Bounds on the Symmetric Cutoff Rate for QAM Transmissions over Time-Correlated Flat-Faded Channels

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***Abstract*— This letter presents new upper and lower bounds on the symmetric cutoff rate for block-coded quadrature-amplitude-modulated (QAM) signaling over links affected by time-correlated Rayleigh-distributed flat-faded phenomena. The proposed bounds assume maximum-likelihood soft-decoding with perfect channel-state-information at the receiving side and hold for *any form* of QAM constellations. These bounds are *quickly* computable and constitute an efficient means to estimate the cutoff rate of systems employing very long codewords, so that the exact evaluation of the cutoff rate results cumbersome. Analytical and numerical evidence of the tightness of the presented bounds is also provided.**

***Index Terms*—Rayleigh channel, symmetric cutoff rate.**

I. Goals of the Work

DATA transmissions on radio channels can be affected by fading phenomena which present high time correlations as in the case of links between low-speed units [2], [3]; in general, such time correlations cannot be fully removed by interleaving/deinterleaving devices because of the limits present on the overall allowable decoding delays. Thus, as a consequence of the memory present in the resulting coding channel, the corresponding cutoff rate *depends* on the length of the transmitted codewords.

The goal of this contribution is twofold. First, in Section II we develop a general formula for the exact computation of the cutoff rate of flat-faded Rayleigh links which generalizes the expression reported in [2, eq. (14)] and holds for *any format* of quadrature-amplitude-modulated (QAM) constellations. Second, in Section III we present tight *easily computable* lower and upper bounds which constitute an attractive tool for a quick estimation of the cutoff rate. The bounds presented *are new* and are the first to be presented in literature for the kind of channels here considered (see [3] and [4] for an extensive overview on this topic and [5] for some generalizations to the case of intersymbol interference (ISI)-impaired faded channels).

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II. Evaluation of the Symmetric Cutoff Rate for General QAM Constellations

Let us assume that an M-ary information stream feeds a block-encoder whose output $\underline{c} \equiv [c(1) \cdots c(N)]^T \in A^N$ is a codeword of N QAM symbols taken from an assigned q-ary complex constellation A. Thus the corresponding (complex) discrete-time sequence $\{r(i) \in C, 1 \leq i \leq N\}$ received at the output of a noisy link affected by flat fading is given by

$$r(i) = g(i)c(i) + \eta(i), \qquad 1 \leq i \leq N \tag{1}$$

where $\{\eta(i)\}$ is a (complex) zero-mean Gaussian thermal-noise sequence with variance N_o. Furthermore, $\{g(i) \equiv g_c(i) + jg_s(i) \in C\}$ is a zero-mean stationary Gaussian fading-sequence whose uncorrelated real components share a common autocorrelation function (a.c.f.) $\{R_g(t), 0 \leq |t| \leq N-1\}$; so, the resulting N-variate Gaussian fading-vector $\underline{G} \equiv [g(1) \cdots g(N)]^T \in C^N$ exhibits a real covariance matrix $\mathrm{cov}_{\underline{G}} \equiv [\mathrm{cov}_{\underline{G}}(k, m), 1 \leq k, m \leq N]$ whose scalar entries are defined as follows: $\mathrm{cov}_{\underline{G}}(k, m) \equiv 2R_g(t = |k-m|)$.

Now, under the assumption of perfect channel-state-information (CSI), the receiving maximum-likelihood (ML) soft-decoder decides for the codeword which minimizes the usual Euclidean-distance from the available observed vector $\underline{r} \equiv [r(1) \cdots r(N)]^T$ (see [2, eq. (6)]); hence, moving from an application of the union–Bhattacharyya bound and then using random-coding argumentations, it can be proved that the symmetric[1] cutoff rate $R_o^*(N)$ for the depicted communication system is given by the general formula [1, eqs. (5.6.1), (5.9.35)], [2, eq. (7)]

$$R_o^*(N) = 2\log q - \frac{1}{N} \cdot \log\left\{\int_{\underline{r}\in C^N}\int_{\underline{G}\in C^N}\left[\sum_{\underline{c}\in A^N}\sqrt{p(\underline{r}/\underline{c}, \underline{G})}\right]^2 p(\underline{G})\, d\underline{r}\, d\underline{G}\right\} \tag{2}$$

where $p(\underline{r}/\underline{c}, \underline{G})$ denotes the Gaussian probability density function (p.d.f.) of the observed vector $\underline{r}$ conditional on $\underline{c}$ and $\underline{G}$ and $p(\underline{G})$ is the Gaussian pdf of the above introduced fading-vector $\underline{G}$ (hereinafter natural logarithms are used). The N-fold integrations present in (2) can be carried out by resorting

[1] According to a current taxonomy [4], the term “symmetric” means that independently-selected and equidistributed codeword–symbols are assumed in the derivation of (2).

to standard formulas for the computation of multivariate Gaussian integrals [2, eq. (B.5) and following text]; so, in our case the final result looks as in the following:

$$R_o^*(N) = 2\log q - \frac{1}{N}\cdot\log\left\{q^N + \sum_{\underline{c}\in A^N}\sum_{\substack{\underline{c}'\in A^N\\ \underline{c}\neq\underline{c}'}}\Phi(\underline{c},\underline{c}';N_o,\text{cov}_{\underline{G}})\right\} \tag{3}$$

with

$$\Phi(\underline{c},\underline{c}';N_o,\text{cov}_{\underline{G}}) \equiv \left\{\det\left[I + \frac{1}{4N_o}\Delta^2(\underline{c},\underline{c}')\,\text{cov}_{\underline{G}}\right]\right\}^{-1} \tag{4}$$

where $\Delta^2(\underline{c},\underline{c}') = \Delta^2(\underline{c}',\underline{c}) \equiv \text{diag}\{\|c(i)-c'(i)\|^2, 1\le i\le N\}$ is the $N\times N$ diagonal matrix which gathers the squared Euclidean distances in between the N symbols constituting the codewords $\underline{c}$ and $\underline{c}'$. Obviously, as it happens for [2, formula (14)], the evaluation of (3) presents a computational complexity which grows *exponentially* with the block-length N. In fact, for general QAM constellations the evaluation of (3) requires the calculation of the $0.5(q^{2N}-q^N)$ Nth-order determinants in (4) and this makes *very expensive* the cutoff rate computation even for moderate values of N and q (at this regard, see also Remark 5 in Section III). However, such a drawback can be effectively overcome through the exploitation of the bounds presented in the next section.

III. The Proposed Upper and Lower Bounds for the Symmetric Cutoff Rate of (3)

Let us indicate with d^2 and D^2 the minimum and maximum squared Euclidean distances in between two constellation points. Therefore, by developing the inner summation present in (3) and then suitably upper-bounding the determinants in (4) we get the following upper bound for the cutoff rate in (3):

$$\begin{aligned}\overline{R}_o^*(N) \equiv \log q - \frac{1}{N}\log\Bigg\{&1 + N(q-1)\left(1+\frac{D^2}{2N_o}R_g(0)\right)^{-1}\\ &+\frac{N(N-1)}{2}(q-1)^2\left[1+\frac{D^2}{N_o}R_g(0)+\left(\frac{D^2}{2N_o}\right)^2\right.\\ &\left.\cdot(R_g(0)^2-|R_g(\min;N-1)|^2)\right]^{-1}\\ &+\left[q^N-1-N(q-1)\left(1+\frac{(q-1)(N-1)}{2}\right)\right]\\ &\cdot(\det[M(N_o;N)])^{-1}\Bigg\}\end{aligned} \tag{5}$$

where

$$|R_g(\min;N-1)| \equiv \min_{1\le t\le N-1}\{|R_g(t)|\}$$
$$M(N_o;N) \equiv I + \frac{1}{4N_o}D^2\,\text{cov}_{\underline{G}}.$$

Furthermore, by lower-bounding the same determinants in (4) we also obtain the lower-bound for the cutoff rate in (3) below reported:

$$\begin{aligned}\underline{R}_o^*(N) \equiv{}& \\ \log q - \frac{1}{N}\log\Bigg\{&1 + N(q-1)\left(1+\frac{d^2}{2N_o}R_g(0)\right)^{-1}\\ &+\frac{N(N-1)}{2}(q-1)^2\left[1+\frac{d^2}{N_o}R_g(0)+\left(\frac{d^2}{2N_o}\right)^2\right.\\ &\left.\cdot(R_g^2(0)-R_g^2(1))\right]^{-1}\\ &+\left[q^N-1-N(q-1)\left(1+\frac{(q-1)(N-1)}{2}\right)\right]\\ &\cdot\left[\left(1+\frac{d^2}{2N_o}R_g(0)\right)^3+2\left(\frac{d^2}{2N_o}\right)^3R_g^2(1)R_g(2)\right.\\ &\left.-\left(1+\frac{d^2}{2N_o}R_g(0)\right)\left(\frac{d^2}{2N_o}\right)^2\cdot(2R_g^2(1)+R_g^2(2))\right]^{-1}\Bigg\}.\end{aligned} \tag{6}$$

The above bounds represent the main results of this contribution and their properties are pointed out in the following remarks.

Remark 1: The computational complexity of the lower-bound in (6) is virtually independent from the values of N and q whereas the evaluation of the upper bound in (5) essentially requires the calculation of the determinant of the $N\times N$ matrix $M(N_o;N)$. However, since this latter is a symmetric positive-definite Toeplitz-type matrix, the computation of the corresponding determinant can be quickly accomplished through the usual Levinson–Durbin algorithm.

Remark 2: Since the following limiting expressions hold:

$$\lim_{N_o\to 0}\overline{R}_o^*(N) = \lim_{N_o\to 0}\underline{R}_o^*(N) \equiv \lim_{N_o\to 0}R_o^*(N) = \log q, \tag{7}$$
$$\lim_{N_o\to\infty}\overline{R}_o^*(N) = \lim_{N_o\to\infty}\underline{R}_o^*(N) \equiv \lim_{N_o\to\infty}R_o^*(N) = 0, \tag{8}$$

we can conclude that the proposed bound are certainly tight for high and low levels of the average received signal-to-noise ratio (SNR) per channel-symbol $\overline{\gamma}_c$. Furthermore, the numerical examples of Figs. 1–3 show that the bounds always differ from the actual cutoff rate less than 8–10% so that they look quite tight even for moderate values of $\overline{\gamma}_c$.

Remark 3: It can be analytically proved that the values assumed by the bounds in (5) and (6) increase when the time-correlation of the fading-phenomena falls and a direct comparison of the plots reported in Figs. 2 and 3 confirms this property. So, in this respect the behavior of the presented bounds *closely mimics* that exhibited by the actual cutoff rate in (3).

Remark 4: Since the derivations of the bounds in (5) and (6) have been carried out by resorting to *dual* bounding argumentations, it could be expected that a "good estimate" of the actual cutoff rate in (3) is simply given by the arithmetical average

$$\hat{R}_o^*(N) \equiv 0.5[\overline{R}_o^*(N) + \underline{R}_o^*(N)] \tag{9}$$

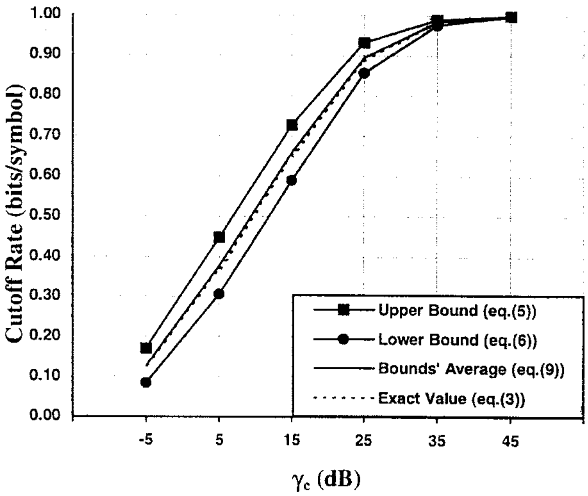

Fig. 1. Behaviors of (3), (5), (6), and (9) versus the average SNR $\overline{\gamma}_c$ for the Rayleigh channel of (1) affected by fading phenomena with time-correlation given by the Gaussian-shaped Watterson-like a.c.f.: $R_g(t) \equiv \exp[-(\pi B_D T t)^2]$. Block-length N of 32 and symmetric BPSK constellation have been considered together with a value of the product Doppler-spread × signaling-period $B_D T$ of 10^{-4} (case of slow-faded channel).

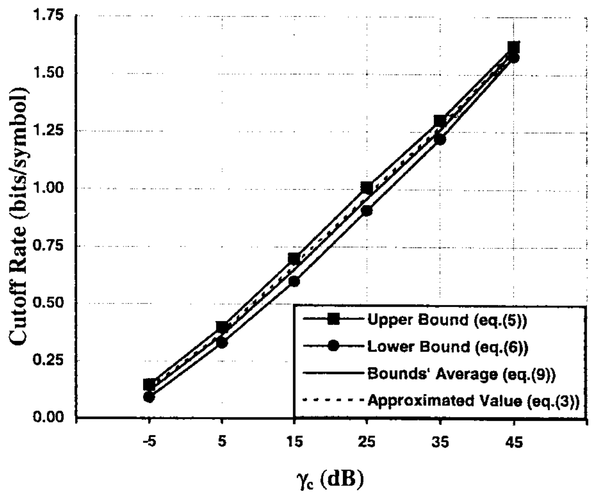

Fig. 2. The same as Fig. 1 for a 4 PSK symmetric constellation ($N = 32$, $B_D T = 10^{-4}$; case of *slow-faded* channel).

of the proposed bounds. Although an analytical proof of this assertion is in general hard to obtain, however the numerical examples reported in Figs. 1–3 directly support this claim.

Remark 5: For symmetric constellations the value assumed by the inner summation in (3) is independent from the picked out codeword $\underline{c}$ which indexes the outer summation; therefore, as in [2, eq. (14)] the numerical evaluation of (3) requires the computation of $(q^N - 1)$ Nth order determinants. For a BPSK constellation with $N = 32$ (see Fig. 1), the overall computation can be carried out within a reasonable short time

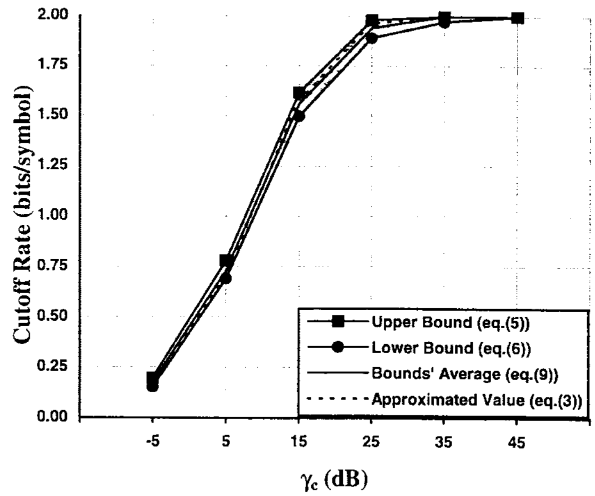

Fig. 3. The same as Fig. 2 for $B_D T = 8 \times 10^{-2}$ ($N = 32$ and 4PSK constellation; case of *fast-faded* channel).

via a standard PC clocked at 200 MHz. However, when the constellation is 4PSK and N is 32 as in Figs. 2 and 3, an exact evaluation of the corresponding $R_o^*(N)$ appears, indeed, very hard to be accomplished since it should require the computation of $(4^{32} - 1)$ determinants.[2] So, in this case, we have introduced some approximations for the numerical evaluation of (3); they consist in computing, at first, only the upper and lower leading terms present in the inner summation of (3) which corresponds to the sets of codewords $\{\underline{c}'\}$ at (squared) Euclidean distances ranging from $32D^2$ to $(32D^2 - K_{\max} d^2)$ and from d^2 to $K_{\max} d^2$ with respects to the codeword $\underline{c}$ picked out as a reference.[3] Afterwards, the so obtained values for the leading terms have been averaged to take into account for the remaining (not explicitly computed) terms present in the inner summation of (3). The resulting approximated cutoff rate curves are plotted in dotted-lines in Figs. 2 and 3.

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[2] The evaluation of the 4^{32} determinants should require more than 58 years when an ultra-fast (and somewhat unrealistic) computer that takes 10^{-10} s to calculate a single determinant is used.

[3] The plots reported in dotted-lines in Figs. 2 and 3 have been numerically evaluated by computing up to 2^{34} leading terms.