r_i}{\partial \boldsymbol{\varphi}}\partial_i \psi) = -\mathrm{Re}\left\{\mathbf{r}\times \psi^\dagger \mathrm{i}\nabla\psi\right\} \,. \tag{85}$$

This expression assumes a particular origin for the axis of rotation of the angle $\boldsymbol{\varphi}$, in contrast to the coordinate-independent spin angular momentum. One could attempt to express orbital angular momentum density as the field whose curl is twice the wave momentum density, but we will not pursue that here.

For densities of total momentum ($\mathbf{P}_T$) and angular momentum ($\mathbf{J}$), we must combine the wave and medium contributions: [12]

$$\mathbf{P}_T = \mathbf{P} + \mathbf{p} = -\mathrm{Re}\left\{\psi^\dagger \mathrm{i}\nabla\psi\right\} + \frac{1}{2}\nabla \times \psi^\dagger \frac{\boldsymbol{\sigma}}{2}\psi \,; \tag{86}$$

$$\mathbf{J} = \mathbf{L} + \mathbf{S} = -\mathrm{Re}\left\{\mathbf{r}\times \psi^\dagger \mathrm{i}\nabla\psi\right\} + \psi^\dagger \frac{\boldsymbol{\sigma}}{2}\psi \,. \tag{87}$$

The expression for total momentum density can also be derived from the symmetrized Belinfante-Rosenfeld stress-energy tensor. [26, 27, 43] Rosenfeld commented that, "Of course, this separation of the total moment of momentum into two terms ... has a direct physical meaning only for physical agencies that are endowed with inertia so that one could attach a system of reference that is at rest with respect to it." [27]

3.2.4. Intrinsic momentum

The total angular momentum operator is well-understood as a generator of rotations, with $\mathbf{L}$ accounting for rotation of the position argument and $\mathbf{s}$ accounting for rotation of the basis states defining the direction of spin density. Since momentum is the generator of translations, the existence of intrinsic momentum implies that translations affect not just the arguments of the wave function but also the bispinor basis states. Applying the intrinsic-momentum transformation $\psi \to \psi + (\mathrm{i}/4)\epsilon\sigma_x\partial_z\psi$ to the wave function in equation (69) represents an infinitesimal displacement in the y-direction and yields the wave function:

$$\psi_{s_x,v_z} = \sqrt{\frac{\omega Q_0}{2}}\begin{bmatrix} 1-(\mathrm{i}/4)\epsilon\left(k\sin(\omega t - kz) + \mathrm{i}\frac{k^3 Q_0}{8\rho}\cos(\omega t - kz)\right) \\ \cos(\omega t - kz) - \mathrm{i}\frac{k^2 Q_0}{4\rho}\sin(\omega t - kz) \\ \cos(\omega t - kz) - \mathrm{i}\frac{k^2 Q_0}{4\rho}\sin(\omega t - kz) \\ 1-(\mathrm{i}/4)\epsilon\left(k\sin(\omega t - kz) + \mathrm{i}\frac{k^3 Q_0}{8\rho}\cos(\omega t - kz)\right) \end{bmatrix} . \tag{88}$$

This wave function yields the same spin density as the original wave function except for an additional constant term. Thus, it still correctly describes the motion of the medium. However, the "translated" wave function does not have the same calculated energies, and does not satisfy the same equation of evolution. The situation is similar to analysis of a mass on a spring with the origin shifted away from the equilibrium position. The spurious displacement adds a term to the Hamiltonian proportional to the offset times the force, and the same is true when adding a constant translation along the displacement axis of a plane wave. The first-order change in the calculated Hamiltonian is:

$$\Delta H = -(1/2)\epsilon f = -(1/2)\epsilon\mu\partial_z^2\xi = -\frac{\epsilon k \omega^2 Q_0}{4}\cos(\omega t - kz) \tag{89}$$

Thus the displacement associated with the conjugate momentum represents a shift of coordinates away from equilibrium along the displacement axis.

Translation along the wave propagation direction simply shifts the coordinate (z), otherwise preserving the Hamiltonian.

Regarded as a function of complementary variables q and p, Hamilton's equations are:

$$\partial_t q = \frac{\partial H}{\partial p}, \quad \partial_t p = -\frac{\partial H}{\partial q}\,. \tag{90}$$

The velocity associated with the wave momentum is thus:

$$\mathbf{v} = \frac{\partial H}{\partial \mathbf{P}} = \frac{\mathrm{Re}(-\mathrm{i}\psi^\dagger c\beta^3 \boldsymbol{\sigma}\cdot\nabla\psi)\mathrm{Re}(\psi^\dagger(-\mathrm{i}\nabla)\psi)}{(\mathrm{Re}(\psi^\dagger(-\mathrm{i}\nabla)\psi))^2} = (0, 0, c) \tag{91}$$

The velocity associated with the intrinsic momentum is found by integrating the rotational kinetic energy term by parts to convert $(1/2)\mathbf{w}\cdot\mathbf{s}$ to $p^2/2\rho$:

$$\mathbf{u} = \frac{\partial H}{\partial \mathbf{p}} = \frac{\mathbf{p}}{\rho} = \frac{1}{4\rho}\nabla\times\psi^\dagger\boldsymbol{\sigma}\psi = (0, \frac{c\,Q_0 k^2}{2\rho}\sin{(\omega t - kz)}, 0) \tag{92}$$

This is of course the velocity of the medium caused by the wave.

These two velocities can be used to compute the slope of the displacement from equilibrium as a function of z:

$$\frac{d\xi_y}{dz} = -\frac{u_y}{v_z} = -\frac{Q_0 k^2}{2\rho}\sin{(\omega t - kz)} \tag{93}$$

Thus we have seen that analysis of classical spin density of elastic waves offers many insights into the physical interpretation of the Dirac equation, including an understanding of the intrinsic momentum required by the Belinfante-Rosenfeld stress-energy tensor.

4. DISCUSSION

We have analyzed a nonlinear Dirac equation based on the simple model of an ideal elastic solid. With proper normalization, momentum and angular momentum densities are computed from the same operators in both classical and quantum physics. Others have also found similarities between quantum mechanics and waves in an elastic solid. [12–14, 21, 44–48] Each of the variables in the Dirac equation has a clear physical interpretation. In particular, spin angular momentum of elementary particles may be regarded as the angular momentum of the vacuum or, equivalently, the "aether".

While it is unclear to what extent classical physics can describe quantum mechanics, it is sensible to suppose that spin density should be described by a single equation valid throughout all space. According to this hypothesis, the Standard Model is a decomposition of spin density waves into so-called "particles". While this hypothesis may be contested, it is incorrect to say that the aether is undetectable. It has been detected according to this hypothesis, consistent with Robert Laughlin's statement that, "Relativity actually says nothing about the existence or nonexistence of matter pervading the universe, only that any such matter must have relativistic symmetry. It turns out that such matter exists." [49]

The equation of evolution of spin density is nonlinear. Nonlinearity is a possible reason for quantized amplitudes, since multiplying a solution by a constant factor need not yield another solution. Many researchers have attempted to quantize the Dirac equation by adding nonlinear terms. [50–59] Particle-like nonlinear wave solutions are sometimes called "breathers" or "solitons". The sine-Gordon equation illustrates particle-like behavior in one dimension, and three-dimensional analogues have also been studied. [60–63]

It is possible that classical wave interactions might explain the Pauli exclusion principle and interaction potentials. In short, adding wave functions of two "independent" particles results in addition of their magnitudes plus unwanted interference terms:

$$(\psi_A + \psi_B)^\dagger(\psi_A + \psi_B) = \psi_A^\dagger\psi_A + \psi_B^\dagger\psi_B + \psi_A^\dagger\psi_B + \psi_B^\dagger\psi_A\,. \tag{94}$$

The conservation law expressed by equation (57) implies that the combined magnitude of the two particles should be conserved. Phase shifts can be introduced to cancel the interference terms without changing the magnitude of each particle. The cancellation of interference terms is equivalent to anticommutation of the wave functions, which is a mathematical statement of the exclusion principle. Derivatives of the phase shifts may be interpreted as interaction potentials.[12]

For a phase shift of the form $\delta = (m\phi - \omega t)$ with integer m, a magnetic vector potential $\mathbf{A} \equiv (\hbar c/e)\nabla\delta$ would have quantized magnetic flux ($\oint(\mathbf{A}\cdot d\ell) = m(hc/e)$) since the phase can only change by multiples of $2\pi m$ when traversing a loop. For $m = 1/2$ this is equal to the magnetic flux quantum of superconductivity (although in that case the electrons are in pairs with $m = 1$ and charge of $2e$). Others have similarly identified the electromagnetic vector potential $\mathbf{A}$ as the gradient of a multivalued field.[64, 65]

For interacting particle-like waves, the magnitude of phase shifts must decrease with distance between the particles. Jehle showed that with a $1/r$ radial dependence of the phase shift (interpreted as a distribution of rotating magnetic flux loops) scaled to yield the electron magnetic field $|B_z| = (\mu_0/2\pi r^3)\mu_B$ in the $z = 0$ plane with $m = 1$ and angular frequency $\omega = 2m_e c^2/\hbar$, the rotating phase shift gives rise to the electron Coulomb potential.[64] Here $\mu_B = (e\hbar/2m_e)$ is the Bohr magneton in SI units. The same result could be obtained with $m = 1/2$ and $\omega = m_e c^2/\hbar$. Jehle also extended the model of quantized flux "loopforms" to other particles. [66]

The elastic solid model might also produce analogues of matter and antimatter. Suppose that elementary particles have spin density vector components that behave like spherical harmonics with parity $(-1)^\ell$. To illustrate the effect of spatial reflection, consider a single vector component $s_x = R(r)(\sin\theta)^\ell \sin(\ell\phi - \omega t)$ for some radial function $R(r)$. Reflection along the x-axis changes ϕ to $\pi - \phi$ to yield $s'_x = R(r)(\sin\theta)^\ell \sin(\ell\pi - \ell\phi - \omega t)$. For odd integer ℓ-values, this yields $s'_x = R(r)(\sin\theta)^\ell \sin(\ell\phi + \omega t)$, which has the same spin density at $t = 0$ but rotates in the opposite direction. Rotation by π radians about the x-axis completes the parity operation by changing the sign of ϕ to yield $P(s_x) = -R(r)(\sin\theta)^\ell \sin(\ell\phi - \omega t)$. This changes the sign of the spin for the same wave propagation direction, resulting in a change in sign of electric charge according to Jehle's model. [64]

For even integer values of ℓ, reflection along the x-axis yields $s'_x = -R(r)(\sin\theta)^\ell \sin(\ell\phi + \omega t)$, and the complete parity operator yields the original function $P(s_x) = R(r)(\sin\theta)^\ell \sin(\ell\phi - \omega t)$.

Thus, it is plausible that particle-like vector waves with odd-integer orbital quantum numbers have distinct mirror images, while particle-like waves with even-integer orbital quantum numbers are their own mirror images. These mirror image wave functions could play the role of antiparticles. This correspondence assumes that electric charge is reversed upon spatial reflection, contrary to the usual assumption but consistent with experiments such as beta decay of Cobalt-60, in which the mirror image process can occur with antimatter but not with matter.

Since vector quantities are bilinear combinations of the bispinor wave functions, the bispinors should transform under rotations with half the phase change of the vector wave functions. Under our assumptions, elementary particle bispinor fields transforming with half-integer orbital quantum numbers would have physically distinct mirror images (antiparticles), whereas elementary particle bispinor fields transforming with whole integer orbital quantum numbers would not have distinct mirror images. These results are consistent with the fact that all elementary fermions have distinct antiparticles, and nearly all elementary bosons are their own antiparticles (an exception is the pair W^+ and W^-, so the connection with spherical harmonics cannot apply to these). This analysis classifies particles on the basis of internal orbital angular momentum rather than spin angular momentum.

The model of the vacuum as an elastic solid also offers a good introduction to general relativity. Gravity, at least when quasi-static, may be interpreted as ordinary refraction of waves toward regions whose wave speed is decreased by the presence of energy. [67–69] Wave speed in an elastic solid may likewise be decreased by stress-induced compression (increased inertial density and decreased shear modulus). Likewise, twisting a rubber band under constant tension tends to shorten it. The increased density is associated with a decreased shear modulus according to the SCG model of a solid under constant pressure. [70]

5. CONCLUSIONS

Classical spin density waves in an ideal elastic solid are modeled by a nonlinear vector wave equation in which temporal changes of spin density are entirely attributable to convection, rotation, and torque. A compatible nonlinear Dirac equation is also derived. Operators for momentum and angular momentum densities are equivalent to those of relativistic quantum mechanics. Vector plane wave solutions are expressed using Dirac bispinors. The Hamiltonian is equal to the total energy, which is a sum of rotational kinetic and potential energies. Rotational kinetic energy is associated with rotation of the wave function with the medium, whereas rotational potential energy is associated with wave propagation and rotation of wave velocity relative to the medium. Intrinsic momentum identical to that derived from the Belinfante-Rosenfeld stress-energy tensor is the generator of translations corresponding to a change of origin of displacements away from the equilibrium position. The usual expression for electron rest energy corresponds to twice the conventional potential energy in the elastic solid model. In sum, waves in an ideal elastic solid provide classical physics analogues for many physical properties of elementary particles.

ACKNOWLEDGMENTS

The author is grateful to Manfred Faber for helpful comments on a draft version of the paper.

DATA AVAILABILITY STATEMENT

No new data were generated or analyzed in this study.

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