or if i is written for ψ

$$i = \sum \frac{E\epsilon^{pt}}{p\frac{dZ}{dp}}. \tag{25}$$

The $\sum$ denotes summation over the roots of the determinental equation. Equation (25) is identical to the equation of subsidence obtained in (11) according to Heaviside's procedure. It has been deduced, however, without the application of the Conjugate Theorem, although the same limitations hold, namely, that there are no null or repeated roots of p.

[1] M. S. Vallarts, "Heaviside's Proof of His Expansion Theorem," *Jl. A. I. E. E*, April, 1926.

[2] Lord Rayleigh, "Scientific Papers," **1**, 176–187; and "Theory of Sound," 103–142.

[3] Oliver Heaviside, "Electrical Papers," **1**, "On Induction between Parallel Wires," 127–129.

[4] *Ibid.*, "Contributions to the Theory of the Propagation of Current in Wires," 142–148.

[5] Loc. cit., 3, **2**, 372–374.

THE DISTRIBUTION OF CHI-SQUARE

By Edwin B. Wilson and Margaret M. Hilferty

Department of Vital Statistics, Harvard School of Public Health

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R. A. Fisher[1] gives a table of χ^2 and states that for large values of n, the number of degrees of freedom in the distribution,

$$\sqrt{2\chi^2} - \sqrt{2n-1} \text{ is normally distributed with } \sigma = 1. \tag{1}$$

It is interesting to ask what other formulas of a similar sort might be used.

When the integrand $f(x)$ of a definite integral vanishes at the limits and has a single maximum, a useful approximation to the value of the integral can sometimes be found by expanding $\log f(x)$ about its maximum $x = m$, writing

$$\varphi(x) = \log f(x) = \varphi(m) + \phi'(x-m) + {}^1/_2\phi''(m)(x-m)^2 + \ldots$$

$$\int_a^b f(x)dx = \int_a^b e^{\varphi(m)} e^{\frac{1}{2}\varphi''(m)(x-m)^2}\, dx \text{ (approx.)}$$

$$\text{or} \int_a^b f(x)dx = e^{\varphi(m)} \frac{\sqrt{2\pi}}{\sqrt{-\varphi''(m)}}$$

$$\text{or } \log \int_b^b f(x)dx = \varphi(m) + \frac{1}{2}\log 2\pi - \frac{1}{2}\log\,[-\varphi''(m)]$$

The assumption is that the higher terms in the expansion of $\varphi(x)$ contribute small amounts to the integral and that the limits a and b are sufficiently remote from the maximum $x = m$ so that the integral may be regarded as a complete probability integral.

The usual application of this method is to the gamma function

$$\Gamma(n) = (n - 1)! = \int_0^\infty x^{n-1}e^{-x}dx = \int_0^\infty py^{np-1}\, e^{-y^p}dy$$

where $x = y^p$. Then

$$\varphi(y) = \log f(y) = \log p + (np - 1)\log y - y^p$$
$$m = (n - 1/p)^{1/p},\ -\varphi''(m) = p^2(n - 1/p)^{1-2/p},$$
$$\varphi(m) = \log p + (n - 1/p)\log(n - 1/p) - (n - 1/p),$$

$$\log\Gamma(n) = \log(n-1)! = \frac{1}{2}\log 2\pi + \left(n - \frac{1}{2}\right)\log\left(n - \frac{1}{p}\right) - \left(n - \frac{1}{p}\right). \quad (2)$$

When $p = 1$ this expression gives the well-known Stirling's formula; when $p = 2$ it gives a somewhat better approximation.[2] One may observe that $\Gamma(n)$ is defined from $n = 0+$ to $n = \infty$, whereas Stirling's formula is defined only from $n = 1+$ on and the alternative formula only from $n = (^1/_2)+$ on. The method gives an approximation to $\Gamma(n)$ over the whole range of the function if $p = \infty$ and

$$\log\Gamma(n) = \log(n - 1)! = \frac{1}{2}\log 2\pi + \left(n - \frac{1}{2}\right)\log n - n$$

or

$$\Gamma(n + 1) = n! = \sqrt{\frac{2\pi}{n+1}}\left(\frac{n+1}{e}\right)^{n+1}.$$

The following table shows the values of $\log_{10}\Gamma(n)$ as computed for some small values of n with $p = 1, 2, 5, \infty$.

	$n = {}^1/_2$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
$p = 1$	imag	∞	−0.03520	0.28076	0.76613	1.37118
$p = 2$	∞	0.03143	0.01178	0.30813	0.78330	1.38420
$p = 5$	0.26080	0.00320	0.00026	0.30097	0.77800	1.38006
$p = \infty$	0.18194	−0.03520	−0.01796	0.28901	0.76912	1.37298
$\log_{10}\Gamma(n)$	0.24857	0.00000	0.00000	0.30103	0.77815	1.38021

It is seen that for small values of n the approximation given by $p = \infty$ is decidedly better than that figured from Stirling's formula ($p = 1$) though not so good as that for $p = 2$ which corresponds to the other formula in common use. It is especially interesting to see that with $p = 5$ the approximation is out of all proportion better than any of the others.

The distribution of χ^2 is governed by the equation for the differential frequency as

$$dF = C\chi^{n-1}\, e^{-\chi^2/2}\, d\chi \quad (3)$$

where C is so adjusted that the integral of dF from 0 to ∞ is 1. The possibility of obtaining so good an approximation to the complete integral, which is $\Gamma(n/2)$ except for a multiplier, suggests that the distribution might be obtained in a similar way.

$$dF = 2^{n/2-1}C\left(\frac{1}{2}\chi\right)^{2(n/2-1)} e^{-(\chi^2/2)}\, d\left(\frac{1}{2}\chi^2\right)$$

$$= 2^{n/2-1}Cpy^{np/2-1}\, e^{-y^p}\, dy \qquad (3')$$

where

$$y^p = (\chi^2/2).$$

The maximum is at

$$y = m = \left(\frac{n}{2} - \frac{1}{p}\right)^{1/p} \quad \text{with } \sigma = \frac{1}{p(n/2-1/p)^{1/2-1/p}}.$$

The question is whether $y = (\chi^2/2)^{1/p}$ may be regarded as a normal variate distributed about the mean m with the indicated value of σ. To obtain the formula analogous to Fisher's one should put $p = 2$ and multiply the mean and standard deviation by 2. We should have

$$\sqrt{2\chi^2} \text{ about mean } \sqrt{2n-2} \text{ with } \sigma = 1.$$

A calculation of a comparative table shows that this result is not so good as his. If one takes $p = 3$ the analysis suggests the normal distribution of

$$\sqrt[3]{\chi^2/2} \text{ about } \sqrt[3]{n/2-1/3} \text{ with } \sigma = \frac{1}{3\sqrt[6]{n/2-1/3}}$$

or

$$\sqrt[3]{\chi^2} \text{ about } \sqrt[3]{n-2/3} \text{ with } \sigma = \frac{\sqrt{2}}{3\sqrt[6]{n-2/3}}. \qquad (4)$$

The result is better than that given by Fisher's formula in some parts of the table and worse in others as shown in table 1. A considerable trial with values of p other than 3 indicates no improvement of significance.

There is another method of attack. One may write

$$\chi^2 = n + e, \quad \chi^{2/p} = (n+e)^{1/p}$$

The expansion by the binomial theorem goes according to powers of e or effectively of e/n, the mean value of e is zero, the mean of e^2 is the second moment of χ^2 about its mean value n and equals $2n$, the mean of e^k is the kth moment of χ^2 about its mean and these moments may all be obtained in terms of the Γ-functions by integration of (3) multiplied by χ^{2k}. The algebra is fairly long but straightforward and the final results are rational in n. Hence the mean value of $\chi^{2/p}$ may be obtained as

$$\text{mean}\left(\frac{\chi^2}{n}\right)^{\frac{1}{p}} = 1 + \frac{1}{n}\frac{1}{p}\left(\frac{1}{p}-1\right) + \frac{4}{3}\frac{1}{n^2}\frac{1}{p}\left(\frac{1}{p}-1\right)\left(\frac{1}{p}-2\right) + \ldots.$$

From this expression and the original expansion the moments of $(\chi^2/n)^{1/p}$ about its mean can be found as

$$\text{2nd moment} = \frac{2}{np^2} + \frac{2}{n^2p^2}\left(\frac{1}{p} - 1\right)\left(\frac{3}{p} - 1\right) + \ldots$$

$$\text{3rd moment} = \frac{4}{n^2}\left(\frac{3}{p} - 1\right) + \ldots$$

From these results it appears that if $p = 3$ the 3rd moment vanishes to the order $1/n^3$, and the term of order $1/n^2$ in the second moment also vanishes. This suggests that to a considerable degree of approximation, increasing as n increases, we may assume that

$$\left(\frac{\chi^2}{n}\right)^{1/3} \text{ is normally distributed about } 1 - \frac{2}{9n} \text{ with } \sigma^2 = \frac{2}{9n}. \quad (5)$$

It is interesting to compare this result with the tabular values of χ^2 and with the results of using (1) given by Fisher or (4) which was obtained by a different method. The comparison is given in table 1.

TABLE 1

VALUES OF χ^2 FOR n = 1, 2, 3, 10, 30, AT P = 0.80, 0.50, 0.20, 0.05, 0.01
True values marked T with those given by formulas (5), (1), (4)

n		P = 0.80	P = 0.50	P = 0.20	P = 0.05	P = 0.01
	T	0.0642	0.455	1.642	3.841	6.635
1	(5)	0.0553	0.470	1.618	3.747	6.586
	(1)	0.0125	0.500	1.696	3.498	5.532
	(4)	0.0102	0.333	1.600	4.287	8.119
	T	0.446	1.386	3.219	5.991	9.210
2	(5)	0.450	1.405	3.195	5.936	9.220
	(1)	0.396	1.500	3.312	5.702	8.235
	(4)	0.378	1.333	3.232	6.222	9.869
	T	1.005	2.366	4.642	7.815	11.341
3	(5)	1.015	2.381	4.622	7.775	11.370
	(1)	0.972	2.500	4.736	7.531	10.171
	(4)	0.946	2.333	4.664	7.995	11.826
	T	6.179	9.342	13.442	18.307	23.209
10	(5)	6.191	9.349	13.419	18.298	23.246
	(1)	6.186	9.500	13.523	18.023	22.346
	(4)	6.155	9.333	13.451	18.372	23.381
	T	23.364	29.336	36.250	43.773	50.892
30	(5)	23.376	29.340	36.237	43.770	50.913
	(1)	23.389	29.500	36.318	43.487	50.074
	(4)	20.984	29.333	36.258	43.815	50.986

Table 1 shows that the distribution (4) obtained from the differential (3′) appears to be better than (1) in some places and worse in others; as it is not so simple it should be rejected. On the other hand the distribution (5) obtained from expansion is decidedly better than (1) in most parts of the tables, and is indeed so good as to make it nearly equivalent to the tabulated values of χ^2 for all values of n from $P = 0.80$ to $P = 0.01$.

Thus a probability integral table taken with the rule (5) can for most purposes replace a table of χ^2. Indeed if the aim is merely to test for "significance" with $P = 0.05$ the point of distinction between non-significance and significance as is customary, we do not need a probability integral table but merely the rule that for significance

$$\left(\frac{\chi^2}{n}\right)^{1/3} - \left(1 - \frac{2}{9n}\right) \text{ shall exceed } 1.65\sqrt{\frac{2}{9n}}.$$

It is somewhat remarkable that (5) gives so good a representation as it does over so wide a range, because the distribution of $\chi^{2/3}$ is really not normal as may be seen from tabulation below, which gives the true mean,[3] the approximate mean, the true value of σ^2 and the approximate value, and the true values of $\beta = \mu_3/\sigma^3$ which has been taken as 0 and of $\beta_2 = \mu_4/\sigma^4$ which has been taken as 3 in the approximation. For small values of n the Charlier A-type expansion would have considerable terms in addition to the first.[4]

n	Mean	$1 - \frac{2}{9n}$	σ^2	$\frac{2}{9n}$	β	β_2
1	0.80238	0.77778	0.18704	0.22222	+0.417	2.68
2	0.89298	0.88889	0.10533	0.11111	+0.168	2.73
3	0.92723	0.92593	0.07226	0.07407	+0.093	2.80
10	0.97782	0.97778	0.02217	0.02222	+0.012	2.97
30	0.99259	0.99259	0.007407	0.007407	−0.004	3.14

[1] R. A. Fisher, *Statistical Methods for Research Workers*, 2nd Edition, 1928, pp. 96–7.

[2] See E. B. Wilson, *Advanced Calculus*, p. 384 and p. 386, Ex. 18.

[3] The true moments of χ^2 may obviously be expressed in terms of the Γ-function.

[4] In many applications of the χ^2-test the number of entries in some cell may be fairly small and in that case the test may not work very well. See "Goodness of Fit," by Wilson, Hilferty and Maher, *J. Amer. Statistical Assoc.*, December, 1931. From a practical point of view this has to be borne in mind in judging how accurate an approximation to the distribution of χ^2 is necessary for practical purposes.