A new tridimensional $3D$ spherical warp drive vector created using the methodology developed by Natario

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Abstract

The Natario warp drive appeared for the first time in 2001.Although the idea of the warp dive as a spacetime distortion that allows a spaceship to travel faster than light predated the Natario work by 7 years Natario introduced in 2001 the new concept of a propulsion vector to define or to generate a warp drive spacetime.Natario defined a warp drive vector for constant speeds in Polar Coordinates but remember that a real warp drive must accelerate or de-accelerate in order to be accepted as a physical valid model so it must possesses variable speeds.We developed the extension for the original Natario warp drive vector that encompasses variable speeds.Also Polar Coordinates uses only two dimensions and we know that a real spaceship is a tridimensional $3D$ object inserted inside a tridimensional $3D$ warp bubble that must be defined in real $3D$ Spherical Coordinates.In this work we present the new warp drive vector in tridimensional $3D$ Spherical Coordinates for both constant or variable speeds.

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1 Introduction:

The Natario warp drive appeared for the first time in 2001.([1]).Although the idea of the warp dive as a spacetime distortion that allows a spaceship to travel faster than light predated the Natario work by 7 years Natario introduced in 2001 the new concept of a propulsion vector to define or to generate a warp drive spacetime.

This propulsion vector nX uses the form $nX = X^i e_i$ where X^i are the shift vectors responsible for the spaceship propulsion or speed and e_i are the Canonical Basis of the Coordinates System where the shift vectors are based or placed.

Natario (See pg 5 in [1]) defined a warp drive vector $nX = vs * (dx)$ where vs is the constant speed of the warp bubble and $*(dx)$ is the Hodge Star taken over the x-axis of motion in Polar Coordinates(See pg 4 in [1]).(see Appendix D about Polar Coordinates).The final form of the original Natario warp drive vector is given by $nX = vs * d(r \cos\theta)$.However Polar Coordinates are not real tridimensional $3D$ coordinates since it uses only the two Canonical Basis $\mathbf{e}_r$ and $\mathbf{e}_\theta$.

The Hodge Star actually must be taken over the product (xvs) giving the expression $nX = *(xvs) = vs * (dx) + x * (dvs)$ but due to a constant speed vs the term $x * d(vs) = 0$.In this work we examine what happens with the Natario vector when the velocity is variable and then the term $x * d(vs)$ no longer vanishes.Remember that a real warp drive must accelerate or de-accelerate in order to be accepted as a physical valid model.

Natario used Polar Coordinates(See pg 4 in [1]) but for a real $3D$ Spherical Coordinates another warp drive vector must be calculated.Remember that a real spaceship is a tridimensional $3D$ object inserted inside a tridimensional $3D$ warp bubble that must be defined in real $3D$ Spherical Coordinates.The final form of the Hodge Star for this warp drive vector is calculated no longer over $*d(r \cos\theta)$ but instead over $*d(r \sin\phi \cos\theta)$ since this form uses all the tridimensional $3D$ Canonical Basis $\mathbf{e}_r$,$\mathbf{e}_\theta$ and $\mathbf{e}_\phi$.(see Appendix E about tridimensional $3D$ Spherical Coordinates).

In this work we present the new warp drive vector in tridimensional $3D$ Spherical Coordinates for both constant $nX = vs * d(x)$ or variable speeds $nX = vs * (dx) + x * (dvs)$.

In order to fully understand the idea presented in this work(a new warp drive vector in tridimensional $3D$ Spherical Coordinates) acquaintance or familiarity with the Natario original warp drive paper is required but we provide all the mathematical demonstration QED(Quod Erad Demonstratum) in the Appendices.

This work is organized as follows:

- A)-Section 2 introduces the original Natario warp drive vector in Polar Coordinates $nX = vs * d(x)$ for constant speeds.
- B)-Section 3 introduces the original Natario warp drive vector in Polar Coordinates $nX = vs * d(x) + x * (dvs)$ for variable speeds.
- C)-Section 4 introduces the new warp drive vector in tridimensional $3D$ Spherical Coordinates $nX = vs * d(x)$ for constant speeds.
- D)-Section 5 introduces the new warp drive vector in tridimensional $3D$ Spherical Coordinates $nX = vs * d(x) + x * (dvs)$ for variable speeds.

We adopted in this work a pedagogical language and a presentation style that perhaps will be considered as tedious,monotonous, exhaustive or extensive by experienced or seasoned readers and we designated this work for novices,newcomers,beginners or intermediate students providing in our work all the mathematical background needed to understand the process Natario used to generate warp drive vectors.

As a matter of fact if a novice,newcomer,beginner or intermediate student not familiarized with the Natario techniques reads the Natario warp drive paper in first place he(or she) will perhaps feel some difficulties.

We hope our paper is suitable to fill this gap.

Although this work was designed to be independent,self-consistent and self-contained it may be regarded as a companion work to our work in [9].

2 The equation of the Natario warp drive vector in polar coordinates with a constant speed vs

The equation of the Natario vector nX(pg 2 and 5 in [1]) is given by:

$$nX = X^r e_r + X^\theta e_\theta \tag{1}$$

With the contravariant shift vector components X^{rs} and X^θ given by:(see pg 5 in [1])(see also Appendix A for details)

$$X^{rs} = 2v_s n(rs) \cos\theta \tag{2}$$

$$X^\theta = -v_s(2n(rs) + (rs)n'(rs)) \sin\theta \tag{3}$$

Considering a valid $n(rs)$ as a Natario shape function being $n(rs) = \frac{1}{2}$ for large rs(outside the warp bubble) and $n(rs) = 0$ for small rs(inside the warp bubble) while being $0 < n(rs) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [1]):

We must demonstrate that the Natario warp drive vector given above satisfies the Natario requirements for a warp bubble defined by:

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nX = vs(t) * dx$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [1])(see Appendix G for an explanation about this statement)

Natario in its warp drive uses the polar coordinates rs and θ.In order to simplify our analysis we consider motion in the $x - axis$ or the equatorial plane rs where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [1]).

In a $1 + 1$ spacetime the equatorial plane we get¿:

$$nX = X^r e_r \tag{4}$$

The contravariant shift vector component X^{rs} is then:

$$X^{rs} = 2v_s n(rs) \tag{5}$$

Remember that Natario(pg 4 in [1]) defines the x axis as the axis of motion.Inside the bubble $n(rs) = 0$ resulting in a $X^{rs} = 0$ and outside the bubble $n(rs) = \frac{1}{2}$ resulting in a $X^{rs} = vs$ and this illustrates the Natario definition for a warp drive spacetime.See Appendix D

3 The equation of the Natario warp drive vector in polar coordinates with a variable speed vs due to a constant acceleration a

The equation of the Natario vector nX is given by:

$$nX = X^t e_t + X^r e_r + X^\theta e_\theta \tag{6}$$

The contravariant shift vector components X^t,X^{rs} and X^θ of the Natario vector are defined by(see Appendices B and C):

$$X^t = 2n(rs)rscos\theta a \tag{7}$$

$$X^{rs} = 2[2n(rs)^2 + rsn'(rs)]atcos\theta \tag{8}$$

$$X^\theta = -2n(rs)at[2n(rs) + rsn'(rs)]\sin\theta \tag{9}$$

Considering a valid $n(rs)$ as a Natario shape function being $n(rs) = \frac{1}{2}$ for large rs(outside the warp bubble) and $n(rs) = 0$ for small rs(inside the warp bubble) while being $0 < n(rs) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [1]):

We must demonstrate that the Natario warp drive vector given above satisfies the Natario requirements for a warp bubble defined by:

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nX = vs(t) * dx + x * dvs$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [1])(see Appendix G for an explanation about this statement)

Natario in its warp drive uses the polar coordinates rs and θ.In order to simplify our analysis we consider motion in the $x - axis$ or the equatorial plane rs where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [1]).

In a $1 + 1$ spacetime the equatorial plane we get¿:

$$nX = X^t e_t + X^r e_r \tag{10}$$

$$X^t = 2n(rs)rsa \tag{11}$$

$$X^{rs} = 2[2n(rs)^2 + rsn'(rs)]at \tag{12}$$

The variable velocity vs due to a constant acceleration a is given by the following equation:

$$vs = 2n(rs)at \tag{13}$$

Remember that Natario(pg 4 in [1]) defines the x axis as the axis of motion.Inside the bubble $n(rs) = 0$ resulting in a $vs = 0$ and outside the bubble $n(rs) = \frac{1}{2}$ resulting in a $vs = at$ as expected from a variable velocity vs in time t due to a constant acceleration a.Since inside and outside the bubble $n(rs)$ always possesses the same values of 0 or $\frac{1}{2}$ then the derivative $n'(rs)$ of the Natario shape function $n(rs)$ is zero and the shift vector $X^{rs} = 2[2n(rs)^2]at$ with $X^{rs} = 0$ inside the bubble and $X^{rs} = 2[2n(rs)^2]at = 2[2\frac{1}{4}]at = at = vs$ outside the bubble and this illustrates the Natario definition for a warp drive spacetime.See Appendix D

4 The equation of the new warp drive vector in tridimensional $3D$ spherical coordinates with a constant speed vs

The equation of the new warp drive vector in tridimensional $3D$ spherical coordinates with a constant speed vs nX is given by:

$$nX = X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{14}$$

With the contravariant shift vector components X^{rs}, X^θ and X^ϕ given by:
(see Appendix J for details)

$$X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] \tag{15}$$

$$X^\theta = -vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] \tag{16}$$

$$X^\phi = [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] \tag{17}$$

Considering a valid $f(r)$ as a shape function being $f(r) = \frac{1}{2}$ for large r(outside the warp bubble) and $f(r) = 0$ for small rs(inside the warp bubble) while being $0 < f(r) < \frac{1}{2}$ in the walls of the warp bubble also known as the warped region:

We must demonstrate that our warp drive vector satisfies the Natario criteria for a warp drive defined by:

any warp drive vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of r defined by Natario as the interior of the warp bubble and $nX = vs(t) * dx$ with $X = vs$ for a large value of r defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [1])(see Appendix G for an explanation about this statement)

Natario in its warp drive uses the polar coordinates r and θ.In order to simplify our analysis we consider motion in the $x-axis$(like Natario did) or the equatorial plane $x-y$ in r where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [1]).Also the equatorial plane $x-y$ makes an angle of 90 degrees with the $z-axis$ so $\sin\phi = 1$ and $cos\phi = 0$.

Then the contravariant components reduces to:

$$X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] \rightarrow X^r = vs(t)[2f(r)] \rightarrow \sin\phi = 1 \rightarrow\ cos\theta = 1 \tag{18}$$

$$X^\theta = -vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] = 0 \rightarrow \sin\phi = 1 \rightarrow \sin\theta = 0 \tag{19}$$

$$X^\phi = [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] = 0 \rightarrow cos\phi = 0 \tag{20}$$

Remember that Natario(pg 4 in [1]) defines the x axis as the axis of motion.Inside the bubble $f(r) = 0$ resulting in a $X^r = 0$ and outside the bubble $f(r) = \frac{1}{2}$ resulting in a $X^r = vs$ and this illustrates the Natario definition for a warp drive spacetime.See Appendix E

Only in tridimensional motion the results becomes different.As a matter of fact we have three different situations to consider in this case:

- 1)-inside the warp bubble $f(r) = 0$
- 2)-outside the warp bubble $f(r) = \frac{1}{2}$
- 3)-in the warp bubble walls $0 < f(r) < \frac{1}{2}$
- A)-situation inside the bubble:

The contravariant components reduces to:

$$X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] = 0 \rightarrow f(r) = 0 \tag{21}$$

$$X^\theta = -vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] = 0 \rightarrow f(r) = 0 \tag{22}$$

$$X^\phi = [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] = 0 \rightarrow f(r) = 0 \tag{23}$$

Inside the bubble the shape function $f(r)$ is constant and always zero so its derivatives vanishes too:

- B)-situation outside the bubble:

The contravariant components reduces to:

$$X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] = vs(t)[\sin\phi][\ cos\theta] \rightarrow f(r) = \frac{1}{2} \tag{24}$$

$$X^\theta = -vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] = -vs(t)[\sin\phi][\sin\theta] \rightarrow f(r) = \frac{1}{2} \tag{25}$$

$$X^\phi = [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] = [vs(t)cos\phi][cot\theta] \rightarrow f(r) = \frac{1}{2} \tag{26}$$

Outside the bubble the shape function $f(r)$ is constant and always $\frac{1}{2}$ so its derivatives vanishes too:

- C)-situation in the warp bubble walls:

The contravariant components are:

$$X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] \tag{27}$$

$$X^\theta = -vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] \tag{28}$$

$$X^\phi = [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] \tag{29}$$

In the warp bubble walls the shape function $f(r)$ varies between $0 < f(r) < \frac{1}{2}$ so its derivatives do not vanishes:

5 The equation of the new warp drive vector in tridimensional $3D$ spherical coordinates with a variable speed vs due to a constant acceleration a

The equation of the new warp drive vector in tridimensional $3D$ spherical coordinates with a variable speed vs due to a constant acceleration a nX is given by:

$$nX = X^t e_t + X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{30}$$

With the contravariant shift vector components X^t,X^{rs}, X^θ and X^ϕ given by: (see Appendices K and L for details)

$$X^t = 2(rf(r)a))(\sin\phi)(\cos\theta) \tag{31}$$

$$X^r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta) \tag{32}$$

$$X^\theta = -(2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\sin\theta) \tag{33}$$

$$X^\phi = (2f(r)at)[2f(r) + (rf'(r))](cos\phi)(cot\theta) \tag{34}$$

Considering a valid $f(r)$ as a shape function being $f(r) = \frac{1}{2}$ for large r(outside the warp bubble) and $f(r) = 0$ for small rs(inside the warp bubble) while being $0 < f(r) < \frac{1}{2}$ in the walls of the warp bubble also known as the warped region:

We must demonstrate that our warp drive vector satisfies the Natario criteria for a warp drive defined by:

any warp drive vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of r defined by Natario as the interior of the warp bubble and $nX = vs(t) * dx + x * dvs(t)$ with $X = vs$ for a large value of r defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [1])(see Appendix G for an explanation about this statement)

Natario in its warp drive uses the polar coordinates r and θ.In order to simplify our analysis we consider motion in the $x - axis$(like Natario did) or the equatorial plane $x - y$ in r where $\theta = 0$ $\sin(\theta) = 0$ and $\cos(\theta) = 1$.(see pgs 4,5 and 6 in [1]).Also the equatorial plane $x - y$ makes an angle of 90 degrees with the $z - axis$ so $\sin\phi = 1$ and $cos\phi = 0$.Then the contravariant components reduces to:

$$X^t = 2(rf(r)a))(\sin\phi)(\cos\theta) \rightarrow X^t = 2(rf(r)a)) \rightarrow \sin\phi = 1 \rightarrow \cos\theta = 1 \tag{35}$$

$$X^r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta) \rightarrow X^r = (2at)[2f(r)^2 + (rf'(r))] \rightarrow \sin\phi = 1 \rightarrow \cos\theta = 1 \tag{36}$$

$$X^\theta = -(2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\sin\theta) = 0 \rightarrow \sin\phi = 1 \rightarrow \sin\theta = 0 \tag{37}$$

$$X^\phi = (2f(r)at)[2f(r) + (rf'(r))](cos\phi)(cot\theta) = 0 \rightarrow \cos\phi = 0 \tag{38}$$

The remaining contravariant components are:

$$X^t = 2(rf(r)a))(\sin\phi)(\cos\theta) \rightarrow X^t = 2(rf(r)a)) \rightarrow \sin\phi = 1 \rightarrow \cos\theta = 1 \tag{39}$$

$$X^r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta) \rightarrow X^r = (2at)[2f(r)^2 + (rf'(r))] \rightarrow \sin\phi = 1 \rightarrow \cos\theta = 1 \tag{40}$$

$$nX = X^t e_t + X^r e_r \tag{41}$$

$$X^t = 2rf(r)a \tag{42}$$

$$X^{rs} = 2[2f(r)^2 + rf'(r)]at \tag{43}$$

The variable velocity vs due to a constant acceleration a is given by the following equation:

$$vs = 2f(r)at \tag{44}$$

Remember that Natario(pg 4 in [1]) defines the x axis as the axis of motion.Inside the bubble $f = 0$ resulting in a $vs = 0$ and outside the bubble $f = \frac{1}{2}$ resulting in a $vs = at$ as expected from a variable velocity vs in time t due to a constant acceleration a.Since inside and outside the bubble $f(r)$ always possesses the same values of 0 or $\frac{1}{2}$ then the derivative $f'(r)$ of the shape function $f(r)$ is zero and the shift vector $X^{rs} = 2[2f(r)^2]at$ with $X^r = 0$ inside the bubble and $X^{rs} = 2[2f(r)^2]at = 2[2\frac{1}{4}]at = at = vs$ outside the bubble and this illustrates the Natario definition for a warp drive spacetime.See Appendix E

Only in tridimensional motion the results becomes different.As a matter of fact we have three different situations to consider in this case:

- 1)-inside the warp bubble $f(r) = 0$
- 2)-outside the warp bubble $f(r) = \frac{1}{2}$
- 3)-in the warp bubble walls $0 < f(r) < \frac{1}{2}$

- A)-situation inside the bubble:

The contravariant components reduces to:

$$X^t = 2(rf(r)a))(\sin\phi)(\cos\theta) = 0 \to f(r) = 0 \tag{45}$$

$$X^r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta) = 0 \to f(r) = 0 \tag{46}$$

$$X^\theta = -(2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\sin\theta) = 0 \to f(r) = 0 \tag{47}$$

$$X^\phi = (2f(r)at)[2f(r) + (rf'(r))](cos\phi)(cot\theta) = 0 \to f(r) = 0 \tag{48}$$

Inside the bubble the shape function $f(r)$ is constant and always zero so its derivatives vanishes too:

- B)-situation outside the bubble:

$$X^t = 2(rf(r)a)(\sin\phi)(\cos\theta) = (ra)(\sin\phi)(\cos\theta) \to f(r) = \frac{1}{2} \tag{49}$$

$$X^r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta) = (at)(\sin\phi)(\cos\theta) \to f(r) = \frac{1}{2} \tag{50}$$

$$X^\theta = -(2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\sin\theta) = -(at)(\sin\phi)(\sin\theta) \to f(r) = \frac{1}{2} \tag{51}$$

$$X^\phi = (2f(r)at)[2f(r) + (rf'(r))](cos\phi)(cot\theta) = (at)(cos\phi)(cot\theta) \to f(r) = \frac{1}{2} \tag{52}$$

Outside the bubble the shape function $f(r)$ is constant and always $\frac{1}{2}$ so its derivatives vanishes too:Hint for the readers:compare these expressions for X^r,X^θ and X^ϕ with its similar counterparts for the situation outside the bubble from the previous section:Can you spot something familiar?or not?.

- C)-situation in the warp bubble walls:

$$X^t = 2(rf(r)a))(\sin\phi)(\cos\theta) \tag{53}$$

$$X^r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta) \tag{54}$$

$$X^\theta = -(2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\sin\theta) \tag{55}$$

$$X^\phi = (2f(r)at)[2f(r) + (rf'(r))](cos\phi)(cot\theta) \tag{56}$$

In the warp bubble walls the shape function $f(r)$ varies between $0 < f(r) < \frac{1}{2}$ so its derivatives do not vanishes:

6 Conclusion

In this work we introduced a new tridimensional $3D$ spherical coordinates warp drive vector using the Natario mathematical techniques.We focused ourselves in the application of the Hodge Star in $3D$ spherical coordinates for both constant and variable speeds.

Our focus was concentrated in the Natario methods to obtain a warp drive vector.We know that we used a language and a presentation method or style that may be regarded as exhaustive tedious and monotonous for experienced or seasoned readers but we are concerned about beginners,newcomers,novices or intermediate students not familiarized with the techniques Natario used to develop warp drive vectors so our extensive mathematical demonstrations QED Quod Erad Demonstratum will benefit this audience at least we hope.We gave our best efforts trying to accomplish this goal but only this audience will tell in the future if we succeeded (or not).

The application of the new tridimensional $3D$ spherical coordinates warp drive vector wether in constant or variable speeds to the ADM(Arnowitt-Dresner-Misner) formalism equations in General Relativity using the approach of MTW(Misner-Thorne-Wheeler)resembling the works [10],[11][12] and [13] will appear in a future work.

A complete study of our new tridimensional $3D$ spherical coordinates warp drive vector wether in constant or variable speeds using the techniques of the rate-of-strain stress tensor as described in pgs 354 and 355 in [8] or Natario in pg 5 in [1] will also appear in a future work.

7 Appendix A:differential forms,Hodge star and the mathematical demonstration of the Natario vectors $nX = -vs * dx$ and $nX = vs * dx$ for a constant speed vs in a R^3 space basis-Polar Coordinates

This appendix is being written for novice or newcomer students on Warp Drive theory still not acquainted with the methods Natario used to arrive at the final expression of the Natario Vector nX

The Canonical Basis of the Hodge Star in spherical coordinates can be defined as follows(see pg 4 in [1],eq 3.72 pg 69$(a)(b)$ in [2]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2 \sin\theta (d\theta \wedge d\varphi) \tag{57}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta(d\varphi \wedge dr) \tag{58}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{59}$$

From above we get the following results

$$dr \sim r^2 \sin\theta(d\theta \wedge d\varphi) \tag{60}$$

$$rd\theta \sim r\sin\theta(d\varphi \wedge dr) \tag{61}$$

$$r\sin\theta d\varphi \sim r(dr \wedge d\theta) \tag{62}$$

Note that this expression matches the common definition of the Hodge Star operator * applied to the spherical coordinates as given by(see eq 3.72 pg 69$(a)(b)$ in [2]):

$$*dr = r^2 \sin\theta(d\theta \wedge d\varphi) \tag{63}$$

$$*rd\theta = r\sin\theta(d\varphi \wedge dr) \tag{64}$$

$$*r\sin\theta d\varphi = r(dr \wedge d\theta) \tag{65}$$

Back again to the Natario equivalence between polar and cartezian coordinates(pg 5 in [1]):

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \sim r^2 \sin\theta\cos\theta d\theta \wedge d\varphi + r\sin^2\theta dr \wedge d\varphi = d\left(\frac{1}{2}r^2 \sin^2\theta d\varphi\right) \tag{66}$$

Look that

$$dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \tag{67}$$

Or

$$dx = d(r\cos\theta) = \cos\theta dr - \sin\theta r d\theta \tag{68}$$

Applying the Hodge Star operator * to the above expression:

$$*dx = *d(r\cos\theta) = \cos\theta(*dr) - \sin\theta(*rd\theta) \tag{69}$$

$$*dx = *d(r\cos\theta) = \cos\theta[r^2\sin\theta(d\theta\wedge d\varphi)] - \sin\theta[r\sin\theta(d\varphi\wedge dr)] \tag{70}$$

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(d\theta\wedge d\varphi)] - [r\sin^2\theta(d\varphi\wedge dr)] \tag{71}$$

We know that the following expression holds true(see eq 3.79 pg 70$(a)(b)$ in [2]):

$$d\varphi\wedge dr = -dr\wedge d\varphi \tag{72}$$

Then we have

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(d\theta\wedge d\varphi)] + [r\sin^2\theta(dr\wedge d\varphi)] \tag{73}$$

And the above expression matches exactly the term obtained by Natario using the Hodge Star operator applied to the equivalence between cartezian and spherical coordinates(pg 5 in [1]).

Now examining the expression:

$$d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{74}$$

We must also apply the Hodge Star operator to the expression above

And then we have:

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{75}$$

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \sim \frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] + \frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] \tag{76}$$

According to eq 3.90 pg 74$(a)(b)$ in [2] the term $\frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] = 0$

This leaves us with:

$$\frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] \sim \frac{1}{2}r^2 2\sin\theta\cos\theta(d\theta\wedge d\varphi) + \frac{1}{2}\sin^2\theta 2r(dr\wedge d\varphi) \tag{77}$$

$$\frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] \sim \frac{1}{2}r^2 2\sin\theta\cos\theta(d\theta\wedge d\varphi) + \frac{1}{2}\sin^2\theta 2r(dr\wedge d\varphi) \tag{78}$$

Because and according to eqs 3.90 and 3.91 pg $74(a)(b)$ in [2],tb 3.2 pg $68(a)(b)$ in [2]:

$$*d(\alpha+\beta) = d\alpha + d\beta \tag{79}$$

$$*d(f\alpha) = df\wedge\alpha + (-1)^p f\wedge d\alpha \dashrightarrow p = 2 \dashrightarrow *d(f\alpha) = df\wedge\alpha + f\wedge d\alpha \tag{80}$$

$$*d(dx) = d(dy) = d(dz) = 0 \tag{81}$$

From above we can see for example that

$$*d[(\sin^2\theta)d\varphi] = d(\sin^2\theta)\wedge d\varphi + \sin^2\theta\wedge dd\varphi = 2sin\theta\cos\theta(d\theta\wedge d\varphi) \tag{82}$$

$$*[d(r^2)d\varphi] = 2rdr\wedge d\varphi + r^2\wedge dd\varphi = 2r(dr\wedge d\varphi) \tag{83}$$

And then we derived again the Natario result of pg 5 in [1]

$$r^2\sin\theta\cos\theta(d\theta\wedge d\varphi) + r\sin^2\theta(dr\wedge d\varphi) \tag{84}$$

Now we will examine the following expression equivalent to the one of Natario pg 5 in [1] except that we replaced $\frac{1}{2}$ by the function $f(r)$:

$$*d[f(r)r^2\sin^2\theta d\varphi] \tag{85}$$

From above we can obtain the next expressions

$$f(r)r^2 * d[(\sin^2\theta)d\varphi] + f(r)\sin^2\theta * [d(r^2)d\varphi] + r^2\sin^2\theta * d[f(r)d\varphi] \tag{86}$$

$$f(r)r^2 2sin\theta\cos\theta(d\theta\wedge d\varphi) + f(r)\sin^2\theta 2r(dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dr\wedge d\varphi) \tag{87}$$

$$2f(r)r^2 sin\theta\cos\theta(d\theta\wedge d\varphi) + 2f(r)r\sin^2\theta(dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dr\wedge d\varphi) \tag{88}$$

$$2f(r)r^2 sin\theta \cos\theta(d\theta \wedge d\varphi) + 2f(r)r\sin^2\theta(dr \wedge d\varphi) + r^2\sin^2\theta f'(r)(dr \wedge d\varphi) \tag{89}$$

Comparing the above expressions with the Natario definitions of pg 4 in [1]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2\sin\theta(d\theta \wedge d\varphi) \tag{90}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta(d\varphi \wedge dr) \sim -r\sin\theta(dr \wedge d\varphi) \tag{91}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{92}$$

We can obtain the following result:

$$2f(r)\ cos\theta[r^2 sin\theta(d\theta \wedge d\varphi)] + 2f(r)\ sin\theta[r\sin\theta(dr \wedge d\varphi)] + f'(r)r\sin\theta[r\sin\theta(dr \wedge d\varphi)] \tag{93}$$

$$2f(r)\ cos\theta e_r - 2f(r)\ sin\theta e_\theta - rf'(r)\sin\theta e_\theta \tag{94}$$

$$*d[f(r)r^2\sin^2\theta d\varphi] = 2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta \tag{95}$$

Defining the Natario Vector as in pg 5 in [1] with the Hodge Star operator * explicitly written :

$$nX = vs(t) * d\left(f(r)r^2\sin^2\theta d\varphi\right) \tag{96}$$

$$nX = -vs(t) * d\left(f(r)r^2\sin^2\theta d\varphi\right) \tag{97}$$

We can get finally the latest expressions for the Natario Vector nX also shown in pg 5 in [1]

$$nX = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{98}$$

$$nX = -2vs(t)f(r)\ cos\theta e_r + vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{99}$$

8 Appendix B:differential forms,Hodge star and the mathematical demonstration of the Natario vectors $nX = -vs * dx$ and $nX = vs * dx$ for a constant speed vs or for the first term $vs * dx$ from the Natario vector $nX = vs * dx + x * dvs$(a variable speed) in a R^4 space basis-Polar Coordinates

This appendix is being written for novice or newcomer students on Warp Drive theory still not acquainted with the methods Natario used to arrive at the final expression of the Natario Vector nX

The Canonical Basis of the Hodge Star in spherical coordinates can be defined as follows(see pg 4 in [1],eqs 3.135 and 3.137 pg 82$(a)(b)$ in [2],eq 3.74 pg 69$(a)(b)$ in [2])(see pg 47 eqs 2.67 to 2.70 and pg 92 in [3]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim dt \wedge (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2 \sin\theta (dt \wedge d\theta \wedge d\varphi) \tag{100}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial \theta} \sim rd\theta \sim dt \wedge (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta (dt \wedge d\varphi \wedge dr) \tag{101}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi} \sim r\sin\theta d\varphi \sim dt \wedge dr \wedge (rd\theta) \sim r(dt \wedge dr \wedge d\theta) \tag{102}$$

From above we get the following results

$$dr \sim r^2 \sin\theta (dt \wedge d\theta \wedge d\varphi) \tag{103}$$

$$rd\theta \sim r\sin\theta (dt \wedge d\varphi \wedge dr) \tag{104}$$

$$r\sin\theta d\varphi \sim r(dt \wedge dr \wedge d\theta) \tag{105}$$

Note that this expression matches the common definition of the Hodge Star operator * applied to the spherical coordinates as given by(see eq 3.74 pg 69$(a)(b)$ in [2]):

$$*dr = r^2 \sin\theta (dt \wedge d\theta \wedge d\varphi) \tag{106}$$

$$*rd\theta = r\sin\theta (dt \wedge d\varphi \wedge dr) \tag{107}$$

$$*r\sin\theta d\varphi = r(dt \wedge dr \wedge d\theta) \tag{108}$$

Back again to the Natario equivalence between polar and cartezian coordinates(pg 5 in [1]):

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \sim r^2 \sin\theta\cos\theta dt \wedge d\theta \wedge d\varphi + r\sin^2\theta dt \wedge dr \wedge d\varphi = d\left(\frac{1}{2} r^2 \sin^2\theta d\varphi\right) \tag{109}$$

Look that

$$dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \tag{110}$$

Or

$$dx = d(r\cos\theta) = \cos\theta dr - \sin\theta r d\theta \tag{111}$$

Applying the Hodge Star operator * to the above expression:

$$*dx = *d(r\cos\theta) = \cos\theta(*dr) - \sin\theta(*rd\theta) \tag{112}$$

$$*dx = *d(r\cos\theta) = \cos\theta[r^2\sin\theta(dt\wedge d\theta\wedge d\varphi)] - \sin\theta[r\sin\theta(dt\wedge d\varphi\wedge dr)] \tag{113}$$

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi)] - [r\sin^2\theta(dt\wedge d\varphi\wedge dr)] \tag{114}$$

We know that the following expression holds true(see eq 3.79 pg 70$(a)(b)$ in [2]):

$$d\varphi\wedge dr = -dr\wedge d\varphi \tag{115}$$

Then we have

$$*dx = *d(r\cos\theta) = [r^2\sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi)] + [r\sin^2\theta(dt\wedge dr\wedge d\varphi)] \tag{116}$$

And the above expression matches exactly the term obtained by Natario using the Hodge Star operator applied to the equivalence between cartezian and spherical coordinates(pg 5 in [1]).

Now examining the expression:

$$d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{117}$$

We must also apply the Hodge Star operator to the expression above

And then we have:

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \tag{118}$$

$$*d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right) \sim \frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] + \frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] \tag{119}$$

According to eq 3.90 pg 74$(a)(b)$ in [2] the term $\frac{1}{2}r^2\sin^2\theta * d[(d\varphi)] = 0$

This leaves us with:

$$\frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] \sim \frac{1}{2}r^2 2\sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi) + \frac{1}{2}\sin^2\theta 2r(dt\wedge dr\wedge d\varphi) \tag{120}$$

$$\frac{1}{2}r^2 * d[(\sin^2\theta)d\varphi] + \frac{1}{2}\sin^2\theta * [d(r^2)d\varphi] \sim \frac{1}{2}r^2 2\sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi) + \frac{1}{2}\sin^2\theta 2r(dt\wedge dr\wedge d\varphi) \quad (121)$$

Because and according to eqs 3.90 and 3.91 pg 74$(a)(b)$ in [2],tb 3.3 pg 68$(a)(b)$ in [2]::

$$*d(\alpha+\beta) = d\alpha + d\beta \quad (122)$$

$$*d(f\alpha) = df\wedge\alpha + (-1)^p f\wedge d\alpha \dashrightarrow p = 3 \dashrightarrow *d(f\alpha) = df\wedge\alpha - f\wedge d\alpha \quad (123)$$

$$*d(dx) = d(dy) = d(dz) = 0 \quad (124)$$

From above we can see for example that

$$*d[(\sin^2\theta)d\varphi] = dt\wedge d(\sin^2\theta)\wedge d\varphi - dt\wedge\sin^2\theta\wedge dd\varphi = 2sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi) \quad (125)$$

$$*[d(r^2)d\varphi] = 2rdt\wedge dr\wedge d\varphi - dt\wedge r^2\wedge dd\varphi = 2r(dt\wedge dr\wedge d\varphi) \quad (126)$$

And then we derived again the Natario result of pg 5 in [1]

$$r^2\sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi) + r\sin^2\theta(dt\wedge dr\wedge d\varphi) \quad (127)$$

Now we will examine the following expression equivalent to the one of Natario pg 5 in [1] except that we replaced $\frac{1}{2}$ by the function $f(r)$:

$$*d[f(r)r^2\sin^2\theta d\varphi] \quad (128)$$

From above we can obtain the next expressions

$$f(r)r^2 * d[(\sin^2\theta)d\varphi] + f(r)\sin^2\theta * [d(r^2)d\varphi] + r^2\sin^2\theta * d[f(r)d\varphi] \quad (129)$$

$$f(r)r^2 2sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi) + f(r)\sin^2\theta 2r(dt\wedge dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dt\wedge dr\wedge d\varphi) \quad (130)$$

$$2f(r)r^2 sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi) + 2f(r)r\sin^2\theta(dt\wedge dr\wedge d\varphi) + r^2\sin^2\theta f'(r)(dt\wedge dr\wedge d\varphi) \quad (131)$$

$$2f(r)r^2sin\theta\cos\theta(dt\wedge d\theta\wedge d\varphi)+2f(r)r\sin^2\theta(dt\wedge dr\wedge d\varphi)+r^2\sin^2\theta f'(r)(dt\wedge dr\wedge d\varphi) \tag{132}$$

Comparing the above expressions with the Natario definitions of pg 4 in [1]):

$$e_r\equiv\frac{\partial}{\partial r}\sim dr\sim dt\wedge(rd\theta)\wedge(r\sin\theta d\varphi)\sim r^2\sin\theta(dt\wedge d\theta\wedge d\varphi) \tag{133}$$

$$e_\theta\equiv\frac{1}{r}\frac{\partial}{\partial\theta}\sim rd\theta\sim dt\wedge(r\sin\theta d\varphi)\wedge dr\sim r\sin\theta(dt\wedge d\varphi\wedge dr)\sim -r\sin\theta(dt\wedge dr\wedge d\varphi) \tag{134}$$

$$e_\varphi\equiv\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\sim r\sin\theta d\varphi\sim dt\wedge dr\wedge(rd\theta)\sim r(dt\wedge dr\wedge d\theta) \tag{135}$$

We can obtain the following result:

$$2f(r)\ cos\theta[r^2sin\theta(dt\wedge d\theta\wedge d\varphi)]+2f(r)\ sin\theta[r\sin\theta(dt\wedge dr\wedge d\varphi)]+f'(r)r\sin\theta[r\sin\theta(dt\wedge dr\wedge d\varphi)] \tag{136}$$

$$2f(r)\ cos\theta e_r-2f(r)\ sin\theta e_\theta-rf'(r)\sin\theta e_\theta \tag{137}$$

$$*d[f(r)r^2\sin^2\theta d\varphi]=2f(r)\ cos\theta e_r-[2f(r)+rf'(r)]\sin\theta e_\theta \tag{138}$$

Defining the Natario Vector as in pg 5 in [1] with the Hodge Star operator * explicitly written :

$$nX=vs(t)*d\left(f(r)r^2\sin^2\theta d\varphi\right) \tag{139}$$

$$nX=-vs(t)*d\left(f(r)r^2\sin^2\theta d\varphi\right) \tag{140}$$

We can get finally the latest expressions for the Natario Vector nX also shown in pg 5 in [1]

$$nX=2vs(t)f(r)\ cos\theta e_r-vs(t)[2f(r)+rf'(r)]\sin\theta e_\theta \tag{141}$$

$$nX=-2vs(t)f(r)\ cos\theta e_r+vs(t)[2f(r)+rf'(r)]\sin\theta e_\theta \tag{142}$$

9 Appendix C:differential forms,Hodge star and the mathematical demonstration of the Natario vector $nX = *(vsx) = vs * dx + x * dvs$ for a variable speed vs and a constant acceleration a in Polar Coordinates

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of r defined by Natario as the interior of the warp bubble and $nX = vs(t) * dx$ with $X = vs$ for a large value of r defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [1])(see Appendix G for an explanation about this statement)

In the Appendices A and B we gave the mathematical demonstration of the Natario vector $nX = vs * dx$ in the R^3 and R^4 space basis when the velocity vs is constant.Hence the complete expression of the Hodge star that generates the Natario vector nX for a constant velocity vs is given by:

$$nX = *(vsx) = vs * (dx) \tag{143}$$

$$*dx = *d(rcos\theta) = *d\left(\frac{1}{2}r^2 \sin^2\theta d\varphi\right) = *d[f(r)r^2 \sin^2\theta d\varphi] \tag{144}$$

The equation of the Natario vector nX(pg 2 and 5 in [1]) is given by:

$$nX = X^r e_r + X^\theta e_\theta \tag{145}$$

$$nX = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{146}$$

With the contravariant shift vector components explicitly given by:

$$X^r = 2v_s f(r)\cos\theta \tag{147}$$

$$X^\theta = -v_s(2f(r) + (r)f'(r))\sin\theta \tag{148}$$

Because due to a constant speed vs the term $x * d(vs) = 0$.Now we must examine what happens when the velocity is variable and then the term $x * d(vs)$ no longer vanishes.Remember that a real warp drive must accelerate or de-accelerate in order to be accepted as a physical valid model.The complete expression of the Hodge star that generates the Natario vector nX for a variable velocity vs is now given by:

$$nX = *(vsx) = vs * (dx) + x * (dvs) \tag{149}$$

In order to study the term $x * d(vs)$ we must introduce a new Canonical Basis for the coordinate time in the R^4 space basis defined as follows:(see eqs 10.102 and 10.103 pgs 363$(a)(b)$ and 364$(a)(b)$ in [2] with the terms $S = u = 1$[1],eq 3.74 pg 69$(a)(b)$ in [2],eqs 11.131 and 11.133 with the term $m = 0$[2] pg 417$(a)(b)$ in [2].)(see pg 47 eqs 2.67 to 2.70 and pg 92 in [3]):

$$e_t \equiv \frac{\partial}{\partial t} \sim dt \sim dr \wedge (rd\theta) \wedge (r \sin\theta d\varphi) \sim r^2 \sin\theta (dr \wedge d\theta \wedge d\varphi) \tag{150}$$

$$dt \sim r^2 \sin\theta (dr \wedge d\theta \wedge d\varphi) \tag{151}$$

The Hodge star operator defined for the coordinate time is given by:(see eq 3.74 pg 69$(a)(b)$ in [2]):

$$*dt = r^2 \sin\theta (dr \wedge d\theta \wedge d\varphi) \tag{152}$$

The valid expression for a variable velocity $vs(t)$ in the Natario warp drive spacetime due to a constant acceleration a must be given by:

$$vs = 2f(r)at \tag{153}$$

Because and considering a valid $f(r)$ as a Natario shape function being $f(r) = \frac{1}{2}$ for large r(outside the warp bubble where $X = vs(t)$ and $nX = vs(t) * dx + x * d(vs(t)))$ and $f(r) = 0$ for small r(inside the warp bubble where $X = 0$ and $nX = 0$) while being $0 < f(r) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pgs 4 and 5 in [1]) and considering also that the Natario warp drive is a ship-frame based coordinates system(a reference frame placed in the center of the warp bubble where the ship resides-or must reside!!) then an observer in the ship inside the bubble sees every point inside the bubble at the rest with respect to him because inside the bubble $vs(t) = 0$ because $f(r) = 0$.

To illustrate the statement pointed above imagine a fish inside an aquarium and the aquarium is floating in the surface of a river but carried out by the river stream.The stream varies its velocity with time.The warp bubble in this case is the aquarium and the walls of the aquarium are the walls of the warp bubble-Natario warped region.An observer in the margin of the river would see the aquarium passing by him at a large speed considering a coordinates system(a reference frame) placed in the margin of the river but inside the aquarium the fish is at the rest with respect to his local neighborhoods.Then for the fish any point inside the aquarium is at the rest with respect to him because inside the aquarium $vs = 2f(r)at$ with $f(r) = 0$ and consequently giving a $vs(t) = 0$.Again with respect to the fish the fish "sees" the margin passing by him with a large relative velocity.The margin in this case is the region outside the bubble "seen" by the fish with a variable velocity $vs(t) = v1$ in the time $t1$ and $vs(t) = v2$ in the time $t2$ because outside the bubble the generic expression for a variable velocity vs is given by $vs = 2f(r)at$ and outside the bubble $f(r) = \frac{1}{2}$ giving a generic expression for a variable velocity vs as $vs(t) = at$ and consequently a $v1 = at1$ in the time $t1$ and a $v2 = at2$ in the time $t2$.Then the variable velocity in not only a function of time alone but must consider also the position of the bubble where the measure is being taken wether inside or outside the bubble.So the velocity must also be a function of r.Its total differential is then given by:

$$dvs = 2[atf'(r)dr + f(r)tda + f(r)adt] \tag{154}$$

[1]These terms are needed to deal with the Robertson-Walker equation in Cosmology using differential forms.We dont need these terms here and we can make $S = u = 1$

[2]This term is needed to describe the Dirac equation in the Schwarzschild spacetime we dont need the term here so we can make $m = 1$.Remember also that here we consider geometrized units in which $c = 1$

Applying the Hodge star to the total differential dvs we get:

$$*dvs = 2[atf'(r) * dr + f(r)t * da + f(r)a * dt] \tag{155}$$

But we consider here the acceleration a a constant.Then the term $f(r)tda = 0$ and in consequence $f(r)t * da = 0$.This leaves us with:

$$*dvs = 2[atf'(r) * dr + f(r)a * dt] \tag{156}$$

$$*dvs = 2[atf'(r) * dr + f(r)a * dt] = 2[atf'(r)r^2 \sin\theta(dt \wedge d\theta \wedge d\varphi) + f(r)ar^2 \sin\theta(dr \wedge d\theta \wedge d\varphi)] \tag{157}$$

$$*dvs = 2[atf'(r) * dr + f(r)a * dt] = 2[atf'(r)e_r + f(r)ae_t] \tag{158}$$

The complete expression of the Hodge star that generates the Natario vector nX for a variable velocity vs is given by:

$$nX = *(vsx) = vs * (dx) + x * d(vs) \tag{159}$$

The term $*dx$ was obtained in the Appendices A and B as follows:(see pg 5 in [1])

$$*dx = 2f(r)\ cos\theta e_r - [2f(r) + rf'(r)] \sin\theta e_\theta \tag{160}$$

The complete expression of the Hodge star that generates the Natario vector nX for a variable velocity vs is now given by:

$$nX = *(vsx) = vs(2f(r)\ cos\theta e_r - [2f(r) + rf'(r)] \sin\theta e_\theta) + x(2[atf'(r)e_r + f(r)ae_t]) \tag{161}$$

But remember that we are in polar coordinates(pg 4 in [1]) in which $x = rcos\theta$(see pg 5 in [1]) (see also Appendix D)and this leaves us with:

$$nX = *(vsx) = vs(2f(r)\ cos\theta e_r - [2f(r) + rf'(r)] \sin\theta e_\theta) + rcos\theta(2[atf'(r)e_r + f(r)ae_t]) \tag{162}$$

But we know that $vs = 2f(r)at$.Hence we get:

$$nX = *(vsx) = 2f(r)at(2f(r)\ cos\theta e_r - [2f(r) + rf'(r)] \sin\theta e_\theta) + rcos\theta(2[atf'(r)e_r + f(r)ae_t]) \tag{163}$$

Then we can start with a warp bubble initially at the rest using the Natario vector shown above and accelerate the bubble to a desired speed of 200 times faster than light.When we achieve the desired speed we turn off the acceleration and keep the speed constant.The terms due to the acceleration now disappears and we are left again with the Natario vector for constant speeds shown below:

$$nX = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r) + rf'(r)] \sin\theta e_\theta \tag{164}$$

Working some algebra with the Natario vector for variable velocities we get:

$$nX = *(vsx) = 2f(r)at(2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta) + rcos\theta(2[atf'(r)e_r + f(r)ae_t]) \tag{165}$$

$$nX = 4f(r)^2at\ cos\theta e_r - 2f(r)at[2f(r) + rf'(r)]\sin\theta e_\theta + 2atf'(r)rcos\theta e_r + 2f(r)rcos\theta ae_t \tag{166}$$

$$nX = 2f(r)rcos\theta ae_t + 4f(r)^2at\ cos\theta e_r + 2atf'(r)rcos\theta e_r - 2f(r)at[2f(r) + rf'(r)]\sin\theta e_\theta \tag{167}$$

$$nX = 2f(r)rcos\theta ae_t + 2[2f(r)^2 + rf'(r)]atcos\theta e_r - 2f(r)at[2f(r) + rf'(r)]\sin\theta e_\theta \tag{168}$$

Then the Natario vector for variable velocities defined using contravariant shift vector components is given by the following expressions:

$$nX = X^t e_t + X^r e_r + X^\theta e_\theta \tag{169}$$

Or being:

$$nX = 2f(r)rcos\theta ae_t + 2[2f(r)^2 + rf'(r)]atcos\theta e_r - 2f(r)at[2f(r) + rf'(r)]\sin\theta e_\theta \tag{170}$$

The contravariant shift vector components are respectively given by the following expressions:

$$X^t = 2f(r)rcos\theta a \tag{171}$$

$$X^r = 2[2f(r)^2 + rf'(r)]atcos\theta \tag{172}$$

$$X^\theta = -2f(r)at[2f(r) + rf'(r)]\sin\theta \tag{173}$$

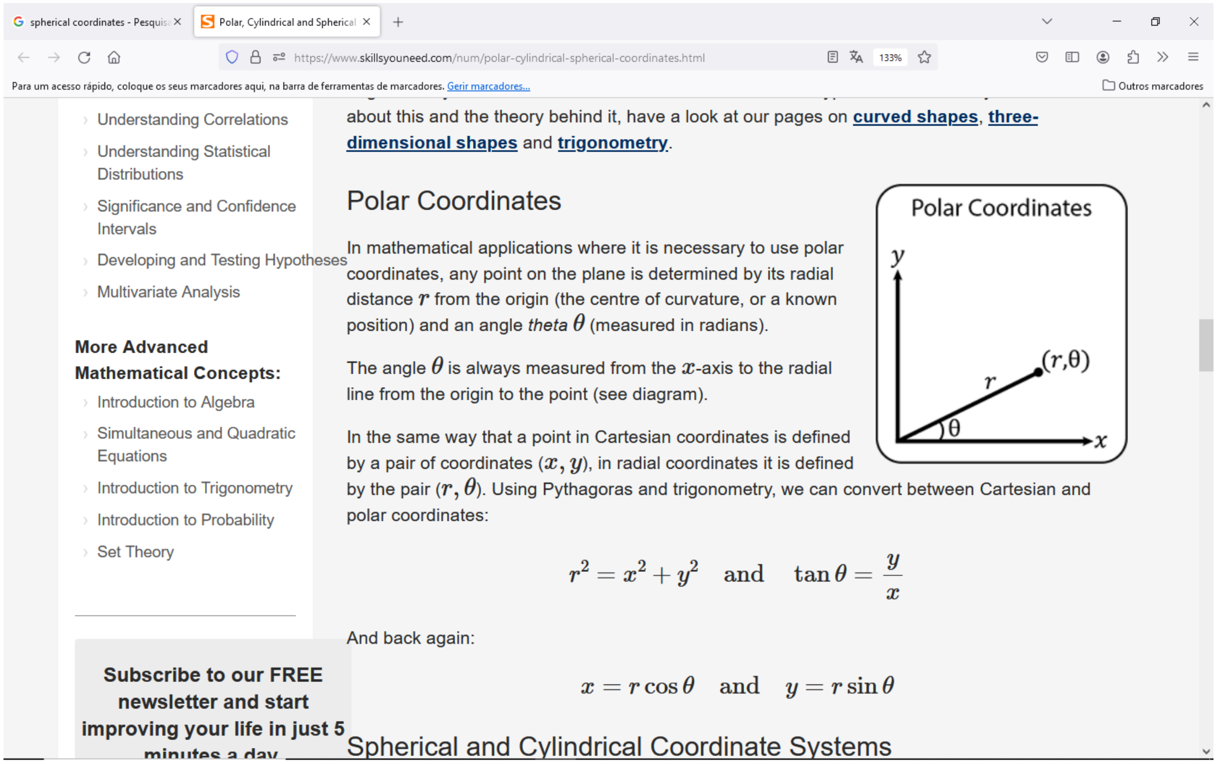

Figure 1: Polar Coordinates.(Source:Internet)

10 Appendix D:Polar Coordinates

Natario (See pg 5 in [1]) defined a warp drive vector $nX = vs * (dx)$ where vs is the **constant** speed of the warp bubble and $*(dx)$ is the Hodge Star taken over the x-axis of motion in **Polar Coordinates**(See pg 4 in [1].(See also Appendices A and B for the detailed calculations).

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \sim \ \sim r^2\sin\theta\cos\theta d\theta \wedge d\varphi + r\sin^2\theta dr \wedge d\varphi = d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right). \tag{174}$$

Consequently if we set exactly what Natario did in pg 5 in [1]:(we adopted the second expression)

$$\mathbf{X} \sim -v_s(t)d\left[f(r)r^2\sin^2\theta d\varphi\right] \sim -2v_s f\cos\theta\mathbf{e}_r + v_s(2f + rf')\sin\theta\mathbf{e}_\theta \tag{175}$$

$$\mathbf{X} \sim v_s(t)d\left[f(r)r^2\sin^2\theta d\varphi\right] \sim 2v_s f\cos\theta\mathbf{e}_r - v_s(2f + rf')\sin\theta\mathbf{e}_\theta \tag{176}$$

$$nX = X^r e_r + X^\theta e_\theta \tag{177}$$

$$X^{rs} = 2v_s f\cos\theta \tag{178}$$

$$X^\theta = -v_s(2f + rf')\sin\theta \tag{179}$$

Considering a valid f as a Natario shape function being $f = \frac{1}{2}$ for large r(outside the warp bubble) and $f = 0$ for small r(inside the warp bubble) while being $0 < f < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [1]):

We must demonstrate that the Natario warp drive vector given above satisfies the Natario requirements for a warp bubble defined by:

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nX = vs(t) * dx$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [1])(see Appendix G for an explanation about this statement)

Inside the bubble $f = 0$ and the Natario vector components are zero too.Outside the bubble $f = \frac{1}{2}$,$X^{rs} = v_s \cos\theta$ and $X^{\theta} = -v_s \sin\theta$.In motion over the x-axis only in the equatorial plane $X^{rs} = v_s$ because $\cos\theta = 1$

Due to a **constant** speed vs the term $x * d(vs) = 0$.Now we must examine what happens when the velocity is **variable** and then the term $x * d(vs)$ no longer vanishes.Remember that a real warp drive must accelerate or de-accelerate in order to be accepted as a physical valid model.The complete expression of the Hodge star that generates the Natario vector nX for a variable velocity vs is now given by(see Appendix C for detailed calculations):

$$nX = *(vsx) = vs * (dx) + x * (dvs) \tag{180}$$

The term $*(dx)$ is **again** taken in **Polar Coordinates**

$$nX = X^t e_t + X^r e_r + X^{\theta} e_{\theta} \tag{181}$$

$$nX = 2f(r)rcos\theta a e_t + 2[2f(r)^2 + rf'(r)]atcos\theta e_r - 2f(r)at[2f(r) + rf'(r)]\sin\theta e_{\theta} \tag{182}$$

$$X^t = 2f(r)rcos\theta a \tag{183}$$

$$X^r = 2[2f(r)^2 + rf'(r)]atcos\theta \tag{184}$$

$$X^{\theta} = -2f(r)at[2f(r) + rf'(r)]\sin\theta \tag{185}$$

The variable velocity vs due to a constant acceleration a is given by the following equation:

$$vs = 2fat \tag{186}$$

Remember that Natario(pg 4 in [1]) defines the x axis as the axis of motion.Inside the bubble $f = 0$ resulting in a $vs = 0$ and outside the bubble $f = \frac{1}{2}$ resulting in a $vs = at$ as expected from a variable velocity vs in time t due to a constant acceleration a.Since inside and outside the bubble f always possesses the same values of 0 or $\frac{1}{2}$ then the derivative f' of the Natario shape function f is zero and the shift vector $X^{rs} = 2[2f^2]at$ with $X^{rs} = 0$ inside the bubble and $X^{rs} = 2[2f^2]at = 2[2\frac{1}{4}]at = at = vs$ outside the bubble and this illustrates the Natario definition for a warp drive spacetime.See Appendix G

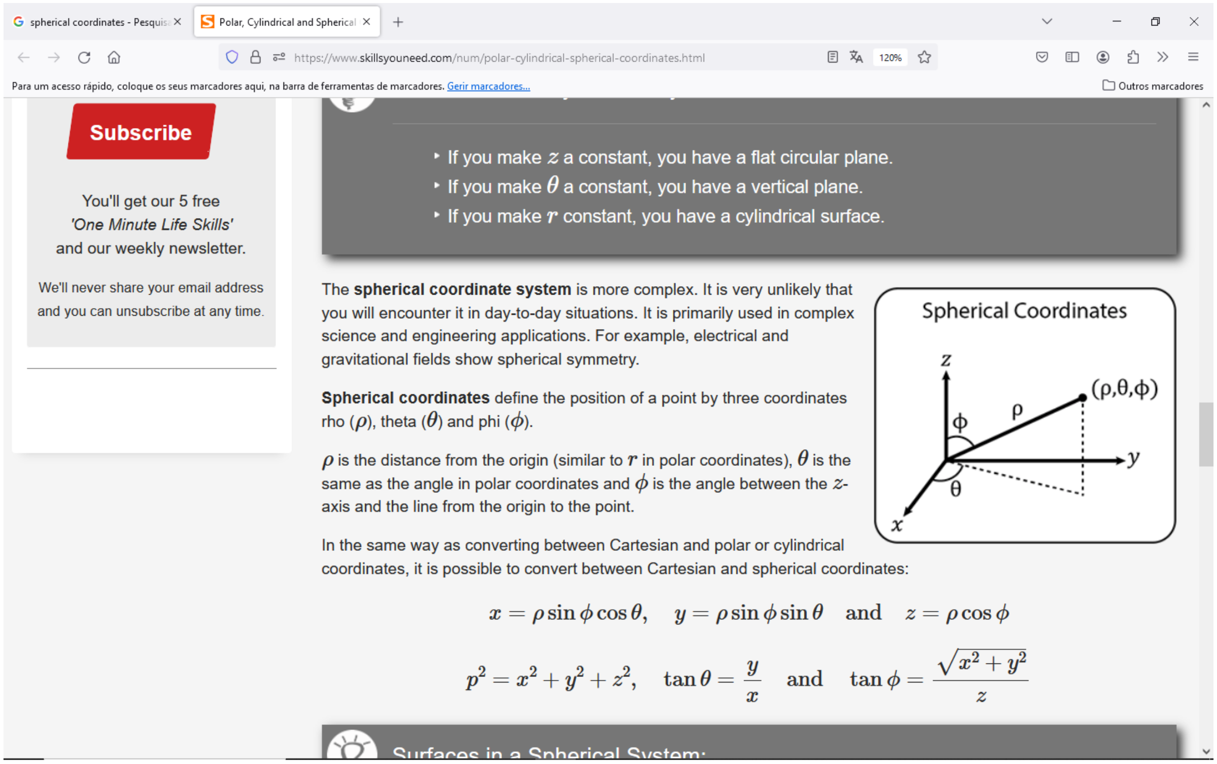

Figure 2: Tridimensional $3D$ Spherical Coordinates.(Source:Internet)

11 Appendix E:Tridimensional $3D$ Spherical Coordinates

Natario (See pg 5 in [1]) defined a warp drive vector $nX = vs * (dx)$ where vs is the **constant** speed of the warp bubble and $*(dx)$ is the Hodge Star taken over the x-axis of motion in **Polar Coordinates**(See pg 4 in [1].(See also Appendix F).

$$\frac{\partial}{\partial x} \sim dx = d(r\cos\theta) = \cos\theta dr - r\sin\theta d\theta \sim \ \sim r^2 \sin\theta\cos\theta d\theta \wedge d\varphi + r\sin^2\theta dr \wedge d\varphi = d\left(\frac{1}{2}r^2\sin^2\theta d\varphi\right). \tag{187}$$

Note that in this case of Tridimensional $3D$ **Spherical Coordinates** the Hodge Star must be taken no longer over $d(r\cos\theta)$ but instead over $d(\rho\sin\phi\cos\theta)$ and this demands more calculations.Replacing ρ by r we have the following expressions for the Hodge Star:(see Appendices J and K)

$$*dx = *d(r\sin\phi\cos\theta) = \sin\phi[*d\left(\frac{1}{2}r^2\sin^2\theta d\phi\right)] + cos\phi[*d[(\frac{1}{2})(r^2)\cot\theta d\theta]] \tag{188}$$

$$\sin\phi[*d[f(r)r^2\sin^2\theta d\phi]] + cos\phi[*d[(f(r))(r^2)\cot\theta d\theta]] \tag{189}$$

Our new tridimensional $3D$ **spherical coordinates** warp drive vector in R^3 with **constant speed** or in R^4 with **constant speed** vs $nX = vs * dx$ or for the first term $vs * dx$ of the new tridimensional $3D$ **spherical coordinates** warp drive vector in R^4 with **variable speed** vs $nX = vs * dx + x * dvs$ is given by:

$$nX = vs(t)[\sin\phi][2f(r)\ cos\theta e_r] - vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta e_\theta] + [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))]e_\phi]] \tag{190}$$

The corresponding shift vectors are:

$$nX = X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{191}$$

$$X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] \tag{192}$$

$$X^\theta = -vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] \tag{193}$$

$$X^\phi = [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] \tag{194}$$

The equation of the **polar coordinates** Natario vector nX in **constant speed** vs(pg 2 and 5 in [1]) is given by:

$$nX = X^r e_r + X^\theta e_\theta \tag{195}$$

With the contravariant shift vector components explicitly written:

$$X^r = 2v_s f(r)\cos\theta \tag{196}$$

$$X^\theta = -v_s(2f(r) + (r)f'(r))\sin\theta \tag{197}$$

Note that Natario in pg 4 in [1] defined the x-axis as the polar axis.if the motion occurs only in the x-axis in **polar coordinates** then the angle between the x-y plane and the z-axis is 90 degrees and in this case $\sin\phi = 1$ and $\cos\phi = 0$ and our new warp drive vector in tridimensional $3D$ **spherical coordinates** reduces to the original Natario warp drive vector in **polar coordinates** both in **constant speed**.

Only in a real tridimensional $3D$ spherical coordinates motion our new warp drive vector accounts for a significant difference

Due to a **constant** speed vs the term $x * d(vs) = 0$.Now we must examine what happens when the velocity is **variable** and then the term $x * d(vs)$ no longer vanishes.Remember that a real warp drive must accelerate or de-accelerate in order to be accepted as a physical valid model.The complete expression of the Hodge star that generates the warp drive vector nX in tridimensional $3D$ **spherical coordinates** for a **variable velocity** vs is now given by(see Appendix L for detailed calculations):

$$nX = A + B \rightarrow A = vs * dx \rightarrow B = x * dvs \tag{198}$$

$$nX = A + B \rightarrow A = vs * dx \rightarrow B = x * dvs \tag{199}$$

$$A = (2f(r)at)(\sin\phi[2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta] + cos\phi[cot\theta[2(f(r)) + (rf'(r))]e_\phi]) \tag{200}$$

$$B = (r\sin\phi\cos\theta)(2[atf'(r)e_r + f(r)ae_t]) \tag{201}$$

Comparing the new warp drive vector for **variable velocities** in real tridimensional $3D$ **spherical coordinates** with the **variable velocities** Natario **polar coordinates** warp drive vector counterpart:

$$nX = X^t e_t + X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{202}$$

$$X^t = 2(rf(r)a))(\sin\phi)(\cos\theta) \tag{203}$$

$$X^r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta) \tag{204}$$

$$X^\theta = -(2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\sin\theta) \tag{205}$$

$$X^\phi = (2f(r)at)[2f(r) + (rf'(r))](cos\phi)(cot\theta) \tag{206}$$

$$nX = X^t e_t + X^r e_r + X^\theta e_\theta \tag{207}$$

$$X^t = 2f(r)r(cos\theta)a \tag{208}$$

$$X^r = 2[2f(r)^2 + rf'(r)]at(cos\theta) \tag{209}$$

$$X^\theta = -2f(r)at[2f(r) + rf'(r)](\sin\theta) \tag{210}$$

Natario defined a motion in the $x-axis$ of **polar coordinates** (pgs 4 and 5 in [1]) then the polar plane $x-y$ makes an angle of 90 degrees with the $z-axis$ and since $\sin\phi = 1$ and $\cos\phi = 0$ it is easy to see that in this case the new warp drive vector for **variable velocities** in real tridimensional $3D$ **spherical coordinates** reduces itself to the **variable velocities** Natario **polar coordinates** warp drive vector counterpart:

The difference occurs only in a real tridimensional motion.

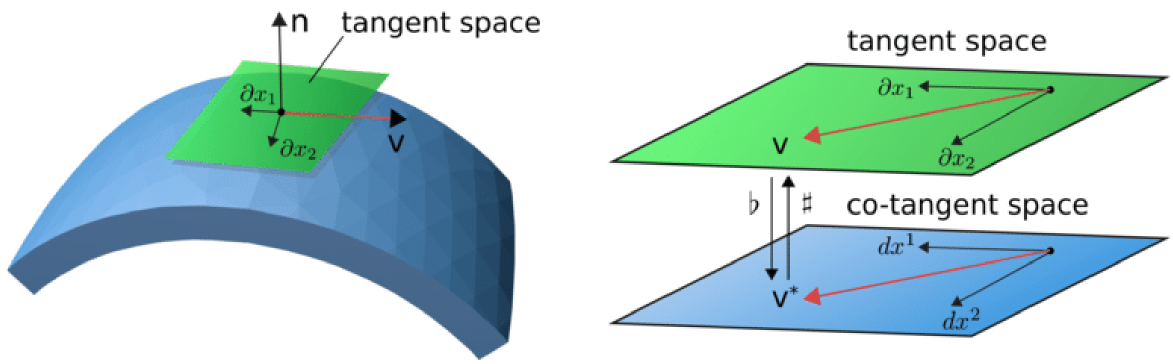

Figure 3: Artistic Presentation of Tangent and Cotangent Spaces I.(Source:Internet)

12 Appendix F:Tangent and Cotangent Spaces I

The Canonical Basis of the Hodge Star * in spherical coordinates in R^3 can be defined as follows(see pg 4 in [1],eq 3.72 pg 69$(a)(b)$ in [2]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2 \sin\theta(d\theta \wedge d\varphi) \tag{211}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta(d\varphi \wedge dr) \tag{212}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{213}$$

The Canonical Basis of the Hodge Star * in spherical coordinates in R^4 can be defined as follows(see pg 4 in [1],eqs 3.135 and 3.137 pg 82$(a)(b)$ in [2],eq 3.74 pg 69$(a)(b)$ in [2])(see pg 47 eqs 2.67 to 2.70 and pg 92 in [3]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim dt \wedge (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2 \sin\theta(dt \wedge d\theta \wedge d\varphi) \tag{214}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim dt \wedge (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta(dt \wedge d\varphi \wedge dr) \tag{215}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dt \wedge dr \wedge (rd\theta) \sim r(dt \wedge dr \wedge d\theta) \tag{216}$$

In order to study the term $x * d(vs)$ we must introduce a new Canonical Basis for the coordinate time in the R^4 space basis defined as follows:(see eqs 10.102 and 10.103 pgs 363$(a)(b)$ and 364$(a)(b)$ in [2] with the terms $S = u = 1$[3],eq 3.74 pg 69$(a)(b)$ in [2],eqs 11.131 and 11.133 with the term $m = 0$[4] pg 417$(a)(b)$ in [2].)(see pg 47 eqs 2.67 to 2.70 and pg 92 in [3]):

$$e_t \equiv \frac{\partial}{\partial t} \sim dt \sim dr \wedge (r d\theta) \wedge (r \sin\theta d\varphi) \sim r^2 \sin\theta (dr \wedge d\theta \wedge d\varphi) \tag{217}$$

As a matter of fact we have for the Canonical Basis and the Hodge Star * in R^4 the following equations (see pg 47 eqs 2.67 to 2.70 in [3]):

$$*e_0 = e_1 \wedge e_2 \wedge e_3 \tag{218}$$

$$*e_1 = e_0 \wedge e_2 \wedge e_3 \tag{219}$$

$$*e_2 = e_0 \wedge e_3 \wedge e_1 \tag{220}$$

$$*e_3 = e_0 \wedge e_1 \wedge e_2 \tag{221}$$

In R^3 the corresponding equations are:(see pg 55 in [5])(see also pg 54 fig 4.2 in [5] for a graphical presentation of the Hodge Star * in R^3)(see pg 18 eq 1.55 in [6]):

$$*e_1 = e_2 \wedge e_3 \tag{222}$$

$$*e_2 = e_3 \wedge e_1 = -e_1 \wedge e_3 \tag{223}$$

$$*e_3 = e_1 \wedge e_2 \tag{224}$$

The Canonical Basis e_i are related to the partial derivatives $\frac{\partial}{\partial x_i}$ or simplifying related to ∂x_i wether in R^3 or R^4 and are graphically represented by the partial derivatives ∂x_i included in the tangent space of the picture given in the beginning of this section.

[3]These terms are needed to deal with the Robertson-Walker equation in Cosmology using differential forms.We dont need these terms here and we can make $S = u = 1$

[4]This term is needed to describe the Dirac equation in the Schwarzschild spacetime we dont need the term here so we can make $m = 1$.Remember also that here we consider geometrized units in which $c = 1$

On the other hand in R^4 we also have the following relations for the Hodge Star *:(see pg 92 in [3])

$$*dt = dx \wedge dy \wedge dz \tag{225}$$

$$*dx = dt \wedge dy \wedge dz \tag{226}$$

$$*dy = dt \wedge dz \wedge dx \tag{227}$$

$$*dz = dt \wedge dx \wedge dy \tag{228}$$

Also for R^4 considering the $((w,v)(\epsilon\Lambda_p^3)(R^{1,3}))$ formalism we may have the following relations:(see pg 382 in [4])($x^1 = x$,$x^2 = y$,$x^3 = z$)

$$*dt = dx^1 \wedge dx^2 \wedge dx^3 \tag{229}$$

$$*dx^1 = dt \wedge dx^2 \wedge dx^3 \tag{230}$$

$$*dx^2 = dt \wedge dx^3 \wedge dx^1 \tag{231}$$

$$*dx^3 = dt \wedge dx^1 \wedge dx^2 \tag{232}$$

In R^3 we would have the following relations:(see pg 117 eqs 4.6 and 4.7 in [7])(see pg 298 in [4])

$$*dx = dy \wedge dz \tag{233}$$

$$*dy = dz \wedge dx \tag{234}$$

$$*dz = dx \wedge dy \tag{235}$$

The differentials dx,dy,dz or dx^1, dx^2 and dx^3 are related to the cotangent space differentials included in the picture given in the beginning of this section.

See the graphical presentations of the relations between tangent and cotangent spaces in pg 55 fig 2.28 and pg 70 fig 3.1 in [4].See pg 168 fig 5.19 for a graphical presentation of $dx \wedge dy$,pg 169 fig 5.20 for a graphical presentation of $dy \wedge dz$ and pg 170 fig 5.21 for a graphical presentation of $dz \wedge dx$ all in [4].

Useful relations to deal with the Hodge Star * are given by eqs 3.90 and 3.91 pg 74$(a)(b)$ in [2],tb 3.3 pg 68$(a)(b)$ in [2]:See also pg 89 in [3],pg 112 in [4],pg 97 in [5],pg 36 eqs 2.21 and 2.22 in [6],pg 70 eq 3.3 in [7].

$$*d(f\alpha) = df \wedge \alpha + (-1)^p f \wedge d\alpha \dashrightarrow p = 3 \dashrightarrow *d(f\alpha) = df \wedge \alpha - f \wedge d\alpha \tag{236}$$

$$*d(f\alpha) = df \wedge \alpha + (-1)^p f \wedge d\alpha \dashrightarrow p = 2 \dashrightarrow *d(f\alpha) = df \wedge \alpha + f \wedge d\alpha \tag{237}$$

$$*d(dx) = *d(dy) = *d(dz) = 0 \tag{238}$$

$p = 3$ stands for the R^4 and $p = 2$ stands for the R^3.

See also Appendix I.

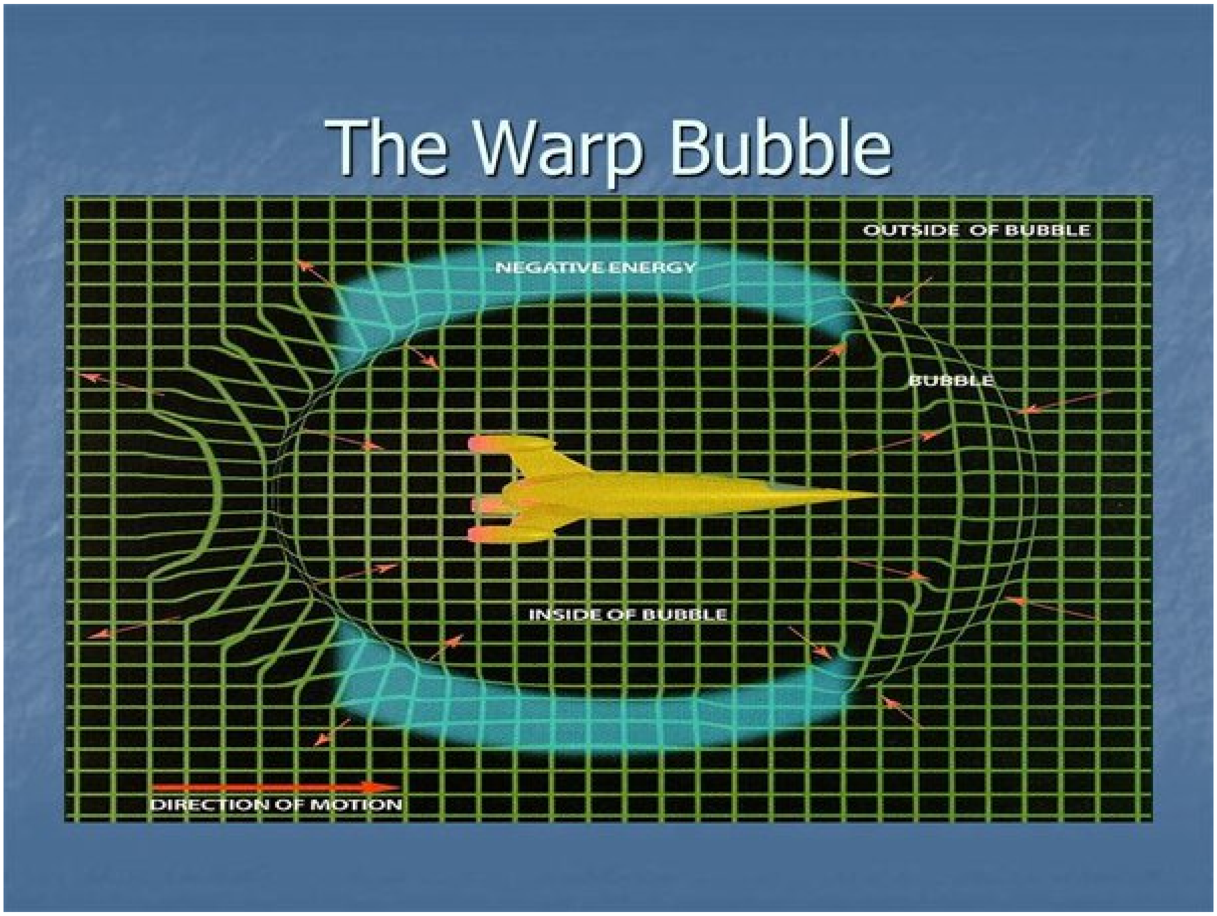

Figure 4: Artistic Presentation of a Warp Bubble.(Source:Internet)

13 Appendix G:Artistic Presentation of a Warp Bubble

In 2001 the Natario warp drive appeared.([1]).This warp drive deals with the spacetime as a "strain" tensor of Fluid Mechanics(pg 5 in [1]). Imagine a fish inside an aquarium and the aquarium is floating in the surface of a river but carried out by the river stream.The warp bubble in this case is the aquarium.An observer at the rest in the margin of the river would see the aquarium passing by him at a large speed but inside the aquarium the fish is at the rest with respect to his local neighborhoods.Since the fish is at the rest inside the aquarium the fish would see the observer in the margin passing by him with a large relative speed since for the fish is the margin that moves with a large relative velocity

any Natario vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of rs defined by Natario as the interior of the warp bubble and $nX = vs(t) * dx$ with $X = vs$ for a large value of rs defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [1])

Lets explain better this statement:Natario considered in this case a coordinates reference frame placed inside the bubble where the fish inside the aquarium or the astronaut in a spaceship inside the bubble depicted above are at the rest with respect to their local neighborhoods.Then any Natario vector must be zero inside the bubble or the aquarium or the spaceship.

On the other hand since the fish sees the margin passing by him with a large relative velocity or the astronaut would see a stationary observer in outer space outside the bubble passing by him with a large relative velocity then any Natario vector outside the bubble must have a value equal to the relative velocity seen by both the fish and the astronaut.

Considering a valid f as a Natario shape function being $f = \frac{1}{2}$ for large r(outside the warp bubble) and $f = 0$ for small r(inside the warp bubble) while being $0 < f < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pg 5 in [1]):The walls of the bubble the Natario warped region corresponds to the distorted region in the picture depicted in this Appendix.

See also Appendix H.

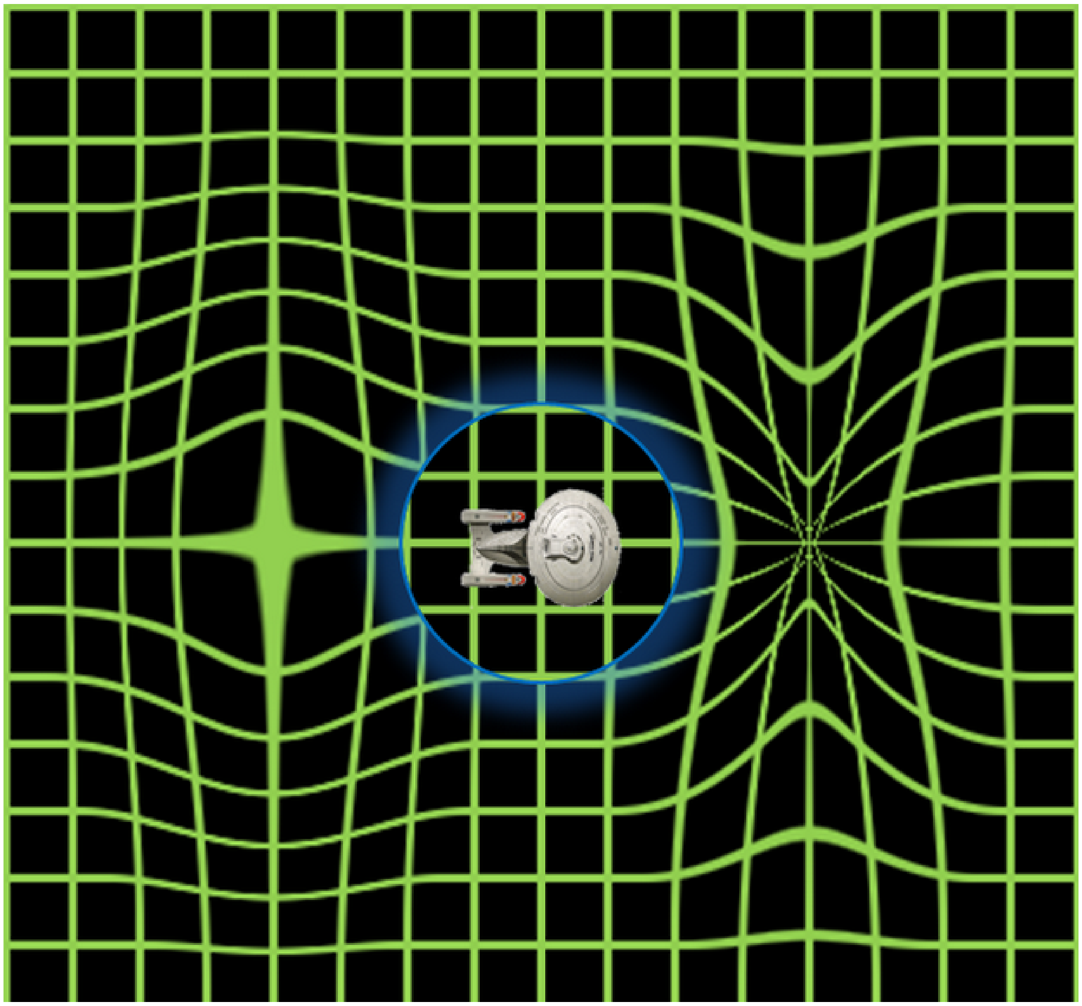

Figure 5: Another Artistic Presentation of a Warp Bubble.(Source:Internet)

14 Appendix H:Another Artistic Presentation of a Warp Bubble

Natario considered a coordinates reference frame placed inside the bubble.Now we must consider a coordinates reference frame placed outside the bubble:In this case the observer at the rest in the margin of the river would see the aquarium passing by him with a large velocity with the fish inside.Also a stationary observer at the rest in outer space would see the spaceship depicted in the picture above passing by him with a large velocity with the astronaut inside.

Now the rules originally defined by Natario are interchanged:

Since the observer in the margin and the observer in outer space are at the rest any Natario vector in this case must be zero outside the bubble.

But since the fish and the spaceship are being seen by the observer at the rest in the margin and the observer at the rest in outer space both fish and spaceship with a large velocity then the Natario vector inside the bubble must have a value equal to the velocity seen by both observers.

Considering a valid f as a Natario shape function being $f = 0$ for large r(outside the warp bubble) and $f = \frac{1}{2}$ for small r(inside the warp bubble) while being $0 < f < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region:The walls of the bubble the Natario warped region corresponds to the distorted region the "blue circle" in the picture depicted in this Appendix.

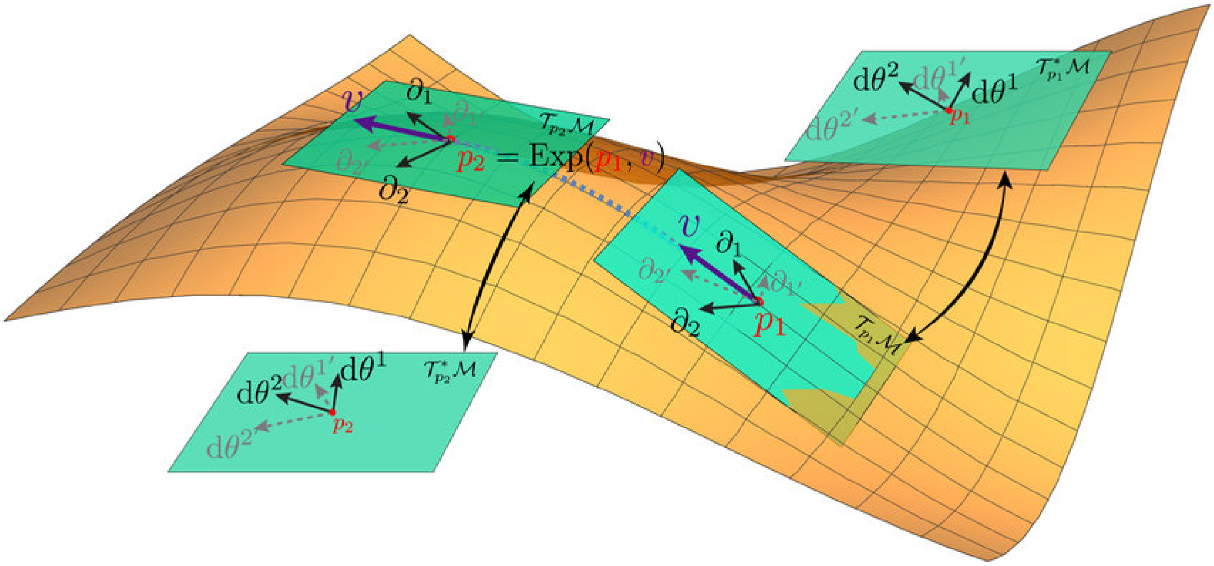

Figure 6: Artistic Presentation of Tangent and Cotangent Spaces II.(Source:Internet)

15 Appendix I:Tangent and Cotangent Spaces II

Consider a curve R in R^4 defined in function of a given set of coordinates u^0,u^1,u^2 and u^3 as being $R = R(u^0, u^1, u^2, u^3)$.

A total derivative of R is given by:

$$dR = \frac{\partial R}{\partial u^0}du^0 + \frac{\partial R}{\partial u^1}du^1 + \frac{\partial R}{\partial u^2}du^2 + \frac{\partial R}{\partial u^3}du^3 \tag{239}$$

Applying the Einstein summing convention:

$$dR = \frac{\partial R}{\partial u^i}du^i = e_i du^i \tag{240}$$

or

$$dR = \frac{\partial R}{\partial u^j}du^j = e_j du^j \tag{241}$$

With $i, j = 0, 1, 2, 3$ as the coordinates,$\frac{\partial R}{\partial u^i}$ and $\frac{\partial R}{\partial u^j}$ as the directional partial derivatives of R with respect to each coordinate and e_i and e_j are the respective Canonical Basis.

Defining $ds^2 = dR \bigotimes dR$ we have:

$$ds^2 = dR \bigotimes dR = \frac{\partial R}{\partial u^i}du^i \bigotimes \frac{\partial R}{\partial u^j}du^j = e_i du^i \bigotimes e_j du^j \tag{242}$$

$$ds^2 = \frac{\partial R}{\partial u^i}\frac{\partial R}{\partial u^j}du^i du^j = e_i e_j du^i du^j = g_{ij} du^i du^j \tag{243}$$

$$g_{ij} = \frac{\partial R}{\partial u^i}\frac{\partial R}{\partial u^j} = e_i e_j \tag{244}$$

The directional partial derivatives of R and their respective Canonical Basis are related to the ∂_i and ∂_j tangent spaces of the picture depicted in the beginning of this section while the differentials du^i and du^j are related to the respective cotangent spaces.See pg 148 problem 17 in [14],pg 132 eq 10.12 pg 133 eqs 10.14a,10.14b and 10.15 in [15].

$g_{ij} = \frac{\partial R}{\partial u^i}\frac{\partial R}{\partial u^j} = e_i e_j$ is the spacetime metric tensor of General Relativity.

16 Appendix J:differential forms,Hodge star and the mathematical demonstration of the warp drive vectors $nX = -vs * dx$ and $nX = vs * dx$ for a constant speed vs in a R^3 space basis-$3D$ Spherical Coordinates

The Canonical Basis of the Hodge Star in spherical coordinates can be defined as follows(see pg 4 in [1],eq 3.72 pg 69$(a)(b)$ in [2]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim (rd\theta) \wedge (r \sin\theta d\varphi) \sim r^2 \sin\theta (d\theta \wedge d\varphi) \tag{245}$$

$$e_\theta \equiv \frac{1}{r} \frac{\partial}{\partial \theta} \sim rd\theta \sim (r \sin\theta d\varphi) \wedge dr \sim r \sin\theta (d\varphi \wedge dr) \tag{246}$$

$$e_\varphi \equiv \frac{1}{r \sin\theta} \frac{\partial}{\partial \varphi} \sim r \sin\theta d\varphi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{247}$$

Back again to the equivalence between $3D$ spherical and cartezian coordinates $d(\rho \sin\phi \cos\theta)$:(See Appendix E)

We will replace ρ by r and φ by ϕ.Then we have:

$$d(r \sin\phi \cos\theta) = sin\phi[d(r \cos\theta)] + (r \cos\theta) d(\sin\phi) \tag{248}$$

$$d(r \sin\phi \cos\theta) = sin\phi[cos\theta dr + r(d \cos\theta)] + (r \cos\theta)(cos\phi d\phi) \tag{249}$$

$$d(r \sin\phi \cos\theta) = sin\phi[cos\theta(dr) - r sin\theta(d\theta)] + (r \cos\theta)[cos\phi(d\phi)] \tag{250}$$

$$d(r \sin\phi \cos\theta) = sin\phi[cos\theta(dr) - sin\theta(rd\theta)] + cos\phi[(r \cos\theta)(d\phi)] \tag{251}$$

Applying the Hodge Star * to the term $[cos\theta(dr) - sin\theta(rd\theta)]$ we will get the same results already shown in the Appendix A and the first part of the $3D$ spherical warp drive vector is the one of the Appendix A multiplied by $\sin\phi$ .Then we must concern ourselves with the term $cos\phi[(r\cos\theta)(d\phi)]$ and the following Canonical Basis for the Hodge Star * since the other two were covered in the Appendix A.

$$e_\phi \equiv \frac{1}{r \sin\theta} \frac{\partial}{\partial \phi} \sim r \sin\theta d\phi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{252}$$

The term $cos\phi[(r\cos\theta)(d\phi)]$ must become compatible with the Canonical Basis for the Hodge Star above and this can be achieved by the following substitution:

$$cos\phi[(r \cos\theta)(d\phi)] = cos\phi[(r \sin\theta \cot\theta)(d\phi)] = cos\phi[\cot\theta(r \sin\theta)(d\phi)] \tag{253}$$

$$cos\phi[\cot\theta * ((r \sin\theta)(d\phi))] = cos\phi[\cot\theta(r(dr \wedge d\theta))] = cos\phi[\cot\theta(e_\phi)] \tag{254}$$

In the Appendix A we used the term $d\left(\frac{1}{2}r^2 \sin^2\theta d\phi\right)$ and its respective Hodge Star $*d\left(\frac{1}{2}r^2 \sin^2\theta d\phi\right)$ also used by Natario in pg 5 in [1] because this term corresponds to the term $[cos\theta(*dr) - sin\theta(*rd\theta)]$ now being multiplied by $sin\phi$.In the $3D$ spherical warp drive this term also appears multiplied by $sin\phi$ but we must look for a corresponding expression concerning the term $cos\phi[\cot\theta * ((r\sin\theta)(d\phi))] = cos\phi[\cot\theta(r(dr \wedge d\theta))]$.

The desired expression is the following one:

$$cos\phi[d[(\frac{1}{2})(r^2)\cot\theta d\theta]] \tag{255}$$

Its respective Hodge Star is:

$$cos\phi[*d[(\frac{1}{2})(r^2)\cot\theta d\theta]] \tag{256}$$

Using the relations in the expression above to deal with the Hodge Star * given by eqs 3.90 and 3.91 pg 74$(a)(b)$ in [2],tb 3.3 pg 68$(a)(b)$ in [2]:See also pg 89 in [3],pg 112 in [4],pg 97 in [5],pg 36 eqs 2.21 and 2.22 in [6],pg 70 eq 3.3 in [7].

$$*d(f\alpha) = df \wedge \alpha + (-1)^p f \wedge d\alpha \dashrightarrow p = 2 \dashrightarrow *d(f\alpha) = df \wedge \alpha + f \wedge d\alpha \tag{257}$$

$$*d(dx) = *d(dy) = *d(dz) = 0 \tag{258}$$

$p = 2$ stands for the R^3.Then we have:

$$*d[(\frac{1}{2})(r^2)\cot\theta d\theta] = (\frac{1}{2})(\cot\theta) * d(r^2 d\theta) + (\frac{1}{2})(r^2) * d(\cot\theta d\theta) + (\frac{1}{2})(r^2)\cot\theta * d(d\theta) \tag{259}$$

$$*d(r^2 d\theta) = d(r^2) \wedge d\theta + r^2 \wedge d(d\theta) = d(r^2) \wedge d\theta = 2rdr \wedge d\theta \tag{260}$$

$$*d(\cot\theta d\theta) = d\cot\theta \wedge d\theta + \cot\theta \wedge d(d\theta) = d\cot\theta \wedge d\theta = -\csc^2\theta d\theta \wedge d\theta = 0 \tag{261}$$

$$*d(d\theta) = 0 \tag{262}$$

$$*d[(\frac{1}{2})(r^2)\cot\theta d\theta] = (\frac{1}{2})(\cot\theta) * d(r^2 d\theta) = (\frac{1}{2})(\cot\theta)(2rdr \wedge d\theta) = (\cot\theta)(rdr \wedge d\theta) \tag{263}$$

And

$$cos\phi[*d[(\frac{1}{2})(r^2)\cot\theta d\theta]] = cos\phi[(\cot\theta)(rdr \wedge d\theta)] = cos\phi[\cot\theta(e_\phi)] \tag{264}$$

Because due to the Canonical Basis of the Hodge Star:

$$e_\phi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \sim r\sin\theta d\phi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{265}$$

Then in the $3D$ spherical coordinates we have the following Hodge Star:

$$*d(r\sin\phi\cos\theta) = \sin\phi[*d\left(\frac{1}{2}r^2\sin^2\theta d\phi\right)] + cos\phi[*d[(\frac{1}{2})(r^2)\cot\theta d\theta]] \tag{266}$$

Also in Appendix A we used the term $*d[f(r)r^2\sin^2\theta d\phi]$ corresponding to the term $*d\left(\frac{1}{2}r^2\sin^2\theta d\phi\right)$ because Natario also used it in pg 5 in [1].Now this term must be multiplied by $\sin\phi$.

From the Appendix A we have:

$$*d[f(r)r^2\sin^2\theta d\phi] = 2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta \tag{267}$$

Defining the Natario Vector as in pg 5 in [1] in polar coordinates with the Hodge Star operator * explicitly written :

$$nX = vs(t) * d\left(f(r)r^2\sin^2\theta d\phi\right) \tag{268}$$

$$nX = -vs(t) * d\left(f(r)r^2\sin^2\theta d\phi\right) \tag{269}$$

We can get finally the latest expressions for the Natario Vector in polar coordinates nX also shown in pg 5 in [1]

$$nX = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{270}$$

$$nX = -2vs(t)f(r)\ cos\theta e_r + vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{271}$$

We choose the polar coordinates Natario vectors $nX = vs(t) * d\left(f(r)r^2\sin^2\theta d\phi\right)$ and

$nX = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta$

But in $3D$ spherical coordinates we have:

$$\sin\phi(*d[f(r)r^2\sin^2\theta d\phi]) = \sin\phi(2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta) \tag{272}$$

Like the term $*d[f(r)r^2\sin^2\theta d\phi]$ is associated to the term $*d\left(\frac{1}{2}r^2\sin^2\theta d\phi\right)$ and now these terms must be multiplied by $\sin\phi$ we must find the corresponding term for $cos\phi[*d[(\frac{1}{2})(r^2)\cot\theta d\theta]]$.

The term we are looking for is the following one:

$$cos\phi[*d[(f(r))(r^2)\cot\theta d\theta]] \tag{273}$$

Solving the Hodge Star we have:

$$*d[(f(r))(r^2)\cot\theta d\theta] \tag{274}$$

$$(f(r))\cot\theta * d(r^2 d\theta) + (f(r))(r^2) * d(\cot\theta d\theta) + (r^2)(\cot\theta) * d(f(r)d\theta)) + ((f(r))(r^2)\cot\theta) * d(d\theta) \tag{275}$$

As already seen before the terms $*d(\cot\theta d\theta) = 0$ and $*d(d\theta) = 0$.Then the Hodge Star becomes:

$$*d[(f(r))(r^2)\cot\theta d\theta] = (f(r))\cot\theta * d(r^2 d\theta) + (r^2)(\cot\theta) * d(f(r)d\theta) \tag{276}$$

$$*d(r^2 d\theta) = d(r^2) \wedge d\theta + r^2 \wedge d(d\theta) = d(r^2) \wedge d\theta = 2rdr \wedge d\theta \tag{277}$$

$$*d(f(r)d\theta) = d(f(r) \wedge d\theta + f(r) \wedge d(d\theta) = d(f(r) \wedge d\theta = f'(r)dr \wedge d\theta \tag{278}$$

Still with the Hodge Star:

$$*d[(f(r))(r^2)\cot\theta d\theta] = (f(r))\cot\theta * d(r^2 d\theta) + (r^2)(\cot\theta) * d(f(r)d\theta) \tag{279}$$

$$*d(r^2 d\theta) = 2rdr \wedge d\theta \tag{280}$$

$$*d(f(r)d\theta) = f'(r)dr \wedge d\theta \tag{281}$$

$$*d[(f(r))(r^2)\cot\theta d\theta] = (f(r))\cot\theta(2rdr \wedge d\theta) + (r^2)(\cot\theta)f'(r))(dr \wedge d\theta) \tag{282}$$

The Canonical Basis for the Hodge Star is:

$$e_\phi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi} \sim r\sin\theta d\phi \sim dr \wedge (rd\theta) \sim r(dr \wedge d\theta) \tag{283}$$

Then the Hodge Star now becomes:

$$*d[(f(r))(r^2)\cot\theta d\theta] = 2(f(r))\cot\theta(rdr \wedge d\theta) + (\cot\theta)rf'(r)(rdr \wedge d\theta) \tag{284}$$

$$*d[(f(r))(r^2)\cot\theta d\theta] = cot\theta[2(f(r)) + (rf'(r))](rdr \wedge d\theta) \tag{285}$$

$$*d[(f(r))(r^2)\cot\theta d\theta] = cot\theta[2(f(r)) + (rf'(r))]e_\phi \tag{286}$$

At last we are ready to present the new tridimensional $3D$ spherical warp drive vector.We already know that in the $3D$ spherical coordinates $d(r\sin\phi\cos\theta)$ we have the following Hodge Star:

$$*d(r\sin\phi\cos\theta) = \sin\phi[*d\left(\frac{1}{2}r^2\sin^2\theta d\phi\right)] + cos\phi[*d[(\frac{1}{2})(r^2)\cot\theta d\theta]] \tag{287}$$

But as we already demonstrated in this section the Hodge Star above can be associated to the following one:

$$\sin\phi[*d[f(r)r^2\sin^2\theta d\phi]] + cos\phi[*d[(f(r))(r^2)\cot\theta d\theta]] \tag{288}$$

With:

$$*d[f(r)r^2\sin^2\theta d\phi] = 2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta \tag{289}$$

$$*d[(f(r))(r^2)\cot\theta d\theta] = cot\theta[2(f(r)) + (rf'(r))]e_\phi \tag{290}$$

Then our tridimensional $3D$ spherical Hodge Star can be given by:

$$\sin\phi[2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta] + cos\phi[cot\theta[2(f(r)) + (rf'(r))]e_\phi] \tag{291}$$

Natario defined two warp drive vectors in pg 5 in [1] as being:(see Appendix A)

$$nX = vs(t) * d\left(f(r)r^2\sin^2\theta d\phi\right) = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{292}$$

$$nX = -vs(t) * d\left(f(r)r^2\sin^2\theta d\phi\right) = -2vs(t)f(r)\ cos\theta e_r + vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{293}$$

$$nX = vs(t) * d\left(f(r)r^2 \sin^2\theta d\phi\right) = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{294}$$

$$nX = -vs(t) * d\left(f(r)r^2 \sin^2\theta d\phi\right) = -2vs(t)f(r)\ cos\theta e_r + vs(t)[2f(r) + rf'(r)]\sin\theta e_\theta \tag{295}$$

We choose this one:$nX = vs(t)*d\left(f(r)r^2 \sin^2\theta d\phi\right) = 2vs(t)f(r)\ cos\theta e_r - vs(t)[2f(r)+rf'(r)]\sin\theta e_\theta$.Then we have the original Natario warp drive vector in polar coordinates:

$$nX = vs(t) * d\left(f(r)r^2 \sin^2\theta d\phi\right) = vs(t)[2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta] \tag{296}$$

Now and finally[5] we can present the final form of our new warp drive vector in tridimensional $3D$ spherical coordinates as being:

$$nX = vs(t)[\sin\phi[*d[f(r)r^2\sin^2\theta d\phi]] + cos\phi[*d[(f(r))(r^2)\cot\theta d\theta]]] \tag{297}$$

$$nX = vs(t)\sin\phi[*d[f(r)r^2\sin^2\theta d\phi]] + vs(t)cos\phi[*d[(f(r))(r^2)\cot\theta d\theta]] \tag{298}$$

$$nX = (\sin\phi)vs(t)[*d[f(r)r^2\sin^2\theta d\phi]] + (cos\phi)vs(t)[*d[(f(r))(r^2)\cot\theta d\theta]] \tag{299}$$

$$*d[f(r)r^2\sin^2\theta d\phi] = 2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta \tag{300}$$

$$*d[(f(r))(r^2)\cot\theta d\theta] = cot\theta[2(f(r)) + (rf'(r))]e_\phi \tag{301}$$

$$nX = vs(t)[\sin\phi[2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta] + cos\phi[cot\theta[2(f(r)) + (rf'(r))]e_\phi]] \tag{302}$$

$$nX = vs(t)[\sin\phi][2f(r)\ cos\theta e_r] - vs(t)[\sin\phi][2f(r)+rf'(r)]\sin\theta e_\theta] + [vs(t)cos\phi][cot\theta[2(f(r))+(rf'(r))]e_\phi]] \tag{303}$$

This is the final form of our new tridimensional $3D$ spherical warp drive vector.Note that Natario in pg 4 in [1] defined the x-axis as the polar axis.if the motion occurs only in the x-axis in polar coordinates then the angle between the x-y plane and the z-axis is 90 degrees and in this case $\sin\phi = 1$ and $\cos\phi = 0$ and our new warp drive vector in tridimensional $3D$ spherical coordinates reduces to the original Natario warp drive vector in polar coordinates.

Only in a real tridimensional $3D$ spherical coordinates motion our new warp drive vector accounts for a significant difference

[5] at last!!!we know that this section is being written in a tedious and monotonous style but we are writing this for beginners or introductory students eagerly needing these mathematical demonstrations QED Quod Erad Demonstratum in order to allow these students to more easily understand the whole process of the obtention of warp drive vectors

For our new tridimensional $3D$ spherical coordinates warp drive vector

$$nX = vs(t)[\sin\phi][2f(r)\ cos\theta e_r] - vs(t)[\sin\phi][2f(r)+rf'(r)]\sin\theta e_\theta] + [vs(t)cos\phi][cot\theta[2(f(r))+(rf'(r))]e_\phi]] \tag{304}$$

The corresponding shift vectors are:

$$nX = X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{305}$$

$$X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] \tag{306}$$

$$X^\theta = -vs(t)[\sin\phi][2f(r)+rf'(r)]\sin\theta] \tag{307}$$

$$X^\phi = [vs(t)cos\phi][cot\theta[2(f(r))+(rf'(r))] \tag{308}$$

17 Appendix K:differential forms,Hodge star and the mathematical demonstration of the warp drive vector $nX = vs * dx$ for a constant speed vs or for the first term $vs * dx$ from the warp drive vector $nX = vs * dx + x * dvs$(a variable speed) in a R^4 space basis-Tridimensional $3D$ Spherical Coordinates

The Canonical Basis of the Hodge Star in spherical coordinates can be defined as follows(see pg 4 in [1],eqs 3.135 and 3.137 pg 82$(a)(b)$ in [2],eq 3.74 pg 69$(a)(b)$ in [2])(see pg 47 eqs 2.67 to 2.70 and pg 92 in [3]):

$$e_r \equiv \frac{\partial}{\partial r} \sim dr \sim dt \wedge (rd\theta) \wedge (r\sin\theta d\varphi) \sim r^2 \sin\theta (dt \wedge d\theta \wedge d\varphi) \tag{309}$$

$$e_\theta \equiv \frac{1}{r}\frac{\partial}{\partial\theta} \sim rd\theta \sim dt \wedge (r\sin\theta d\varphi) \wedge dr \sim r\sin\theta(dt \wedge d\varphi \wedge dr) \tag{310}$$

$$e_\varphi \equiv \frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi} \sim r\sin\theta d\varphi \sim dt \wedge dr \wedge (rd\theta) \sim r(dt \wedge dr \wedge d\theta) \tag{311}$$

Useful relations to deal with the Hodge Star * are given by eqs 3.90 and 3.91 pg 74$(a)(b)$ in [2],tb 3.3 pg 68$(a)(b)$ in [2]:See also pg 89 in [3],pg 112 in [4],pg 97 in [5],pg 36 eqs 2.21 and 2.22 in [6],pg 70 eq 3.3 in [7].

$$*d(f\alpha) = df \wedge \alpha + (-1)^p f \wedge d\alpha \dashrightarrow p = 3 \dashrightarrow *d(f\alpha) = df \wedge \alpha - f \wedge d\alpha \tag{312}$$

$$*d(f\alpha) = df \wedge \alpha + (-1)^p f \wedge d\alpha \dashrightarrow p = 2 \dashrightarrow *d(f\alpha) = df \wedge \alpha + f \wedge d\alpha \tag{313}$$

$$*d(dx) = *d(dy) = *d(dz) = 0 \tag{314}$$

$p = 3$ stands for the R^4 and $p = 2$ stands for the R^3.

Back again to the equivalence between $3D$ spherical and cartezian coordinates $d(\rho\sin\phi\cos\theta)$:(See Appendix E)

We will replace ρ by r and φ by ϕ.Then we have:

$$d(r\sin\phi\cos\theta) = sin\phi[d(r\cos\theta)] + (r\cos\theta)d(\sin\phi) \tag{315}$$

$$d(r\sin\phi\cos\theta) = sin\phi[cos\theta(dr) - sin\theta(rd\theta)] + cos\phi[(r\cos\theta)(d\phi)] \tag{316}$$

Applying the Hodge Star * to the terms above we will get the same results already shown in the Appendix J.As a matter of fact comparing the Appendices A and B the given final result is the same in both Appendices except for the fact that in Appendix A the Hodge Star is taken over R^3 and in Appendix B the Hodge Star is taken over R^4.

So the expressions for the Hodge Star of the term $d(r\sin\phi\cos\theta)$ covered in the last(and gigantic or enormous) Appendix J taken over R^3 that uses the terms

$$*d(r\sin\phi\cos\theta) = \sin\phi[*d\left(\frac{1}{2}r^2\sin^2\theta d\phi\right)] + cos\phi[*d[(\frac{1}{2})(r^2)\cot\theta d\theta]] \tag{317}$$

$$\sin\phi[*d[f(r)r^2\sin^2\theta d\phi]] + cos\phi[*d[(f(r))(r^2)\cot\theta d\theta]] \tag{318}$$

Will appear in identical form if we compute the Hodge Star for the same term

$$d(r\sin\phi\cos\theta)$$

in R^4.The only difference is the term in R^4

$$*d(f\alpha) = df\wedge\alpha + (-1)^p f\wedge d\alpha \dashrightarrow p = 3 \dashrightarrow *d(f\alpha) = df\wedge\alpha - f\wedge d\alpha \tag{319}$$

Different than its counterpart in R^3

$$*d(f\alpha) = df\wedge\alpha + (-1)^p f\wedge d\alpha \dashrightarrow p = 2 \dashrightarrow *d(f\alpha) = df\wedge\alpha + f\wedge d\alpha \tag{320}$$

But since the term $f\wedge d\alpha = 0$ wether in R^4 or R^3 the final result of the Hodge Star is the same wether in R^4 or R^3 and we do not need to repeat here the tedious and monotonous piles of calculations shown in the (monster)Appendix J since the results are the same ones.

Our new tridimensional $3D$ spherical coordinates warp drive vector in R^4 with constant speed $vs\ nX = vs * dx$ or for the first term $vs * dx$ of the new tridimensional $3D$ spherical coordinates warp drive vector in R^4 with variable speed $vs\ nX = vs * dx + x * dvs$ is given by:

$$nX = vs(t)[\sin\phi][2f(r)\ cos\theta e_r] - vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta e_\theta] + [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))]e_\phi]] \tag{321}$$

The corresponding shift vectors are:

$$nX = X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{322}$$

$$X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] \tag{323}$$

$$X^\theta = -vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] \tag{324}$$

$$X^\phi = [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] \tag{325}$$

18 Appendix L:differential forms,Hodge star and the mathematical demonstration of the new warp drive vector $nX = *(vsx) = vs * dx + x * dvs$ for a variable speed vs and a constant acceleration a in Tridimensional $3D$ Spherical Coordinates

any warp drive vector nX generates a warp drive spacetime if $nX = 0$ and $X = vs = 0$ for a small value of r defined by Natario as the interior of the warp bubble and $nX = vs(t) * dx$ with $X = vs$ for a large value of r defined by Natario as the exterior of the warp bubble with $vs(t)$ being the speed of the warp bubble.(pg 4 in [1])(see Appendix G for an explanation about this statement)

In the Appendices J and K we gave the mathematical demonstration of the new warp drive vector nX in the R^3 and R^4 space basis in tridimensional $3D$ spherical coordinates where the velocity vs is constant.Hence the complete expression of the Hodge star that generates the warp drive vector $nX = vs * dx$ for a constant velocity vs is given by:

$$nX = *(vsx) = vs * (dx) \tag{326}$$

$$*dx = *d(r \sin\phi \cos\theta) = \sin\phi[*d\left(\frac{1}{2}r^2 \sin^2\theta d\phi\right)] + cos\phi[*d[(\frac{1}{2})(r^2)\cot\theta d\theta]] \tag{327}$$

$$\sin\phi[*d[f(r)r^2 \sin^2\theta d\phi]] + cos\phi[*d[(f(r))(r^2)\cot\theta d\theta]] \tag{328}$$

Our new tridimensional $3D$ spherical coordinates warp drive vector in R^4 with constant speed vs $nX = vs * dx$ or for the first term $vs * dx$ of the new tridimensional $3D$ spherical coordinates warp drive vector in R^4 with variable speed vs $nX = vs * dx + x * dvs$ is given by:

$$nX = vs(t)[\sin\phi][2f(r)\ cos\theta e_r] - vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta e_\theta] + [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))]e_\phi]] \tag{329}$$

The corresponding shift vectors are:

$$nX = X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{330}$$

$$X^r = vs(t)[\sin\phi][2f(r)\ cos\theta] \tag{331}$$

$$X^\theta = -vs(t)[\sin\phi][2f(r) + rf'(r)]\sin\theta] \tag{332}$$

$$X^\phi = [vs(t)cos\phi][cot\theta[2(f(r)) + (rf'(r))] \tag{333}$$

Because due to a constant speed vs the term $x * d(vs) = 0$.Now we must examine what happens when the velocity is variable and then the term $x * d(vs)$ no longer vanishes.Remember that a real warp drive must accelerate or de-accelerate in order to be accepted as a physical valid model.The complete expression of the Hodge star that generates the warp drive vector nX for a variable velocity vs is now given by:

$$nX = *(vsx) = vs * (dx) + x * (dvs) \tag{334}$$

In order to study the term $x * d(vs)$ we must introduce a new Canonical Basis for the coordinate time in the R^4 space basis defined as follows:(see eqs 10.102 and 10.103 pgs 363$(a)(b)$ and 364$(a)(b)$ in [2] with the terms $S = u = 1$[6],eq 3.74 pg 69$(a)(b)$ in [2],eqs 11.131 and 11.133 with the term $m = 0$[7] pg 417$(a)(b)$ in [2].)(see pg 47 eqs 2.67 to 2.70 and pg 92 in [3]):

$$e_t \equiv \frac{\partial}{\partial t} \sim dt \sim dr \wedge (rd\theta) \wedge (r \sin\theta d\phi) \sim r^2 \sin\theta (dr \wedge d\theta \wedge d\phi) \tag{335}$$

The Hodge star operator defined for the coordinate time is given by:(see eq 3.74 pg 69$(a)(b)$ in [2]):

$$*dt = r^2 \sin\theta (dr \wedge d\theta \wedge d\phi) \tag{336}$$

The valid expression for a variable velocity $vs(t)$ in the Natario warp drive spacetime due to a constant acceleration a must be given by:

$$vs = 2f(r)at \tag{337}$$

Because and considering a valid $f(r)$ as a Natario shape function being $f(r) = \frac{1}{2}$ for large r(outside the warp bubble where $X = vs(t)$ and $nX = vs(t) * dx + x * d(vs(t))$) and $f(r) = 0$ for small r(inside the warp bubble where $X = 0$ and $nX = 0$) while being $0 < f(r) < \frac{1}{2}$ in the walls of the warp bubble also known as the Natario warped region(pgs 4 and 5 in [1]) and considering also that the Natario warp drive is a ship-frame based coordinates system(a reference frame placed in the center of the warp bubble where the ship resides-or must reside!!) then an observer in the ship inside the bubble sees every point inside the bubble at the rest with respect to him because inside the bubble $vs(t) = 0$ because $f(r) = 0$.

To illustrate the statement pointed above imagine a fish inside an aquarium and the aquarium is floating in the surface of a river but carried out by the river stream.The stream varies its velocity with time.The warp bubble in this case is the aquarium and the walls of the aquarium are the walls of the warp bubble-Natario warped region.An observer in the margin of the river would see the aquarium passing by him at a large speed considering a coordinates system(a reference frame) placed in the margin of the river but inside the aquarium the fish is at the rest with respect to his local neighborhoods.Then for the fish any point inside the aquarium is at the rest with respect to him because inside the aquarium $vs = 2f(r)at$ with $f(r) = 0$ and consequently giving a $vs(t) = 0$.Again with respect to the fish the fish "sees" the margin passing by him with a large relative velocity.The margin in this case is the region outside the bubble "seen" by the fish with a variable velocity $vs(t) = v1$ in the time $t1$ and $vs(t) = v2$ in the time $t2$ because outside the bubble the generic expression for a variable velocity vs is given by $vs = 2f(r)at$ and outside the bubble $f(r) = \frac{1}{2}$ giving a generic expression for a variable velocity vs as $vs(t) = at$ and consequently a $v1 = at1$ in the time $t1$ and a $v2 = at2$ in the time $t2$.Then the variable velocity in not only a function of time alone but must consider also the position of the bubble where the measure is being taken wether inside or outside the bubble.So the velocity must also be a function of r.Its total differential is then given by:

$$dvs = 2[atf'(r)dr + f(r)tda + f(r)adt] \tag{338}$$

[6]These terms are needed to deal with the Robertson-Walker equation in Cosmology using differential forms.We dont need these terms here and we can make $S = u = 1$

[7]This term is needed to describe the Dirac equation in the Schwarzschild spacetime we dont need the term here so we can make $m = 1$.Remember also that here we consider geometrized units in which $c = 1$

Applying the Hodge star to the total differential dvs we get:

$$*dvs = 2[atf'(r) * dr + f(r)t * da + f(r)a * dt] \tag{339}$$

But we consider here the acceleration a a constant.Then the term $f(r)tda = 0$ and in consequence $f(r)t * da = 0$.This leaves us with:

$$*dvs = 2[atf'(r) * dr + f(r)a * dt] \tag{340}$$

$$*dvs = 2[atf'(r) * dr + f(r)a * dt] = 2[atf'(r)r^2 \sin\theta(dt \wedge d\theta \wedge d\phi) + f(r)ar^2 \sin\theta(dr \wedge d\theta \wedge d\phi)] \tag{341}$$

$$*dvs = 2[atf'(r) * dr + f(r)a * dt] = 2[atf'(r)e_r + f(r)ae_t] \tag{342}$$

The complete expression of the Hodge star that generates the warp drive vector nX for a variable velocity vs is given by:

$$nX = *(vsx) = vs * (dx) + x * d(vs) \tag{343}$$

The term $*dx$ was obtained in the Appendices J and K as follows:

$$*dx = *d(r \sin\phi \cos\theta) = \sin\phi[*d\left(\frac{1}{2}r^2 \sin^2\theta d\phi\right)] + cos\phi[*d[(\frac{1}{2})(r^2)\cot\theta d\theta]] \tag{344}$$

$$\sin\phi[*d[f(r)r^2 \sin^2\theta d\phi]] + cos\phi[*d[(f(r))(r^2)\cot\theta d\theta]] \tag{345}$$

$$\sin\phi[2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta] + cos\phi[cot\theta[2(f(r)) + (rf'(r))]e_\phi] \tag{346}$$

The complete expression of the Hodge star that generates the warp drive vector nX for a variable velocity vs is now given by:

$$nX = vs(\sin\phi[2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta] + cos\phi[cot\theta[2(f(r)) + (rf'(r))]e_\phi]) + x(2[atf'(r)e_r + f(r)ae_t]) \tag{347}$$

But remember that we are in tridimensional $3D$ spherical coordinates(see Appendix E) in which $x = r \sin\phi \cos\theta$ and this leaves us with:

$$nX = A + B \rightarrow A = vs * dx \rightarrow B = x * dvs \tag{348}$$

$$A = vs(\sin\phi[2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta] + cos\phi[cot\theta[2(f(r)) + (rf'(r))]e_\phi]) \tag{349}$$

$$B = (r \sin\phi \cos\theta)(2[atf'(r)e_r + f(r)ae_t]) \tag{350}$$

But we know that $vs = 2f(r)at$.Hence we get:

$$nX = A + B \rightarrow A = vs * dx \rightarrow B = x * dvs \tag{351}$$

$$A = (2f(r)at)(\sin\phi[2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta] + cos\phi[cot\theta[2(f(r)) + (rf'(r))]e_\phi]) \tag{352}$$

$$B = (r\sin\phi\cos\theta)(2[atf'(r)e_r + f(r)ae_t]) \tag{353}$$

Then we can start with a warp bubble initially at the rest using the warp drive vector shown above and accelerate the bubble to a desired speed of 200 times faster than light.When we achieve the desired speed we turn off the acceleration and keep the speed vs constant.The term B due to the acceleration $x * (dvs)$ now disappears the speed vs is no longer $vs = 2f(r)at$ and we are left again with the warp drive vector for constant speeds shown below:

$$nX = A \rightarrow A = vs * dx \tag{354}$$

$$A = vs(\sin\phi[2f(r)\ cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta] + cos\phi[cot\theta[2(f(r)) + (rf'(r))]e_\phi]) \tag{355}$$

Working some algebra with the new warp drive vector for variable velocities we get:[8]

$$nX = A + B \rightarrow A = vs * dx \rightarrow B = x * dvs \tag{356}$$

$$A = (2f(r)at)(\sin\phi[2f(r)\cos\theta e_r - [2f(r) + rf'(r)]\sin\theta e_\theta] + cos\phi[cot\theta[2(f(r)) + (rf'(r))]e_\phi]) \tag{357}$$

$$B = (r\sin\phi\cos\theta)(2[atf'(r)e_r + f(r)ae_t]) \tag{358}$$

$$A = (2f(r)at)\sin\phi[2f(r)\cos\theta e_r] - (2f(r)at)\sin\phi[2f(r) + rf'(r)]\sin\theta e_\theta + (2f(r)at)cos\phi[cot\theta[2(f(r)) + (rf'(r))]e_\phi]] \tag{359}$$

$$B = 2(r\sin\phi\cos\theta)atf'(r)e_r + 2(r\sin\phi\cos\theta)f(r)ae_t \tag{360}$$

$$A = 4(f(r)^2at)(\sin\phi)(\cos\theta)e_r - (2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\sin\theta)e_\theta + (2f(r)at)[2(f(r)) + (rf'(r))](cos\phi)(cot\theta)e_\phi \tag{361}$$

$$B = 2(at)(rf'(r))(\sin\phi)(\cos\theta)e_r + 2(rf(r)a))(\sin\phi)(\cos\theta)e_t \tag{362}$$

[8]again:we know that we are being tedious monotonous and repetitive but we are writing this mainly for beginners or introductory students

Rearranging the terms we have:

$$A = 4(f(r)^2 at)(\sin\phi)(\cos\theta)e_r - (2f(r)at)[2f(r)+rf'(r)](\sin\phi)(\sin\theta)e_\theta + (2f(r)at)[2(f(r))+(rf'(r))](cos\phi)(cot\theta)e_\phi \tag{363}$$

$$A = (2f(r)at)\sin\phi[2f(r)\cos\theta e_r] - (2f(r)at)\sin\phi[2f(r)+rf'(r)]\sin\theta e_\theta + (2f(r)at)cos\phi[cot\theta[2(f(r))+(rf'(r))]e_\phi]] \tag{364}$$

$$(2f(r)at)[2f(r)](\sin\phi)(\cos\theta)e_r - (2f(r)at)[2f(r)+rf'(r)](\sin\phi)(\sin\theta)e_\theta + (2f(r)at)[2f(r)+(rf'(r))](cos\phi)(cot\theta)e_\phi \tag{365}$$

$$B = 2(at)(rf'(r))(\sin\phi)(\cos\theta)e_r + 2(rf(r)a))(\sin\phi)(\cos\theta)e_t \tag{366}$$

Working the terms with e_r

$$(2f(r)at)\sin\phi[2f(r)\cos\theta e_r] + 2(at)(rf'(r))(\sin\phi)(\cos\theta)e_r \tag{367}$$

$$(2f(r)at)[2f(r)](\sin\phi)(\cos\theta)e_r + 2(at)(rf'(r))(\sin\phi)(\cos\theta)e_r \tag{368}$$

$$(2at)[2f(r)^2](\sin\phi)(\cos\theta)e_r + 2(at)(rf'(r))(\sin\phi)(\cos\theta)e_r \tag{369}$$

$$(2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta)e_r \tag{370}$$

At last we can give now the new warp drive vector for variable velocities in real tridimwensional $3D$ spherical coordinates using its respective contravariant shift vector components:[9]

$$nX = X^t e_t + X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{371}$$

$$X^t = 2(rf(r)a))(\sin\phi)(\cos\theta) \tag{372}$$

$$X^r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta) \tag{373}$$

$$X^\theta = -(2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\sin\theta) \tag{374}$$

$$X^\phi = (2f(r)at)[2f(r) + (rf'(r))](cos\phi)(cot\theta) \tag{375}$$

[9]again:the section is extensive but a beginner needs all these QED Quod Erad Demonstratum mathematical demonstrations

Comparing the new warp drive vector for variable velocities in real tridimensional $3D$ spherical coordinates with the Natario polar coordinates warp drive vector counterpart:

$$nX = X^t e_t + X^r e_r + X^\theta e_\theta + X^\phi e_\phi \tag{376}$$

$$X^t = 2(rf(r)a))(\sin\phi)(\cos\theta) \tag{377}$$

$$X^r = (2at)[2f(r)^2 + (rf'(r))](\sin\phi)(\cos\theta) \tag{378}$$

$$X^\theta = -(2f(r)at)[2f(r) + rf'(r)](\sin\phi)(\sin\theta) \tag{379}$$

$$X^\phi = (2f(r)at)[2f(r) + (rf'(r))](cos\phi)(cot\theta) \tag{380}$$

$$nX = X^t e_t + X^r e_r + X^\theta e_\theta \tag{381}$$

$$X^t = 2f(r)r(cos\theta)a \tag{382}$$

$$X^r = 2[2f(r)^2 + rf'(r)]at(cos\theta) \tag{383}$$

$$X^\theta = -2f(r)at[2f(r) + rf'(r)](\sin\theta) \tag{384}$$

Natario defined a motion in the $x - axis$ of polar coordinates (pgs 4 and 5 in [1]) then the polar plane $x - y$ makes an angle of 90 degrees with the $z - axis$ and since $\sin\phi = 1$ and $\cos\phi = 0$ it is easy to see that in this case the new warp drive vector for variable velocities in real tridimensional $3D$ spherical coordinates reduces itself to the Natario polar coordinates warp drive vector counterpart:

The difference occurs only in a real tridimensional motion.

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