A Proposed Resolution to Dark Energy and Dark Matter: Replacing Euclidean and non-Euclidean Geometry Via Axiomatization of Torricelli's Homogeneous Infinitesimals

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Abstract

The conundrum of Dark Matter coupled with the discovery in 1998 that the universe is paradoxically accelerating its expansion has led some cosmologists to question the correctness of the non-Euclidean geometric theory of gravity, General Relativity. In the 17th century, there was also a great paradox between two views for the geometric constituents of a line, heterogeneous (made of points) versus homogeneous (made of infinitesimal segments). Evangelista Torricelli, a protege of Galileo, elucidated his logical reasoning on why lines must be made of segments and not points and created one particular fundamental example among many. In this paper, I produce unknown corollaries to Torricelli's argument allowing me to falsify the relationship between infinitesimals and the Archimedean axiom, resolve L'Hôpital's paradox, rewrite the Fundamental Theorem of Calculus and derive Gaussian curvature. I hypothesize that the intractability of Dark Energy and Dark Matter is due to the points of coordinate systems within General Relativity actually being a logically flawed heterogeneous interpretation with basis vectors as a stand-in for the properties of homogeneous infinitesimals. I propose a novel but geometrically logical model for gravity based on the changing area of "surfaces" that suffers from no Cosmological Constant but can model red-shift of light. I present hypothetical arguments to demonstrate that there is sufficient compelling similarities to support the investigation of rewriting Euclidean/non-Euclidean geometry, the Calculus and all the laws of physics with axiomatic homogeneous infinitesimals of three components that each follow the theory of proportionality. They are relative cardinality, homogeneous infinitesimal and lastly their sum.

Keywords: Cosmological Constant, Dark Energy, General Relativity, quintessence, Unified Theory

1 Introduction

The Dark Energy Task Force, a committee of scientists tasked with advising the DOE, NASA and NSF on Dark Energy, has stated [1], "The acceleration of the Universe is, along with dark matter, the observed phenomenon which most directly demonstrates that our fundamental theories of particles and gravity are either incorrect or incomplete." The theoretical value for the Cosmological Constant (CC) is well known by now as the worst prediction ever made in physics for good reason:

> An alternative explanation of the accelerating expansion of the Universe is that general relativity or the standard cosmological model is incorrect. We are driven to consider this prospect by potentially deep problems with the other options. A cosmological constant leaves unresolved one of the great mysteries of quantum gravity and particle physics: If the cosmological constant is not zero, it would be expected to be 10^{120} times larger than is observed.

If these problems are fundamental enough for the Task Force to advise that General Relativity (GR) itself could be incomplete or incorrect then it also begs the question: *How* could it be either? I propose an answer: GR could be incorrect if our concept of infinitesimals has always been incomplete.

2 Background

The meaning behind dx (invented by Leibniz[1] for the Calculus but also ubiquitous in GR[2], can be traced back to concepts from over 2500 years ago and more rigidly to Bonaventura Cavalieri in 1635[3]. One of the great debates during his time was whether lines were made of non-dimensional points (heterogeneous) or made of infinitesimal segments of lines (homogeneous)[4]. In the same vein, it was also debated whether area would be composed of infinitesimally thin slices of area versus stacked lines and whether volume was made of infinitesimally thin sheets of volume versus stacked planes. Evangelista Torricelli, a brilliant scientist and inventor in his own right and well known to Galileo, is also known in these debates for his talent at taking a difficult concept and explaining it in many different ways. This has been said to have enabled the transfer of fundamental concepts more so than the voluminous writings

[1] see [2] for a discussion on who invented the term "infinitesimal"

[2] It would seem to me it is taken for granted. Often the Einstein field equation in compact form doesn't even bother to include the infinitesimal notation $dx_\mu dx_\nu$ with the metric notation $g_{\mu\nu}dx_\mu dx_\nu$ such as $R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = kT_{\mu\nu}$.

[3] see [3] p303 for timeline

[4] see [4] p. 4 for discussion

of Cavalieri. Torricelli's analysis of the heterogeneous/homogeneous debate [5] landed him firmly on the infinitesimal segment[5] side as recent authors have pointed out[6].

> All indivisibles seem equal to one another, that is, points are equal to points, lines are equal in thickness to lines, and surfaces are equal in depth to surfaces is an opinion that in my judgment is not only difficult to prove, but false.

By this he meant that it would seem we should be using infinitesimal segments[7] instead of non-dimensional points. Whereas points can't be distinguished from each other, the segments can have infinitesimal length and that length isn't necessarily the same from one segment to another thus distinguishing them (as would be his similar argument for area and volume).

One example in particular that he used to demonstrate his reasoning, prior to his early death at the age of 39, has been called by Francois De Gandt the "condensed" "fundamental example" for Torricelli's view on the heterogeneous/homogeneous paradox[8]. While I have come to very much agree with the sentiment that this is a "condensed paradox", my examination of Torricelli's example has also revealed startling unknown similarities with the chain of logic that was used to create non-Euclidean geometry and ultimately General Relativity.

3 Flatness, Curvature and HIs

Imagine that you could have a single line and it is itself composed of "infinitesimal" line segments. Suppose that the magnitude $|SEG^n|$ of a segment n could either be of equal relative magnitude (as in Eqn.1) to an adjacent segment $n-1$ or could have a different value (as in Eqn.2) *within the line itself.* A simple to state introductory hypothesis is whether Euclidean geometry can be derived from

$$|SEG^n| - |SEG^{n-1}| = 0 : \text{intrinsically flat} \tag{1}$$

which I will call *intrinsically flat*[9] and whether non-Euclidean geometry can be derived from

$$|SEG^n| - |SEG^{n-1}| \neq 0 : \text{intrinsically curved} \tag{2}$$

which I will call *intrinsically curved.*

Pairing the properties of these equations with that of Torricelli's homogeneous infinitesimal (HI) concept creates an interesting perspective. If the logically true portions of non-Euclidean straight and curved lines were actually based on properties of HIs then curved space-time could provide certain accurate predictions within a region yet still suffer from paradoxes as a flawed interpretation of the underlying geometry. By this I mean that Torricelli's HIs have a striking resemblance to both coordinate

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[5] Torricelli may have not always written "point" but he certainly was of the opinion that they were not non-dimensional

[6] [3] and [4]p125

[7] there was a philosophical distinction between indivisibles and infinitesimals. I do not expand upon the indivisible concept as I view this a geometrical and philosophical red herring. See p. 24[6]

[8] see [5] p. 164

[9] Note that Euclid's definition of a straight line (Euclid's Elements, Book I, Definition 4) is one that "lies evenly upon itself" and in this case both terms are equal or even.

systems and basis vectors. For coordinate systems, I can derive the real number line with HIs and show how it can contract and extend. For basis vectors, HIs also possess direction and relative magnitude, but with HIs their absolute magnitudes $|SEG|$ are arbitrary[10] whereas with basis vectors their flat boundary condition magnitude seems to be defined as "1" within GR[11]. I hypothesize that by using HIs instead and inverting the perfect fluid analogy that GR incorporates (i.e. using $\Delta\rho$ instead of ρ for energy density[12]), a more logical and predictive model will result. Using GR notation with this research, I propose an equation (see Eqn. 61) where the Newtonian approximation is akin to the left side rather than the right side of

$$|\Lambda - \Lambda 2\phi| \approx |1 - 2\phi| \tag{3}$$

with

$$\Lambda = \rho_{vac} \tag{4}$$

and

$$\Delta\rho_{vac} = \textbf{energy density}. \tag{5}$$

If Λ, sometimes referred to as a constant of integration, is not uniform throughout the universe (for Dark Matter it would have a different value within galaxies), nor during its development (for Dark Energy the overall value would have changed as the universe has evolved), then a simple analogy is that I am re-establishing the boundary conditions with Equation 3.

As a historical analogy, HIs would be to curved and straight lines underlying spacetime as ellipses were to the perfect circles upon which epicycles and deferents relied[13]. Since infinitesimals have been almost entirely replaced by the concept of the mathematical limit[14][15][16], then the CC problem would effectively stem from premature abandonment of HI research by the time the Calculus was developed and thus never made it in to the consideration of non-Euclidean geometry nor relativity.

4 Structure

The logical order for this work is not the same as for historical order of the development of Euclidean/Calculus/non-Euclidean geometry so some portions may seem out of sync initially. To simplify this, the structure of this paper is in two sections. Firstly a geometrical introduction to the Calculus of HIs and secondly to philosophical hypotheses of the geometry. The geometrical introduction (some of which is in the appendix) consists of:

1. Explain Torricelli's Parallelogram and his logic for HIs.

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[10] similar to Riemann's "two magnitudes can only be compared when one is a part of the other; in which case also we can only determine the more or less and not the how much"[7]

[11] i.e. see Fig 2.3 [8]

[12] see change of state paradox as discussed by DETF [1]

[13] this was inspired by free Yale University ASTR 160 Lect. 24 class video of C. Bailyn @15:30 https://oyc.yale.edu/astronomy/astr-160/lecture-24

[14] see [4] p. 359-364 for discussion

[15] [9] "banished"

[16] [10] p. xii

2. Produce two new corollaries to his example and derive the concept of intrinsic flatness, curvature, relative strain, Relative Cardinality, absolute strain and line length.
3. Demonstrate that flat HIs with RC are compatible with the Archimedean axiom using L'Hôpital's paradox and non-standard analysis as proofs that I can derive the real number line.
4. Introduce the name and acronym for this research, set the HI as a primitive notion and give postulates.
5. Analyze lineal, areal, and voluminal lines.
6. Introduce Background and Foreground geometry
7. Using a flat background, relax the point postulate and hypothesize how to derive the Fundamental Theorem of Calculus: Use Relative Cardinality to define Euclidean geometry, Leibniz's $\frac{dy}{dx}$ and the process of integration.
8. Using foreground geometry, describe the similarities between voluminal lines and principal curvature K of Gaussian curvature.
9. Introduce concept of RC functions and HI functions.

The philosophical introduction consists of equating the HIs philosophically and my hypothesis that background and foreground geometry, along with RC and HI functions, should replace the concepts of coordinate systems, field potentials and tensors.

1. Give hypothesis for the physical philosophy similarities and differences between energy density and perfect fluid change of state paradoxes in GR and elastic medium relative strain philosophy of this research.
2. Hypothesize that length contractions and time dilations can be approximated to a HI function
3. Give hypothesis of how red shift is a HI function of relative strain that creates a gravitational well and a HI function of relative strain change universe wide.
4. Give hypothesis that the change of momentum/energy density and time dilation/length contraction for a particle is best described using a HI wave function with voluminal line paths instead of world lines of GR.
5. Hypothesize that physical events can be traced to either periodic minima and maxima of intrinsic curvature (space) or periodic temporal events (time)[17]
6. Describe the geometric similarities and differences between RC functions in Newtonian gravity, curved coordinate system of GR, and HI function. Describe hypothetical path to unify GR and quantum mechanics by reinterpreting both using HI functions.

One obvious section that should be included in this paper is an analysis of prior research by Gauss, Riemann, Bolyai and Lobachevsky etc. during the development of non-Euclidean geometry when they analyzed Torricelli's HIs. Unfortunately, I only have access to commonly published works and not perhaps unpublished notes. I currently find no published evidence that any of them analyzed Torricelli's work. However, absence of evidence is not evidence of absence and thus will have to rely upon the peer review process to enlist mathematical historians. My efforts at assistance prior to submission of this paper has not been fruitful.

[17] the "events" within *Gravitation*[8], such as p. 6, seem more to do with the Relative Cardinality

I call the framework for this research the Calculus, Philosophy and Notation of Axiomatic Homogeneous Infinitesimals (CPNAHI). In light of the breadth of this research versus the readability of an introductory paper, I have decided to follow Torricelli's example and try to err on the side of simplicity. His known simplifications have proven to be more effective for the initial spread of ideas than a dense but unread "Geometria Indivisibilibus".

5 Geometry: The Calculus of Homogeneous Infinitesimals

5.1 Historical Analysis of Torricelli's Parallelogram

In order to give background to an uninitiated reader, I could recreate the historical explanation for Torricelli's parallelogram involving area and area of lines[18] but it isn't necessary to get to the crux of his philosophy. Condensing this example[19] and adding in a bit more notation:

Assume that a line is made of points and that the number of points in a line determine the length. Two lines that are of the same length have the same number of points. A shorter line has less points and a longer line has more points.

Now take a parallelogram with the four corner points labeled A,B,C, and D. Draw a line BD down the diagonal of it as shown in Figure 1. Let us make a point E on the diagonal line BD. Now draw perpendicular lines from E to a point F on AD, and a second line to a point G on CD. Move these two lines point by point simultaneously so that E moves toward D until they meet, keeping the lines EF and EG always parallel to AB and BC respectively. When we move the lines EF and EG, we are moving their ends simultaneously from point to point on AD, CD and BD.

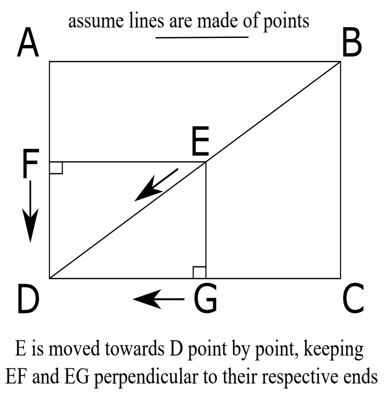

Fig. 1 Torricelli's parallelogram paradox

[18] some of the arguments Torricelli made are how lines logically seemed to have non-zero width but hopefully my argument makes the resolution of all those trivial

[19] it may be possible that I read a notationless explanation similar to the one presented here. If so, I am unable to find it again and my apologies to that author.

Since line AD is shorter than the line CD, the number of points that the line AD contains is less than the number of points that line CD contains. However, this creates a paradox. Since we moved the lines point by point and with both points F and G ending up together at point D then this shows that lines AD and CD must also have the same number of points as shown in the equations in Fig. 2.

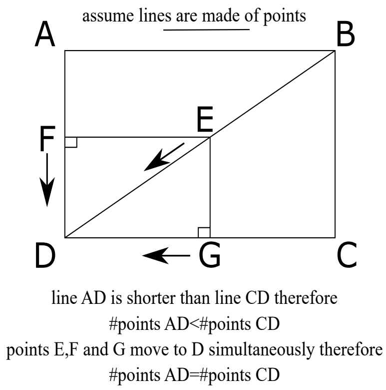

Fig. 2 Torricelli's parallelogram paradox

Torricelli's meaning was that the lines AD and CD must be made up of infinitesimal segments (and not dimensionless points) and that these segments must consist of the same number in each line even if they are not of the same magnitude.

The current most advanced explanation for this paradox is said to be the difference between cardinality and magnitude[20]. While I very much agree with the presence of these fundamental properties, let us introduce some notation to gain further insight.

Specifically avoiding Leibniz's notation I need something basic to designate that I am referring to the cardinal number of segments $\#SEG$ so that I can write

$$\#SEG_{AD} = \#SEG_{CD}. \tag{6}$$

Expressing that the magnitudes of these segments thus cannot be equal I write

$$|SEG_{AD}| \neq |SEG_{CD}|. \tag{7}$$

While this can and has been said to have lead to the development of the Calculus, let us dig further.

5.2 Homogeneous Infinitesimal Magnitudes: Their Sums, Differences, Strains and Relative Cardinality

Assume that the relative length of a line is defined by the sum of the magnitudes of the segments within it (i.e. as opposed to Bernhard Riemann's definition [7] that the length of every line is "measurable by every other line" which includes no mention of the infinitesimals of which it is composed)

[20] see [4] p. 125

$$\sum |SEG_{AD}| \equiv \mathbf{length}_{AD}. \tag{8}$$

Now let us define that the magnitude of any segment *within the same line* can be compared to the magnitude of any other segment such that I can write the equation or inequality of one segment n vs an adjacent segment $n-1$

$$|SEG^{n}_{AD}| - |SEG^{n-1}_{AD}| = 0 : \text{intrinsically flat} \tag{9}$$

which I will call *intrinsically flat* and

$$|SEG^{n}_{AD}| - |SEG^{n-1}_{AD}| \neq 0 : \text{intrinsically curved} \tag{10}$$

which I will call *intrinsically curved. I am unable to find an equivalent definition for curvature within any theory since the advent of the infinitesimal or indivisible concept.* While the most immediate subject to discuss concerning these equations would be the Archimedean axiom[21] (and perhaps Bernhard Riemann's definition of curvature and flatness [7]), let us hold off for a bit.

Just to enhance clarification, let us bring in line BD into our consideration also. Figures 3,4 and 5 are a visual aid for understanding the previous two equations. If by the property of congruence we can lay the lines BD, CD and AD next to each other and they are of unequal length, let us then imagine that we can use the vertical dividing lines to help denote the segments within each line. Torricelli's example is represented by Fig. 3 so that we could understand that the magnitudes of the segments within BD are all the same (intrinsically flat). The same for CD and AD. However, the magnitudes of the segments within BD must not be the same as CD, nor AD (Again, this is what Torricelli meant when he said that points are indistinguishable whereas segments can differ by their magnitude). We can then also understand that the cardinality (this can be thought of as the "number" for now) within AD must be the same as BD as well as for CD.

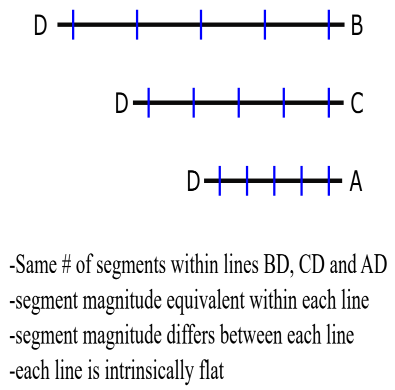

Fig. 3 Intrinsically Flat Lines With Equal Cardinality

[21]see [4] p. 50 for discussion of Euclid, Elements, book V, definition IV

5.2.1 Relative Strain

Although strain is normally considered a physical concept, we will consider it first as a geometric one. Figure 3 is an example of relative strain ϵ_{rel} if we consider the concept that line AD is being stretched out to the length of line BD but our measurement system is *also* being stretched out, relative to each other. If our measurement system is another line AD stretching to BD then the cardinality stays the same relative to our measurement system.

5.3 Torricelli's Parallelogram Theorem

Let us refer to his parallelogram setup as Torricelli's Parallelogram Theorem (as it will become within a new axiomatic framework) so that I can assign these equations as a description of it. I use the term "parallelogram" instead of "rectangle" since his Italian use of "parallelogrammo" seems to translate to the former.

$$\#\text{segments}_{BD} = \#\text{segments}_{CD} = \#\text{segments}_{AD} \tag{11}$$

$$|SEG^{n}_{BD}| - |SEG^{n-1}_{BD}| = 0 \tag{12}$$

$$|SEG^{n}_{CD}| - |SEG^{n-1}_{CD}| = 0 \tag{13}$$

$$|SEG^{n}_{AD}| - |SEG^{n-1}_{AD}| = 0 \tag{14}$$

$$|SEG_{BD}| > |SEG_{CD}| > |SEG_{AD}| \tag{15}$$

5.4 First Corollary to Torricelli's Theorem

However, this also means that another way to compare lines (which are not representative of Torricelli's example) would be to set the magnitudes of all the segments within the lines equivalent not only within the lines (intrinsically flat) but between the lines also. The longer the line is, the more segments it has (as opposed to dimensionless points) as in Figure 4.

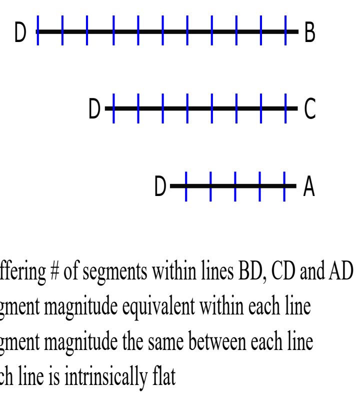

Fig. 4 Intrinsically Flat Lines With Differing Cardinality

I assign these equations as a description of the First Corollary:

$$\#\text{segments}_{BD} > \#\text{segments}_{CD} > \#\text{segments}_{AD} \tag{16}$$

$$|SEG_{BD}^{n}| - |SEG_{BD}^{n-1}| = 0 \tag{17}$$
$$|SEG_{CD}^{n}| - |SEG_{CD}^{n-1}| = 0 \tag{18}$$
$$|SEG_{AD}^{n}| - |SEG_{AD}^{n-1}| = 0 \tag{19}$$

$$|SEG_{BD}| = |SEG_{CD}| = |SEG_{AD}| \tag{20}$$

5.4.1 Absolute Strain

Figure 4 is an example of absolute strain ϵ_{abs} if we consider the concept that line AD is being stretched out to the length of line BD but our measurement system is another AD that is *not* being stretched out. The magnitudes of the segments stay the same but the cardinality increases relative to our measurement system line. Thus *relative* strain is from *changing magnitude* and *absolute* strain is from *changing cardinality.*

5.5 Second Corollary to Torricelli's Theorem

Likewise, we could also set the magnitudes of the segments to differ *within each line* (intrinsically curved). It may be helpful to imagine that the line is either compressed or stretched out in different places and to understand my hypothesis is that Riemann's example of areas that are stretched out [7] can be derived from intrinsic curvature. I do not bother with cardinality in this example as it may obfuscate the simple meaning I am attempting to demonstrate (although it is no less important). See Figure 5.

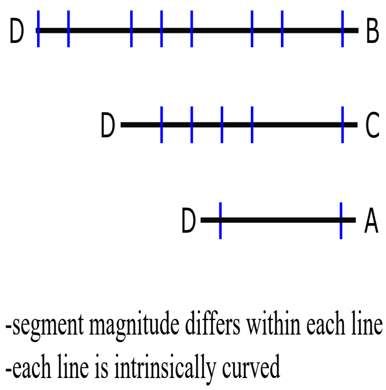

Fig. 5 Intrinsically Curved Lines

$$\#\text{segments}_{BD} <=> \#\text{segments}_{CD} <=> \#\text{segments}_{AD} \tag{21}$$
$$|SEG_{BD}^{n}| - |SEG_{BD}^{n-1}| <=> 0 \tag{22}$$

$$|SEG^n_{CD}| - |SEG^{n-1}_{CD}| <=> 0 \tag{23}$$
$$|SEG^n_{AD}| - |SEG^{n-1}_{AD}| <=> 0 \tag{24}$$

$$|SEG^{BD}| <> |SEG^{CD}| <> |SEG^{AD}| \tag{25}$$

Note that for the limited example here, it may be possible for the line to be flat between some HIs and curved between others, unlike the Theorem and First Corollary where they are always flat within the line.

5.6 Line Length: Sum of Segments and Relative Cardinality of Flat Lines

If I choose to build a line out of equal magnitude segments (flat), then I can represent Eqn. 8 as the number of segments or *Relative Cardinality* (RC),

$$\#SEG \equiv RC, \tag{26}$$

times the magnitude of a representative segment $|SEG|$ so that I can write

$$\sum |SEG| = RC * |SEG| \equiv \textbf{line length}. \tag{27}$$

Thus, in our normal sense of line length, if we wanted to compare the length of the three lines from Torricelli's Theorem we would have to define that each line is composed of segments of equal magnitude and that the line with the greatest Relative Cardinality is the longer line within Fig. 4.

5.6.1 Segment Notation

See Appendix Section D for my argument that Leibniz's notation $\frac{dy}{dx}$ can be derived from

$$\frac{\Delta RC_y}{\Delta RC_x} \equiv \frac{dy}{dx}. \tag{28}$$

In order to distinguish HIs, I borrow his notation and modify it to

$$|SEG| \equiv \overleftrightarrow{dx} \tag{29}$$

where the double ended arrow above dx indicates that it is indicating the *magnitude* of a HI.

In order to have notation to indicate that we are examining the difference of magnitude between two HIs, I introduce the notation in the equation

$$\overleftrightarrow{dx}_a - \overleftrightarrow{dx}_b \equiv \overset{\overleftrightarrow{\Delta}}{dx} \tag{30}$$

where the Δ combined with the double ended arrow indicates the difference of magnitude between two HIs. This allows us to rewrite Eqn. 9 as

$$\overset{\overset{\leftrightarrow}{\Delta}}{dx} = 0 \tag{31}$$

for intrinsically flat and Eqn. 10 as

$$\overset{\overset{\leftrightarrow}{\Delta}}{dx} \neq 0 \tag{32}$$

for intrinsically curved.

5.6.2 Archimedean Axiom

It has been said that infinitesimals do not follow the Archimedean axiom[22]. As a counterargument, I am going to falsify the following statement (bold mine)[23]:

> it shows that for any o infinitesimal relative to a, the ratio $a + o : a$ determines the same upper set as $a : a$ but differs from the latter in having an empty "middle set";**since the quantities no are obviously all infinitesimal in relation to a**

Let us define the real line by a combination of RC and segment magnitude $|SEG|$ and rewrite a definition given in Ref.[13]:

Assume segment magnitudes, $[o, p]$, are said to have a ratio with respect to one another and Relative Cardinalities, $[n, m]$, are said to have a ratio with respect to one another. In the simplifying case of $o = p$, no is capable of exceeding mp. We can easily set the inequality to $no > mp$ and choose $m = 1$ where $n > 1$ and $mp = a$.

Stating that any number n times an infinitesimal o cannot be greater than a, $no > a$, is simply a demonstration of the confusion of trying to compare RC, magnitudes and line length with poor definitions. These are three different types of proportion. Eudoxus' Theory of Proportions must apply to cardinality n, homogeneous infinitesimals o AND homogeneous sums such as line length a, areas, volume etc.[24]. Otherwise, is a supposed to be the length of a line, magnitude of an infinitesimal or a cardinal number? Is n modifying the magnitude of o or is it the cardinality of segments that are of o infinitesimal length that sum up to equal a line length that is greater than the line length a?

The modern mathematical notation[13] of

$$(\forall \epsilon > 0)(\exists n \in \mathbb{N})[n\epsilon > 1] \tag{33}$$

or

$$(\exists \epsilon > 0)(\forall n \in \mathbb{N})[\epsilon \leq \frac{1}{n}] \tag{34}$$

appears to not have been helpful to others in understanding the distinction between a segment magnitude ϵ, a cardinal number n and a line length of 1. It also does not

[22] see [11] for discussion
[23] see p.171 [12]
[24] I may refer to this as the Theorem of Homogeneous Infinitesimal Relativity in future papers

appear to have led to the concept that infinite sets can have differing cardinality similar to finite sets nor that infinitesimals ϵ can have different magnitudes and multiple directions.

5.6.3 L'Hôpital's Paradox and non-standard analysis

As a proof, let us use the definition for line length to resolve L'Hôpital's paradox[25] and non-standard analysis. L'Hôpital's paradox is given by the equation

$$X + dx = X \tag{35}$$

where X is a non-infinitesimal line segment and dx is an infinitesimal segment. We can instead write

$$X = X \tag{36}$$

becomes

$$RC_1 \overset{\bowtie}{dx}_1 = RC_2 \overset{\bowtie}{dx}_2 \tag{37}$$

with

$$RC_1 = RC_2 \tag{38}$$

and

$$\overset{\bowtie}{dx}_1 = \overset{\bowtie}{dx}_2. \tag{39}$$

If for every $\overset{\bowtie}{dx}_1$ that is added to the left side of Eqn.37, increasing the Relative Cardinality in comparison to the right side, the magnitude of the left hand side segments are also correspondingly reduced. I can then write

$$RC_1 \overset{\bowtie}{dx}_1 + 1\overset{\bowtie}{dx}_1 = RC_2 \overset{\bowtie}{dx}_2 \tag{40}$$

which gives

$$\frac{RC_1 + 1}{RC_2} = \frac{\overset{\bowtie}{dx}_2}{\overset{\bowtie}{dx}_1}. \tag{41}$$

This proves that ratios of relative cardinalities can be algebraically compared to ratios of homogeneous infinitesimals in accordance with the theory of proportions.

For non-standard analysis, "Two hyperreal numbers are infinitely close if their difference is an infinitesimal"[11] can be proven if we do *not* resize Relative Cardinality or $\overset{\bowtie}{dx}_1$. Eqn.37 allows Eqn.40 to be rewritten as

$$RC_2 \overset{\bowtie}{dx}_2 - RC_1 \overset{\bowtie}{dx}_1 = 1\overset{\bowtie}{dx}_1. \tag{42}$$

Regardless of this paper, note that non-standard analysis and the above statement that infinitesimals are non-Archimedean are saying the exact opposite thing. **They cannot both be correct.**

[25] see [4] p. 13 for discussion

5.7 Pandora's Box of HIs

With all of this we can understand that Torricelli's actual argument was that he was choosing to define and examine the lines as having equivalent RC with differing segment magnitude[26].

While this hopefully seems to be a fairly simple analysis, I also view it as opening Pandora's box as it casts suspicion on the derivation of everything[27] that is based upon infinitesimals. Although it may not be obvious yet, I am not only saying that the $\frac{dy}{dx}$ of Leibniz's notation can be defined as $\frac{\Delta RC_y}{\Delta RC_x} \equiv \frac{dy}{dx}$ but also that infinity can have different magnitudes (I hypothesize that Cantor's transfinite numbers [14] can be derived from this), that Hilbert's "betweenness" postulate, basis vectors[28] and even tensors are all flawed representations of the magnitude of HIs. This path could obviously meander forever so instead let us restart by enlisting the aid of primitive notions within an axiomatic framework.

6 CPNAHI: The Calculus, Philosophy and Notation of Axiomatic Homogeneous Infinitesimals

As one author states[15], axioms should be given without justification. However, I see no way to simply launch into a proof using CPNAHI and feel I must include a brief defense concerning the structure of this paper. I know of nothing within the body of knowledge of geometry nor mathematics[29] that already contains analysis of both Torricell's work and non-Euclidean geometry from which to launch my hypotheses. I instead will attempt to justify my primitive notions and postulates and list similarities between these and certain topics to help motivate individuals to my view that CPNAHI has compelling features. I fear letting the perfect be the enemy of the good.

I have chosen the term CPNAHI because of my view that while *Notation* provides economy of thought, my actual equations are of geometric concepts such as summation, differences, ratios, etc. of HIs (*Calculus*) equated to *Philosophical* concepts such as money, population, space, time, force, velocity, change in wavelength, change in clock rate, strain, pressure, momentum, density and change in density, energy, *ad infinitum*. In simpler words, HIs are a *language*.[30] It is also my view that if the notation giving us economy of thought does not properly represent the underlying geometry, then the philosophical interpretations will be of poor and misleading value.

6.1 Primitive Notion and Postulates

CPNAHI Primitive Notion

Let a homogeneous infinitesimal (HI) be a primitive notion.

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[26] As we will see, later in this paper, there is another way to view the actual area of the rectangles based only on RC

[27] see [11] for examples just viewed through the lense of Calculus alone

[28] see [8] p. 52 and p. 229 for examples

[29] nor alternate axiom systems. See Chapter 15 [16]

[30] it may be informative if you consider the hypothetical argument between a "hard-nosed physicist" and a "hard-nosed mathematician" [8] p. 230 if neither views geometry as a language

CPNAHI Postulates

1. HIs can have the property of length, area, volume etc.[31] but have no shape[32]
2. HIs can be adjacent or non-adjacent to other HIs
3. a set of HIs can be a closed set
4. a lineal line is defined as a closed set of adjacent HIs (path) with the property of length. These HIs have one direction.
5. an areal line is defined as a closed set of adjacent HIs (path) with the property of area. These HIs possess two orthogonal directions.
6. a voluminal line is defined as a closed set of adjacent HIs (path) with the property of volume. These HIs possess three orthogonal directions.
7. the cardinality of these sets is infinite
8. the cardinality of these sets can be relatively less than, equal to or greater than the cardinality of another set and is called Relative Cardinality (RC)
9. Postulate of HI proportionality: RC, HI magnitude and the sum each follow Eudoxus' theory of proportion.
10. the magnitudes of a HI can be relatively less than, equal to or the same as another HI
11. the magnitude of a HI can be null
12. if the HI within a line is of the same magnitude as the corresponding adjacent HI, then that HI is intrinsically flat relative to the corresponding HI
13. if the HI within a line is of a magnitude other than equal to or null as the corresponding adjacent HI, then that HI is intrinsically curved relative to the corresponding HI
14. a HI that is of null magnitude in the same direction as a path is defined as a point
15. the coordinates of any coordinate system are the Relative Cardinality of Homogeneous Infinitesimals.

7 CPNAHI Lines

7.1 Lineal Lines

A lineal line is defined as a path consisting of lineal HIs (i.e the segments). See Appendix Section A for a discussion and graphical aids in understanding a lineal line. A point in a lineal line is defined as a HI null in the direction of the line. There are no other directions. If each adjacent non-null HI is of equivalent magnitude then the line is intrinsically flat. If an adjacent non-null HI is of differing magnitude then the line is intrinsically curved.

Since one-dimensional $\mathbb{R}^1$ can be functionally identical to a lineal line, this is why Torricelli's example works when examining the line segments comparatively. Although his parallelogram possessed area, geometrically he was simply examining one of the issues that arise from not defining relative cardinality and magnitudes. Lineal line points are similar to points within $\mathbb{R}$ since both have null infinitesimal magnitude. See

[31] this is also in accordance with Eudoxus' theory of proportions which I view as equivalent to not being possible to sum heterogeneous infinitesimals

[32] no "protruding parts" to paraphrase Descartes. See [4] p. 169 for discussion

Lineal Line Pseudo-Point Geometry, Appendix Section C for a discussion of $\mathbb{R}^n$ for $n \geq 2$.

7.2 Areal Lines

An areal line is defined as a path consisting of areal HIs. See Appendix Section B for a discussion and graphical aids in understanding an areal line. A point in an areal line is null in the direction of the line but is non-null in the orthogonal direction. If an adjacent non-null HI is of equivalent magnitude in the direction of the line, it is intrinsically flat. If an adjacent HI is of differing non-null magnitude then the line is intrinsically curved.

7.3 Voluminal Lines

A voluminal line is defined as a path consisting of voluminal HIs. Figure 6 is a graphical aid in understanding a voluminal line. A point in a voluminal line is null in the direction of the line (z) but is non-null in the two orthogonal directions (x and y). If an adjacent HI is of equivalent magnitude in the direction of the line, it is intrinsically flat. If an adjacent HI is of differing non-null magnitude then the line is intrinsically curved.

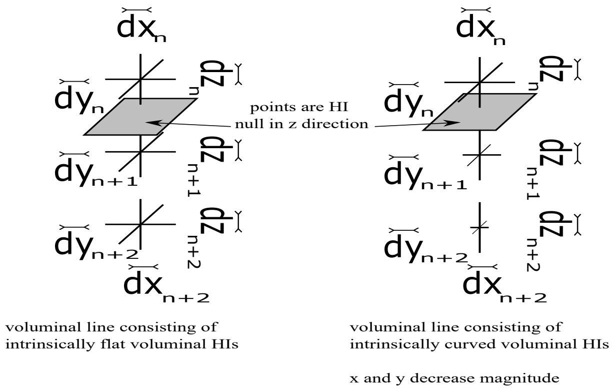

Fig. 6 Intrinsically Straight and Curved Voluminal Lines

In Figure 6, the image on the left is intrinsically flat so that we can write

$$\overleftrightarrow{dx}_{\,n} - \overleftrightarrow{dx}_{\,n+1} = \overset{\Delta}{dx} = 0, \tag{43}$$

$$\overleftrightarrow{dy}_{\,n} - \overleftrightarrow{dy}_{\,n+1} = \overset{\Delta}{dy} = 0, \tag{44}$$

and

$$\overleftrightarrow{dz}_{\,1} - \overleftrightarrow{dz}_{\,n+1} = \overset{\Delta}{dz} = 0. \tag{45}$$

The one on the right is intrinsically curved in the orthogonal directions. As we examine down in the z direction we write

$$\overset{\bigtriangleup}{dx} <> 0, \tag{46}$$

$$\overset{\bigtriangleup}{dy} <> 0, \tag{47}$$

and

$$\overset{\bigtriangleup}{dz} = 0. \tag{48}$$

In both we have a voluminal point represented by the shaded parallelogram. It is null in the z direction. I could also show a voluminal line that is intrinsically curved in the z direction but that isn't necessary for the similarity we are examining[33].

I find it has the same properties as the constituent of a sphere as areal lines do when they constitute a circle (see App. Sec. B). If the points of adjacent voluminal lines make up the "surface" of a sphere then each concentric surface would have the same cardinality as the radius increases because the points now have the property of area and that area increases as the radius increases.[34]

With this understanding that voluminal lines can be intrinsically curved not only in the direction of the line but also with respect to the orthogonal directions of their HIs, let us examine the properties of a set of these as compared to Gaussian curvature.

7.3.1 Similarity Between Principal Curvature of Points on a Surface and Intrinsic Curvature Across Points Within Voluminal Lines

There are four basic types of points on surfaces within Gaussian curvature: elliptic, hyperbolic, parabolic and planar. If we examine the principal curvature K properties for each type of point, we can see a pattern in Table 1 that matches the orthogonal intrinsic curvature of voluminal lines within CPNAHI: I propose that Gaussian curvature is a flawed view of voluminal lines.[35]

Table 1 Gaussian Principal Curvature Vs Voluminal Line Orthogonal Direction Intrinsic Curvature

	principal curvature K_1 and K_2	intrinsic curvature of $\overset{\bigtriangleup}{dx}$ and $\overset{\bigtriangleup}{dy}$
elliptic	same sign	same sign
hyperbolic	opposite sign	opposite sign
parabolic	one is 0 and other is pos or neg	one is flat and other is pos or neg
planar	both are 0	both are flat

[33] although examining z is necessary when deriving Newton's gravity as this is his flat background dr radius

[34] the affine connection of rolling a "Euclidean plane" along the surface would seem to be derivable from this definition

[35] it may also be helpful to view the similarity with Fig. 9.2 p.232 [8]

I am assuming for now that the need for the two infinitesimal terms $dx_\mu dx_\nu$ in GR is that these can be derived from $\overset{\asymp}{dx}$ and $\overset{\asymp}{dy}$ and that $\overset{\asymp}{dz}$ is considered to be the same magnitude[36].

8 Background and Foreground Geometry

The following images are a schematic attempt to relay a concept of the meaning of geometry as a language. They are not meant to be quantitative, only descriptive. I use a single line to aid in this introduction as a voluminal line would be difficult for me to draw with my limited skills and isn't quite necessary for this introduction. For a flat background, this line would be the line radiating out from the center (the radius of Newton's equations). For CPNAHI, these lines are the same as the z direction within Figure 6.

8.1 Background Geometry: Flat HIs

Let us consider a concept of lines and what I call "background" geometry. In Figure 7 I have a comparison. The three lines give three separate concepts of flat lines or space: made of equidistant points, made of basis vectors and made of lineal lines. A position, denoted by the black dot, can be defined on the line (coordinate system). Note that there are no real distinguishing features and seem to be similar.

8.1.1 Philosophical Equivalence of Background Flat HIs

Essentially any and all "field" potentials, such as with Newton's $\frac{d\phi}{dx}$, are just defined as using one direction of a flat HI to represent an infinitesimal distance dx in absolute space and another HI direction as an infinitesimal of field potential $d\phi$. Notationally,

$$\phi = RC_\phi * \overset{\asymp}{d\phi} \tag{49}$$

and

$$X = RC_x * \overset{\asymp}{dz} \tag{50}$$

but it is the change in relative cardinality

$$\frac{\Delta RC_\phi}{\Delta RC_z} \tag{51}$$

that is the equivalent of gravitational force[37].

8.2 Foreground Geometry: Curved and Flat HIs

Within CPNAHI, let us define a geometry as foreground. By this I mean that no "absolute objects" philosophically can exist. Only HIs that are either flat or curved. I can define a "geometrical object" as the maxima or minima of the intrinsic curvature.

[36] in a future paper I will explore if this is what is meant by a "Riemannian" metric
[37] In a future paper I will attempt to derive Maxwell's equations via CPNAHI.

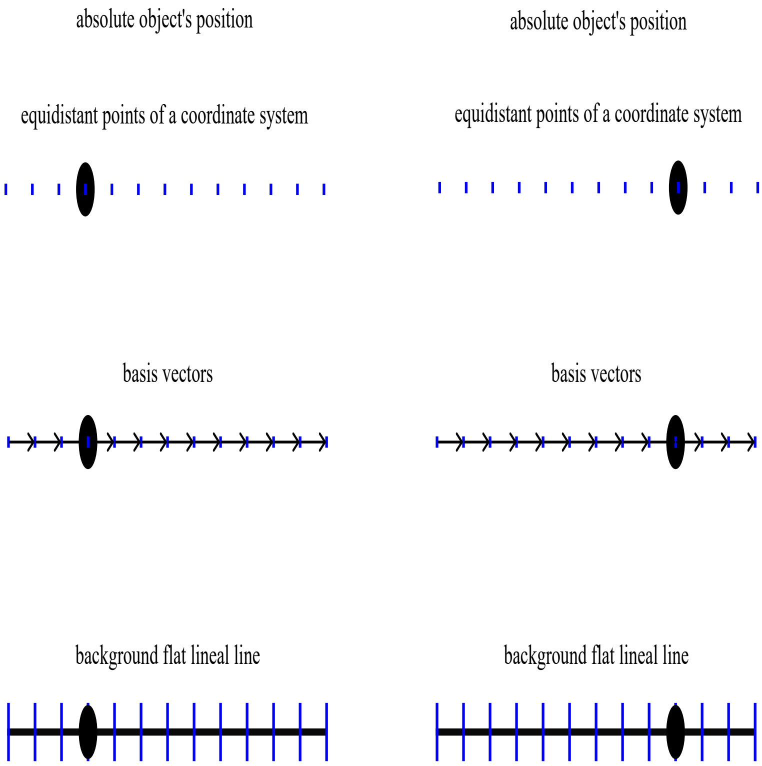

Fig. 7 Schematic comparison of background lines

These maxima and minima can be thought of as geometric waves. I do not yet know whether a position can be defined although cardinality of the maxima/minima would seem to be a good place to start.

8.2.1 HI functions in Curved and Flat foreground HIs

Let another HI, not adjacent to the foreground HIs, exist as a HI function of the foreground HIs. By this I mean:

$$\overset{\triangle}{dy} = f(\overset{\triangle}{dx}_{\text{foreground}}). \quad (52)$$

The intrinsic curvature of $\overset{\triangle}{dy}$ is a function of the intrinsic curvature of $\overset{\triangle}{dx}$. If the intrinsic curvature of x is zero, then so too the intrinsic curvature of y. If the intrinsic curvature of x is not zero, then neither is the intrinsic curvature of y.

8.2.2 Philosophical Equivalence of Foreground HIs

Let us now bring in physical philosophy to HIs. Assume that we are going to equate $\overset{\triangle}{dy}$ to the change in wavelength of a test particle/wave $\overset{\triangle}{dx}_{wl}$. If $\overset{\triangle}{dx}_{wl} = 0$ then the wavelength is not changing (no length-contraction). Also assume that we are going to philosophically equate $\overset{\triangle}{dy}$ to the change in process time $\overset{\triangle}{dt}$ (i.e. half-life) of a particle/wave. If $\overset{\triangle}{dt} = 0$ then the radioactive decay periodicity is constant (no time dilation). Thus if $\overset{\triangle}{dx}_{\text{foreground}}$ is zero there is no time dilation nor length contraction but if it is not zero then the amount of time dilation and length contraction is a HI function of the intrinsic curvature of the foreground HIs.

8.3 Foreground Geometry Comparison with GR

The top two lines in the Figure 8 shows a basic concept for what I think General Relativity is describing: a mixture of geometry and physical philosophy. A physical "object" can affect a foreground geometry that is composed of points or basis vectors. It is shown having moved from left to right. This object affects the spacing of the points[38] or similarly the magnitude of the basis vectors. The points nearest to the object are closer together and more equidistant away from the object in the top lines. The magnitude of the distance between the points seems to be taken from the concept of basis vectors, in that the magnitude equals "1". As the points get closer, the value becomes less than 1. If the points get close enough, they overlap and the value becomes 0. In the middle line, the basis vectors are smaller nearer the object but become larger and of constant magnitude away from the object.

The bottom line is a lineal foreground line within CPNAHI. There is no "physical object" as this is geometry only. The object is defined as where the HIs reach minimum magnitude. Away from the object, the magnitudes become equivalent (intrinsically flat). I term this local relative strain ϵ_{rel} *local curvature.* (A side note in that magnitudes getting larger instead of smaller could be a more consistent and understandable framework.)

We could analyze this concept against that of a "metric"[39] and tensors but this would seem to be far too intense for an introductory paper without prior acceptance of my primitive notion and postulates.

9 Schematic comparison of perfect fluid model within GR and elastic material model within CPNAHI

On the left and right hand side of figure 9 is a conceptual schematic comparison between what I hypothesize GR is attempting to physically model versus a philosophical model of an elastic medium being equated to HIs. I say attempt because the Cosmological Constant Λ is included. The terms in the left graph with question marks are from the Report of the Dark Energy Task Force[1].

On the left side, as a beginner explanation, for GR the physical object is defined as having energy density that is analogous to the particle density ρ of a perfect fluid[40].

[38] see [8] p. 23 for discussion
[39] see Chapter 13[8]
[40] see Box 5.1 [8]

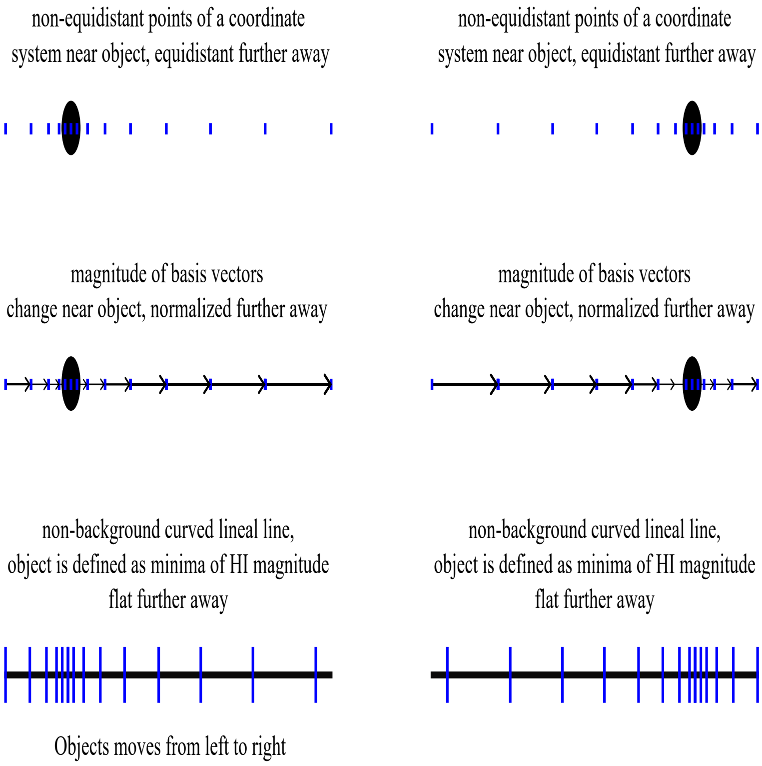

Fig. 8 Schematic comparison of points, basis vectors and lineal HIs in foreground geometry

It also is defined as having momentum that is analogous to the pressure p of a perfect fluid. The farther away from the object, the limiting value of the distance between the points is "1". As the ρ of the object increases, the smaller the distance between the points that are closest to the object, becoming less than 1. If the density increases enough (or "volume"[41] decreases enough.), the distance between the points will go to zero and thus they will overlap. The overlap would seem to correspond to the Schwarzschild radius

$$1 - 2\phi = 0 \tag{53}$$

where

$$\phi = .5. \tag{54}$$

It also has the difficulty in that away from this object (that is using the perfect fluid analogy), there was supposed to be nothing other than the equidistant points with a spacing of 1 (or basis vectors of magnitude 1). Instead there now seems to be theoretical attempts to also place a "dynamical fluid"[1][42] in the regions away from

[41] I put quotation marks because this volume is based on equidistant points. In other words, the distance between the points surrounding the energy density decrease but the points that define the spatial volume of the energy density are constant. In CPNAHI terms, changing absolute strain would change the surrounding relative strain. I find this and any other "constants" that rely solely upon absolute strain definitions as dubious.

[42] searching Arxiv.org returned 2,495 results for "quintessence", 1,832 results for "dark energy fluid", 1,232 for "Cosmological Constant fluid" on 2024-08-18

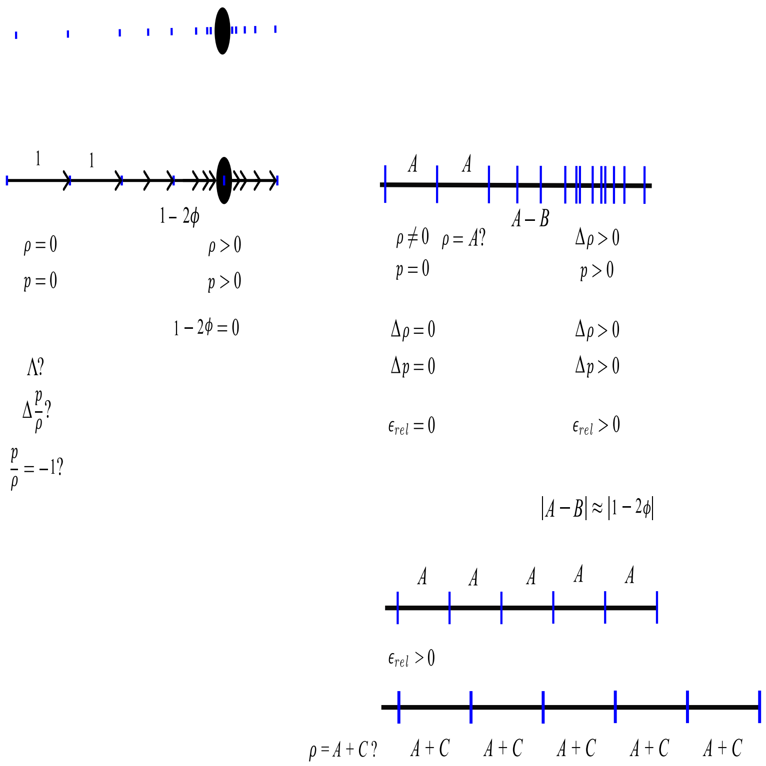

Fig. 9 Schematic comparison of perfect fluid model within GR and elastic material model within CPNAHI

the object where nothing was and to represent this value using the Greek letter Λ which represents a "scalar multiple of the metric $g_{\mu\nu}$".

On the right hand side I have a lineal line within CPNAHI. *Equated* to this geometric line is the physical model of an elastic medium. The flat HIs represent an *unstrained* ($\epsilon_{rel} = 0$) elastic medium. Where the HIs reduce in magnitude I have geometric strain of the medium that increases ($\epsilon_{rel} > 0$). Near the minima for the HI magnitude, I model this as the greatest change in density of the elastic medium. At this stage in my research, I am currently assuming that I can still model this area as having the greatest pressure (momentum) and not greatest change in Δp. I have more research to do in order to ensure which is the most logical representations of the properties of fluid models. Geometrically, I define another HI that is a HI function (see Section E) to the magnitudes of the HIs within the line. Philosophically I overlap this functional HI as a representation of the magnitude of temporal process and the relative magnitude of wavelength. The larger the intrinsic curvature within the line, the larger the change in the temporal process and change in wavelength. Notationally, I mean the HI function

$$\overset{\times\!\Delta\!\times}{dt}_{\;tp} = f(\overset{\times\!\Delta\!\times}{dx}_{\;\text{medium}}) \tag{55}$$

for change of temporal process (time dilation) and

$$\overset{\triangle}{dx}_{wl} = f(\overset{\triangle}{dx}_{\text{medium}}) \tag{56}$$

for change in wavelength (length contraction). The greater the strain in the medium $\overset{\triangle}{dx}_{\text{medium}}$, the greater the change in the temporal process and wavelength. Thus, GR uses tensors where "time" is a coordinate system on equal footing with a three dimensional coordinate system of space. It views the change in the spacing of the points of this coordinate system as a "curvature" of space-time. Geometrically, CPNAHI uses HI functions of a voluminal HIs that are summed to create and define volume. Philosophically CPNAHI models time dilation and length contractions as HI functions of a three directional relative strain of an elastic medium representing the volume. Although relativity considers the speed of light constant, I find it very doubtful that $\frac{\overset{\triangle}{dx}_{wl}}{\overset{\triangle}{dt}_{tp}}$ actually is and have a hypothetical experiment that could confirm this. I find it more compelling to define $\frac{\overset{\vdash\dashv}{dx}}{\overset{\vdash\dashv}{dt}}$ as a relative flex rate of the medium where the ratio $\frac{\overset{\triangle}{dx}}{\overset{\triangle}{dt}}$ is an approximately constant HI function of the non-zero strain $\overset{\triangle}{dx}_{\text{medium}}$. I will expand on this in a future paper.

As for the equation $|\Lambda - \Lambda 2\phi| \approx |1 - 2\phi|$, this will also require a future paper to determine the most correct philosophical distinction between whether this is

$$\overset{\vdash\dashv}{dx}_{\text{medium}} = \Lambda = \rho_{vac} \tag{57}$$

and

$$\overset{\triangle}{dx}_{\text{medium}} = \Lambda - \Lambda 2\phi = \Delta\rho_{vac} = \textbf{energy density} \tag{58}$$

or whether

$$\overset{\triangle}{dx}_{\text{medium}} = \textbf{strain}. \tag{59}$$

Looking at the bottom line on the right side, I also see no reason that the lineal line itself couldn't have a symmetric change in magnitudes of the HIs. I call this *universal curvature*, meaning that there is a overall change in magnitude but it remains locally flat everywhere. This is shown in the figure as the magnitude changing from A to $A+C$. If I model this as a change in the density ρ_{vac}, then this would seem conceptually very close to the change in state of a dynamic fluid as discussed by the DETF [1]. As with local curvature, a change in wavelength would also result from universal curvature. I view it as probable that the intractability of the GR view of Λ as a scalar multiple of the metric $g_{\mu\nu}$ is the inability to conceptually equate these.

Thus for the sake of simple comparison, I consider it logical and probable to write

$$\Lambda \equiv \rho_{vac} \tag{60}$$

and that if the intrinsic curvature of z is very small then it is almost flat and the spherical approximation of the right side of Fig. B5 can be approximated to the left side giving us

$$\overset{\bowtie}{dz}^{\text{foreground}} = \Delta\Lambda_z = \Delta\rho_{vac_z} \approx ||\Lambda_z - \Lambda 2\phi_z|| \approx ||\frac{\Delta RC_{\phi}^{\text{background}}}{\Delta RC_{r}^{\text{background}}}|| \equiv \frac{d\phi_{Newton}}{dr} \quad (61)$$

with the z being the direction of the voluminal line in Fig.6 and r as the radius of a spherical coordinate system in LLPP. I propose this equation to represent an approximation for gravitational force[43]. I propose that gravitational time dilation and length contractions are HI functions of $\overset{\bowtie}{dz}^{\text{foreground}}$. If ρ_{vac} is not constant throughout the universe nor during its development, then CPNAHI would seem a more logical way forward in examining the Cosmological Constant problem.

10 Summary

This paper has introduced the homogeneous infinitesimal as an axiomatic primitive notion and viewing it through the concept of RC functions on background geometry and HI functions on foreground geometry. By this I mean that I have attempted:

1. to provide compelling evidence that Euclidean and non-Euclidean geometry, the Calculus, as well as our laws of physics are all based upon the undocumented properties of homogeneous infinitesimals within this axiomatic framework. Euclidean geometry and the Calculus would be based upon a flat HI background where lines are composed of LLPP geometry. Functions are defined based on the Relative Cardinality of these lines. Non-Euclidean geometry can be derived from flat and intrinsically curved foreground HIs where lines are lineal, areal, voluminal etc.
2. to show the background for future research on the distinction and similarities between a four component tensor and HI function that has one component as a function of another three.
3. to show the importance of philosophy in that these geometrical HIs can be equated to distinct but similar concepts such as flat $\overset{\asymp}{dt}$ being equated to both the infinitesimal passage of time and to the lack of time dilation.
4. to elucidate my view that the Cosmological Constant paradox is the Gordian Knot that results from of the confusion between homogeneous infinitesimals and basis vectors/points on the geometric side and a perfect fluid model of particles to an inverted elastic medium wave model on the philosophical side.

10.1 Hypotheses

The following hypotheses will be proven or falsified in a future paper. These area included to aid in generating interest in CPNAHI.

[43]it would seem more logical at the moment to model $\Delta\rho_{vac}$ as causing ϵ_{rel} but I will require more time to study perfect fluid theory and a possible radial limit to the intrinsic curvature

1. Define CPNAHI Geometric Conservation (GC) as conservation of HI magnitude and/or Relative Cardinality
2. Conservation of energy and momentum within GR can be derived from GC
3. Covariance and Contravariance can be derived from GC
4. Events within MTW (Box 1.1) are equivalent to the points within HIs. One event is one point.
5. HIs do not have the boundary condition flaw inherent in basis vector concept.
6. 0-spheres are made of lineal lines, 1-spheres of areal lines and 2-spheres of voluminal lines. Non-null directions within the points make up the surface of the sphere. 0-spheres would have points of null direction, 1-spheres of length and 2-spheres of area.
7. Riemann's n-ply extended magnitude can be derived from a HI
8. The Calculus was initially developed utilizing the infinitesimal segment concept (i.e. Newton's fluxion and Leibniz's differential dx). Hypothesis: The Leibnizian concept of a derivative can be derived from the change of RC of lineal lines on a flat background. If the RC does not change, then the derivative is zero, if the RC does change then the derivative is non-zero.
9. Differential Geometry is the mathematical discipline that studies the geometry of smooth shapes and smooth spaces, otherwise known as smooth manifolds. Hypothesis: The derivative of a lineal line can be proven as examining the infinitesimal difference between magnitudes of lineal HIs. If the derivative difference is zero, then the line is intrinsically flat. If the derivative difference is non-zero then the line is intrinsically curved. Prove that differential geometry can be derived from this.
10. Euclid's Parallel Postulate Hypothesis: The parallel postulate can be defined from the change in RC between areal HIs bounded by LLPP lineal lines (using GC).
11. Coordinate systems utilize points. Heterogeneous lines use points. Hypothesis: The metric tensor utilized in General Relativity is actually a measure of the relative distance between points which is a flawed view of the magnitude of HIs.
12. The tensoral representation of co-moving with a particle in a Lorentz frame is given by

$$||\eta|| \equiv || \begin{Bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{Bmatrix} ||. \qquad (62)$$

    Hypothesis: Co-moving with periodic intrinsic curvature and using HI functions can approximate this tensor. The foreground intrinsic curvature is non-zero as a particle/wave passes, but the maxima and minima of HIs within a wave function as we co-move with it would be flat[44]. I will expand on the CPNAHI alternative to Special Relativity in a future paper.
13. George Cantor defined that a number could have a magnitude between finite and infinite called "transfinite" numbers [14]. Hypothesis: Relative Cardinality is the concept that infinity can have different magnitudes. Transfinite numbers are a conception of this.[45]

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[44] see [8] p. 53 for discussion on Lorentz frame
[45] as opposed to Galileo, see [4] pp. 92-93 for discussion

14. Hilbert's paper on his axioms[17] states

    > The axioms of this group define the idea expressed by the word "between,"

    Hypothesis: The "between" within Hilbert's axioms are a flawed view of the magnitude of a HI.
15. Bernhard Riemann described a surface as being stretched [7]. Hypothesis: Voluminal points that are adjacent to other voluminal points in the non-null direction can be thought of as creating a surface. If the magnitude of these non-null directions are equivalent, then HIs can be proven to be a more logically consistent description of flat. If they are not equivalent, then HIs can be proven to be a more logically consistent description of "stretching". I will attempt to show in future work that Riemann's paper matches the description of the properties of HIs.
16. Hypothesis: If foreground voluminal intrinsic curvature is defined as a magnitude with direction, this can be shown to change the directional components of a voluminal line which can be equated to the path of a wave function representing a physical particle. This can define and derive a gravitational "force" vector at that location. The changing of the components of the wave function in response to the foreground curvature can be equated to a change in wavelength, temporal process, momentum and energy density.
17. Partial derivatives of time and/or space are present within equations which describe quantum mechanics. Hypothesis: GR and Quantum mechanics can be written using one equation by rewriting both governing equations with partial derivatives of $\breve{dt}$ and $\breve{dx}$ as well as redefining energy density and momentum.

Appendix A CPNAHI lineal lines

Let us switch notation from x to y. Define a "line" as a set or "path" of adjacent HIs that possess length and an "intrinsically straight line" (Fig. A1) as a path of "adjacent" HIs with the property of

$$\breve{dy}_{n} - \breve{dy}_{n+1} = \overset{\Delta}{\breve{dy}} = 0. \tag{A1}$$

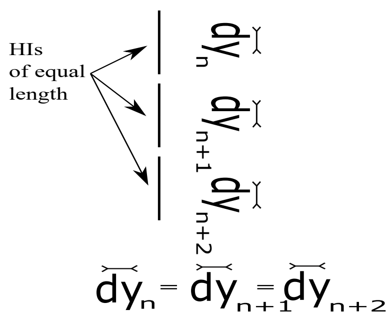

Fig. A1 Intrinsically straight lineal line made of equal HIs

An intrinsically curved line made of HIs of length (Fig. A2) could be represented by HI $\overset{\rightarrowtail\leftarrowtail}{dy}_{\,n}$ having greater infinitesimal length than HI $\overset{\rightarrowtail\leftarrowtail}{dy}_{\,n+1}$. This allows us to write the inequality

$$\overset{\rightarrowtail\leftarrowtail}{dy}_{\,n} > \overset{\rightarrowtail\leftarrowtail}{dy}_{\,n+1} \tag{A2}$$

which gives us

$$\overset{\rightarrowtail\leftarrowtail}{dy}_{\,n} - \overset{\rightarrowtail\leftarrowtail}{dy}_{\,n+1} = \overset{\overset{\Delta}{\rightarrowtail\leftarrowtail}}{dy} \neq 0. \tag{A3}$$

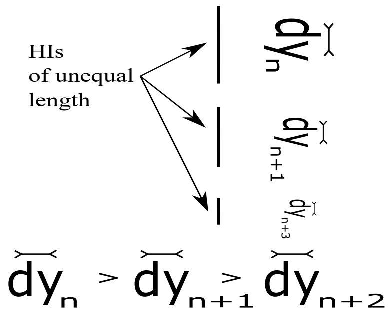

Fig. A2 Intrinsically curved lineal line made of unequal HIs

Appendix B Euclidean lines vs CPNAHI areal lines

Figure B3 demonstrates the conception of columns of intrinsically flat areal HIs that makeup areal lines that can be summed to create area. This is in opposition to the heterogeneous view of stacked lines of zero width being summed to create area.

B.1 Euclidean lines vs CPNAHI area lines for a circle

Figure B4 demonstrates that an areal line composed of HIs of area can define the radius of a circle[46]. Areal points, which are HIs that are null in the line direction (radial here), can be defined as forming the area and circumference of circles as in Figure B4. Every circle has the same number of points. The reason that the circumference grows as the radius increases is due to the increase in the relative magnitude of the HIs in the orthogonal direction[47]. Note that the circumference of every circle is intrinsically flat. A note in the margin of Torricelli's Opere essentially describes the properties of these areal lines ("tapering") as they make up a circle[48]. I hypothesize that a 1-sphere is the same as Figure B4.

Figure B5 is a conceptual comparison between multiple lines penetrating the circumferences of concentric circles versus single lines that penetrate and form the circumference of radial circles within CPNAHI. Understand that the dual "lines" in the

[46] also see [4] p. 28 Al-Ghazalis wheel
[47] Kepler viewed these as triangles. See [4] p. 62.
[48] see [4] p. 126 for discussion

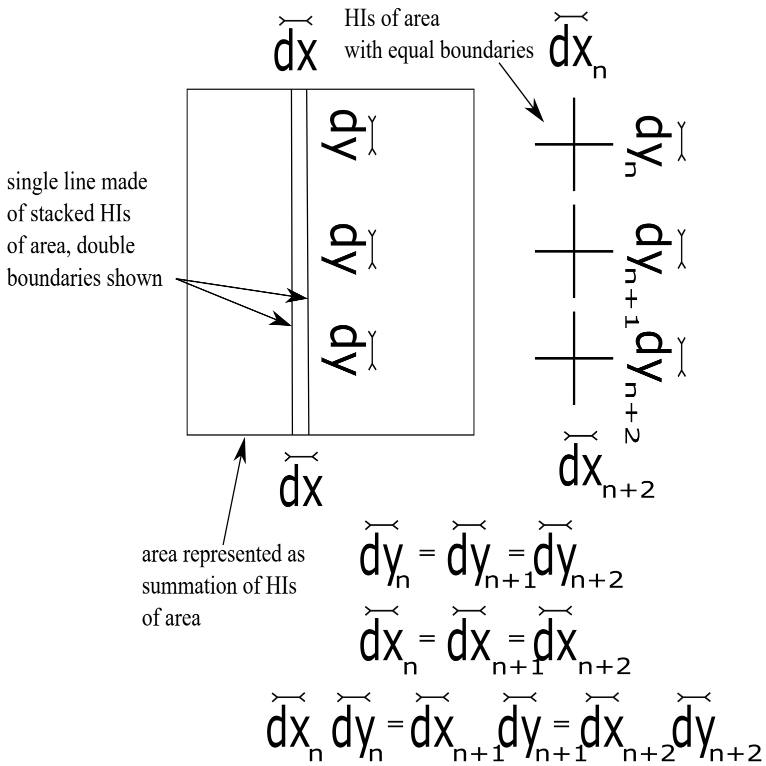

Fig. B3 Intrinsically Flat Areal Lines Summed To Create Area Of A Parallelogram

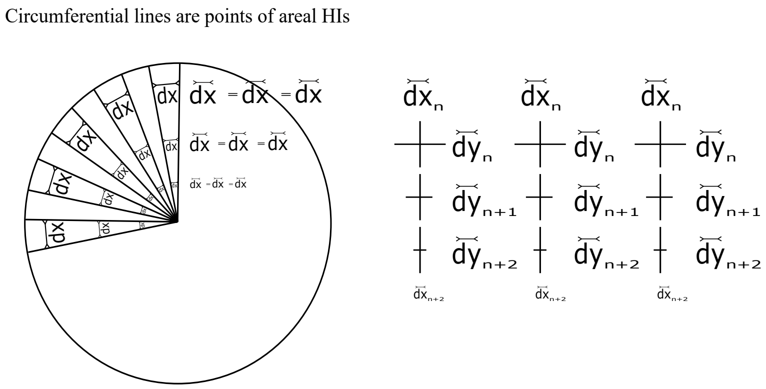

Fig. B4 Intrinsically Orthogonal-Curved Areal Lines Summed To Create Area Of A Circle

figure on the right indicating the increasing magnitudes of the HIs can be graphically misleading. Only a single line is represented in the figure on the right and it has the property of area. The lines radiating out from the figure on the left possess no width but only length.

It may also be helpful to understand we can view the figure on the left as a radial coordinate system of background geometry. An objects position can be described using this coordinate system. The figure on the right is made of HIs using foreground geometry. A geometric object can be define via changes in the magnitudes of the HIs (geometrical waves).

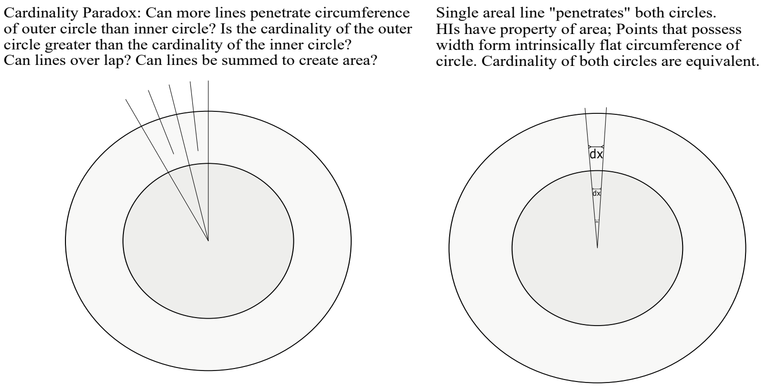

Fig. B5 Euclidean Circle Paradox Versus CPNAHI Areal Lines

B.2 Manifolds and 2-spheres vs CPNAHI voluminal point surfaces

By the same token, compare the following quote and the pictorial concept of rays emanating from a source within a differentiable manifold as shown in the 2-sphere from Figure 9.3 p. 241 in *Gravitation*[8]. Note the similarities of "rays" and voluminal lines. I hypothesize that a "2 sphere" is equivalent to saying the surface is made of the area of the voluminal points that are contained within the voluminal lines radiating as "rays" from the origin. Note that I do not show the z direction in these drawings that could also reduce in magnitude along with x and y.

Thus S^2 is a manifold, and the rays $\mathcal{P}$ are the points of S^2.

If one considers that the voluminal points that make up the area of the sphere and the points are within the voluminal lines, then there is a similarity between the "rays" and voluminal lines.

Appendix C Lineal Line PseudoPoint (LLPP) Geometry Versus Leibnizian Tangent

C.1 LLPP on Flat Background HIs

Let us consider a background geometry consisting of areal HIs. For this introduction we will consider only areal lines and that they are all flat. If we relax the point postulate for lines so that they are always null in all directions, then this is equivalent to points in Euclidean geometry and I call this a pseudopoint. We allow a lineal line to be drawn on this background so that any and all drawn lineal lines are flat and can now intersect at the null points. See Figure B5 for the issue this causes with cardinality. Let us call these special conditions Lineal Line PseudoPoint geometry (LLPP).

Philosophically, an "absolute object" can exist "on" this background and line lengths as defined by Equation 27 can be used to define a coordinate system. In other words, an objects position can be defined via the coordinate system created from the background areal HIs. Note that the second Corollary this does not apply because I intentionally analyzed the lines from their components comparatively without background areal HIs limiting our ability to change the magnitude of the segments.

Appendix D Areal Homogeneous Infinitesimals Versus Leibnizian/Newtonian Differentiation

In Figure D6 I have Keisler's example[9] of a tangent line drawn to a line $y = f(x)$. I purposefully use his example as I hypothesize that non-standard analysis can be derived from CPNAHI. Extrinsic curvature in LLPP[49] is defined as a change in length of an areal line segment with respect to two lines. This must be a change in RC since the background is flat. Any lines drawn upon this flat background are in and of themselves flat from segment to segment within the path of the line.

Hypothesis: This image can be described as using LLPP to describe a change in the RC of areal HIs.

In Figure D7 I have placed what look like the + signs in Keisler's image to indicate that the background consists solely of flat areal HIs. These signs represent the $\overset{\bowtie}{dy}$ and $\overset{\bowtie}{dx}$ of the background areal HIs. The drawn lines all use LLPP to exist. From Equation 27, we can see that within this diagram

$$\Delta y = RC_y \overset{\bowtie}{dy} \tag{D4}$$

and

$$\Delta x = RC_x \overset{\bowtie}{dx} \; . \tag{D5}$$

Since the background areal HIs, $\overset{\bowtie}{dy}$ and $\overset{\bowtie}{dx}$ are defined as flat then we know that

$$\overset{\bowtie}{dy} = \overset{\bowtie}{dx} \tag{D6}$$

and thus we get the equality of ratios

$$\frac{\Delta y}{\Delta x} = \frac{RC_y \overset{\bowtie}{dy}}{RC_x \overset{\bowtie}{dx}} . \tag{D7}$$

[49] which does not appear to be the same as MTW [8] 21.5 after a perfunctory examination

Δy = change in y along curve
dy = change in y along curve tangent line

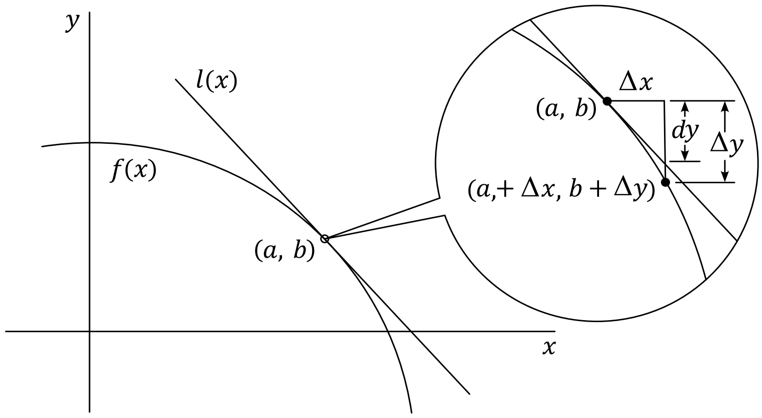

Fig. D6 Keisler differential Figure 2.2.3

Δy = change in y along curve
dy = change in y along tangent line

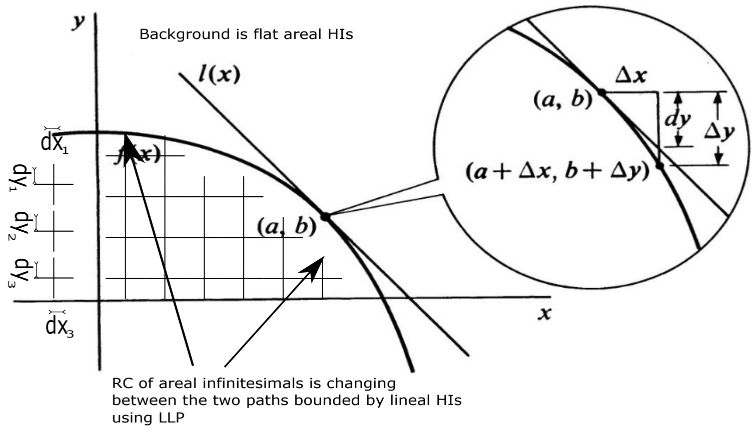

Fig. D7 Keisler Differential Explained Via LLPP

Note that Δx and Δy are line segment representing a change in a and b. So although I could write

$$\frac{\Delta RC_y}{\Delta RC_x} \equiv \frac{dy}{dx} \tag{D8}$$

it would be correct to also set $\Delta RC_x = 1$ and to view differentiation as the column of areal HIs that is ΔRC_y high.

We can then understand that the following non-CPNAHI notation is a special case of flat HIs where every segment is divided into equal segments:

$$\lim_{1 \to \infty} \Delta x_n = dx. \tag{D9}$$

By definition, on a flat background, any line with which you would compare the length with would have identical segment magnitude and thus the line length depends solely on RC.

D.1 Leibniz Notation, Integration, Fundamental Theorem of Calculus, Euclid's Parallel Postulate, Straight Voluminal Lines in a Curved Foreground

It may be obvious by now that basic integration can be derived as the summation of columns of areal HIs $RC_y \overset{\rtimes\!\ltimes}{dy}$ high by $RC_x \overset{\rtimes\!\ltimes}{dx}$ wide,

$$\sum RC_y \overset{\rtimes\!\ltimes}{dy}\, RC_x \overset{\rtimes\!\ltimes}{dx}\ . \tag{D10}$$

Since $RC_x = 1$ for a column and with the RC function $y = f(x)$[50], we can see the notational difficulties that necessitated new notation different from Leibniz since

$$\sum RC_y \overset{\rtimes\!\ltimes}{dy}\ \overset{\rtimes\!\ltimes}{dx}\ = \int f(x) dx. \tag{D11}$$

I do not put in here a discussion of the length of $y = RC_y \overset{\rtimes\!\ltimes}{dy}$ from the origin as a function of the length of $x = RC_x \overset{\rtimes\!\ltimes}{dx}$ from the origin as it should be obvious from the previous equations.

D.1.1 Fundamental Theorem of Calculus

Thus, within CPNAHI, we can understand the Fundamental Theorem of Calculus is defined by using LLPP on a flat background: As a simple explanation, for differentiation it is the change in cardinality of columns of areal HIs bounded between two lines. For integration it is the summation of columns of areal HIs bounded between two lines. Leibniz's notation seems to work because we are always comparing the Relative Cardinality of y with the Relative Cardinality of x but RC_x is defined as 1 in differential notation of $\frac{dy}{dx}$. This can easily be seen by changing a graph from x and y and using a Cartesian Coordinate plot instead.

[50] see [11] for a discussion of when Euler helped move the Calculus from being about variables of geometry to functions

D.1.2 Euclid's Parallel Postulate

Similarly, Euclid's parallel postulate can be defined through Geometric Conservation, in that two LLPP lines are parallel provided the Relative Cardinality of the column of flat background areal HIs that they bound between them does not change.

D.1.3 Straight Voluminal Lines in a Curved Foreground

In foreground geometry straight voluminal lines can be defined, using Geometric Conservation, by the magnitude of the three orthogonal components as shown in Fig. 6. Even if $\overset{\bowtie}{dy}_n$ and $\overset{\bowtie}{dx}_n$ are not equal in magnitude relative to each other but do not change relative to themselves, $\overset{\bowtie}{dy}_n = \overset{\bowtie}{dy}_{n+1}$ and $\overset{\bowtie}{dx}_n = \overset{\bowtie}{dx}_{n+1}$ along the voluminal line path, then this is how a line can be straight in a curved foreground. I will expand upon this and the similarities/differences between GR world lines in a future paper.

D.1.4 Torricelli's Theorem Within CPNAHI

Using the definition of a LLPP, we can understand that Torricelli's parallelogram paradox as historically examined is actually NOT a comparison of HI magnitudes but instead of differing RC for areal lines. My examination of it simply gives us a geometric scenario to a use and compare different values for both infinitesimal magnitudes and relative cardinality.

D.1.5 Conservation and Change of RC*HI

Without proof, allow me to assert that two lineal lines that are flat and of equal magnitude segments are of equal length if they both have the same RC. Changing the magnitude of the segments does not change their ratio. By the same token, the area bounded by parallel lines in Torricelli's parallelogram consists of flat equal HIs of area and both will contain the same RC as if the HIs in each were stacked into area lines. Changing the magnitude of the areal HIs will not change their ratio. Using CPNAHI, I hypothesize that the left side of the Einstein Field Equation is an attempt to match the 4 volume sum of intrinsically curved HIs with 4 volume of flat HIs that philosophically represent 3 spatial dimensions and 1 time dimension without spelling out their RC nor magnitudes. In other words, the 3-volume change represented by the Newtonian gravitational equation plus 1 dimension for time is attempting to match the 4 volume change that would occur by keeping the RC constant but changing the HI magnitudes. The Newtonian equation changes cardinality to change 4-volume, the EFE is attempting to represent that with change of HI magnitude.

Appendix E HI functions versus RC functions

Suppose that we are using LLPP, and that the length of one line is dependent on the length of another line orthogonal to it. I call this an RC function and write

$$\Delta RC_y = f(\Delta RC_x). \tag{E12}$$

I detect no difference between this and $y = f(x)$. Essentially the length of y is a dependent on the length of x.

Suppose that we have a lineal HI $\overset{\smile}{dx}$ and that the magnitude of second HI $\overset{\smile}{dy}$ is dependent upon the magnitude of the first. I write

$$\overset{\triangle}{dy} = f(\overset{\triangle}{dx}) \tag{E13}$$

and call this a HI function.

E.1 Proposed Overview of HI functions versus RC functions

Imagine two sets of flat voluminal HIs that sum up to create two volumes, one background and one foreground. I hypothesize that upon the background flat voluminal HIs, LLPP is used to create $\mathbb{R}$, Euclidean geometry and the Calculus with RC functions upon which Newtonian physics relies. With the foreground we are free to change magnitudes of the HIs themselves, creating intrinsic curvature and geometric waves (maxima and minima). Upon this foreground we are free to map on these waves philosophical models of an elastic medium that utilize HI functions to describe physical phenomena present in our universe such as energy density, momentum, time dilation and length contractions.

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