Performance Bounds and Cutoff Rates for Data Channels Affected by Correlated Randomly Time-Variant Multipath Fading

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***Abstract*—The symmetric cutoff rate is derived for quadrature-amplitude-modulated (QAM) block-coded data transmissions over noisy links affected by time-correlated multipath-phenomena which introduce intersymbol interference (ISI) in the received signal. Maximum likelihood (ML) decoding supported by perfect channel-state information (CSI) is assumed. Due to the presence of time-correlated ISI phenomena, the resulting transmission link constitutes an example of a continuous-state channel with memory.**

***Index Terms*—Cutoff rate, multipath channel, random coding, Rayleigh fading.**

I. Motivation for the Work

The potential benefits arising from coding operations for data transmissions over noisy links are well summarized by the channel cutoff rate which is representative of the performance (measured in terms of error probability exponent) achievable through codes of assigned rates and block lengths [2], [4], [8, Sec. 4.4], [13, Sec. II]. As a consequence, in the past years a large body of published works has concerned the computation of the cutoff rate for several classes of real-life channels. In detail, comprehensive results for memoryless links are summarized in [8, Sec. 4.5] while in the recent contribution [13] additional insight into the cutoff rate of the companding channel is provided. Furthermore, the important case of data channels affected by time-invariant deterministic intersymbol interference (ISI) has also been widely investigated [2], [6], [7].

The case of randomly faded data-links has also been examined in a lot of works, mainly by resorting to the *simplifying* assumption of *perfect interleaving* [5], [8, Sec. 4.5 and references therein], [14]. However, when such an assumption falls short [3], [4], [11], [12], the time correlation present in the fading phenomena introduces memory in the link which becomes *continuous-state*;[1] in this case, Gallager's approach of [1, Sec. 5.9] for the computation of the error-probability exponent must be, indeed, generalized. As a matter of fact, to date the coding capability of data links affected by time-correlated fading phenomena has been only partially explored. In particular, in [3], [4], the cutoff rates and the related error exponents have been computed for flat-faded (i.e., ISI-free) channels and some generalizations of these results have been presented more recently in [9]–[12].

In this letter, we deal with the most general case of data links affected by *time-correlated Rayleigh fadings* which introduce randomly *time-variant ISI* in the received signals. The cutoff rate and the error exponent for such channels are presented in Section II while several illustrative numerical results are reported in the conclusive Section III.

II. The Symmetric Cutoff Rate for QAM Block-Coded Data Transmissions on Multipath-Faded ISI Channels

Let us assume that an M-ary random source message feeds a block encoder of rate $R \equiv \log(M)/N$ that outputs a codeword $\underline{x}_m \equiv [x_m(1) \cdots x_m(N)]^T \in A^N$ constituted by N symbols from an assigned q-ary QAM constellation $A \equiv \{\alpha_1, \cdots, \alpha_q\} \subset C^1$ (hereafter natural logarithms are used). So, the corresponding sequence $\{y(i) \in C^1, 1 \le i \le N\}$ observed at the output of the ISI-corrupted noisy transmission channel can be modeled as

$$y(i) = \sum_{k=0}^{L_C-1} g(i;k)x_m(i-k) + n(i) \equiv \overline{G}^T(i)\overline{s}(i) + n(i) \equiv z(i) + n(i), \qquad 1 \le i \le N \tag{1}$$

where

$$\overline{s}(i) \equiv [x_m(i) \cdots x_m(i-L_C+1)]^T \in A^{L_C} \equiv A_S \equiv \{\overline{\sigma}_1, \cdots, \overline{\sigma}_{N_S}\}, \qquad 1 \le i \le N, \, (N_S \equiv q^{L_C}) \tag{2}$$

is the system state at the ith epoch [1, Sec. 4.6], and

$$\overline{G}(i) \equiv [g(i;0) \cdots g(i;L_C-1)]^T \equiv \overline{G}_C(i) + j\overline{G}_s(i) \in C^{L_C}, \quad 1 \le i \le N \tag{3}$$

is the L_C-variate impulse-response vector of the channel at the ith signaling period.[2] Now, as pointed out in [1, Sec. 4.6], due to the ISI effects the system state $\overline{s}(i)$ depends *both* on the transmitted codeword $\underline{x}_m$ *and* on the initial state $\overline{s}(0)$ so that we refer to it by adopting the more complete notation $\overline{s}(i) \equiv \overline{s}(i; \underline{x}_m, \overline{s}(0))$. Furthermore, we also assume that the L_C-variate impulse-response sequence $\{\overline{G}(i) \in \mathbb{C}^{L_C}, 1 \le i \le N\}$ in (3) constitutes a complex stationary Gaussian random sequence which takes into account the Rayleigh-distributed multipath phenomena affecting the transmission link; so, its real components $\{\overline{G}_C(i)\}$ and $\{\overline{G}_S(i)\}$

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[1] According to [1, Sec. 5.9], we can define the state of the channel as the corresponding time-variant impulse-response; since the latter is a *continuous-valued* time-correlated random process in multipath faded environments, the resulting channel can be qualified as *continuous state* in accordance with the taxonomy introduced in [1, Sec. 5.9].

[2] In (1), the sequence $\{g(i;k) \in \mathbb{C}^1, 1 \le i \le N, 0 \le k \le L_C - 1\}$ is the usual discrete-time time-varying impulse response of the overall low-pass (complex) system sampled at the chip-period T_c while $\{n(i) \in \mathbb{C}^1\}$ is a complex zero-mean Gaussian noise sequence with variance of $N_o/2$ per component. Finally, $z(i)$ is defined as $z(i) \equiv \overline{G}^T(i)\overline{s}(i)$ and L_C is the length of the impulse response of the channel in multiples of the chip period.

are assumed uncorrelated and share a common autocorrelation sequence $\{\overline{R}_G(t),\, 0 \le t \le N-1\}$ with real matrix lags of dimensions $(L_C x L_C)$. Therefore, by indicating as $\underline{\Gamma}(N) \equiv [\overline{G}(1)^T \cdots \overline{G}^T(N)]^T$ the $L_C N$-variate complex vector which collects the channel impulse responses until step N, the resulting N-variate observed vector $\underline{y} \equiv [y(1) \cdots y(N)]^T \in \mathbb{C}^N$ corresponding to the transmitted codeword $\underline{x}_m$ is given by the following vector relationship [see (1)]:

$$\underline{y} \equiv \Lambda(\underline{\Gamma}(N))\underline{s}(\underline{x}_m, \overline{s}(0)) + \underline{n} \equiv \underline{z}(\Gamma(N); \underline{x}_m, \overline{s}(0)) + \underline{n}. \quad (4)$$

In (4), $n \equiv [n(1) \cdots n(N)]^T$ is an N-variate complex Gaussian noise vector and $\underline{s}(\underline{x}_m, \overline{s}(0)) \equiv [\overline{s}(1; \underline{x}_m, \overline{s}(0))^T \cdots \overline{s}(N; \underline{x}_m, \overline{s}(0))^T]^T$ is the NL_C-variate system super-state which collects the state-sequence of (2) generated by the transmission of the codeword x_m. Furthermore, $\Lambda(\underline{\Gamma}(N)) \equiv \mathrm{diag}\{\overline{G}^T(1), \cdots, \overline{G}^T(N)\}$ is an $N \times N$ diagonal block matrix with (matrix) entries along the main diagonal given by the above-introduced response sequence $\{\overline{G}(i),\, 1 \le i \le N\}$.

The relationship in (4) describes the transmission link on a *codeword basis* and represents the basic system model adopted for the computation of the cutoff rate. In fact, by assuming equiprobable codewords with independently selected symbols [1, Sec. 5.9], [2, Sec. III] and maximum likelihood (ML) decoding with perfect channel-state information (CSI), an application of the union-Bhattacharyya bound followed by random-coding argumentations [1, Sec. 5.3] allows us to bound the block-error probability $\overline{P}_E(\underline{\Gamma}(N))$ conditioned on the fading realization $\Gamma(N)$ as

$$\begin{aligned}\overline{P}_E(\underline{\Gamma}(N)) \le \frac{M}{q^{2N}} &\sum_{\underline{x} \in A^N} \sum_{\overline{s}(0) \in A_S} \sum_{\underline{x}' \in A^N} \sum_{\overline{s}(0)' \in A_S} \\ &\times \sqrt{P(\overline{s}(0)) P(\overline{s}(0)')} \\ &\times \exp\left\{ -\frac{1}{4N_o} \|\underline{z}(\underline{\Gamma}(N); \underline{x}, \overline{s}(0)) \right. \\ &\left. - \underline{z}(\underline{\Gamma}(N); \underline{x}', \overline{s}(0)')\|^2 \right\}. \quad (5)\end{aligned}$$

Furthermore, the averaging of the above bound over the statistics of the fading process can be carried out by noting that $\underline{\Gamma}(N)$ in (5) is an $L_C N$-variate zero-mean complex Gaussian random variable that exhibits the Toeplitz-type real covariance block-matrix reported below:

$$\begin{aligned}&\underline{\mathbb{R}}_G(N) \\ &\equiv \frac{1}{2} E\{\underline{\Gamma}(N)\underline{\Gamma}^H(N)\} \\ &= \begin{vmatrix} \overline{R}_G(0) & \overline{R}_G(1)^T & \cdots & \overline{R}_G(N-1)^T \\ \overline{R}_G(1) & \overline{R}_G(0) & \cdots & \overline{R}_G(N-2)^T \\ \vdots & \vdots & \ddots & \vdots \\ \overline{R}_G(N-1) & \overline{R}_G(N-2) & \cdots & \overline{R}_G(0) \end{vmatrix}. \quad (6)\end{aligned}$$

So, after computing the expectation of (5) with respect to $\underline{\Gamma}(N)$, we obtain the following upper bound for the resulting averaged block-error probability:

$$\overline{P}_E \equiv E_{\underline{\Gamma}(N)}\{\overline{P}_E(\underline{\Gamma}(N))\} \le \exp\{-N[R_o^*(N) - R]\}, \quad R \le R_o^*(N) \quad (7)$$

where the symmetric cutoff rate $R_o^*(N)$ for codewords of length N is defined as

$$\begin{aligned}R_o^*(N) \equiv -\frac{1}{N} \log\Bigg\{ q^{-(2N+L_C)} &\cdot \sum_{\underline{x} \in A^N} \sum_{\overline{s}(0) \in A_S} \sum_{\underline{x}' \in A^N} \sum_{\overline{s}(0)' \in A_S} \\ &\cdot \psi(N_o, \underline{\mathbb{R}}_G(N); \underline{x}, \overline{s}(0); \underline{x}', \overline{s}(0)') \Bigg\} \quad (8)\end{aligned}$$

with the additional positions reported below:[3]

$$\begin{aligned}&\psi(N_o, \underline{\mathbb{R}}_G(N); \underline{x}, \overline{s}(0); \underline{x}', \overline{s}(0)') \\ &\equiv \left\{ \det\left[I + \frac{1}{2N_o} \underline{\Delta}^2(\underline{x}, \overline{s}(0); \underline{x}', \overline{s}(0)') \underline{\mathbb{R}}_G(N) \right] \right\}^{-1} \quad (9)\end{aligned}$$

$$\begin{aligned}\underline{\Delta}^2(\underline{x}, \overline{s}(0); \underline{x}', \overline{s}(0)') \equiv \mathrm{diag}\{&[\overline{s}(i; \underline{x}, \overline{s}(0)) - \overline{s}(i; \underline{x}', \overline{s}(0)')] \\ &\times [\overline{s}(i; \underline{x}, \overline{s}(0)) - \overline{s}(i; \underline{x}', \overline{s}(0)')]^H, \\ &1 \le i \le N\}. \quad (10)\end{aligned}$$

Some remarks are in order concerning the ensemble-averaged performance bound in (7).

Remark 1: Due to the time-correlation exhibited by the fading phenomena, the cutoff rate in (8) generally *depends* on the length N of the transmitted codewords. Such a dependence may modify substantially any exponential behavior of the resulting error probability in (7), especially in static or quasi-static multipath faded environments. To this regard, we also note that for static fading channels the covariance block matrix $\mathbb{R}_G(N)$ in (6) becomes singular; in this limiting case it can be shown that the cutoff rate formula in (8) still holds with the $\psi(\cdot, \cdot; \cdot, \cdot; \cdot, \cdot)$ functional replaced by $\{\det[I + (1/2N_o)\, \underline{\mathrm{Tr}}\{\underline{\Delta}^2(\underline{x}, \overline{s}(0); \underline{x}', \overline{s}(0)')\}\overline{R}_G(0)]\}^{-1}$, where $\underline{\mathrm{Tr}}\{\underline{\Delta}^2(\cdot)\}$ indicates the $L_C x L_C$ matrix obtained by summing the N block entries of $\underline{\Delta}^2(\cdot)$ disposed along the main diagonal.

Remark 2: When the time correlation exhibited by the fading phenomena vanishes, the matrix $\mathbb{R}_G(N)$ in (6) boils down to a diagonal (block) matrix. It can be shown that, in this case, the cutoff rate in (8) no longer depends on N and can be computed by resorting to a suitable application of Frobenius' theorem [15, eq. (15)].

III. Some Illustrative Examples and Conclusive Remarks

The cutoff rate formula in (8) has been numerically evaluated for some multipath-faded test channels. The obtained results are reported in this section; they clarify the actual dependence of the cutoff rate on the time correlation exhibited by the fading phenomena and also point out the effects on the cutoff rate of several important system parameters as the codeword length N, the Doppler bandwidth B_D, the signaling period T_c, and the constellation size q.

[3] The matrix in (10) is an Hermitian semidefinite-positive diagonal block matrix whose entries are $(L_C x L_C)$ matrices (obviously, the off-diagonal entries are null blocks).

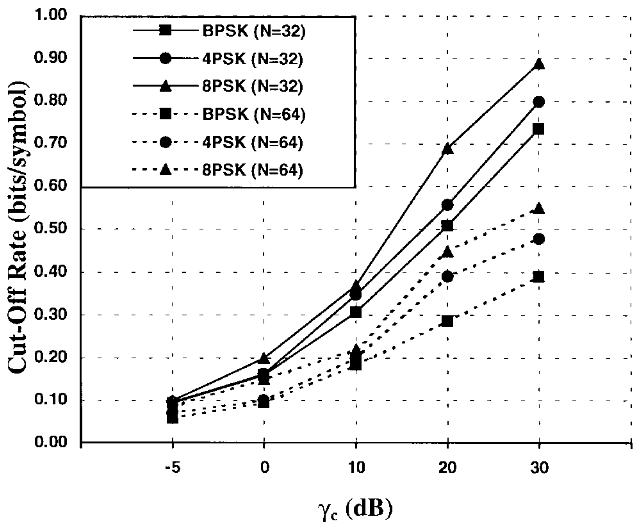

Fig. 1. Behavior of the cutoff rate versus the SNR $\overline{\gamma}_c$ for the two-path channel of (11) with vanishing $B_D T_C$. $\overline{\gamma}_c$ is the average received SNR per channel symbol.

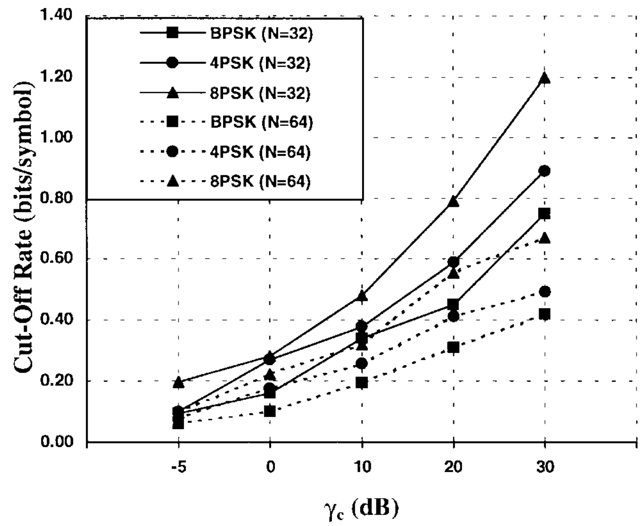

Fig. 2. The same as in Fig. 1 for $B_D T_C = 5 \cdot 10^{-5}$.

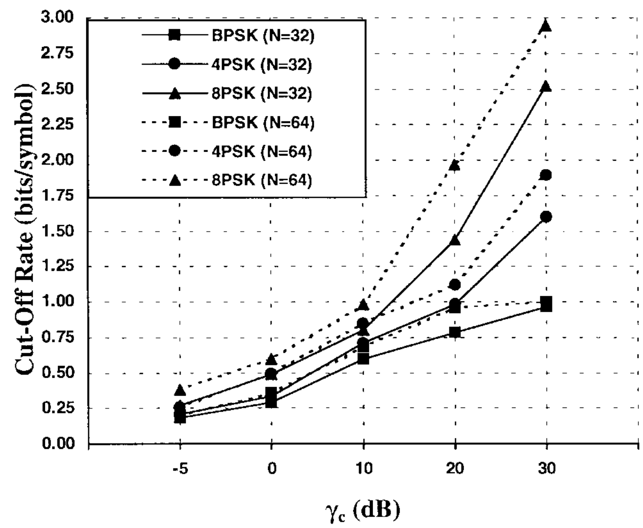

Fig. 3. The same as in Fig. 1 for $B_D T_C = 5 \cdot 10^{-2}$.

A. Dependence of the Cutoff Rate on the Time Correlation of the Multipath Fadings

Let us consider the land–mobile test channel constituted by two equal-power uncorrelated taps impaired by Rayleigh fadings described by the (matrix) autocorrelation sequence

$$\overline{R}_G(t) = I_{2\times 2} J_o(2\pi B_D T_C t), \qquad 0 \le t \le N-1 \tag{11}$$

where $J_o(\cdot)$ is the usual zeroth-order Bessel function of the first kind. The obtained values for the cutoff rates are reported in Figs. 1–3 for codewords of length $N = 32$ and $N = 64$ and for values of the product $B_D T_C$ equal to zero, $5{\cdot}10^{-5}$ and $5{\cdot}10^{-2}$, respectively. Now, Figs. 1 and 2 point out that, when the fading phenomena are "static" or "quasi-static," the cutoff rate performance is poor and substantial improvements cannot be obtained by expanding the constellation size. Moreover, according to the observation that channels affected by slow fading phenomena exhibit vanishing Shannon's capacity [4, Sec. II.B], the plots of Figs. 1 and 2 also show that the cutoff rate values decrease when the block length N increases. However, an opposite trend is exhibited by the numerical results reported in Fig. 3 which refer to the channel of (11) with $B_D T_C = 5 \cdot 10^{-2}$. In fact, for such a fast-faded channel, improvements on the cutoff rate performance are experienced when the codeword length N increases; moreover, at a unitary information rate, reductions of the signal-to-noise ratio (SNR) $\overline{\gamma}_c$ over 15 dB can be obtained by expanding the constellation size q from two to eight.

To illustrate the dependence of the cutoff rate on the shape of the Doppler power spectrum which characterizes the fading phenomena, we have considered a second test channel constituted by two equal-powered mutually uncorrelated taps affected by Watterson-type fadings described by the following Gaussian-shaped autocorrelation sequence:

$$\overline{R}_G(t) = I_{2\times 2} \exp[-(\pi B_D T_C t)^2], \qquad 0 \le t \le N-1. \tag{12}$$

Since Watterson-type fading channels exhibit more time diversity than land-mobile multipath links, the cutoff rate values reported in Fig. 4 for the Watterson-type channel outperform the corresponding ones drawn in Fig. 3 for the above considered land-mobile links. Moreover, the performance plots of Fig. 4 also point out that for the channel of (12) reductions in the SNR of about 17–18 dB can be achieved at a unitary transmission rate by using 8PSK constellations in place of the BPSK ones.

B. Dependence of the Cutoff Rate on the Correlation between the Channel Taps

In order to gain some insight into the behavior of the cutoff rate when the channel paths are correlated, we consider the two-tap link described by the following (matrix) autocorrelation sequence:

$$\overline{R}_G(t) = \begin{vmatrix} 1 & 0.9999 \\ 0.9999 & 1 \end{vmatrix} J_o(2\pi B_D T_C t), \quad 0 \le t \le N-1. \tag{13}$$

Numerical results for the resulting cutoff rate are reported in Fig. 5 together with the corresponding curves obtained for

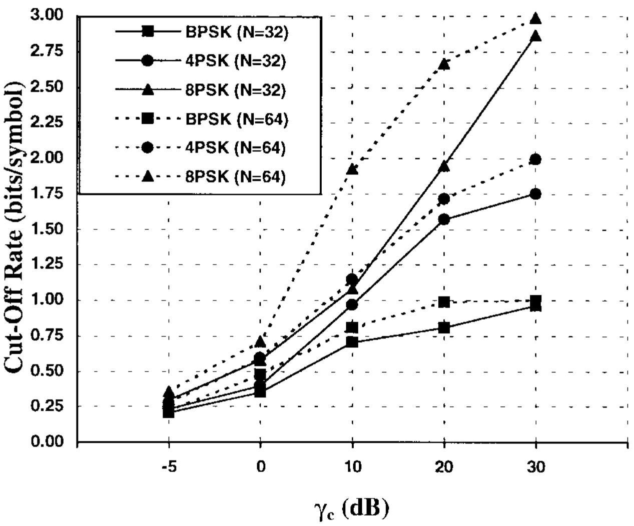

Fig. 4. Cutoff rate values versus $\overline{\gamma}_c$ for the two-path Watterson-like multipath-faded channel of (12) with $B_D T_C = 5 \cdot 10^{-2}$.

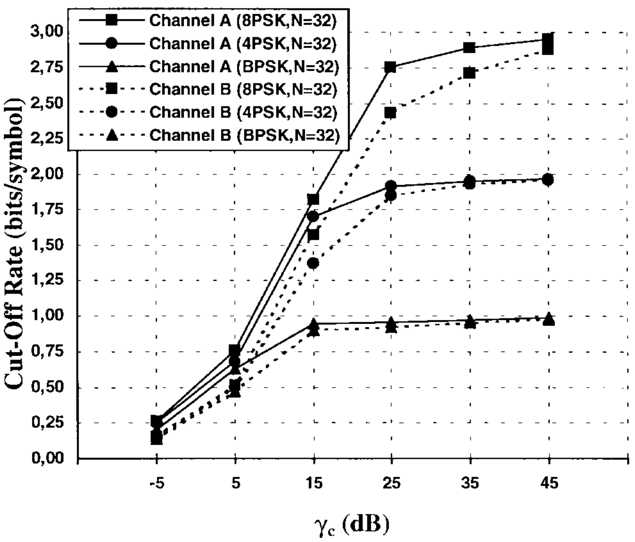

Fig. 5. Cutoff rate versus $\overline{\gamma}_c$ for the two-tap channels of (11) (Channel A) and (13) (Channel B) at $B_D T_C = 8 \cdot 10^{-2}$. BPSK, 4PSK, and 8PSK modulation formats have been considered for a codeword length $N = 32$.

the uncorrelated tap channel of (11). An examination of these plots shows that at transmission rates of about 0.5 bit/symbol the performance loss suffered by the correlated tap channel of (13) with respect to the uncorrelated tap one of (11) is over 3 dB for BPSK constellations while the penalty resulting for 4PSK and 8PSK modulation formats amounts to about 2.6–2.7 dB at a unitary transmission throughput.

The degrading effects induced by the path correlation appear again more evident from the numerical results of Fig. 6, which refer to the three-tap channels with Butterworth-type Doppler spectra described by the following (matrix) autocorrelation sequences:

$$\overline{R}_G(t) = I_{3\times 3} \exp(-2\pi B_D T_C t), \qquad 0 \le t \le N-1 \quad (14)$$

$$\overline{R}_G(t) = \begin{vmatrix} 1 & 0.9999 & 0.9998 \\ 0.9999 & 1 & 0.9999 \\ 0.9998 & 0.9999 & 1 \end{vmatrix} \exp[-2\pi B_D T_C t], \quad 0 \le t \le N-1. \quad (15)$$

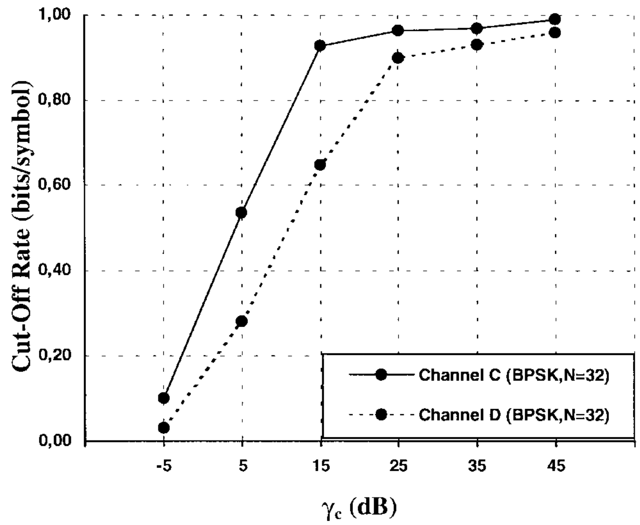

Fig. 6. Cutoff rate versus $\overline{\gamma}_c$ for the three-tap channels of (14) (Channel C) and (15) (Channel D) with first-order Butterworth-type Doppler spectra at $B_D T_C = 10^{-4}$. The reported plots refer to the case of a BPSK constellation with a codeword length $N = 32$.

In fact, an examination of Fig. 6 shows, indeed, that in this case at an information rate of about 0.5 bit/symbol the penalty induced by the tap correlation amounts to about 6.8–7 dB.

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