A Solution to Einstein's Field Equations that Results in a Sign Change to the Analogous Friedmann Acceleration Equation

Greg P. Proper, P.E.*
Professional Engineer (ret.)

(Dated: March 25, 2024)

This brief paper demonstrates that a mathematical solution to Einstein's field equations exists that features a sign change in what proves to be the analog to the Friedmann 2 (acceleration) equation. The purpose here is not to physically challenge the FLRW (Friedmann Lemaitre Robertson-Walker) solution, but to demonstrate that this sign change is mathematically possible. In order to achieve this result, the metric is modified so that the temporal increment is affected by the expansion in the same manner as the spatial increments. Unlike the Schwarzschild, there is no theorem that states that the R-W is mathematically unique. Therefore, from a purely mathematical perspective there is no prohibition on the exploration of other alternatives to the R-W. In this particular instance the four principal equations of cosmology reduce to two mathematically consistent equations and total energy is conserved.

Keywords: Einstein's field equations, Λ

I. INTRODUCTION

Observations of Type 1a Supernova in distant galaxies have resulted in the conclusion that the Universe's expansion is accelerating. These observations have required the addition of a negative energy density term (Λ) into the Friedmann equations - specifically the negative sign in Friedmann's acceleration equation precludes an accelerated expansion without the addition of Λ. Consequently, a solution to the field equations that changes sign in the analog to this equation should be of interest. As stated in the abstract, this sign change is achieved by modifying the metric so that the temporal increment is affected by the expansion in the same manner as the spatial increments. The immediately obvious objection is that the author has merely employed a coordinate transformation, and this new metric is nothing more than the Robertson-Walker (R-W) in disguise. Two points are offered to counter this objection. The first point is that, unlike the Schwarzschild, there is no mathematical theorem that states that the R-W geometry is mathematically unique. Therefore, from a purely mathematical standpoint it would be incorrect to state that this alternative geometry must be the R-W that has undergone a coordinate transformation, as it could also represent a hypothetical model in which both time passes and clocks speed up commensurate with the expansion. The second point is that the author has not employed the mathematics of coordinate transformation and substitution, but rather has utilized the lengthy process of solving the Einstein equations directly in order to obtain this solution. Again, the primary purpose here is to demonstrate that this sign change is mathematically possible. The sign change that occurs in the Friedmann 2 analog seems counterintuitive, but the one feasible solution to these analog equations has the Friedmann 2 analog reducing to an equation of state (*i.e* $\rho \propto p$), and the Friedmann 1 analog reducing to $\rho \propto a^{-4}$. The calculations herein demonstrate that the field equations allow for the possibility that each instant of time has its own unique set of frame parameters (*i.e.* each temporal instant with its own unique measure, clock, and unit mass).

II. AN ALTERNATIVE SOLUTION TO THE FLRW

The FLRW solution is presented first followed by the alternative solution in considerably more detail. This will allow the reader to compare and contrast the differing results. This purely mathematical approach demonstrates that the unaltered field equations themselves do not require the addition of Λ, but rather the FLRW solution requires it. The sign change that occurs in the Friedmann 2 analog is consistent with the hypothetical speeding-up of clocks throughout cosmological time which can produce an effect on the expansion similar to that which is observed.

The following calculations have been simplified by assuming that space is Euclidean. For reference the Einstein field equations (sans Λ)are

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi \mathrm{G}T_{\mu\nu}. \tag{1}$$

The flat-space Robertson-Walker (R-W) geometry is

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + sin^2\theta \mathrm{d}\phi^2)), \tag{2}$$

* gproper@arvig.net

(in which c = 1 and $a = R/R_0$). The metric of the R-W is

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & a^2 & & \\ & & a^2r^2 & \\ & & & a^2r^2sin^2\theta \end{pmatrix}, \quad (2a)$$

and the e-m tensor is

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & & & \\ 0 & & g_{ij}p & \\ 0 & & & \end{pmatrix}. \quad (3)$$

Using (2a) and (3) as inputs into (1) then give Friedmann's equations. They are:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho, \quad (4)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p). \quad (5)$$

(4) and (5) are Friedmann 1 and 2 respectively (flat-space versions). Detailed calculations of these equations can be readily found elsewhere.[1]

The basic approach to solving the Einstein equations was presented above and will now be presented in more detail with slight modifications to the inputs. Rather than utilizing the R-W geometry (2) as primary input, consider (mathematically) the slightly modified geometry,

$$ds^2 = a^2(-dt^2 + dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2)). \quad (6)$$

The corresponding metric to (6) is

$$g_{\mu\nu} = \begin{pmatrix} -a^2 & & & \\ & a^2 & & \\ & & a^2r^2 & \\ & & & a^2r^2sin^2\theta \end{pmatrix}, \quad (6a)$$

and its companion e-m tensor is

$$T_{\mu\nu} = \begin{pmatrix} a^2\rho & 0 & 0 & 0 \\ 0 & & & \\ 0 & & g_{ij}p & \\ 0 & & & \end{pmatrix}. \quad (7)$$

Since it is the rest frame of the Universe that evolves, it is trivial to demonstrate that (6) requires clocks to speed-up commensurate with a,

$$d\tau^2 = -ds^2 \rightarrow d\tau = a dt. \quad (8)$$

In lieu of (8) it will be of interest to observe how the derived analogs differ from their Friedmann counterparts, and whether these differences will preclude the Field equations from delivering mathematically consistent results.

Before proceeding, a brief discussion of Birkhoff's Theorem may prove helpful. Given the spherical symmetry of the Schwarzschild metric Birkhoff examined the question, "How many possible solutions are there in which the 4-index Riemann tensor is non-zero and its contraction to $R_{\mu\nu}$ is equal to zero (*i.e.* the vacuum solution)?". Birkhoff proved that in this instance there is only one solution, that being the Schwarzschild. This has relevance to this discussion because (6) resembles the Conformal-Time R-W metric and some Reviewers will no doubt state that the Author has merely employed a coordinate transformation of the R-W (t as defined in this paper would allegedly be conformal-time). However, it is not appropriate to conflate that which is true for the Schwarzschild to be that which must be true for the R-W. That is, Birkhoff's uniqueness theorem only applies to the Schwarzschild and no such similar theorem exists for the R-W; and while it may be true that (6) could be construed as the Conformal-Time R-W, it is also true that it could be construed as a cosmological model in which $d\tau = a dt$ (i.e. clocks speed up commensurate with the expansion) as both situations are represented by similar mathematics. Note that in the following paragraphs a direct solution of the Einstein equations has been utilized as opposed to coordinate transformation and substitution.

The first step in solving the Einstein equations using (6a) and (7) as inputs is to calculate the lefthand (geometric) side of the Einstein equations. The Christoffel symbols (along with their symmetric counterparts - refer Appendix A) are:

$$\Gamma^0_{00} = \tfrac{\dot{a}}{a}, \quad \Gamma^1_{01} = \Gamma^1_{10} = \tfrac{\dot{a}}{a}, \quad \Gamma^2_{02} = \Gamma^2_{20} = \tfrac{\dot{a}}{a},$$

$$\Gamma^0_{11} = \tfrac{\dot{a}}{a}, \quad \Gamma^1_{22} = -r, \quad \Gamma^2_{12} = \Gamma^2_{21} = \tfrac{1}{r},$$

$$\Gamma^0_{22} = \tfrac{\dot{a}}{a}r^2,$$

$$\Gamma^0_{33} = \tfrac{\dot{a}}{a}r^2sin^2\theta, \; \Gamma^1_{33} = -rsin^2\theta, \; \Gamma^2_{33} = -\sin\theta\cos\theta,$$

$$\Gamma^3_{03} = \Gamma^3_{30} = \tfrac{\dot{a}}{a}, \; \Gamma^3_{13} = \Gamma^3_{31} = \tfrac{1}{r}, \; \Gamma^3_{23} = \Gamma^3_{32} = cot\theta.$$

(Note that only the $\Gamma^0_{\mu\nu}$ Christoffel symbols differ from FLRW. The FLRW has $\Gamma^0_{00} = 0$, $\Gamma^0_{11} = a\dot{a}$, $\Gamma^0_{22} = a\dot{a}r^2$, $\Gamma^0_{33} = a\dot{a}r^2\sin^2\theta$.)

The pertinent Ricci tensors are (refer Appendix B):

$$R_{00} = -3\left[\tfrac{\ddot{a}}{a} - \left(\tfrac{\dot{a}}{a}\right)^2\right],$$

$$R_{11} = \left[\tfrac{\ddot{a}}{a} + \left(\tfrac{\dot{a}}{a}\right)^2\right],$$

$$R_{22} = \left[\tfrac{\ddot{a}}{a} + \left(\tfrac{\dot{a}}{a}\right)^2\right] r^2,$$

$$R_{33} = \left[\tfrac{\ddot{a}}{a} + \left(\tfrac{\dot{a}}{a}\right)^2\right] r^2 sin^2\theta.$$

The Ricci scalar is (refer Appendix B)

$$R = 6\frac{\ddot{a}}{a^3}.$$

The left-hand side of the $_{00}$ Einstein equation is

$$R_{00} - \frac{1}{2}Rg_{00} = -3\left[\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2\right] - \frac{1}{2}\left(6\frac{\ddot{a}}{a^3}\right)(-a^2)$$
$$= 3\left(\frac{\dot{a}}{a}\right)^2.$$

The $_{00}$ Einstein equation is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi \mathrm{G}}{3}\rho a^2, \tag{9}$$

which is the equation analogous to Friedmann 1 (*i.e.* (4)).

The left-hand side of the $_{11}$ Einstein equation is

$$R_{11} - \frac{1}{2}Rg_{11} = \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] - \frac{1}{2}\left(6\frac{\ddot{a}}{a^3}\right)(a^2)$$
$$= -2\frac{\ddot{a}}{a} + \frac{8\pi \mathrm{G}}{3}\rho a^2.$$

The $_{11}$ Einstein equation is

$$\frac{\ddot{a}}{a} = \frac{4\pi \mathrm{G}}{3}(\rho - 3p)\,a^2, \tag{10}$$

which is the equation analogous to Friedmann 2 (*i.e.* (5)).

Differentiating (9) and eliminating $\ddot{a}$ by equating the result to (10) gives this alternative's analog of the fluid equation

$$(\dot{a})^2 = \frac{8\pi \mathrm{G}}{3}\rho a^4 \rightarrow 2\dot{a}\ddot{a} = \frac{8\pi \mathrm{G}}{3}(\dot{\rho}a^4 + 4\rho a^3\dot{a})$$

$$\rightarrow 2\dot{a}\ddot{a} = \frac{8\pi \mathrm{G}}{3}\dot{a}(\rho - 3p)\,a^3 = \frac{8\pi \mathrm{G}}{3}(\dot{\rho}a^4 + 4\rho a^3\dot{a})$$

$$\rightarrow \rho\dot{a} - 3p\dot{a} = \dot{\rho}a + 4\rho\dot{a}$$

$$\rightarrow \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0, \tag{11}$$

whose basic form is unchanged from FLRW cosmology.

Comparing and contrasting Friedmann 2 (5) with its analog (10), one notes that there is a sign change in ρ from minus to plus. Upon first review, it would thus appear that (10) is not physically consistent as ρ would in all cases be repulsive. However, before dismissing these results entirely, one might pursue the one possible solution to (10) which might be true; the possibility of $\ddot{a} = 0$ (*i.e.*, $\dot{a}$ is constant throughout time).

Notwithstanding SN1a observational data to the contrary, there is a particularly elegant mathematical solution to these equations which occurs when $\dot{a}$ is constant. The analog to Friedman 1 (9) reduces to

$$\rho \propto \frac{1}{a^4}. \tag{12}$$

Furthermore, observe that when $\dot{a}$ is constant, (10) reduces to an equation of state similar to radiant energy

$$p/\rho = 1/3. \tag{13}$$

(12) can also be derived by combining (13) with the fluid equation (11) (which is unchanged from FLRW),

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + \frac{1}{3}\rho\right) = 0 \rightarrow \left[\frac{\dot{\rho}}{\rho} = -4\frac{\dot{a}}{a}\right] \rightarrow \rho \propto a^{-4}.$$

Consequently, in this instance the analog equations reduce to two mathematically consistent equations, (12) and (13).

Since $\dot{a}$ is constant, time itself is directly related to the expansion. Each instant of t now constitutes its own unique physical frame of reference as $\mathrm{d}\tau \propto a$, $R \propto a$ and (12) implies that

$$\rho \propto \frac{1}{a^4} \rightarrow m \propto \frac{1}{a},$$

in which m is a unit of mass.

It still remains that SN1a observations have demonstrated that $\dot{a}$ is not constant which again seemingly render the above calculations superfluous. However, these observations are based on the assumption that clocks run at a constant speed throughout time. Specifically, SN1a observations have shown that light travel times from past events in faraway galaxies are clearly longer than anticipated (rendering lower observed luminosities and consequently, an $\dot{a}$ less than its present value). However, the same effect (*i.e.*, longer light travel times) would also occur if clocks ran slower in the past than they do in the present.

III. CONCLUSION

Unlike the Schwarzschild there is no theorem that states that the R-W is mathematically unique. This allows the mathematical consideration of other evolving geometries. Directly solving the Einstein equations using (6a) and (7) as inputs then gives the Friedmann analogs (9) and (10).

Since conventional energy is unmistakably attractive there is likely only one feasible physical solution to these analog equations; that is, $\dot{a}$ must be constant. In this instance (10) reduces to

$$p/\rho = 1/3. \tag{13}$$

Not surprising, this requires

$$\rho \propto \frac{1}{a^4} \rightarrow m \propto \frac{1}{a}. \tag{12}$$

In this instance there are only two principal cosmology equations.

Each instant of $t(a)$ now constitutes its own unique cosmological frame as $d\tau \propto a$, $R \propto a$ and $m \propto \frac{1}{a}$. The unaltered Einstein equations themselves allow the possibility of a universe in which mass decreases monotonically and in which gravitational potential becomes monotonically less negative as time passes, space expands, and clocks speed up. Total energy is therefore conserved.

Interestingly, such a universe is also necessarily indeterministic. Consider, if space, clocks, and mass (energy) vary parametrically with $t(a)$ as described in the previous paragraph then quantities such as velocity and angular momentum would remain invariant, while momentum and energy would vary with m, *i.e.* inversely to space and clocks. This would necessarily lead to a form of uncertainty anytime these divergent quantities were paired in experimental calculations.

SN1a observations are based on the assumption that clocks run at a constant speed throughout time. However, the same effect (*i.e.*, longer light travel times) could also be achieved if clocks ran slower in the past than they do in the present. This, along with the change in sign in (10) would negate the requirement for adding Λ into the Einstein equations.[2]

In conclusion, the author reiterates that the intent of this article is to provide insight into the mathematics of the Einstein equations with respect to evolving geometries.

Appendix A CHRISTOFFEL SYMBOL CALCULATIONS

For reference the metric is:

$$g_{\mu\nu} = \begin{pmatrix} -a^2 & & & \\ & a^2 & & \\ & & a^2 r^2 & \\ & & & a^2 r^2 sin^2\theta \end{pmatrix}, \tag{6a}$$

and the equation for calculating Christoffell symbols is:

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} \left[\frac{\partial g_{\mu\beta}}{\partial x^\nu} + \frac{\partial g_{\nu\beta}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\beta} \right].$$

The non-zero $\Gamma^0_{\mu\nu}$ values are:

$$\Gamma^0_{00} = \frac{1}{2} g^{00} \left[\frac{\partial g_{00}}{\partial x^0} + \frac{\partial g_{00}}{\partial x^0} - \frac{\partial g_{00}}{\partial x^0} \right] = \frac{1}{2} \left(\frac{-1}{a^2} \right) [-2a\dot{a}] = \frac{\dot{a}}{a},$$

$$\Gamma^0_{11} = \frac{1}{2} g^{00} \left[\frac{\partial g_{10}}{\partial x^1} + \frac{\partial g_{10}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^0} \right] = \frac{1}{2} \left(\frac{-1}{a^2} \right) [-2a\dot{a}] = \frac{\dot{a}}{a},$$

$$\Gamma^0_{22} = \frac{1}{2} g^{00} \left[\frac{\partial g_{20}}{\partial x^2} + \frac{\partial g_{20}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^0} \right] = \frac{1}{2} \left(\frac{-1}{a^2} \right) [-2a\dot{a}r^2] = \frac{\dot{a}}{a} r^2,$$

$$\Gamma^0_{33} = \frac{1}{2} g^{00} \left[\frac{\partial g_{30}}{\partial x^3} + \frac{\partial g_{30}}{\partial x^3} - \frac{\partial g_{33}}{\partial x^0} \right] = \frac{1}{2} \left(\frac{-1}{a^2} \right) [-2a\dot{a}r^2 \sin^2\theta] = \frac{\dot{a}}{a} r^2 \sin^2\theta.$$

The non-zero $\Gamma^1_{\mu\nu}$ values are:

$$\Gamma^1_{01} = \frac{1}{2} g^{11} \left[\frac{\partial g_{01}}{\partial x^1} + \frac{\partial g_{11}}{\partial x^0} - \frac{\partial g_{01}}{\partial x^1} \right] = \frac{1}{2} \left(\frac{1}{a^2} \right) [2a\dot{a}] = \frac{\dot{a}}{a},$$

$$\Gamma^1_{10} = \frac{\dot{a}}{a},$$

$$\Gamma^1_{22} = \frac{1}{2} g^{11} \left[\frac{\partial g_{21}}{\partial x^2} + \frac{\partial g_{21}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^1} \right] = \frac{1}{2} \left(\frac{1}{a^2} \right) [-2a^2 r] = -r,$$

$$\Gamma^1_{33} = \frac{1}{2} g^{11} \left[\frac{\partial g_{31}}{\partial x^3} + \frac{\partial g_{31}}{\partial x^3} - \frac{\partial g_{33}}{\partial x^1} \right] = \frac{1}{2} \left(\frac{1}{a^2} \right) [-2a^2 r \sin^2\theta] = -r \sin^2\theta.$$

The non-zero $\Gamma^2_{\mu\nu}$ values are:

$$\Gamma^2_{02} = \frac{1}{2} g^{22} \left[\frac{\partial g_{02}}{\partial x^2} + \frac{\partial g_{22}}{\partial x^0} - \frac{\partial g_{02}}{\partial x^2} \right] = \frac{1}{2} \left(\frac{1}{a^2 r^2} \right) [2a\dot{a}r^2] = \frac{\dot{a}}{a},$$

$$\Gamma^2_{20} = \frac{\dot{a}}{a},$$

$$\Gamma^2_{12} = \frac{1}{2} g^{22} \left[\frac{\partial g_{12}}{\partial x^2} + \frac{\partial g_{22}}{\partial x^1} - \frac{\partial g_{12}}{\partial x^2} \right] = \frac{1}{2} \left(\frac{1}{a^2 r^2} \right) [2a^2 r] = \frac{1}{r},$$

$$\Gamma^2_{21} = \frac{1}{r},$$

$$\Gamma^2_{33} = \frac{1}{2} g^{22} \left[\frac{\partial g_{32}}{\partial x^3} + \frac{\partial g_{32}}{\partial x^3} - \frac{\partial g_{33}}{\partial x^2} \right] = \frac{1}{2} \left(\frac{1}{a^2 r^2} \right) [-a^2 r^2 (2 \sin\theta \cos\theta)] = -\sin\theta \cos\theta.$$

The non-zero $\Gamma^3_{\mu\nu}$ values are:

$$\Gamma^3_{03} = \frac{1}{2} g^{33} \left[\frac{\partial g_{03}}{\partial x^3} + \frac{\partial g_{33}}{\partial x^0} - \frac{\partial g_{03}}{\partial x^3} \right] = \frac{1}{2} \left(\frac{1}{a^2 r^2 \sin^2\theta} \right) [2a\dot{a} r^2 \sin^2\theta] = \frac{\dot{a}}{a},$$

$$\Gamma^3_{30} = \frac{\dot{a}}{a},$$

$$\Gamma^3_{13} = \frac{1}{2} g^{33} \left[\frac{\partial g_{13}}{\partial x^3} + \frac{\partial g_{33}}{\partial x^1} - \frac{\partial g_{13}}{\partial x^3} \right] = \frac{1}{2} \left(\frac{1}{a^2 r^2 \sin^2\theta} \right) [2a^2 r \sin^2\theta] = \frac{1}{r},$$

$$\Gamma^3_{31} = \frac{1}{r},$$

$$\Gamma^3_{23} = \frac{1}{2} g^{33} \left[\frac{\partial g_{23}}{\partial x^3} + \frac{\partial g_{33}}{\partial x^2} - \frac{\partial g_{23}}{\partial x^3} \right] = \frac{1}{2} \left(\frac{1}{a^2 r^2 \sin^2\theta} \right) [a^2 r^2 (2 \sin\theta \cos\theta)] = \cot\theta,$$

$$\Gamma^3_{32} = \cot\theta.$$

Appendix B RICCI TENSOR AND SCALAR CALCULATIONS

For reference the non-zero Christoffell symbols are:

$$\Gamma^0_{00} = \frac{\dot{a}}{a}, \quad \Gamma^1_{01} = \Gamma^1_{10} = \frac{\dot{a}}{a}, \quad \Gamma^2_{02} = \Gamma^2_{20} = \frac{\dot{a}}{a},$$

$$\Gamma^0_{11} = \frac{\dot{a}}{a}, \quad \Gamma^1_{22} = -r, \quad \Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r},$$

$$\Gamma^0_{22} = \frac{\dot{a}}{a} r^2,$$

$$\Gamma^0_{33} = \frac{\dot{a}}{a} r^2 sin^2\theta, \quad \Gamma^1_{33} = -r sin^2\theta, \quad \Gamma^2_{33} = -\sin\theta\cos\theta,$$

$$\Gamma^3_{03} = \Gamma^3_{30} = \frac{\dot{a}}{a}, \quad \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}, \quad \Gamma^3_{23} = \Gamma^3_{32} = cot\theta.$$

and the Riemann tensor equation is:

$$R^\alpha{}_{\beta\mu\nu} = \frac{\partial}{\partial x_\mu} \Gamma^\alpha_{\nu\beta} - \frac{\partial}{\partial x_\nu} \Gamma^\alpha_{\mu\beta} + \Gamma^\alpha_{\mu\gamma} \Gamma^\gamma_{\nu\beta} - \Gamma^\alpha_{\nu\gamma} \Gamma^\gamma_{\mu\beta}.$$

The R_{00} Ricci equation is

$$R_{00} = R^0{}_{000} + R^1{}_{010} + R^2{}_{020} + R^3{}_{030},$$

and the pertinent Riemann tensors are:

$$R^0{}_{000} = 0,$$

$$R^1{}_{010} = 0 - \frac{\partial}{\partial x^0} \left(\frac{\dot{a}}{a} \right) + \left(\frac{\dot{a}}{a} \right)^2 - \left(\frac{\dot{a}}{a} \right)^2 = \left(\frac{\dot{a}}{a} \right)^2 - \frac{\ddot{a}}{a},$$

$$R^2{}_{020} = 0 - \frac{\partial}{\partial x^0} \left(\frac{\dot{a}}{a} \right) + \left(\frac{\dot{a}}{a} \right)^2 - \left(\frac{\dot{a}}{a} \right)^2 = \left(\frac{\dot{a}}{a} \right)^2 - \frac{\ddot{a}}{a},$$

$$R^3{}_{030} = 0 - \frac{\partial}{\partial x^0} \left(\frac{\dot{a}}{a} \right) + \left(\frac{\dot{a}}{a} \right)^2 - \left(\frac{\dot{a}}{a} \right)^2 = \left(\frac{\dot{a}}{a} \right)^2 - \frac{\ddot{a}}{a}.$$

$$\rightarrow R_{00} = -3 \left[\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a} \right)^2 \right].$$

The R_{11} Ricci equation is

$$R_{11} = R^0{}_{101} + R^1{}_{111} + R^2{}_{121} + R^3{}_{131},$$

and the pertinent Riemann tensors are:

$$R^0{}_{101} = \frac{\partial}{\partial x^0} \left(\frac{\dot{a}}{a} \right) - 0 + \left(\frac{\dot{a}}{a} \right)^2 - \left(\frac{\dot{a}}{a} \right)^2 = \frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a} \right)^2,$$

$$R^1{}_{111} = 0,$$

$$R^2{}_{121} = 0 - \left(\frac{-1}{r^2} \right) + \left(\frac{\dot{a}}{a} \right)^2 - \left(\frac{1}{r^2} \right) = \left(\frac{\dot{a}}{a} \right)^2,$$

$$R^3{}_{131} = 0 - \left(\frac{-1}{r^2} \right) + \left(\frac{\dot{a}}{a} \right)^2 - \left(\frac{1}{r^2} \right) = \left(\frac{\dot{a}}{a} \right)^2.$$

$$\rightarrow R_{11} = \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a} \right)^2 \right].$$

The R_{22} Ricci equation is

$$R_{22} = R^0{}_{202} + R^1{}_{212} + R^2{}_{222} + R^3{}_{232},$$

and the pertinent Riemann tensors are:

$$R^0{}_{202} = \frac{\partial}{\partial x^0}\left(\frac{\dot{a}}{a}r^2\right) - 0 + \left(\frac{\dot{a}}{a}\right)\left(\frac{\dot{a}}{a}r^2\right) - \left(\frac{\dot{a}}{a}r^2\right)\left(\frac{\dot{a}}{a}\right) = \left[\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2\right] r^2,$$

$$R^1{}_{212} = \frac{\partial}{\partial x^1}(-r) - 0 + \left(\frac{\dot{a}}{a}\right)\left(\frac{\dot{a}}{a}r^2\right) - (-r)\left(\frac{1}{r}\right) = \left(\frac{\dot{a}}{a}\right)^2 r^2,$$

$$R^2{}_{222} = 0,$$

$$R^3{}_{232} = 0 - \frac{\partial}{\partial x^2}\cot\theta + \left[\left(\frac{\dot{a}}{a}\right)\left(\frac{\dot{a}}{a}r^2\right) + \left(\frac{1}{r}\right)(-r)\right] - \cot^2\theta$$
$$= \csc^2\theta + \left(\frac{\dot{a}}{a}\right)^2 r^2 - 1 - \cot^2\theta = \left(\frac{\dot{a}}{a}\right)^2 r^2.$$

$$\rightarrow R_{22} = \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] r^2.$$

The R_{33} Ricci equation is

$$R_{33} = R^0{}_{303} + R^1{}_{313} + R^2{}_{323} + R^3{}_{333},$$

and the pertinent Riemann tensors are:

$$R^0{}_{303} = \frac{\partial}{\partial x^0}\left(\frac{\dot{a}}{a}r^2\sin^2\theta\right) - 0 + \left(\frac{\dot{a}}{a}\right)\left(\frac{\dot{a}}{a}r^2\sin^2\theta\right) - \left(\frac{\dot{a}}{a}r^2\sin^2\theta\right)\left(\frac{\dot{a}}{a}\right) = \left[\frac{2\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2\right] r^2\sin^2\theta,$$

$$R^1{}_{313} = \frac{\partial}{\partial x^1}(-r\sin^2\theta) - 0 + \left(\frac{\dot{a}}{a}\right)\left(\frac{\dot{a}}{a}r^2\sin^2\theta\right) - (-r\sin^2\theta)\left(\frac{1}{r}\right) = \left(\frac{\dot{a}}{a}\right)^2 r^2\sin^2\theta,$$

$$R^2{}_{323} = \frac{\partial}{\partial x^2}(-\sin\theta\cos\theta) - 0$$
$$+\left[\left(\frac{\dot{a}}{a}\right)\left(\frac{\dot{a}}{a}r^2\sin^2\theta\right) + \left(\frac{1}{r}\right)(-r\sin^2\theta)\right] - [(-\sin\theta\cos\theta)(\cot\theta)]$$
$$= (-\cos^2\theta + \sin^2\theta)$$
$$+ \left(\frac{\dot{a}}{a}\right)^2 r^2\sin^2\theta - \sin^2\theta + \cos^2\theta$$
$$= \left(\frac{\dot{a}}{a}\right)^2 r^2\sin^2\theta,$$

$$R^3{}_{333} = 0.$$

$$\rightarrow R_{33} = \left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] r^2\sin^2\theta.$$

The Ricci scalar is:

$$R = R^0{}_0 + R^1{}_1 + R^2{}_2 + R^3{}_3.$$

$$R = -\frac{1}{a^2}\left[-3\frac{\ddot{a}}{a} + 3\left(\frac{\dot{a}}{a}\right)^2\right] + \frac{1}{a^2}\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right]$$
$$+\frac{1}{a^2r^2}\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] r^2 + \frac{1}{a^2r^2\sin^2\theta}\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right] r^2\sin^2\theta$$
$$= 6\frac{\ddot{a}}{a^3}.$$

[1] Carroll, Sean M., *Spacetime and Geometry.* San Francisco: Pearson—Addison Wesley 2004, pgs. 332-333.

[2] Proper, Greg P., *A model of a simple, baryon-dominated universe that expands at an ever-increasing rate without relying on vacuum energy (Λ).* Mod. Phys. Lett. A., Volume 26, Issue 33, 2501, 30 October 2011 .