From Complex Dynamics to Foundational Physics (Part 1)

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Abstract

As of today, Quantum Field Theory (QFT) and General Relativity (GR) are broadly recognized paradigms of foundational physics. There are, however, growing suspicions that both paradigms fail to hold somewhere above the Standard Model (SM) scale and in the realm of primordial cosmology. Evidence collected on multiple fronts indicates that *emergence* and *complexity* are universal features of far-from-equilibrium systems with many degrees of freedom. In line with these findings, Part 1 of this report explores the complex dynamics of evolving dimensional fluctuations beyond the SM scale. Part 2 outlines the role of complex dynamics in the nonintegrable sector of particle physics, Dark Matter condensation and the gravitational regime of the early Universe.

Key words: Non-equilibrium critical phenomena, Self-Organized Criticality, Standard Model, fractal spacetime, multifractals, emergence, complex dynamics.

Cautionary remarks

We caution from the outset that the sole intent of this report is to lay the groundwork for further exploration of the topic. Exclusively presented in a preliminary form, our analysis is far from completion and far from meeting the quality standards of a formal research project. The style and presentation do not comply with traditional academic standards. Independent work is needed to develop, validate, or reject the hypotheses presented here. Readers unfamiliar with the topic are encouraged to carefully review the enclosed references prior to drawing premature conclusions.

1. Introduction and Motivation

Extensive evidence exists nowadays that large systems of nonlinearly interacting components are prone to slide outside thermodynamic equilibrium and become *nonintegrable* in the long run. Reliable modeling of such systems requires new concepts and methods inspired by chaos theory, multifractal geometry, non-equilibrium critical phenomena, self-organized criticality (SOC), and fractional calculus. These non-conventional tools form the mathematical basis of *complex dynamics* [1, 6-7, 9-14, 16-17, 29].

A fundamental assumption of QFT is that its vacuum state is a large many-body system of quantum fluctuations whose dynamics follows the framework of equilibrium thermodynamics. There is no guarantee that this conjecture will continue to hold above the low-energy scale of the SM. On the contrary, many condensed matter studies consistently suggest the opposite: on high-energy scales, unbalanced vacuum fluctuations are likely to slide outside equilibrium and the perturbative treatment of conventional QFT is prone to break down. When driven far away from thermodynamic equilibrium, complex systems are known to exhibit emergent dynamics stemming from the interplay between *nonlinear interaction* of components and *steady dissipation*. As paradigm of this type of emergent behavior, SOC has a vast range of applications extending from astrophysics, natural hazards and magnetospheric physics, to complex networks, internet dynamics, biophysics, and social sciences [15, 24].

According to [1], there is a tentative path leading from Dimensional Regularization procedure of QFT to *fractal spacetime*, on the one hand, and to *fractional dynamics*, on the other. Both concepts require passage to a spacetime endowed with *continuous dimensions,* which are conjectured to flow with the observation scale. This viewpoint offers unforeseen insights on the deep ultraviolet extension of Effective Field Theory (EFT). At the same time, they are manifestly at odds with most SM extensions, Quantum Gravity theories or field unification models.

In a nutshell, fractals and multifractals define the underlying geometry of complex behavior for systems evolving in far-from-equilibrium conditions. To appreciate the alleged impact of fractal spacetime beyond the boundaries of EFT, we bring up two textbook examples:

1) **Perturbative QFT** is an archetype of EFT which lies at the foundation of the SM for particle physics. Although astonishingly successful in matching results from collider data, QFT is confronted by many drawbacks. It is known, for instance, that amplitude computations are at best *asymptotic expansions*, meaning that Feynman diagrams fail to uncover the true dynamics of the theory. Non-perturbative QFT methods have been partially successful, yet it is fair to say that, for the most part, the

underlying physics of QFT remains unknown. In addition, there are several caveats and patches needed to secure self-consistency demanded by the principles of QFT.

2) Another often cited paradigm of an EFT is **General Relativity (GR)**, whose validity rests on the principle of *general (or diffeomorphism) covariance*. This principle asserts that all physical laws must take the same mathematical form regardless of the coordinate system used by observers in arbitrary relative motion. Stated differently, general covariance means invariance of physical laws under all possible coordinate transformations. An implicit assumption of general covariance is that any coordinate transformation and its inverse are smooth/analytic functions that can be differentiated arbitrarily many times. However, it is known that there is a plethora of *non-differentiable curves and surfaces* in Nature, as abundantly discovered since the introduction of fractal geometry in 1983. The unavoidable conclusion is GR assigns a preferential status to differentiable transformations, which is at odds with the very spirit of general covariance. In addition, the singularity problem, the relentless instability

of the transient regime of GR, and the N-body problem are likely tied to the transition from deterministic to a self-sustained complex evolution in the primordial stages of Universe formation [28 - 30, 33-34].

Experimental and numerical evidence of recent years indicates that *emergence* and *complexity* are universal features of far-from-equilibrium systems having many degrees of freedom. In line with these findings, several independent research works point to direct and indirect hints for a spacetime endowed with *minimal fractality*, likely to develop somewhere above the SM scale [5, 12-13, 17-22, 25-27].

The report is divided into two parts. Part 1 explores the complex dynamics of evolving dimensional fluctuations above the SM scale. In support of this scenario, the Appendix section bridges the gap between universal route to chaos and the flavor composition of the SM. Part 2 outlines the role of complex dynamics in the nonintegrable sector of particle physics, the formation of Dark Matter as large-scale dimensional condensate and the gravitational regime of the primordial Universe.

2. From Dimensional Regularization to fractal spacetime

There are several ways of introducing the concept of spacetime continuum whose nonintegral dimensions evolve with the observation scale [1, 20-21]. To make the paper self-contained and for the sake of clarity, we begin by iterating here the arguments for fractal spacetime inspired by Dimensional Regularization program of QFT [1].

It is known that Euclidean formulation of the Path Integral in QFT enables a useful analogy between QFT and critical phenomena. To this end, consider the two-point function of massive scalar field theory. The Euclidean propagator in momentum space is given by

$$D_E(p) = \frac{1}{p^2 + m^2} \tag{1}$$

and the correlation function of the corresponding statistical system is

$$\langle \varphi(x)\varphi(0) \rangle = \int \frac{d^4 p}{(2\pi)^4} \frac{\exp(ipx)}{p^2 + m^2} \tag{2}$$

in which $|p^2| = p_\mu p_\mu$ and $px = p_\mu x_\mu$. In the limit $m|x| >> 1$, (2) is well approximated by,

$$\langle \varphi(x)\varphi(0) \rangle \approx \frac{1}{|x|^2}\exp(-m|x|) \tag{3}$$

Let us assume that the field is placed on a four-dimensional lattice of points separated by a fixed spacing $a = \Lambda_{UV}^{-1}$, in which Λ_{UV} is the cutoff scale. The spatial coordinate is then,

$$|x| = Na = N\Lambda_{UV}^{-1}, \quad N >> 1 \tag{4}$$

and (3) can be written as,

$$\langle \varphi_N \varphi_0 \rangle \propto \exp[-N(m/\Lambda_{UV})] \tag{5}$$

By analogy with statistical mechanics, (5) defines the dimensionless correlation length according to,

$$\langle \varphi_N \varphi_0 \rangle \propto \exp(-N/\xi) \tag{6}$$

where,

$$\xi = \frac{\Lambda_{UV}}{m} \tag{7}$$

Dimensional Regularization in momentum space sets up a relationship between the cutoff scale Λ_{UV} and the dimensional deviation from four-space dimensions $\varepsilon(\mu) = 4 - d(\mu) << 1$ as in

$$\Lambda_{UV}^2 = \mu^2 \exp(1/\varepsilon) >> \mu^2 \tag{8}$$

where μ is the running scale. The asymptotic limit $m = O(\mu) << \Lambda_{UV}$ leads to the raw estimate,

$$\boxed{\varepsilon(\mu) = 4 - d(\mu) = O[m^2(\mu)/\Lambda_{UV}^2) << 1} \tag{9}$$

By (7) and (9), we arrive at the effective approximation,

$$\boxed{\xi(\mu) \propto [\varepsilon(\mu)]^{-1/2}} \tag{10}$$

As a diverging correlation length is a characteristic feature of critical phenomena, (10) indicates that removing the dimensional regulator in QFT (that is, taking the classical continuum limit $\varepsilon \to 0$) is *analogous to tuning the corresponding statistical system towards the critical point* [8, 17, 20-21]. In this sense, (10) underlies the idea of *criticality in continuous dimension* $d(\mu)$, conjectured to play a key role in the ultraviolet regime of field theory and primordial cosmology. Since, by definition, fractal structures are characterized by continuous dimensions and are the underlying geometry of both critical phenomena and chaotic behavior, (10) leads to the conclusion that taking the limit $\varepsilon \to 0$ in Dimensional Regularization turns the classical spacetime into a *minimal fractal manifold (MFM)*.

3. Complex dynamics of evolving spacetime dimensions

Reaction-diffusion processes are a subset of complex phenomena defined within the framework of Non-equilibrium Statistical Physics [2–3, 16]. These models are typically formulated in $d+1$ dimensions, where d is the dimension of the Euclidean manifold representing the physical space and t is the time coordinate. Reaction-Diffusion models on discrete manifolds (called lattices) are characterized by the following features,

a) local variables reside at lattice sites,

b) reaction chains are driven by probabilistic transition rules among sites.

We consider below a toy Reaction-Diffusion model acting on a two-dimensional lattice $(d=2)$, whose local variables are time-varying *dimensional deviations* $[\delta\varepsilon(t)]$, referred herein to as "*dimensional pixels*". The model is built on four premises, namely,

A1) At any given moment "t", a pixel consists of a pair of lattice sites that are either *occupied* (1) or *empty* (0) and are located horizontally adjacent to each other.

A2) The representative pixel states are listed as,

$$[\delta\varepsilon(t)]=\{[0,1];\ [1,0];\ [1,1];\ [0,0]\}_t$$

A3) There are four transpositions among these binary states from time "t" to time "$t+dt$", that is,

$$[1]\rightarrow[0] \quad \textit{self-annihilation} \tag{11a}$$

$$[1]\rightarrow[1]+[1] \quad \textit{decay/percolation} \tag{11b}$$

$$[1]+[1]\rightarrow[0] \quad \textit{pair annihilation} \tag{11c}$$

$$[1]+[1] \to [1] \;\; clustering \tag{11d}$$

A4) Following ref. [1] and figs. 1 - 2 below, dimensional pixels undergo transition events between "t" and "$t+dt$" as described by,

$$[1,0] \xrightarrow[D]{} [0,1] \tag{12a}$$

$$[1,1] \xrightarrow[u]{} [1,0],[0,1] \tag{12b}$$

$$[1,0],[0,1] \xrightarrow[\kappa]{} [1,1] \tag{12c}$$

Here, (12a) denotes a *scattering* event at rate D, (12b) a *clustering* event at rate u and (12c) a *decay* (or *percolation*) event at rate $\kappa = \lambda - \lambda_c$, with λ being a control parameter approaching its critical value λ_c.

Up to a leading order approximation, the macroscopic attributes of Reaction-Diffusion processes may be encoded in a *mean-field* (MF) equation [2], which quantifies the competition between losses and gains in density $\rho(t)$. In particular, the decay/percolation process occurs with a rate proportional to $\kappa\rho(t)$ and leads to a gain in density. By contrast, the clustering process (12b) drops the density with a rate proportional to $u\rho^2(t)$. Ignoring diffusion (12a), the resulting MF equation takes the form

$$\boxed{\frac{\partial\rho(t)}{\partial t} = \kappa\rho(t) - u\rho^2(t)} \tag{14}$$

In the context of our paper, the control parameter $\lambda(t) = \lambda[\delta\varepsilon(t)]$ represents the *density of dimensional pixels* $\delta\varepsilon(t) << 1$ while $\rho(t)$ denotes the *density of active (or unstable) lattice sites.*

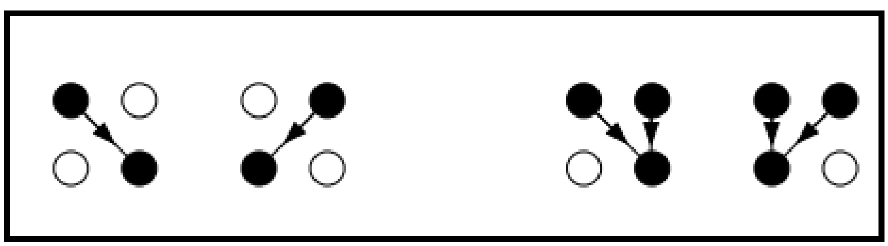

Fig. 1: *Left panel*: Scattering, *Right panel*: Clustering ([2])

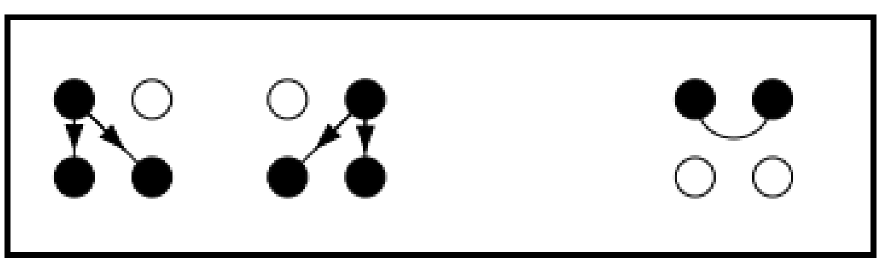

Fig. 2 *Left panel*: Decay/Percolation, *Right panel*: Annihilation ([2])

The MF equation (14) exhibits a two-phase configuration: an *absorbing phase* with a vanishing density of active sites ($\rho = 0$) below the critical point $\lambda < \lambda_c$ and an *active phase* with a steady-state density $\rho = \kappa/u \neq 0$ above the critical point $\lambda > \lambda_c$.

If the starting point of the time evolution is a fully occupied lattice $\rho(0) = 1$, the solution of (14) reads,

$$\rho(t) = \frac{\kappa}{u - (u - \kappa)\exp(-\kappa t)} \tag{15}$$

Relation (15) shows that, when the percolation rate vanishes at the critical point $\kappa = \lambda - \lambda_c = 0$, the density of unstable states drops asymptotically as in

$$\lim_{\kappa \to 0} \rho(t) = (1 + u t)^{-1} \tag{16a}$$

or,

$$\rho(t) \propto t^{-1} \tag{16b}$$

By (16a) and (16b), the number of unstable/active sites eventually goes to zero and the dynamics of dimensional deviations slows down. It follows that, in the far infrared regime ($t \to \infty$), spacetime settles in a stationary state matching the classical limit $\delta\varepsilon = O(\varepsilon) \to 0$.

Before proceeding further, a few cautionary remarks are in order:

a) Clearly, by design, the scenario embodied in (A1) – (A4) is merely a convenient simplification. Any realistic model of pixels defined in continuous spatial dimensions must necessarily include infinite sets of non-integer pairs having the form,

$$[\delta\varepsilon(t)] = \{[\alpha_1, \beta_1];\ [\alpha_2, \beta_2];....[\alpha_N, \beta_N]\}_t \tag{17}$$

in which $N \to \infty$ and α_i, β_i are arbitrary numbers with

$$0 \le \alpha_i \le 1;\quad 0 \le \beta_i \le 1;\quad \alpha_i \ne \beta_i \tag{18}$$

b) It can be argued that, on Euclidean spacetime endowed with continuous dimensions, the deviations ε and their fluctuations $\delta\varepsilon$ play an identical role with the coefficients $g_{\mu\nu}$ of a corresponding non-Euclidean metric [see e.g. 5]. Stated differently, a flat spacetime endowed with a fractal structure may be considered as *dual* to a curved manifold.

On this basis, a reasonable assumption is that classical gravitation is *implicitly accounted for* in the Reaction-Diffusion model detailed above.

c) The stability of the MF solution with respect to perturbations can be studied in a variety of ways. In a general scenario, for example, one accounts for the combined effects of diffusion (D), Gaussian noise $\varsigma(t)$ and random fluctuations $\eta(t)$ on the percolation parameter,

$$\kappa \rightarrow \kappa + \eta(t) \tag{19}$$

In this scenario, (14) gets upgraded to a Langevin type equation, namely,

$$\frac{\partial \rho(t)}{\partial t} = \kappa\rho(t) - u\rho^2(t) + D\Delta\rho(t) + \varsigma(t) + \rho(t)\eta(t) \tag{20}$$

As discussed in [2–3, 16] and the complex dynamics literature, the generic model (20) has applications across a wide range of topics, including (but not limited to) percolation phenomena, epidemics spreading, forest fires, earthquake propagation, lattice dynamics with long-range correlations, spin glasses, spatiotemporal patterns in condensed matter physics, random graph theory, galaxy clustering, large network dynamics and so on.

4. From Reaction-Diffusion processes to Self-organized Criticality (SOC)

Of particular interest is the relationship between the Reaction-Diffusion model previously outlined and Self-organized Criticality (SOC). Among the many ways to unveil this connection a straightforward approach is to supplement (14) with a *driving source* whose function is to boost dimensional instability and prevent relaxation. Adding to the percolation rate a time-independent source term E turns (14) into [4]

$$\frac{\partial \rho(t)}{\partial t} = (\kappa + E)\,\rho(t) - u\rho^2(t) \tag{21}$$

According to [4], since E is a conserved quantity ($\dot{E} = 0$), it can be conveniently used as control parameter instead of κ. Following this scenario, the critical point is reached at $E_c = -\kappa$ and (21) replicates the *fixed sandpile model* of SOC, whose steady state is determined by E. The impact of

SOC on the non-integrable sector of particle physics and early Universe cosmology is covered in Part 2 of this report.

In closing Part 1, we again emphasize that the hypotheses detailed here, despite offering encouraging leads forward, are strictly provisional thoughts. Much remains to be done to debunk, confirm, or place these ideas on a more robust foundation.

APPENDIX

Feigenbaum's Universality and the dynamic structure of the SM

The Reaction-Diffusion model (14) with $u=1$ may be analyzed as an iterated map operating in discrete time $t_n = n\Delta t$, $n = 1, 2, ... N$. Setting

$$\frac{\partial \rho}{\partial t} \Rightarrow \frac{\rho_{n+1} - \rho_n}{\Delta t} \tag{22}$$

turns (14) into

$$\rho_{n+1} = (1 + \kappa \Delta t) \rho_n [1 - \frac{\Delta t}{1 + \kappa \Delta t} \rho_n] \tag{23}$$

Under the change of variable

$$v_n = \frac{\Delta t}{1+\kappa\Delta t}\rho_n \tag{24}$$

(23) takes the form of the *logistic map*

$$v_{n+1} = \gamma v_n(1-v_n) \tag{25}$$

in which the critical parameter is

$$\gamma = (1+\kappa\Delta t) \tag{26}$$

In doing so, one recovers the *universal route to chaos of unimodal maps* via the well-known scenario of period-doubling bifurcations [31-32]. One is led to the Feigenbaum's sequence of critical parameters as function of the number of iterations

$$\gamma_n - \gamma_0 \Rightarrow \kappa_n - \kappa_0 \Rightarrow \lambda_n - \lambda_0 \propto \delta^{-n} \tag{27}$$

where δ stands for the Feigenbaum constant.

According to [31-32], iterated maps of the unit interval are generic models of dynamical systems in discrete time. The standard representation of these models is based on first order difference equations having the form

$$x_{n+1} = f(x_n, \gamma) \tag{28}$$

The dynamics of iterated maps can be either *conservative* or *dissipative*. In the former case, the function (28) is monotonic and describes a one-to-one mapping, whereas in the latter case is non-monotonic and describes a two-to-one mapping. Typical examples of dissipative systems include the *quadratic map* and *unimodal maps* of the type (25). In 1978, Feigenbaum discovered that the onset of chaos in quadratic maps occurs through *period-doubling bifurcations* driven by changes of the critical parameter γ. Unimodal maps exhibit the following behavior: for small values of γ, (28) has a single stable fixed point and all nearby points converge to it under multiple iterations of (22)-(23). Ramping up γ to a critical value γ_1 makes the fixed point unstable and produces a new stable pair of points of period 2. Further increasing γ to another value (γ_2) bifurcates this cycle into a cycle of period 4. The bifurcation process continues with a new sequence of cycles of period 2^j, $j \geq 3$, eventually leading to a Cantor set structure that attracts almost all the points of the interval $[-1,1]$. On letting γ increase beyond an endpoint value γ_∞, stable periodic orbits surface again and split up in a similar way. In the new sequence, γ scans another series of critical values corresponding to cycles of period $3 \cdot 2^j$, $j = 0,1,2,...$ and so on. When applied to the flavor composition of SM, this bifurcation scenario leads to the content of Tab. 1 below. The first branch of the fermion sector represents the set of ***3 left-handed neutrinos*** occurring at $j = 0$, a finding which naturally accounts for both triplication of fermion families and the fermion chirality in the SM [see e.g. 24]. It is instructive to note that, according to this scenario, Dark Matter appears to echo the properties of *anyons in three-dimensional space* [33].

Parameter	**Flavor Content**	**Bifurcation Pattern**	**Spin**
γ_1	Higgs scalar	2^j	0
$\gamma_n < \gamma_{DM}$	Gauge Bosons	2^j	1
$\gamma_{DM} < \gamma_\infty$	Dark Matter	2^j	undefined
$\gamma_\infty < \gamma_n$	Fermions	$3 \cdot 2^j$	½

Tab. 1: The flavor structure of visible and Dark Matter sectors

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