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NORMAL FORMS FOR SEMILINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS IN BANACH SPACES AND APPLICATIONS. PART II

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Abstract. A normal form theory for functional differential equations in Banach spaces of retarded type is addressed. The theory is based on a formal adjoint theory for the linearized equation at an equilibrium and on the existence of center manifolds for perturbed inhomogeneous equations, established in the first part of this work under weaker hypotheses than those that usually appear in the literature. Based on these results, an algorithm to compute normal forms on finite dimensional invariant manifolds of the origin is presented. Such normal forms are important in obtaining the ordinary differential equation giving the flow on center manifolds explicitly in terms of the original functional differential equation. Applications to Bogdanov-Takens and Hopf bifurcations are presented.

1. **Introduction.**

The aim of the present paper is to construct and show applications of a normal form theory on center or other invariant manifolds at equilibria for semilinear functional differential equations (FDEs) in Banach spaces. Here, we consider autonomous linear FDEs of retarded type in the form

$$\dot{u}(t) = A_T u(t) + L(u_t), \tag{1.1}$$

where X is a Banach space, $r > 0$, $C := C([-r, 0]; X)$ is the Banach space of continuous mappings from $[-r, 0]$ to X equipped with the sup norm, $u_t \in C$ is defined by $u_t(\theta) = u(t + \theta)$ for $t \in [-r, 0]$, $L : C \longrightarrow X$ is a bounded linear operator, and $A_T : D(A_T) \subset X \longrightarrow X$ is the infinitesimal generator of a compact C_0-semigroups of linear operators on X. We also consider semilinear FDEs of type

$$\dot{u}(t) = A_T u(t) + L(u_t) + F(u_t), \tag{1.2}$$

where F is regular enough and $F(0) = 0, DF(0) = 0$.

In applications, it is of particular interest to consider the ordinary differential equation (ODE) giving the flow on center manifolds, since the qualitative behaviour of the solutions can be described by the flow on these manifolds. With the present approach, we give *explicit* normal forms (in the usual sense for ODEs) for the equation giving the flow on the center manifold of equilibria for equations in the form (1.2), without having to compute that manifold beforehand. These normal forms are also applicable to determine the flow on other finite dimensional invariant manifolds, for instance center-unstable manifolds, provided their existence. Situations with parameters will also be treated, since the normal form theory developed here is particularly powerful in the study of bifurcation problems.

The normal form theory presented here on one hand relies on the existence of center manifolds for (1.2), and, on the other hand, on a complete formal adjoint theory for equations (1.1) established in Part I of the present work (Faria, Huang

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and Wu [7]), where ideas in Arino and Sanchez [1], Busenberg and Huang [2], Huang [15], Travis and Webb [22] were pursued. We should mention that in [2] a formal adjoint theory was already derived for a particular model. In [7] the formal duality was used to decompose the phase space C by a finite set of characteristic values, and results on the existence and regularity of center manifolds for perturbed FDEs (1.2) were established. These tools enable us to construct a normal form algorithm along lines similar to the ones considered in previous works of Faria and Magalhães [8], [9] on normal forms for autonomous retarded FDEs in finite dimensional spaces.

We point out that normal forms have already been constructed for particular classes of FDEs in Banach spaces in Faria [5], [6], but under some strong assumptions that restrict their application. In fact, for the normal form construction in Faria [6], as well as for the adjoint theory for linear equations of type (1.1) and invariant manifolds results in Lin, So and Wu [17], Memory [18], Wu [25], it was assumed, first, that the eigenvectors of A_T formed a basis $\{\beta_k\}_{k=1}^{\infty}$ for X, and secondly, that the linear operator L did not mix the modes of eigenspaces of A_T (i.e., $L(\varphi\beta_k) \in span\{\beta_k\}$, for all $\varphi \in C([-r,0];\mathbb{R})$ and all eigenvectors β_k). This last condition was relaxed in Faria [5], where it was sufficient to impose that the eigenvectors of A_T could be organized in blocks, in such a way that the modes of the generalized eigenspace for A_T generated by them were not mixed by L. In both cases, the previous approachs for developing a normal form theory rely on the eigenspaces of A_T, through which linear FDEs of type (1.1) or perturbed FDEs of type (1.2) are decomposed as sequences of FDEs in finite dimensional spaces $\mathbb{R}^n$ (all of them being scalar FDEs for the situation in [6], [17] and [18], and possibly non-scalar under the weaker condition imposed in [5]), to which the standard formal adjoint theory for FDEs of Hale [12] can be applied. The approach followed here is completely different, since there are no hypotheses on the eigenvectors of A_T nor on relating the linear operators A_T, L. Therefore, it is necessary to reconstruct a normal form theory solely based on the formal adjoint theory presented in [7], which enables us to decompose the phase space C by a nonempty finite set of characteristic values of (1.1).

The paper is organized as follows. In Section 2, we recall some relevant results in [7] and [22], that will be used in what follows. The option of presenting a detailed background section was made so that the reader could follow easily the exposition in Sections 3 and 4 and have the necessary results to understand clearly the illustrations in the last section. In Section 3, we introduce an enlarged phase space where (1.2) can be written as an abstract ODE in a Banach space. Section 4 is dedicated to the theory of normal forms. Finally in Section 5, application of normal forms to the study of Bogdanov-Takens and Hopf bifurcations are presented, and illustrated with examples.

We now set some notation that will be used throughout the paper. For a given Banach space X and for a linear operator A from its domain in X to X, we will use $D(A)$, $R(A)$ and $N(A)$ to denote the domain, range and kernel of A, respectively. The spectrum, the point spectrum and resolvent of A are considered as subsets of $\mathbb{C}$, and are denoted by $\sigma(A)$, $\sigma_P(A)$ and $\rho(A)$, respectively. If $\lambda \in \sigma_P(A)$, $\mathcal{M}_\lambda(A)$ is the generalized eigenspace associated with λ.

2. **Preliminaries.**

Consider an autonomous linear retarded FDE (1.1) in the phase space $C = C([-r,0];X)$, X a Banach space, with $A_T : D(A_T) \subset X \longrightarrow X$, $L : C \longrightarrow X$ linear operators. We require the following assumptions:

(H1) A_T generates a C_0-semigroup of linear operators $\{T(t)\}_{t\geq 0}$ on X, with $\|T(t)\| \leq Me^{\omega t}$ $(t \geq 0)$ for some $M \geq 1, \omega \in \mathbb{R}$.

(H2) $T(t)$ is a compact operator for each $t > 0$.

(H3) there is $\eta : [-r, 0] \longrightarrow \mathcal{L}(X, X)$ of bounded variation such that $L(\varphi) = \int_{-r}^{0} d\eta(\theta)\varphi(\theta), \varphi \in C$, where $\mathcal{L}(X, X)$ denotes the Banach space of bounded linear operators from X into X.

Under (H1)-(H2), it was shown in [22] that the initial value problem

$$u(t) = T(t)\varphi(0) + \int_0^t T(t-s)L(u_s)ds, \quad u_0 = \varphi \tag{2.1}$$

for $\varphi \in C$, has a unique solution $u(\varphi)(t), t \geq -r$. Moreover, defining $U(t), t \geq 0$, by $U(t) : C \longrightarrow C$, $U(t)\varphi = u_t(\varphi)$, $\{U(t)\}_{t\geq 0}$ is a C_0-semigroup of bounded linear operators on C, with $U(t)$ a compact operator for $t > r$. Its infinitesimal generator $A_U : C \longrightarrow C$ is given by

$$\begin{aligned} &A_U\varphi = \dot{\varphi} \\ &D(A_U) = \{\varphi \in C : \dot{\varphi} \in C, \varphi(0) \in D(A_T), \dot{\varphi}(0) = A_T\varphi(0) + L(\varphi)\}, \end{aligned} \tag{2.2}$$

and has only point spectrum, $\sigma(A_U) = \sigma_P(A_U)$. Futhermore, for any $\alpha \in \mathbb{R}$ the set $\{\lambda \in \sigma(A_U) : Re\lambda \geq \alpha\}$ is finite. For $\lambda \in \mathbb{C}$, we note that $\lambda \in \sigma(A_U)$ iff λ is a *characteristic value* for (1.1), that is, if λ satisfies the *characteristic equation*

$$\Delta(\lambda)x := A_Tx + L(e^{\lambda \cdot}x) - \lambda x = 0, \quad \text{for some } \ x \in D(A_T) \setminus \{0\}, \tag{2.3}$$

where $\Delta(\lambda) : D(A_T) \subset X \longrightarrow X$ and $e^{\lambda \cdot}x \in C$ is given by $(e^{\lambda \cdot}x)(\theta) = e^{\lambda\theta}x$ for $\theta \in [-r, 0]$ and $x \in X$. Clearly, $N(A_U - \lambda I) = \{e^{\lambda \cdot}x : x \in N(\Delta(\lambda))\}$. If $\lambda \in \sigma(A_U)$, then the ascent and descent of $A_U - \lambda I$ are both finite and equal, and

$$C = N[(A_U - \lambda I)^m] \oplus R[(A_U - \lambda I)^m], \tag{2.4}$$

with $N[(A_U - \lambda I)^m] = \mathcal{M}_\lambda(A_U)$ finite dimensional and $R[(A_U - \lambda I)^m]$ a closed subspace of C.

The results concerning a formal adjoint theory obtained by Hale [12] for linear FDEs in finite dimensional spaces of the form $\dot{u}(t) = L(u_t)$, with $L : C([-r, 0]; \mathbb{R}^n) \longrightarrow \mathbb{R}^n$ linear bounded, remain valid for (1.1) without essential modifications. These results are summarized here, and the reader should consult [7], [15] and [22] for details.

Let X^* be the dual of X, $C^* := C([0, r]; X^*)$, and define a *formal duality* as the bilinear form $<< \cdot, \cdot >>$ from $C^* \times C$ to the scalar field given by

$$<< \alpha, \varphi >>=< \alpha(0), \varphi(0) > - \int_{-r}^{0}\int_{0}^{\theta} < \alpha(\xi - \theta), d\eta(\theta)\varphi(\xi) > d\xi \tag{2.5}$$

for $\alpha \in C^*, \varphi \in C$, where $< \cdot, \cdot >$ is the usual duality between X^* and X. We remark that

$$<< fu^*, \varphi >>=< u^*, f(0)\varphi(0) > - < u^*, L\Big(\int_0^\theta f(\xi - \theta)\varphi(\xi)d\xi\Big) >, \tag{2.6}$$

where $f \in C([0, r]; \mathbb{R}), u^* \in X^*, \varphi \in C$, and we use fu^* to denote $f \otimes u^*$ in C^*, i.e., $(fu^*)(s) = f(s)u^*$ for $0 \leq s \leq r$. Here and throughout this paper, for the sake of simplicity, we abuse the notation and often write $L(\varphi(\theta))$ instead of $L(\varphi)$, for $\varphi \in C$. We also define the *formal adjoint operator* ${}^*L$ of L by

$${}^*L : C^* \longrightarrow X^*, \quad {}^*L(\alpha) = \int_{-r}^{0} d\eta^*(\theta)\alpha(-\theta),$$

and the *formal adjoint equation* for (1.1) by

$$\dot{\alpha}(t) = -A_T^*\alpha(t) - {}^*L(\alpha^t), \quad t \le 0, \tag{2.7}$$

where: $\eta^*(\theta)$ is the adjoint of $\eta(\theta) \in \mathcal{L}(X^*, X^*)$, A_T^* is the adjoint of A_T and $\alpha^t \in C^*$ is given by $\alpha^t(s) = \alpha(t+s)$ for $s \in [0, r]$. Similarly to what was done for (1.1), the solutions of (2.7) are associated with a C_0-semigroup of linear operators $\{{}^*U(t)\}_{t\ge 0}$ on C^* , whose infinitesimal generator ${}^*A_U$ is given by

$$\begin{aligned} &{}^*A_U\alpha = -\dot{\alpha},\\ &D({}^*A_U) = \{\alpha \in C^* : \dot{\alpha} \in C^*, \alpha(0) \in D(A_T^*), -\dot{\alpha}(0) = A_T^*\alpha(0) + {}^*L(\alpha)\}. \end{aligned} \tag{2.8}$$

The concept of *adjointness* relative to the formal duality $<< \cdot, \cdot >>$ is justified since $<< {}^*A_U\alpha, \varphi >> = << \alpha, A_U\varphi >>$, for $\alpha \in D({}^*A_U), \varphi \in D(A_U)$.

For $\lambda \in \mathbb{C}, j \in \mathbb{N}_0, m \in \mathbb{N}$, and similarly to what was done in [1] and [12, Section 7.3], in [7] the following linear operators were considered:

$$L_\lambda^j : X \longrightarrow X,\ L_\lambda^j(x) = L\Big(\frac{\theta^j}{j!}e^{\lambda\theta}x\Big),$$

$$\mathcal{L}_\lambda^{(m)} : X^m \longrightarrow X^m,\ \mathcal{L}_\lambda^{(m)} = \begin{pmatrix} \Delta(\lambda) & L_\lambda^1 - I & L_\lambda^2 & \dots & L_\lambda^{m-1} \\ 0 & \Delta(\lambda) & L_\lambda^1 - I & \dots & L_\lambda^{m-2} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & \Delta(\lambda) & L_\lambda^1 - I \\ 0 & 0 & \dots & 0 & \Delta(\lambda) \end{pmatrix},$$

$$\mathcal{R}_\lambda^{(m)} : C \longrightarrow X^m,\ \mathcal{R}_\lambda^{(m)}(\psi) = \begin{pmatrix} -L\Big(\int_0^\theta e^{\lambda(\theta-\xi)}\frac{(\theta-\xi)^{m-1}}{(m-1)!}\psi(\xi)d\xi\Big) \\ \vdots \\ -L\Big(\int_0^\theta e^{\lambda(\theta-\xi)}(\theta-\xi)\psi(\xi)d\xi\Big) \\ \psi(0) - L\Big(\int_0^\theta e^{\lambda(\theta-\xi)}\psi(\xi)d\xi\Big) \end{pmatrix}.$$

Proposition 2.1. *([7]) Assume (H1)-(H3) and let $\lambda \in \mathbb{C}, m \in \mathbb{N}$. Then,*
(i) $\varphi \in N[(A_U - \lambda I)^m]$ if and only if

$$\varphi(\theta) = \sum_{j=0}^{m-1} \frac{\theta^j}{j!}e^{\lambda\theta}u_j,\ \theta \in [-r, 0],\ \text{with } \begin{pmatrix} u_0 \\ \vdots \\ u_{m-1} \end{pmatrix} \in N(\mathcal{L}_\lambda^{(m)});$$

(ii) $\psi \in R[(A_U - \lambda I)^m]$ if and only if $\mathcal{R}_\lambda^{(m)}(\psi) \in R(\mathcal{L}_\lambda^{(m)})$;
(iii) $\alpha \in N[({}^*A_U - \lambda I)^m]$ if and only if

$$\alpha(s) = \sum_{j=0}^{m-1} \frac{(-s)^j}{j!}e^{-\lambda s}x_{m-j-1}^*, s \in [0, r],\ \text{with } (x_0^*, \dots, x_{m-1}^*)^T \in N((\mathcal{L}_\lambda^{(m)})^*).$$

Some spectral properties and the Fredholm alternative related to this formal duality are referred to in the next statement:

Proposition 2.2. *([7]) (i) $\sigma_P(A_U) = \sigma_P({}^*A_U)$; moreover, if $\lambda \in \sigma_P(A_U)$, the ascent of $A_U - \lambda I$ and ${}^*A_U - \lambda I$ are equal and dim $N[(A_U - \lambda I)^m] =$ dim $N[({}^*A_U - \lambda I)^m], m \in \mathbb{N}$;*

(ii) for $\lambda \in \sigma(A_U)$ and $m \in \mathbb{N}$, then $\psi \in R[(A_U - \lambda I)^m]$ if and only if $<< \alpha, \psi >>= 0$ for all $\alpha \in N[({}^*A_U - \lambda I)^m]$;

(iii) for $\lambda, \mu \in \sigma(A_U), \lambda \neq \mu$ and $m, r \in \mathbb{N}$, $<< \alpha, \varphi >>= 0$ for all $\alpha \in N[({}^*A_U - \lambda I)^m]$ and $\varphi \in N[(A_U - \mu I)^r]$.

For $\lambda \in \sigma(A_U)$, let $\Phi_\lambda = (\varphi_1, \dots, \varphi_{p_\lambda})$, $\Psi_\lambda = (\psi_1, \dots, \psi_{p_\lambda})^T$ be bases of the generalized eigenspaces $\mathcal{M}_\lambda(A_U)$ and $\mathcal{M}_\lambda({}^*A_U)$, respectively, where $p_\lambda = \dim \mathcal{M}_\lambda(A_U) = dim\, \mathcal{M}_\lambda({}^*A_U)$. From Proposition 2.2 and (2.4), we can choose $\Phi_\lambda, \Psi_\lambda$ such that $<< \Psi_\lambda, \Phi_\lambda >>:= [<< \psi_i, \varphi_j >>]_{i,j=1,\dots,p_\lambda} = I_{p_\lambda}$. As for linear FDEs in $\mathbb{R}^n$ (cf. [12]), there is a $p_\lambda \times p_\lambda$ constant matrix B_λ, with $\sigma(B_\lambda) = \{\lambda\}$ and such that $\dot{\Phi}_\lambda = \Phi_\lambda B_\lambda$, $-\dot{\Psi}_\lambda = B_\lambda \Psi_\lambda$, and $U(t) = \Phi_\lambda e^{B_\lambda t}$, $t > 0$. For $\Lambda = \{\lambda_1, \dots, \lambda_s\} \subset \sigma(A_U)$, define $\Phi_\Lambda = (\Phi_{\lambda_1}, \dots, \Phi_{\lambda_s}), \Psi_\Lambda = (\Psi_{\lambda_1}, \dots, \Psi_{\lambda_s})^T$, where $\Phi_{\lambda_j}, \Psi_{\lambda_j}$ are bases for $\mathcal{M}_{\lambda_j}(A_U), \mathcal{M}_{\lambda_j}({}^*A_U)$, respectively, such that $<< \Psi_\Lambda, \Phi_\Lambda >>= I_p$, where $p = p_{\lambda_1} + \cdots + p_{\lambda_s}$.

Proposition 2.3. *([7]) Assume (H1)–(H3), let* $\Lambda = \{\lambda_1, \dots, \lambda_s\} \subset \sigma(A_U)$, *define*

$$P_\Lambda = \mathcal{M}_{\lambda_1}(A_U) \oplus \cdots \oplus \mathcal{M}_{\lambda_s}(A_U),\ P_\Lambda^* = \mathcal{M}_{\lambda_1}({}^*A_U) \oplus \cdots \oplus \mathcal{M}_{\lambda_s}({}^*A_U),$$

and consider bases $\Phi_\Lambda, \Psi_\Lambda$ *for* P_Λ, P_Λ^* *such that* $<< \Psi_\Lambda, \Phi_\Lambda >>= I_p,\ p = dim\, P_\Lambda$. *Then, there exists a subspace* Q_Λ *of* C, *invariant under* A_U *and* $U(t),\ t \geq 0$, *such that*

$$C = P_\Lambda \oplus Q_\Lambda \tag{2.9}$$

with $Q_\Lambda = \{\varphi \in C\ : << \Psi_\Lambda, \varphi >>= 0\}$. *Moreover,* $\varphi \in C$ *is written according to decomposition (2.9) as* $\varphi = \varphi_{P_\Lambda} + \varphi_{Q_\Lambda}$, *where* $\varphi_{P_\Lambda} = \Phi_\Lambda << \Psi_\Lambda, \varphi >>$ *and* $\varphi_{Q_\Lambda} \in Q_\Lambda$.

We refer to (2.9) as *the decomposition of* C *by* Λ, or *by the generalized eigenspace* P_Λ.

3. **The Enlarged Phase Space.** Consider an equation with an equilibrium at zero of the form

$$\dot{u}(t) = A_T u(t) + L(u_t) + F(u_t),\ t \geq 0, \tag{3.1}$$

where A_T, L are as in (1.1), $F : C \longrightarrow X$ is a C^k function ($k \geq 2$) with $F(0) = 0, DF(0) = 0$. In this section, we always assume hypotheses (H1)–(H3).

Let Λ be a nonempty finite subset of $\sigma(A_U)$ (e.g., $\Lambda = \{\lambda \in \sigma(A_U) : Re\, \lambda = 0\} \neq \emptyset$), and consider the decomposition (2.9) of C by Λ. For the sake of simplicity and according to Proposition 2.3, we write $\Lambda = \{\lambda_1, \dots, \lambda_s\}, P := P_\Lambda, Q := Q_\Lambda$,

$$\begin{aligned} &\Phi := \Phi_\Lambda = (\Phi_1, \dots, \Phi_s) \\ &\Psi := \Psi_\Lambda = (\Psi_1, \dots, \Psi_s)^T,\ \text{with}\ << \Psi, \Phi >>= I_p, \end{aligned} \tag{3.2}$$

and define

$$B := diag\,(B_1, \dots, B_s),$$

where B_i are $p_i \times p_i$ matrices such that $\dot{\Phi}_i = \Phi_i B_i, -\dot{\Psi}_i = B_i \Psi_i,\ p := \sum_{i=1}^s p_i,\ p_i := dim\, \mathcal{M}_{\lambda_i}(A_U)$. Clearly $\dot{\Phi} = \Phi B, -\dot{\Psi} = B\Psi$. Recall also that an element $\varphi \in C$ is decomposed according to $C = P \oplus Q$ as

$$\varphi = \varphi_P + \varphi_Q,\ \text{with}\ \varphi_P = \Phi << \Psi, \varphi >>,\ \varphi_Q \in Q. \tag{3.3}$$

To develop a normal form theory for abstract FDEs, we now follow closely the work in [8] and [9]. First, it is necessary to enlarge the phase space C in such a way that Eq. (3.1) is written as an abstract ODE. An adequate phase space to accomplish this is the space BC,

$$BC := \{\psi : [-r, 0] \longrightarrow X \mid \psi \text{ is continuous on } [-r, 0), \exists \lim_{\theta \to 0^-} \psi(\theta) \in X\},$$

with the sup norm. The elements of BC have the form $\psi = \varphi + X_0 \alpha, \varphi \in C, \alpha \in X$, where

$$X_0(\theta) = \begin{cases} 0, & -r \leq \theta < 0 \\ I, & \theta = 0, \quad (I : X \longrightarrow X \text{ is the identity}), \end{cases}$$

so that BC is identified with $C \times X$, with the norm $|\varphi + X_0\alpha| = |\varphi|_C + |\alpha|_X$.

In BC we define an extension of the infinitesimal generator A_U, denoted by $\tilde{A}_U$,

$$\begin{aligned} &\tilde{A}_U : C_0^1 \subset BC \longrightarrow BC \\ &\tilde{A}_U\varphi = \dot{\varphi} + X_0[A_T\varphi(0) + L(\varphi) - \dot{\varphi}(0)], \end{aligned} \tag{3.4}$$

where $D(\tilde{A}_U) = C_0^1 := \{\varphi \in C \mid \dot{\varphi} \in C, \varphi(0) \in D(A_T)\}$. We also define

$$\pi : BC \longrightarrow P, \quad \pi(\varphi + X_0\alpha) = \Phi\big(<< \Psi, \varphi >> + < \Psi(0), \alpha > \big). \tag{3.5}$$

Lemma 3.1. *π is a continuous projection onto P, which commutes with $\tilde{A}_U$ in C_0^1.*

Proof. Clearly $R(\pi) = P$. From (2.6) and Proposition 2.1(iii), it follows that π is continuous. Write Φ, Ψ given in (3.2) as $\Phi = (\varphi_1, \dots, \varphi_p), \Psi = (\psi_1, \dots, \psi_p)^T$. Since $<< \Psi, \Phi >>= I_p$, then $\pi(\varphi_i) = \varphi_i, i = 1, \dots, p$ and hence $\pi \circ \pi = \pi$.

For $\varphi \in C_0^1$, we have

$$\pi\tilde{A}_U\varphi = \Phi\big(<< \Psi, \dot{\varphi} >> + < \Psi(0), A_T\varphi(0) + L(\varphi) - \dot{\varphi}(0) > \big). \tag{3.6}$$

Integrating by parts, we obtain

$$\begin{aligned} << \Psi, \dot{\varphi} >> &=< \Psi(0), \dot{\varphi}(0) > - \int_{-r}^0 \int_0^\theta < \Psi(\xi - \theta), d\eta(\theta)\dot{\varphi}(\xi) > d\xi \\ &=< \Psi(0), \dot{\varphi}(0) > - \int_{-r}^0 < \Psi(0), d\eta(\theta)\varphi(\theta) > \\ &\quad + \int_{-r}^0 < \Psi(-\theta), d\eta(\theta)\varphi(0) > + \int_{-r}^0 \int_0^\theta < \dot{\Psi}(\xi - \theta), d\eta(\theta)\varphi(\xi) > d\xi \\ &=< \Psi(0), \dot{\varphi}(0) > - < \Psi(0), L(\varphi) > + < {}^*L(\Psi), \varphi(0) > \\ &\quad + \int_{-r}^0 \int_0^\theta < \dot{\Psi}(\xi - \theta), d\eta(\theta)\varphi(\xi) > d\xi. \end{aligned}$$

Since $\psi_j \in D({}^*A_U), j = 1, \dots, p$, then $-\dot{\Psi}(0) = A_T^*\Psi(0) + {}^*L(\Psi)$. From (3.6), we derive

$$\begin{aligned} \pi\tilde{A}_U\varphi &= \Phi\Big[< -\dot{\Psi}(0), \varphi(0) > + \int_{-r}^0 \int_0^\theta < \dot{\Psi}(\xi - \theta), d\eta(\theta)\varphi(\xi) > d\xi \Big] \\ &= \Phi << -\dot{\Psi}, \varphi >>= \Phi << B\Psi, \varphi >>= \dot{\Phi} << \Psi, \varphi >>= \tilde{A}_U\pi\varphi. \end{aligned}$$

■

Decomposition $C = P \oplus Q$ and the above lemma allow us to decompose BC as a topological direct sum,

$$BC = P \oplus N(\pi), \tag{3.7}$$

where the subspace Q is contained in the null space of π. Therefore, Eq. (3.1) can be decomposed as a system of abstract ODEs in $\mathbb{R}^p \times N(\pi) \equiv BC$, as follows. Setting $v(t) = u_t \in C$, from (3.1) we have

$$\frac{dv}{dt}(0) = A_T v(0) + L(v) + F(v), \quad \frac{dv}{dt}(\theta) = \frac{dv}{d\theta}(\theta) \text{ for } \theta \in [-r, 0),$$

or simply

$$\frac{dv}{dt} = \tilde{A}_U v + X_0 F(v). \tag{3.8}$$

Note that (3.8) is the abstract ODE in BC associated with (3.1). Using (3.7), we write $v(t) \in C_0^1$ as $v(t) = \Phi x(t) + y(t)$, with $x(t) =<< \Psi, v(t) >>\in \mathbb{R}^p, y(t) =$

$(I-\pi)v(t) \in N(\pi) \cap C_0^1 = Q \cap C_0^1 = \{\varphi \in Q : \dot{\varphi} \in C, \varphi(0) \in D(A_T)\} := Q_0^1$. Thus, $v(t)$ is a solution of (3.8) iff

$$\begin{aligned}\Phi \frac{dx}{dt}(t) + \frac{dy}{dt}(t) &= \tilde{A}_U \Phi x(t) + (I-\pi)\tilde{A}_U y(t) \\ &\quad + \Phi < \Psi(0), F(\Phi x(t) + y(t)) > + (I-\pi) X_0 F(\Phi x(t) + y(t)).\end{aligned}$$

Since $\tilde{A}_U \Phi = \Phi B$, $\tilde{A}_U \pi = \pi \tilde{A}_U$ in C_0^1 and $\frac{dy}{dt}(t) \in N(\pi)$, the above equation is equivalent to the system on $\mathbb{R}^p \times N(\pi)$

$$\begin{cases} \dot{x}(t) = Bx(t) + < \Psi(0), F(\Phi x(t) + y(t)) > \\ \dot{y}(t) = A_1 y(t) + (I-\pi) X_0 F(\Phi x(t) + y(t)), \quad x(t) \in \mathbb{R}^p, y(t) \in Q_0^1, \end{cases} \tag{3.9}$$

(here the dot denotes the derivative with respect to t), where A_1 is the restriction of $\tilde{A}_U$ to Q_0^1 interpreted as an operator acting in the Banach space $N(\pi)$, i.e.,

$$A_1 : Q_0^1 \subset N(\pi) \longrightarrow N(\pi), \ A_1 \varphi = \tilde{A}_U \varphi, \quad \text{for } \varphi \in Q_0^1.$$

The spectrum of A_1 will be an important tool for the construction of normal forms. This is the reason why it is crucial to restrict the range of $\tilde{A}_U|_{Q_0^1}$, by considering A_1 in the space $N(\pi)$, rather than the full space BC.

Lemma 3.2. *With the notations above,* $\sigma(\tilde{A}_U) = \sigma_P(\tilde{A}_U) = \sigma(A_U)$.

Proof. It is obvious that $\sigma_P(\tilde{A}_U) = \sigma_P(A_U)$. On the other hand, it is known that $\sigma_P(A_U) = \sigma(A_U)$. Consider now $\lambda \in \rho(A_U)$. From Proposition 2.4 in [7], we have $R(\Delta(\lambda)) = X$; hence, for each $\psi \in C, \alpha \in X$ there is $b \in D(A_T)$ such that

$$\Delta(\lambda) b = \psi(0) - L\big(\int_0^\theta e^{\lambda(\theta-\xi)} \psi(\xi) d\xi\big) + \alpha.$$

Define $\varphi(\theta) = e^{\lambda\theta} b + \int_0^\theta e^{\lambda(\theta-\xi)} \psi(\xi) d\xi$. Then, $\varphi \in C_0^1$, $\dot{\varphi} - \lambda\varphi = \psi$ and

$$\begin{aligned} A_T b + L(e^{\lambda\theta} b) + L\big(\int_0^\theta e^{\lambda(\theta-\xi)} \psi(\xi) d\xi\big) - \dot{\varphi}(0) \\ = \Delta(\lambda) b - \psi(0) + L\big(\int_0^\theta e^{\lambda(\theta-\xi)} \psi(\xi) d\xi\big) = \alpha, \end{aligned}$$

proving that $(\tilde{A}_U - \lambda I)\varphi = \psi + X_0 \alpha$. We conclude then that $R(\tilde{A}_U - \lambda I) = BC$ and since $\tilde{A}_U$ is a closed operator in the Banach space BC this justifies that $\lambda \in \rho(\tilde{A}_U)$. ■

Lemma 3.3. *With the notations above,* $\sigma(A_1) = \sigma_P(A_1) = \sigma(A_U) \setminus \Lambda$.

Proof. Using arguments as in [8, Lemma (5.2)], one can prove the following claims:

Claim 1: $\sigma_P(A_1) = \sigma(A_U) \setminus \Lambda$.

Claim 2: $\sigma(A_1) \subset \sigma(\tilde{A}_U)$.

From the previous lemma, it is now sufficient to show that

Claim 3: if $\lambda \in \Lambda$, then $R(A_1 - \lambda I) = N(\pi)$.

Let $\lambda \in \Lambda$ and consider $f \in N(\pi)$. As $f = (I-\pi)f$, $\tilde{A}_U$ commutes with π in its domain and $C_0^1 \cap N(\pi) = Q_0^1$, then $f \in R(A_1 - \lambda I)$ iff $f \in R(\tilde{A}_U - \lambda I)$. Hence to justify Claim 3 it is sufficient to show that for each $f = \phi + X_0 \alpha \in BC$ with

$$<< \Psi, \phi >> + < \Psi(0), \alpha >= 0, \tag{3.10}$$

there exists $h \in C_0^1$ such that $(\tilde{A}_U - \lambda I)h = \phi + X_0\alpha$, which is equivalent to

$$\begin{cases} \dot{h} - \lambda h = \phi \\ A_T h(0) + L(h) - \dot{h}(0) = \alpha. \end{cases} \tag{3.11}$$

The solution of the first equation is $h(\theta) = e^{\lambda\theta}b + \int_0^\theta e^{\lambda(\theta-\xi)}\phi(\xi)d\xi$, where $b = h(0)$. Moreover we have $\dot{h}(0) = \lambda b + \phi(0)$. By substituting these expressions into the second equation of (3.11) we conclude that there is $h \in C_0^1$ satisfying (3.11) iff there is $b \in D(A_T)$ such that

$$\Delta(\lambda)b = L\Big(\int_0^\theta e^{\lambda(\theta-\xi)}\phi(\xi)d\xi\Big) + \phi(0) + \alpha. \tag{3.12}$$

Let $\lambda = \lambda_i$ for some $i \in \{1, 2, \cdots, s\}$ and let $\{\psi_1^{\lambda_i}, \cdots, \psi_k^{\lambda_i}\}$ $(k \le p_i)$ be a basis of $N({}^*A_U - \lambda_i I)$. It follows from Proposition 2.1(iii) that

$$\psi_j^{\lambda_i}(s) = e^{-\lambda_i s}x_j^*, \quad s \in [0, r], \quad j = 1, 2, \cdots, k,$$

where $\{x_1^*, \cdots, x_k^*\}$ is a basis of $N(\Delta(\lambda)^*)$. Now (3.10) and (2.6) clearly imply that for $j = 1, \cdots, k$,

$$\begin{aligned} 0 &= << \psi_j^{\lambda_i}, \phi >> + < \psi_j^{\lambda_i}(0), \alpha > \\ &= < x_j^*, \phi(0) > - < x_j^*, L\Big(\int_0^\theta e^{\lambda_i(\theta-\xi)}\phi(\xi)d\xi\Big) > + < x_j^*, \alpha > \\ &= < x_j^*, -L\Big(\int_0^\theta e^{\lambda_i(\theta-\xi)}\phi(\xi)d\xi\Big) + \phi(0) + \alpha > . \end{aligned}$$

That is, $-L\Big(\int_0^\cdot e^{\lambda_i(\cdot-\xi)}\phi(\xi)d\xi\Big) + \phi(0) + \alpha \in [N(\Delta(\lambda)^*)]^\perp = \overline{R(\Delta(\lambda))} = R(\Delta(\lambda))$ (see [7, Lemma 2.6]). It follows that (3.12) has a solution $b \in D(A_T)$. ■

4. **Normal Forms on Center Manifolds or Other Invariant Manifolds.**

For the sake of applications, we are particularly interested in obtainig normal forms for equations giving the flow on center manifolds. Therefore, we now fix Λ as the set of eigenvalues for A_U on the imaginary axis. However, we shall consider and analyse situations corresponding to other choices of Λ. In the following, we always assume (H1)–(H3) and use formal series, although in applications only a few terms of those series are computed.

Suppose then that $\Lambda = \{\lambda \in \sigma(A_U) : Re\,\lambda = 0\}$, consider Eq. (3.1) written in the form (3.9) and expand F in Taylor series as

$$F(v) = \sum_{j \ge 2} \frac{1}{j!}F_j(v), \ v \in C,$$

where F_j is jth Fréchet derivative of F. Eq. (3.9) becomes

$$\begin{cases} \dot{x} = Bx + \sum_{j\ge2} \frac{1}{j!} f_j^1(x, y) \\ \dot{y} = A_1 y + \sum_{j\ge2} \frac{1}{j!} f_j^2(x, y), \end{cases} \tag{4.1}$$

with $f_j := (f_j^1, f_j^2), j \ge 2$, defined by

$$f_j^1(x, y) = < \Psi(0), F_j(\Phi x + y) >, \quad f_j^2(x, y) = (I - \pi)X_0 F_j(\Phi x + y). \tag{4.2}$$

As for autonomous FDEs in $\mathbb{R}^n$ (cf. [8], [9]), normal forms are obtained by a recursive process of changes of variables. At each step, the change of variables has the form

$$(x, y) = (\bar{x}, \bar{y}) + \frac{1}{j!}(U_j^1(\bar{x}), U_j^2(\bar{x})), \tag{4.3$_j$}$$

where $x, \bar{x} \in \mathbb{R}^p$, $y, \bar{y} \in Q_0^1$ and $U_j^1 : \mathbb{R}^p \longrightarrow \mathbb{R}^p, U_j^2 : \mathbb{R}^p \longrightarrow Q_0^1$ are homogeneous polynomials of degree j in $\bar{x}$. For each j, the aim is to choose (4.3_j) in such a way that all the *non-resonant* terms of degree j vanish in the transformed equation.

From Theorem 4.2 in [7], the existence of a locally center manifold is guaranteed under the present circumstances. We want now to linearize the function giving the center manifold, and simplify the ODE giving the flow on it, by removing all the non-resonant terms — which means that this ODE should be in normal form.

We describe now the algorithm for computing such normal forms. Suppose that the changes of variables $(4.3_\ell), 2 \leq \ell \leq j-1$ have already been performed. Denote by $\tilde{f}_j = (\tilde{f}_j^1, \tilde{f}_j^2)$ the terms of order j in (x, y) obtained after these transformations, and effect then (4.3_j). This recursive process transforms (4.1) into

$$\begin{cases} \dot{\bar{x}} = B\bar{x} + \sum_{j\geq 2} \frac{1}{j!} g_j^1(\bar{x}, \bar{y}) \\ \dot{\bar{y}} = A_1\bar{y} + \sum_{j\geq 2} \frac{1}{j!} g_j^2(\bar{x}, \bar{y}), \end{cases} \tag{4.4}$$

where $g_j := (g_j^1, g_j^2)$ are the new terms of order j given by

$$g_j^1(x, y) = \tilde{f}_j^1(x, y) - [DU_j^1(x)Bx - BU_j^1(x)]$$
$$g_j^2(x, y) = \tilde{f}_j^2(x, y) - [DU_j^2(x)Bx - A_1(U_j^2(x))], \; j \geq 2.$$

We introduce now some notation: for $j \in \mathbb{N}$ and Y a normed space, let $V_j^p(Y)$ denote the space of homogeneous polynomials of degree j in p variables, $x = (x_1, \dots, x_p)$, with coefficients in Y, $V_j^p(Y) = \{\sum_{|q|=j} c_q x^q : q \in \mathbb{N}_0^p, c_q \in Y\}$, with the norm $|\sum_{|q|=j} c_q x^q| = \sum_{|q|=j} |c_q|_Y$. Define also the operators $M_j = (M_j^1, M_j^2), j \geq 2$, by

$$\begin{gathered} M_j^1 : V_j^p(\mathbb{R}^p) \longrightarrow V_j^p(\mathbb{R}^p), \; (M_j^1 h_1)(x) = Dh_1(x)Bx - Bh_1(x) \\ M_j^2 : V_j^p(Q_0^1) \subset V_j^p(N(\pi)) \longrightarrow V_j^p(N(\pi)), \; (M_j^2 h_2)(x) = D_x h_2(x)Bx - A_1(h_2(x)). \end{gathered} \tag{4.5}$$

Setting $U_j = (U_j^1, U_j^2)$, it is clear that

$$g_j = \tilde{f}_j - M_j U_j. \tag{4.6}$$

The ranges of M_j^1, M_j^2 contain exactly the terms that can be removed from the equation. They are determined (in general not in a unique way) by the choices of complementary spaces for $R(M_j)$. Naturally, the situation $R(M_j^2) = V_j^p(N(\pi)), j \geq 2$, is of particular interest, since it allows us to choose U_j^2 such that $\tilde{f}_j^2(x, 0) = (M_j^2 U_j^2)(x)$, so that the center manifold has equation $y = 0$. Hence, it is important to characterize the spectrum of $M_j^2, j \geq 2$.

Lemma 4.1. *The linear operators $M_j^2, j \geq 2$, are closed and their spectra are*

$$\sigma(M_j^2) = \sigma_P(M_j^2) = \{(q, \bar{\lambda}) - \mu : \mu \in \sigma(A_1), q \in \mathbb{N}_0^p, |q| = j\},$$

where: $\bar{\lambda} = (\lambda_1, \dots \lambda_p)$, $\lambda_1, \dots, \lambda_p$ are the elements of Λ, counting multiplicities, and $(q, \bar{\lambda}) = q_1\lambda_1 + \dots + q_p\lambda_p$, $|q| = q_1 + \dots + q_p$, for $q = (q_1, \dots, q_p)$.

Proof. Using the arguments for finite dimensional ODEs in Chow and Hale [4, pp. 408-410], we obtain

$$\sigma_P(M_j^2) = \{(q, \bar{\lambda}) - \mu : \mu \in \sigma(A_1), q \in \mathbb{N}_0^p, |q| = j\}.$$

To show that $\sigma(M_j^2) = \sigma_P(M_j^2)$, we follow the proof of Theorem (5.4) in [8]. In both cases, the proofs are algebraic and include an inductive reasoning which is straightforward to adapt to the present situation, so it is omitted. ■

Proposition 4.2. *Let* $\Lambda = \{\lambda \in \sigma(A_U) : Re\,\lambda = 0\} \neq \emptyset$ *and consider the space* BC *decomposed by* Λ, $BC \equiv \mathbb{R}^p \times N(\pi)$. *Then, there exists a formal change of variables* $(x, y) = (\bar{x}, \bar{y}) + O(|\bar{x}|^2)$, *such that:*

(i) Eq. (4.1) is transformed into Eq. (4.4), where $g_j^2(\bar{x}, 0) = 0, j \geq 2$*;*

(ii) a locally center manifold for Eq. (3.1) at zero satisfies $\bar{y} = 0$*; futhermore, the flow on it is given by the ODE*

$$\dot{\bar{x}} = B\bar{x} + \sum_{j \geq 2} \frac{1}{j!} g_j^1(\bar{x}, 0), \quad \bar{x} \in \mathbb{R}^p, \tag{4.7}$$

which is in normal form (in the usual sense of normal forms for ODEs).

Proof. From Lemma 3.3, $\sigma(A_1) = \sigma(A_U) \setminus \Lambda$. Then, for $\mu \in \sigma(A_1), q \in \mathbb{N}_0^p, |q| = j$, we have $Re\,[(q, \bar{\lambda}) - \mu] = -Re\,\mu \neq 0$, and Lemma 4.1 implies that $0 \in \rho(M_j^2)$ and $R(M_j^2) = V_j^p(N(\pi)), j \geq 2$. It is then possible to choose U_j^2 so that $\tilde{f}_j^2(x, 0) = (M_j^2 U_j^2)(x)$, and (i) follows from (4.6). Clearly, for (4.4) a locally center manifold is now given by $\bar{y} = 0$, and (4.7) describes the flow on it. For adequate choices of $U_j^1, j \geq 2$, this ODE in $\mathbb{R}^p$ is in normal form, since the operators M_j^1 defined in (4.5) coincide with those operators defined for computing normal forms for ODEs in $\mathbb{R}^p$ (cf. e.g. [4] and [11]). ■

Suppose now that another nonempty finite subset Λ of $\sigma(A_U)$ is chosen, and consider decomposition (3.7) of BC by Λ. Assume that there exists a locally invariant manifold $\mathcal{M}_{\Lambda,F}$ for Eq. (3.1) tangent to P at zero. For instance, if $\Lambda = \{\lambda \in \sigma(A_U) : Re\,\lambda \geq 0\} \neq \emptyset$, P is the center-unstable space for the linear equation $\dot{u}(t) = A_T u(t) + L(u_t)$ and $\mathcal{M}_{\Lambda,F}$ is the center-unstable manifold for Eq. (3.1) at zero. In this case, the existence and regularity of $\mathcal{M}_{\Lambda,F}$ were proven in [16]. In general, provided the existence and regularity of $\mathcal{M}_{\Lambda,F}$, we obtain a similar result to the one stated above for the case of center manifolds, if some additional non-resonance conditions are assumed.

Definition 4.1. *Let* Λ *be a nonempty finite subset of* $\sigma(A_U)$. *Eq. (4.1) (or Eq. (3.1)) is said to satisfy the* non-resonance conditions relative to Λ *if*

$$(q, \bar{\lambda}) \neq \mu, \quad \text{for all } \mu \in \sigma(A_U) \setminus \Lambda, q \in \mathbb{N}_0^p, |q| \geq 2. \tag{4.8}$$

From Lemmas 3.3 and 4.1, if (4.8) holds then $0 \in \rho(M_j^2)$ and $R(M_j^2) = V_j^p(N(\pi))$, for all $j \geq 2$, and we can state the following:

Proposition 4.3. *If (4.8) is satisfied, the statements in Proposition 4.2 are valid for other invariant manifolds associated with other nonempty finite subsets* Λ *of* $\sigma(A_U)$*, assuming that these manifolds exist. In particular, they are valid for the case of center-unstable manifolds.*

For $\Lambda = \{\lambda \in \sigma(A_U) : Re\,\lambda = 0\} \neq \emptyset$ as before, or in a more general setting for Λ such that (4.8) holds, we give now the definition of normal forms relative to Λ.

Definition 4.2. *Eq. (4.4) is said to be a* normal form for Eq. (4.2) (or Eq. (3.1)) relative to Λ *if* $g_j = (g_j^1, g_j^2)$ *are defined by (4.6), with* $U_j^2(x) = (M_j^2)^{-1} \tilde{f}_j^2(x, 0)$ *and* U_j^1 $(j \geq 2)$ *are chosen in such a way that Eq. (4.7) is an ODE in normal form.*

Remark 4.1. From the method of normal forms for finite dimensional ODEs, Eq. (4.7) is in normal form if $U_j^1(x) = (M_j^1)^{-1} P_j^1 \tilde{f}_j^1(x, 0), j \geq 2$, where P_j^1 is the projection of $V_j^p(\mathbb{R}^p)$ onto $R(M_j^1)$ and $(M_j^1)^{-1}$ is a right inverse of M_j^1, with P_j^1, M_j^1 depending on the choices of complementary spaces for $R(M_j^1), N(M_j^1)$ in $V_j^p(\mathbb{R}^p)$, respectively (see [4, Chap. 12] and [8]).

Remark 4.2. Consider Eq. (3.1), with $F \in C^k$, for some $k \geq 2$, and assume that the non-resonance conditions (4.8) are fulfilled, but only for $|q| = j,\ 2 \leq j \leq k$ (instead of $|q| \geq 2$). Using the algorithm described above, steps of order $j,\ 2 \leq j \leq k$, can be performed through changes of variables of the form (4.3_j). We obtain then a *normal form relative to* Λ *up to k-order terms*:

$$\begin{cases} \dot{\bar{x}} = B\bar{x} + \sum_{j=2}^{k} \frac{1}{j!} g_j^1(\bar{x}, \bar{y}) + h.o.t. \\ \dot{\bar{y}} = A_1\bar{y} + \sum_{j=2}^{k} \frac{1}{j!} g_j^2(\bar{x}, \bar{y}) + h.o.t., \end{cases}$$

where *h.o.t* stands for higher order terms. The first equation at $\bar{y} = 0$ gives the normal form up to k-order terms on the invariant manifold associated with Λ, if it exists.

Remark 4.3. The terms $g_j^1(x, 0)$ in (4.7) are recursively given in terms of the coefficients of the original FDE (3.1), according to the following scheme (see Remark 4.1 for notation):

First step $(j = 2)$: $\tilde{f}_2^1 = f_2^1$; $g_2^1(x, 0) = (I - P_2^1) f_2^1(x, 0)$.

Second step $(j = 3)$: $U_2^1(x) = (M_2^1)^{-1} P_2^1 f_2^1(x, 0)$; $U_2^2(x) = (M_2^2)^{-1} f_2^2(x, 0)$; $\tilde{f}_3^1(x, 0) = f_3^1(x, 0) + \frac{3}{2}[(D_x f_2^1)U_2^1 + (D_y f_2^1)U_2^2 - (D_x U_2^1) g_2^1](x, 0)$; $g_3^1(x, 0) = (I - P_3^1)\tilde{f}_3^1(x, 0)$. $\quad \ldots$

For studying bifurcation problems, we need to consider situations with parameters:

$$\dot{u}(t) = A_T u(t) + L(\alpha)(u_t) + F(u_t, \alpha), \tag{4.9}$$

where $\alpha \in V$, V a neighbourhood of zero in $\mathbb{R}^m$, $L : V \longrightarrow \mathcal{L}(C; X), F : C \times V \longrightarrow X$ are C^k functions, $k \geq 2, F(0, \alpha) = 0, D_1F(0, \alpha) = 0$, for all $\alpha \in V$. Introducing the parameter α as a variable by adding $\dot{\alpha} = 0$, we write (4.9) as

$$\begin{aligned} &\dot{u}(t) = A_T u(t) + L_0(u_t) + (L(\alpha) - L_0)(u_t) + F(u_t, \alpha) \\ &(\dot{\alpha}(t) = 0), \end{aligned} \tag{4.10}$$

where $L_0 := L(0)$. In an obvious way, the above procedure can be repeated for (4.10), noting however that the term $(L(\alpha) - L_0)(u_t)$ is no longer of the first order, since α is taken as a variable. On the other hand, as for Eq. (1.1), the infinitesimal generator of the C_0-semigroup associated with the flow of the linear equation $\dot{u}(t) = A_T u(t) + L_0(u_t), \dot{\alpha}(t) = 0$ has only point spectrum, given by $\sigma(A_U) \cup \{0\}$ (A_U being the infinitesimal generator for $\dot{u}(t) = A_T u(t) + L_0(u_t)$). Now, $\lambda = 0$ is always an eigenvalue, whose associated generalized eigenspace is $\mathcal{M}_0(A_U) \times \mathbb{R}^m$, with the notation $\mathcal{M}_0(A_U) = \{0\}$ if $0 \in \rho(A_U)$. In order to consider the entire generalized eigenspace associated with $\lambda = 0$, the assumption

$$0 \in \Lambda, \text{ whenever } 0 \in \sigma(A_U) \tag{4.11}$$

is required; and the *non-resonance conditions relative to* Λ read now as

$$(q, \bar{\lambda}) \neq \mu, \quad \text{for all } \mu \in \sigma(A_U) \setminus \Lambda, q \in \mathbb{N}_0^p, |q| \geq 0. \tag{4.12}$$

Writing the Taylor expansion $L(\alpha) = L_0 + L_1(\alpha) + \frac{1}{2}L_2(\alpha) + \cdots$, we note that $f_j = (f_j^1, f_j^2), j \geq 2$, are now defined by (see [9] for details)

$$\begin{aligned} f_j^1(x, y, \alpha) &= < \Psi(0), jL_{j-1}(\alpha)(\Phi x + y) + F_j(\Phi x + y, \alpha) > \\ f_j^2(x, y, \alpha) &= (I - \pi)X_0[jL_{j-1}(\alpha)(\Phi x + y) + F_j(\Phi x + y, \alpha)]. \end{aligned} \tag{4.13}$$

5. **Applications to Bifurcation Problems.**

Consider a delayed equation with spatial diffusion of type

$$\begin{aligned}\frac{\partial u(t,x)}{\partial t} &= d\frac{\partial^2 u(t,x)}{\partial x^2} + a(x,\alpha)u(t,x) + b(x,\alpha)u(t-1,x)\\ &\quad + f(u(t,x),u(t-1,x),x,\alpha), \quad t>0,\ x\in(\ell_1,\ell_2)\end{aligned} \tag{5.1}$$

where: $d>0, \ell_2>\ell_1$, $\alpha\in V$, $a,b:[\ell_1,\ell_2]\times V\longrightarrow\mathbb{R}$ are continuous functions and C^k relative to α, $f:\mathbb{R}\times\mathbb{R}\times[\ell_1,\ell_2]\times V\longrightarrow\mathbb{R}$ is continuous and $f(z_1,z_2,\cdot,\alpha)$ is a C^{k+1} function such that $f(0,0,x,\alpha)=D_1f(0,0,x,\alpha)=D_2f(0,0,x,\alpha)=0$ for $(x,\alpha)\in[\ell_1,\ell_2]\times V$, where $V\subset\mathbb{R}^m$ $(m\geq 1)$ is a neighbourhood of zero and $k\geq 1$. We also require the solutions u to satisfy either Neumann or Dirichlet conditions:

$$\frac{\partial u}{\partial x}(t,\ell_1) = \frac{\partial u}{\partial x}(t,\ell_2) = 0, \quad \text{or} \tag{5.2}$$

$$u(t,\ell_1) = u(t,\ell_2) = 0. \tag{5.3}$$

Let $X=L^2[\ell_1,\ell_2]$, and consider the operator A_T defined by $A_Tv=dv''$ and domain $D:=D(A_T)=\{v\in W^{2,2}[\ell_1,\ell_2]: v'(\ell_1)=v'(\ell_2)=0\}$ if (5.2), or $D:=D(A_T)=\{v\in W^{2,2}[\ell_1,\ell_2]: v(\ell_1)=v(\ell_2)=0\}$ if (5.3). Then, A_T generates a C_0-semigroup of compact operators. We note that other choices were possible: for instance, we could consider $X=C[\ell_1,\ell_2]$, $D(A_T)=\{v\in C^2[\ell_1,\ell_2]: v'(\ell_1)=v'(\ell_2)=0\}$ in the case of (5.2), and $D(A_T)=\{v\in C^2[\ell_1,\ell_2]: v(\ell_1)=v(\ell_2)=0\}$ in the case of (5.3).

In the phase space $C=C([-1,0];X)$, Eq. (5.1) is written as

$$\dot{u}(t) = A_Tu(t) + L(\alpha)u_t + F(u_t,\alpha) \tag{5.4}$$

where $u(t)=u(t,\cdot)\in X$, $L(\alpha):C\longrightarrow X, F:C\times V\longrightarrow X$ are defined by $L(\alpha)(\varphi)=a(\cdot,\alpha)\varphi(0)+b(\cdot,\alpha)\varphi(-1)$, $F(\varphi,\alpha)=f(\varphi(0),\varphi(-1),\cdot,\alpha)$.

For $a(x,0)=a_0(x), b(x,0)=b_0(x)$, then $L_0:=L(0)$ is given by $L_0(\varphi)=a_0(\cdot)\varphi(0)+b_0(\cdot)\varphi(-1)$. The linearized equation at $u=0,\alpha=0$ is

$$\dot{u}(t) = A_Tu(t) + L_0u_t, \tag{5.5}$$

with characteristic equation

$$\Delta(\lambda)u = 0 \text{ for some } u\in D\setminus\{0\},$$

where

$$\Delta(\lambda)u = du'' + a_0u + e^{-\lambda}b_0u - \lambda u.$$

5.1. **A Bogdanov-Takens Bifurcation.**

Suppose that $\alpha=(\alpha_1,\alpha_2)\in V\subset\mathbb{R}^2$, $a(x,\alpha)=a_0(x)+\alpha_1a_1(x)+O(|\alpha|^2), b(x,\alpha)=b_0(x)+\alpha_2b_1(x)+O(|\alpha|^2)$, and $k=1$ in (5.1). For (5.5), we now assume the following hypotheses:

(5.6) $\lambda=0$ is a double characteristic value of (5.5) and the ascent of A_U is 2.

(5.7) all other characteristic values of (5.5) have non-zero real parts.

Assumption (5.6) means that (see Proposition 2.1 and [7])

$$\dim N(A_U) = 1,$$

$$\dim N[(A_U)^2] = 2,$$

$$\mathcal{M}_0(A_U) = N[(A_U)^2] = \{v+\theta u : \Delta(0)u=0, \Delta(0)v + L_0(\theta u) - u = 0, u,v\in D\}.$$

As usual, here and in the sequel we abuse the notation and write $L_0(\varphi(\theta))$ for $L_0(\varphi)$.

Let $\Lambda = \{0\}$ and consider the enlarged phase space BC decomposed by Λ as $BC = P \oplus N(\pi)$, where $P = \mathcal{M}_0(A_U)$ is the center space for (5.5). Then, (5.6)-(5.7) imply that there exist functions $u_0 \in D \setminus \{0\}, v_0 \in D$ such that

$$P = \operatorname{span} \Phi,\ \Phi(\theta) = [\varphi_1(\theta)\ \varphi_2(\theta)] = [u_0\ v_0 + \theta u_0],\ \theta \in [-1, 0],$$

and

$$du_0'' + (a_0 + b_0)u_0 = 0,\ dv_0'' + (a_0 + b_0)v_0 - (b_0 + 1)u_0 = 0. \tag{5.8}$$

We note that $\dot{\Phi} = \Phi B$, where B is the 2×2 matrix $B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

The formal duality $<< \cdot, \cdot >>$ associated with the adjoint equation for (5.5) is given by (2.6), where in this case $r = 1$ and $< \cdot, \cdot >$ is the duality in $X^* \times X$ (X considered as a Banach space, rather than a Hilbert space), i.e., $< u, v >= \int_{\ell_1}^{\ell_2} u(x)v(x)dx$, and η is such that

$$L_0(\varphi) = a_0(\cdot)\varphi(0) + b_0(\cdot)\varphi(-1) = \int_{-1}^{0} d\eta(\theta)\varphi(\theta).$$

From Proposition 2.1, a basis Ψ for the adjoint space P^* satisfying $-\dot{\Psi} = B\Psi$ has the form

$$P^* = \operatorname{span} \Psi,\ \Psi(\theta) = \begin{pmatrix} \psi_1(s) \\ \psi_2(s) \end{pmatrix} = \begin{pmatrix} x_0 - sy_0 \\ y_0 \end{pmatrix},\ s \in [0, 1],$$

where $x_0 \in D, y_0 \in D \setminus \{0\}$ are such that

$$dy_0'' + (a_0 + b_0)y_0 = 0,\ dx_0'' + (a_0 + b_0)x_0 - (b_0 + 1)y_0 = 0.$$

Since (5.6) holds, then $x_0 = \beta_1 u_0 + \beta_2 v_0, y_0 = \beta_2 u_0$ for some contants $\beta_1, \beta_2 \in \mathbb{R}$. Using (2.6), (5.8) and noting that $\Delta(0)^* = \Delta(0)$, it is easy to see that condition $<< \Psi, \Phi >>= (\psi_i, \phi_j)_{i,j=1}^2 = I_2$ implies that

$$\Psi(s) = \begin{pmatrix} \beta_1 u_0 + \beta_2(v_0 - su_0) \\ \beta_2 u_0 \end{pmatrix},\ s \in [0, 1],$$

where $\beta_1, \beta_2 \in \mathbb{R}$ are determined by

$$\begin{aligned} &< u_0, (1 + b_0)v_0 - b_0 u_0/2 > \beta_2 = 1, \\ &< u_0, (1 + b_0)v_0 - b_0 u_0/2 > \beta_1 \\ &\qquad + (< v_0, (1 + b_0)v_0 - b_0 u_0 > + < u_0, b_0 u_0/6 >)\beta_2 = 0. \end{aligned} \tag{5.9}$$

Remark 5.1. Clearly, β_1, β_2 are determined by (5.9), since $< u_0, (1 + b_0)v_0 - b_0 u_0/2 >\neq 0$. In fact, from (5.8) we obtain $< u_0, (1 + b_0)u_0 >=< u_0, \Delta(0)v_0 >=< \Delta(0)u_0, v_0 >= 0$. In order to get a contradiction, suppose $< u_0, (1 + b_0)v_0 - b_0 u_0/2 >= 0$. Then

$$< c_1 \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ u_0 \end{pmatrix}, \begin{pmatrix} -b_0 u_0/2 \\ (1 + b_0)u_0 \end{pmatrix} >= 0, \quad \text{for all } c_1, c_2 \in \mathbb{R}.$$

Since $\left\{ \begin{pmatrix} 0 \\ u_0 \end{pmatrix}, \begin{pmatrix} u_0 \\ v_0 \end{pmatrix} \right\}$ is a basis for $N[(\mathcal{L}_0^{(2)})^*]$, we derive from this that

$$\begin{pmatrix} -b_0 u_0/2 \\ (1 + b_0)u_0 \end{pmatrix} \in N[(\mathcal{L}_0^{(2)})^*]^\perp = \overline{R(\mathcal{L}_0^{(2)})} = R(\mathcal{L}_0^{(2)}),$$

where the last equality follows from [7, Lemma 2.6]. On the other hand,

$$\begin{pmatrix} -b_0 u_0/2 \\ (1 + b_0)u_0 \end{pmatrix} = \mathcal{R}_0^{(2)}(\varphi_1), \quad \text{where } \varphi_1 = u_0.$$

Proposition 2.1 and (2.4) imply now $\varphi_1 \in R[(A_U)^2] \cap N[(A_U)^2] = \{0\}$, a contradiction.

Write (5.4) as $\dot{u}(t) = A_T u(t) + L_0 u_t + (L(\alpha) - L_0)(u_t) + F(u_t, \alpha)$. Decomposing $u_t = \Phi x(t) + y_t, x(t) \in \mathbb{R}^2, y_t \in Q_0^1$ as for (3.9), (5.4) is decomposed as

$$\begin{cases} \dot{x} = Bx + < \Psi(0), (L(\alpha) - L_0)(\Phi x + y) + F(\Phi x + y, \alpha) > \\ \dot{y} = A_1 y + (I - \pi) X_0 [(L(\alpha) - L_0)(\Phi x + y) + F(\Phi x + y, \alpha)]. \end{cases} \tag{5.10}$$

According to (4.13), we write

$$\begin{aligned} < \Psi(0), (L(\alpha) - L_0)(\Phi x + y) &+ F(\Phi x + y, \alpha) >= \\ &\frac{1}{2} f_2^1(x, y, \alpha) + O(|\alpha|^2 |(x, y)| + |\alpha| |(x, y)|^2) \end{aligned}$$

where f_2^1 is a homogeneous polynomial in (x, y, α) of degree 2 with coefficients in $\mathbb{R}^2$.

To show the application of normal forms, suppose now that $f(z_1, z_2, x, \alpha) = c_0(x) z_1 z_2 + O(|\alpha||z|^2 + |z|^3)$, $c_0 : [\ell_1, \ell_2] \longrightarrow \mathbb{R}$ a C^2 function. This means that (5.1) has the form

$$\begin{aligned} \frac{\partial u(t,x)}{\partial t} = d \frac{\partial^2 u(t,x)}{\partial x^2} &+ a(x,\alpha) u(t,x) + b(x,\alpha) u(t-1,x) \\ &+ c_0(x) u(t,x) u(t-1,x) + h.o.t, \quad t > 0,\ x \in (\ell_1, \ell_2) \end{aligned} \tag{5.11}$$

where $h.o.t$ contains only terms of order higher than three in $(u, \alpha) \in C \times V$.

Consider the problem (5.11), with either boundary conditions (5.2) or (5.3), in its abstract form (5.4) and initial condition $u_0 = \varphi \in C = C([-1, 0]; X)$. For $X = L^2[\ell_1, \ell_2]$ as above, this problem is not well-posed, since $L^2[\ell_1, \ell_2]$ is not a Banach algebra. In order to guarantee the existence of solutions, the state space should be restricted to an appropriate space of functions from $[\ell_1, \ell_2]$ to $\mathbb{R}$ invariant under products. For instance, we could consider $X = C[\ell_1, \ell_2]$ or $X = W^{2,2}[\ell_1, \ell_2]$ (see e.g. [5], [17], [18] and [25]). Another possibility is to restrict the initial-history space, i.e., the space for initial conditions φ. This latter approach is one usually chosen in the literature dealing with parabolic differential equations, without or with delay. To overcome the difficulty, one can consider a fractional power $(A_T)^\beta$ of the operator A_T for an adequate $0 < \beta < 1$ (see Henry [14]). Then, the fractional power space $X_\beta = D(A^\beta)$ with the norm $\|v\|_\beta = \|(A_T)^\beta v\|$ is taken as the Banach state space, and $C_\beta = C([-1, 0]; X_\beta)$ as the new phase space. In the present situation, in order to simplify the computations and use Φ, Ψ as above, it is convenient to keep $X = L^2[\ell_1, \ell_2]$ and the duality $< \cdot, \cdot >$ in $X^* \times X$. Since $A_T = d\frac{d^2}{dx^2}$ with domain D, it is sufficient to take $\beta = \frac{1}{2}$ and consider $C_{\frac{1}{2}}$, $F : C_{\frac{1}{2}} \times V \longrightarrow X$, $F(\varphi, \beta) = c_0(\cdot)\varphi(0)\varphi(-1) + h.o.t.$. See [10], [13], [14], [19], [23] and [24] for details. A different framework of investigating the existence of solutions of partial FDEs with delay was considered in [20] and [21]. In particular, a system similar to (5.11) with Dirichlet conditions on the boundary was studied in [21]. In this paper, the authors considered $X = L^2[\ell_1, \ell_2]$, initial conditions chosen in a "natural" initial-history space, and proved existence of solutions by exploring different techniques and properties, such as the accretivity of the negative Laplacian. Here, we proceed with the computation of normal forms, without further considerations on the existence of solutions for the initial value problem, since this is not the aim of this paper.

With the notations of Section 4, for f given as above, we have

$$\begin{aligned}\frac{1}{2}f_2^1(x,0,\alpha) =& < \Psi(0), \alpha_1 a_1\Phi(0)x + \alpha_2 b_1\Phi(-1)x > + < \Psi(0), c_0(\Phi(0)x)(\Phi(-1)x) > \\ =& < \Psi(0), (\alpha_1 a_1 + \alpha_2 b_1)u_0 x_1 + (\alpha_1 a_1 v_0 + \alpha_2 b_1(v_0 - u_0))x_2 > \\ &+ < \Psi(0), c_0(u_0x_1 + v_0x_2)(u_0x_1 + (v_0 - u_0)x_2) >,\end{aligned}$$

where $\Psi(0) = col(\beta_1 u_0 + \beta_2 v_0, \beta_2 u_0)$. The normal form for (5.10) on the center manifold of the origin at $\alpha = 0$ as the form

$$\dot{x} = Bx + \frac{1}{2}g_2^1(x,0,\alpha) + h.o.t.,$$

where $g_2^1(x,0,\alpha) = (I - P_2^1)f_2^1(x,0,\alpha)$ (see (4.6) and Remark 4.3) and *h.o.t.* stands for higher order terms.

Recall the operators M_j^1 given by (4.5). In this case, we have

$$M_2^1\begin{pmatrix}p_1\\p_2\end{pmatrix} = \begin{pmatrix}\frac{\partial p_1}{\partial x_1}x_2 - p_2\\ \frac{\partial p_2}{\partial x_1}x_2\end{pmatrix}.$$

It is easy to check that one can choose the decomposition $V_2^2(\mathbb{R}^2) = R(M_2^1) \oplus (R(M_2^1))^c$, with complementary space $(R(M_2^1))^c$ defined by

$$(R(M_2^1))^c = span\Big\{\begin{pmatrix}0\\x_1\alpha_1\end{pmatrix},\begin{pmatrix}0\\x_1\alpha_2\end{pmatrix},\begin{pmatrix}0\\x_2\alpha_1\end{pmatrix},\begin{pmatrix}0\\x_2\alpha_2\end{pmatrix},\begin{pmatrix}0\\x_1^2\end{pmatrix},\begin{pmatrix}0\\x_1x_2\end{pmatrix}\Big\}.$$

Note that $g_2^1(x,0,\alpha) = Proj_{(R(M_2^1))^c}f_2^1(x,0,\alpha)$. The decomposition above and the definition of M_2^1 yield

$$\frac{1}{2}g_2^1(x,0,\alpha) = \begin{pmatrix}0\\ \lambda_1x_1 + \lambda_2x_2\end{pmatrix} + \begin{pmatrix}0\\ A_1x_1^2 + A_2x_1x_2\end{pmatrix},$$

where

$$\begin{aligned}&A_1 = \beta_2 < u_0, c_0u_0^2 > \\ &A_2 = 2\beta_1 < u_0, c_0u_0^2 > + \beta_2 < u_0, c_0u_0(4v_0 - u_0) >\end{aligned} \tag{5.12}$$

and the bifurcating parameters are given by

$$\begin{aligned}\lambda_1 =& (< u_0, a_1u_0 > \alpha_1 + < u_0, b_1u_0 > \alpha_2)\beta_2 \\ \lambda_2 =& \big(< u_0, a_1u_0 > \beta_1 + 2 < v_0, a_1u_0 > \beta_2\big)\alpha_1 \\ &+ \big(< u_0, b_1u_0 > \beta_1 + < u_0, b_1(2v_0 - u_0) > \beta_2\big)\alpha_2.\end{aligned} \tag{5.13}$$

These results lead to the following statement:

Theorem 5.1. *Consider Eq. (5.1) with $\alpha = (\alpha_1, \alpha_2) \in V \subset \mathbb{R}^2$, $a(x,\alpha) = a_0(x) + \alpha_1a_1(x) + O(|\alpha|^2), b(x,\alpha) = b_0(x) + \alpha_2b_1(x) + O(|\alpha|^2)$, where $a_0, b_0, a_1, b_1 : [\ell_1, \ell_2] \longrightarrow \mathbb{R}$ are continuous, and $f(z_1, z_2, x, \alpha) = c_0(x)z_1z_2 + O(|\alpha||z|^2 + |z|^3)$, with $c_0 : [\ell_1, \ell_2] \longrightarrow \mathbb{R}$ a C^2 function. Assume also either (5.2) or (5.3), and that (5.5) and (5.6) hold. Then, there is 2-dimensional locally center manifold of the origin at $\alpha = 0$, on which the flow is given by*

$$\begin{cases}\dot{x}_1 = x_2 + h.o.t\\ \dot{x}_2 = \lambda_1x_1 + \lambda_2x_2 + A_1x_1^2 + A_2x_1x_2 + h.o.t,\end{cases} \tag{5.14}$$

where the coefficients $\lambda_1, \lambda_2, A_1, A_2$ are given by (5.12) and (5.13). If $A_1A_2 \neq 0$ and

$$< u_0, a_1u_0 >< u_0, b_1(2v_0 - u_0) > -2 < u_0, b_1u_0 >< v_0, a_1u_0 > \neq 0,$$

then (5.14) exhibits a generic Bogdanov-Takens bifurcation.

Proof. It remains to prove the last statement. Note that the above inequality means that λ_1, λ_2 given by (5.13) are linearly independent. Therefore, $\dot{x}_1 = x_2,\ \dot{x}_2 = \lambda_1 x_1 + \lambda_2 x_2 + A_1 x_1^2 + A_2 x_1 x_2$ is a versal unfolding for (5.14) (see [3], [4], [11]). ■

Example 5.1. As a particular case, suppose that the above hypotheses hold with $b_0(x) \equiv -1$. In this situation, we can choose $v_0 = 0$ and conditions (5.8) reduce to

$$du_0'' + (a_0 - 1)u_0 = 0, \quad \text{for some } u_0 \in D \setminus \{0\}.$$

Consequently,

$$\Phi(\theta) = [u_0\ \theta u_0],\ \theta \in [-1, 0], \quad \Psi(s) = \begin{pmatrix} (\beta_1 - s\beta_2)u_0 \\ \beta_2 u_0 \end{pmatrix},\ s \in [0, 1],$$

where from (5.9) the coefficients β_1, β_2 are given by

$$\beta_1 = \frac{2}{3 < u_0, u_0 >}, \quad \beta_2 = \frac{2}{< u_0, u_0 >}.$$

Thus, the flow on the center manifold of the origin at $\alpha = 0$ is given by (5.14) , with the following coefficients and bifurcating parameters:

$$\begin{aligned} A_1 &= \frac{2 < u_0, c_0 u_0^2 >}{< u_0, u_0 >}, \quad A_2 = -\frac{2 < u_0, c_0 u_0^2 >}{3 < u_0, u_0 >}, \\ \lambda_1 &= \frac{2}{< u_0, u_0 >}\Big(< u_0, a_1 u_0 > \alpha_1 + < u_0, b_1 u_0 > \alpha_2 \Big), \\ \lambda_2 &= \frac{2}{3 < u_0, u_0 >}\Big(< u_0, a_1 u_0 > \alpha_1 - 2 < u_0, b_1 u_0 > \alpha_2 \Big). \end{aligned}$$

If $< u_0, a_1 u_0 >< u_0, b_1 u_0 >< u_0, c_0 u_0^2 > \neq 0$, then (5.14) undergoes a generic Bogdanov-Takens bifurcation on the center manifold of the origin. Futhermore, we have $A_1 A_2 < 0$. If $\int_{\ell_1}^{\ell_2} c_0(x) u_0^3(x) dx < 0$, then $A_1 < 0, A_2 > 0$. In this case and in the (λ_1, λ_2)-bifurcation diagram, the Hopf bifurcation curve H and the homoclinic bifurcation curve HL lie in the region $\lambda_1 > 0, \lambda_2 < 0$, with H to the left of HL; both the homoclinic loop and the periodic orbit are asymptotically stable ([3], [4]). The case $A_1 > 0, A_2 < 0$ is analogous.

5.2. **A Hopf Bifurcation.** Consider again (5.1), and suppose now that $a(x, \alpha) = a_0(x) + \alpha a_1(x) + O(\alpha^2), b(x, \alpha) = b_0(x) + \alpha b_1(x) + O(\alpha^2)$, for $\alpha \in V \subset \mathbb{R}$. Let (5.5) be its linearized equation at $u = 0, \alpha = 0$. For $\alpha = 0$, we now assume the following:

(5.15) there is a pair of simple characteristic values of (5.5) on the imaginary axis, $\pm i\omega$ $(\omega \neq 0)$;

(5.16) all the other characteristic values of (5.5) have nonzero real parts.

Considering X as a complex Banach space, let u_0 be such that

$$du_0'' + (a_0 + b_0 e^{-i\omega} - i\omega)u_0 = 0, \quad u_0 \in D \setminus \{0\};$$

then, *span* $\{u_0\} = N(\Delta(i\omega))$ and $\mathcal{M}_{i\omega}(A_U) = N(A_U - i\omega I) = \{ce^{i\omega \cdot} u_0 : c \in \mathbb{C}\}$. We can choose

$$\Phi(\theta) = [e^{i\omega\theta} u_0\ \ e^{-i\omega\theta} \bar{u}_0] \text{ for } \theta \in [-1, 0], \quad \Psi(s) = \begin{pmatrix} \beta e^{-i\omega s} u_0 \\ \bar{\beta} e^{i\omega s} \bar{u}_0 \end{pmatrix} \text{ for } s \in [0, 1],$$

with $<< \Psi, \Phi >>= I$ if

$$\beta =< u_0, (1 + b_0 e^{-i\omega}) u_0 >^{-1} . \tag{5.17}$$

Clearly $<< e^{-i\omega\cdot}u_0, e^{i\omega\cdot}u_0 >> \neq 0$, otherwise Proposition 2.2(ii) would imply that $e^{i\omega\cdot}u_0 \in R(A_U - i\omega I)$, a contradition by (2.4). Thus, β is well-defined by (5.17).

For $B = diag\,(i\omega, -i\omega)$, we have $\dot{\Phi} = \Phi B, -\dot{\Psi} = B\Psi$. In the enlarged phase space BC, we decompose Eq. (5.1) with boundary conditions (5.2) or (5.3) by $\Lambda = \{i\omega, -i\omega\}$, getting Eq. (3.9), where $x = (x_1, x_2) \in \mathbb{C}^2$. Let

$$\begin{aligned} F_0(\varphi, \alpha) &= (L(\alpha) - L_0)(\varphi) + F(\varphi, \alpha) \\ &= \alpha\big(a_1(\cdot)\varphi(0) + b_1(\cdot)\varphi(-1)\big) + f(\varphi(0), \varphi(-1), \alpha) + O(\alpha^2). \end{aligned}$$

Considering $k = 2$ in (5.1), thus $F_0 \in C^3$, we write the Taylor formula

$$\begin{aligned} < \Psi(0), F_0(\Phi x + y, \alpha) > &= \frac{1}{2} f_2^1(x, y, \alpha) + \frac{1}{3!} f_3^1(x, y, \alpha) + h.o.t. \\ (I - \pi)X_0 F_0(\Phi x + y, \alpha) &= \frac{1}{2} f_2^2(x, y, \alpha) + \frac{1}{3!} f_3^2(x, y, \alpha) + h.o.t. \end{aligned}$$

where $f_j^1(x, y, \alpha), f_j^2(x, y, \alpha)$ are homogeneous polynomials in (x, y, α) of degree $j,\ j = 2, 3$, with coefficients in $\mathbb{C}^2, Ker\,\pi$, respectively. It will turn out that the procedure described in Section 4 gives a normal form on the center manifold of the origin at $\alpha = 0$ written as

$$\dot{x} = Bx + \frac{1}{2} g_2^1(x, 0, \alpha) + \frac{1}{3!} g_3^1(x, 0, \alpha) + h.o.t., \tag{5.18}$$

where

$$\frac{1}{2} g_2^1(x, 0, \alpha) = \begin{pmatrix} A_1 x_1 \alpha \\ B_1 x_2 \alpha \end{pmatrix}, \quad \frac{1}{3!} g_3^1(x, 0, \alpha) = \begin{pmatrix} A_2 x_1^2 x_2 \\ B_2 x_1 x_2^2 \end{pmatrix} + O(|x|\alpha^2),$$

with $B_1 = \overline{A}_1, B_2 = \overline{A}_2$, because the coefficients in (5.1) are real. Thus, the change to real coordinates w, where $x_1 = w_1 - iw_2, x_2 = w_1 + iw_2$, followed by the use of polar coordinates (ρ, ξ), $w_1 = \rho\cos\xi, w_2 = \rho\sin\xi$, transforms the normal form (5.18) into

$$\left\{ \begin{array}{l} \dot{\rho} = K_1 \alpha \rho + K_2 \rho^3 + O(\alpha^2 \rho + |(\rho, \alpha)|^4) \\ \dot{\xi} = -\omega + O(|(\rho, \alpha)|), \end{array} \right. \tag{5.19}$$

with $K_1 = Re\, A_1,\ K_2 = Re\, A_2$.

If $K_2 \neq 0$, which is the case of the generic Hopf bifurcation, the direction of the bifurcation and the stability of the nontrivial periodic orbits are determined by the sign of $K_1 K_2$ and of K_2 (e.g. [4]). The computation of K_1, K_2 requires the resolution of ODEs and PDEs that are difficult to handle for the general case (5.1). Nevertheless, we shall present here explicit formulas for the calculus of such coefficients for particular functions a, b, f appearing in (5.1), and the complete calculus for some examples.

We continue with the computation of g_2^1, g_3^1, omiting some details. Consider the operators M_j^1 defined in (4.5). For the present situation, in particular we get $M_j^1(\alpha^\ell x^q e_k) = i\omega(q_1 - q_2 + (-1)^k)\alpha^\ell x^q e_k,\ \ell + q_1 + q_2 = j, k = 1, 2$, for $j = 1, 2,\ q = (q_1, q_2) \in \mathbb{N}_0^2,\ \ell \in \mathbb{N}_0$ and $\{e_1, e_2\}$ the canonical basis for $\mathbb{C}^2$. Hence,

$$\begin{aligned} N(M_2^1) &= span\Big\{ \begin{pmatrix} x_1\alpha \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2\alpha \end{pmatrix} \Big\} \\ N(M_3^1) &= span\Big\{ \begin{pmatrix} x_1^2 x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} x_1\alpha^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1 x_2^2 \end{pmatrix}, \begin{pmatrix} 0 \\ x_2\alpha^2 \end{pmatrix} \Big\}. \end{aligned} \tag{5.20}$$

For equation (5.1), the second order terms in (α, x) of the normal form on the center manifold are given by

$$\begin{aligned}\frac{1}{2}g_2^1(x,0,\alpha) =& \frac{1}{2}Proj_{N(M_2^1)}f_2^1(x,0,\alpha)\\ &= Proj_{N(M_2^1)} < \Psi(0), \alpha\big(a_1\Phi(0)x + b_1\Phi(-1)x\big) > .\end{aligned}$$

Therefore, this gives

$$\frac{1}{2}g_2^1(x,0,\alpha) = \begin{pmatrix} A_1 x_1 \alpha \\ \overline{A_1} x_2 \alpha \end{pmatrix}, \tag{5.21}$$

with

$$A_1 = \beta < u_0, (a_1 + b_1 e^{-i\omega})u_0 > .$$

In order to guarantee the existence of a Hopf bifurcation on the center manifold of the origin, we further assume the following Hopf condition:

(5.22) $< u_0, (a_1 + b_1 e^{-i\omega})u_0 > \neq 0.$

For the sake of simplicity, and to illustrate how to compute the cubic terms, here we only consider the situation $f(z_1, z_2, x, \alpha) = cb(x,\alpha)z_1^n z_2$, with $c \in \mathbb{R}$ and $n = 1$ or $n = 2$, which corresponds to a PFDE of the form

$$\begin{aligned}\frac{\partial u(t,x)}{\partial t} &= d\frac{\partial^2 u(t,x)}{\partial x^2} + a(x,\alpha)u(t,x) + b(x,\alpha)u(t-1,x)[1 + cu(t,x)^n]\\ & t > 0,\ x \in (\ell_1, \ell_2)\end{aligned} \tag{5.23}$$

with $n = 1$ or $n = 2$. As for (5.11), (5.23) is not well-defined if $X = L^2[\ell_1, \ell_2]$. For the existence of solutions of the correspondent Cauchy problem, we refer the reader for the discussion and references presented in section 5.1.

We observe that (see Remark 4.3 and (5.20)) $g_3^1(x,0,\alpha) = Proj_S \tilde{f}_3^1(x,0,0) + O(|x|\alpha^2)$, where

$$S := span\left\{ \begin{pmatrix} x_1^2 x_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x_1 x_2^2 \end{pmatrix} \right\}$$

and the term $\tilde{f}_3^1(x,0,0)$ is defined as

$$\tilde{f}_3^1(x,0,0) = f_3^1(x,0,0) + \frac{3}{2}[(D_x f_2^1)U_2^1 - (D_x U_2^1)g_2^1](x,0,0) + \frac{3}{2}[(D_y f_2^1)h](x,0,0), \tag{5.24}$$

for $U_2^1(x,0) = (M_2^1)^{-1}Proj_{R(M_2^1)}f_2^1(x,0,0) = (M_2^1)^{-1}f_2^1(x,0,0)$ and $h = h(x)(\theta)$ such that

$$h(x) = U_2^2(x,0), (M_2^2 U_2^2)(x,0) = f_2^2(x,0,0). \tag{5.25}$$

For (5.23) with $n = 2$, we have

$$\begin{aligned}& f_2^1(x,y,0) = 0, f_2^2(x,y,0) = 0,\\ & f_3^1(x,y,0) = 3!c < \Psi(0), b_0(\Phi(-1)x + y(-1))(\Phi(0)x + y(0))^2 >;\end{aligned}$$

hence, $\tilde{f}_3^1(x,0,0) = f_3^1(x,0,0)$ and

$$\begin{aligned}\frac{1}{3!}g_3^1(x,0,0) &= cProj_S \begin{pmatrix} \beta < u_0, b_0(e^{-i\omega}u_0x_1 + e^{i\omega}\bar{u}_0x_2)(u_0x_1 + \bar{u}_0x_2)^2 > \\ \bar{\beta} < \bar{u}_0, b_0(e^{-i\omega}u_0x_1 + e^{i\omega}\bar{u}_0x_2)(u_0x_1 + \bar{u}_0x_2)^2 > \end{pmatrix}\\ &= \begin{pmatrix} A_2 x_1^2 x_2 \\ \overline{A_2} x_1 x_2^2 \end{pmatrix},\end{aligned}$$

where

$$A_2 = c\beta < u_0, b_0(2e^{-i\omega} + e^{i\omega})u_0|u_0|^2 >,$$

and $|u_0|^2 =< u_0, \bar{u}_0 >= \int_{\ell_1}^{\ell_2} |u_0(x)|^2 dx$. Thus, the normal form (5.18) indeed has the form

$$\dot{x} = Bx + \begin{pmatrix} A_1 x_1 \alpha \\ \overline{A_1} x_2 \alpha \end{pmatrix} + \begin{pmatrix} A_2 x_1^2 x_2 \\ \overline{A_2} x_1 x_2^2 \end{pmatrix} + O(|x|\alpha^2 + |x|^4).$$

Using (5.17), the above considerations lead to the following result:

Theorem 5.2. *Consider Eq. (5.23) with $n = 2$ and boundary conditions (5.2) or (5.3), and suppose that (5.15), (5.16) and (5.22) hold. Then a Hopf bifurcation occurs at $\alpha = 0$ on a locally 2-dimensinal center manifold of the origin. On this manifold, the flow is given in polar coordinates by equation (5.19), with*

$$K_1 = Re\Big(\frac{< u_0, (a_1 + b_1 e^{-i\omega})u_0 >}{< u_0, (1 + b_0 e^{-i\omega})u_0 >}\Big),\ K_2 = cRe\Big(\frac{< u_0, b_0(2e^{-i\omega} + e^{i\omega})u_0|u_0|^2 >}{< u_0, (1 + b_0 e^{-i\omega})u_0 >}\Big).$$

Example 5.2. Consider the problem:

$$\begin{aligned} &\frac{\partial u(t,x)}{\partial t} = d\frac{\partial^2 u(t,x)}{\partial x^2} + b(x,\alpha)u(t-1,x)[1 + u(t,x)^2], \quad t > 0,\ x \in (0,\pi) \\ &\frac{\partial u}{\partial x}(t,0) = \frac{\partial u}{\partial x}(t,\pi) = 0, \end{aligned} \tag{5.26}$$

with $b(x,\alpha) = b_0 + \alpha b_1(x) + O(\alpha^2)$ and b_0 constant. Define as before $L_0(\varphi) := L(0)(\varphi) = b_0\varphi(-1)$. It is easy to see that the linearized equation at $u = 0, \alpha = 0$, $\dot{u}(t) = d\Delta u(t) + L_0(u_t)$, has simple eigenvalues $\pm i\frac{\pi}{2}$ iff $b_0 = -\frac{\pi}{2}$; in this case, all the other eigenvalues have nonzero real parts (e.g. [6], [17]). Let $\Lambda = \{i\frac{\pi}{2}, -i\frac{\pi}{2}\}$. With the notations above, we can choose $u_0 = 1$ (a constant function) and $\beta = \frac{2(2-i\pi)}{\pi(4+\pi^2)}$. Assuming also that $\int_0^\pi b_1(x)dx \neq 0$, hypothesis (5.22) is satisfied, and the flow on the center manifold of the origin is given by Eq. (5.19) with

$$K_1 = Re\ A_1 = -\frac{2}{4+\pi^2}\int_0^\pi b_1(x)dx, \quad K_2 = Re\ A_2 = \frac{\pi^2}{4+\pi^2}.$$

If $\int_0^\pi b_1(x)dx > 0$ (respectively < 0), the Hopf bifurcation is supercritical (respectively subcritical). In both cases, the nontrivial periodic orbits bifurcating from $\alpha = 0$ are unstable.

Example 5.3. Consider the problem:

$$\begin{aligned} &\frac{\partial u(t,x)}{\partial t} = d\frac{\partial^2 u(t,x)}{\partial x^2} + a(x,\alpha)u(t,x) + b(x,\alpha)u(t-1,x)[1 - u(t,x)^2], \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad t > 0,\ x \in (0,\pi) \\ &u(t,0) = u(t,\pi) = 0, \end{aligned} \tag{5.27}$$

where $a(x,\alpha) = a_0 + \alpha a_1(x) + O(\alpha^2), b(x,\alpha) = b_0 + \alpha b_1(x) + O(\alpha^2)$ are C^1 functions and a_0, b_0 constants. In the space $C = C([-1,0];X)$, $X = L^2[0,\pi]$, consider $\dot{u}(t) = d\Delta u(t) + a_0 u(t) + b_0 u(t-1)$, the linearized equation for $u = 0, \alpha = 0$. One can prove that this equation has pure imaginary simple eigenvalues $\pm i\omega$, $\omega \neq 0$, iff a_0, b_0 are such that $b_0 \cos\omega = dk^2 - a_0, b_0 \sin\omega = -\omega$, for some $k \in \mathbb{N}$. In this situation, for $\Lambda = \{i\omega, -i\omega\}$ and $k \in \mathbb{N}$ fixed, one can choose $u_0(x) = \sin(kx)$ in the definition of the bases Φ, Ψ. It is easy to see that the characteristic equation is equivalent to the sequence of equations $a_0 + b_0 e^{-\lambda} - \lambda = dm^2$, $m \in \mathbb{N}$. Let $a_0 = a_k := dk^2 (k \in \mathbb{N})$ and $b_0 = -\frac{\pi}{2}$; in this situation, suppose also that d, k are such that $\pm i\omega = \pm i\frac{\pi}{2}$ are the only roots of these equations with zero real parts. For this, it is sufficient to assume that $2d(2k-1) > \pi$. Assuming also that $\int_0^\pi (a_1(x) - ib_1(x))\sin^2(kx)dx \neq 0$, we conclude that for (5.27) a Hopf bifurcation occurs in the center manifold of the origin. With the previous notations,

$\beta = \frac{4(2-i\pi)}{\pi(4+\pi^2)}$, $A_1 = \beta \int_0^\pi (a_1(x) - ib_1(x)) \sin^2(kx)dx$ and $A_2 = -\frac{\beta\pi i}{2} \int_0^\pi \sin^4(kx)dx$. Theorem 5.2 implies that the flow on this manifold is given by (5.19), with

$$\begin{aligned} K_1 &= \frac{4}{\pi(4+\pi^2)} \int_0^\pi [2a_1(x) - \pi b_1(x)] \sin^2(kx)dx \\ K_2 &= -\frac{2\pi}{4+\pi^2} \int_0^\pi \sin^4(kx)dx < 0. \end{aligned}$$

Hence, the non-trivial periodic orbits near $\alpha = 0$ arrising from the Hopf bifurcation are always stable. The direction of the Hopf bifurcation depends on the sign of K_1.

As we saw, in the situation of Eq. (5.23) with $n = 2$ and either Neumann or Dirichlet conditions, the computation of the cubic terms of the normal form on the center manifold are quite easy, since $f_2(x, y, 0) = 0$. It is more complicated to apply the algorithm of normal forms in the case $f_2(x, y, 0) \neq 0$. To see how it works, consider now (5.23) with $n = 1$. Then, the quadratic and cubic terms in (φ, α) for $F_0(\varphi, \alpha)$ have the form

$$\frac{1}{2}F_2(\varphi, \alpha) = \alpha a_1\varphi(0) + \alpha b_1\varphi(-1) + cb_0\varphi(0)\varphi(-1), \ \frac{1}{3!}F_3(\varphi, 0) = 0,$$

respectively. Thus,

$$\begin{aligned} \frac{1}{2}f_2^1(x, y, 0) =& c < \Psi(0), b_0(\Phi(-1)x + y(-1))(\Phi(0)x + y(0)) > \\ =& c \begin{pmatrix} \beta < u_0, b_0(e^{-i\omega}u_0x_1 + e^{i\omega}\bar{u}_0x_2 + y(-1))(u_0x_1 + \bar{u}_0x_2 + y(0)) > \\ \bar{\beta} < \bar{u}_0, b_0(e^{-i\omega}u_0x_1 + e^{i\omega}\bar{u}_0x_2 + y(-1))(u_0x_1 + \bar{u}_0x_2 + y(0)) > \end{pmatrix} \end{aligned}$$

From (4.6), (5.20) and the definition of F_0, we have $f_3^1(x, 0, 0) = 0$, $g_2^1(x, 0, 0) = 0$, and

$$\frac{1}{3!}g_3^1(x, 0, 0) = \frac{1}{4}Proj_S[(D_xf_2^1)U_2^1 + (D_yf_2^1)h](x, 0, 0), \tag{5.29}$$

where U_2^1 and h are as in (5.24), (5.25). After computing f_2^1 and U_2^1, we get

$$Proj_S[(D_xf_2^1)U_2^1](x, 0, 0) = \begin{pmatrix} C_1x_1^2x_2 \\ \overline{C}_1x_1x_2^2 \end{pmatrix}$$

where

$$Re\ C_1 = -\frac{8c^2}{\omega}Im\ \Big(\beta^2e^{-i\omega}Re\,(e^{i\omega}) < u_0, b_0u_0^2 >< u_0, b_0|u_0|^2 > \Big). \tag{5.30}$$

To determine $Proj_S[(D_yf_2^1)h](x, 0, 0)$, and using the definitions of π, A_1 and M_2^2 in Sections 3 and 4, we start by noting that $h(x)$ is evaluated by the system

$$\dot{h}(x) - D_xh(x)Bx = 2c\Phi < \Psi(0), b_0(\Phi(0)x)(\Phi(-1)x) > \tag{5.31_a}$$

$$\dot{h}(x)(0) - d\Delta h(x)(0) - L_0h(x) = 2cb_0(\Phi(0)x)(\Phi(-1)x), \tag{5.31_b}$$

where $\dot{h}$ denotes the derivative of $h(x)(\theta)$ relative to θ. Writing h, which is a homogeneous second order polynomial in $(x_1, x_2) \in \mathbb{C}^2$ and coefficients in $\operatorname{Ker}\pi$, as

$$h(x) = h_{20}x_1^2 + h_{11}x_1x_2 + h_{02}x_2^2,$$

from ($5.31_{a,b}$) we get that $h_{11} = 0$ and $h_{02} = \overline{h_{20}}$. A few computations give us

$$Proj_S[(D_yf_2^1)h](x, 0, 0) = \begin{pmatrix} C_2x_1^2x_2 \\ \overline{C}_2x_1x_2^2 \end{pmatrix}$$

where

$$C_2 = 2c\beta < u_0, b_0[u_0(h_{11}(-1) + e^{-i\omega}h_{11}(0)) + \bar{u}_0(h_{20}(-1) + e^{i\omega}h_{20}(0))] > . \tag{5.32}$$

On the other hand, $(5.31_{a,b})$ implies that h_{02} and h_{11} are determined respectively by

$$\left\{\begin{array}{l} \dot{h}_{20}(\theta) - 2i\omega h_{20}(\theta) = 2ce^{-i\omega}[\beta < u_0, b_0 u_0^2 > e^{i\omega\theta} u_0 + \bar{\beta} < \bar{u}_0, b_0 u_0^2 > e^{-i\omega\theta}\bar{u}_0] \\ \dot{h}_{20}(0) - d\Delta h_{20}(0) - a_0 h_{20}(0) - b_0 h_{20}(-1) = 2ce^{-i\omega} b_0 u_0^2 \end{array}\right. \tag{5.33}$$

and

$$\left\{\begin{array}{l} \dot{h}_{11}(\theta) = 4cRe\,(e^{i\omega})[\beta < u_0, b_0|u_0|^2 > e^{i\omega\theta} u_0 + \bar{\beta} < \bar{u}_0, b_0|u_0|^2 > e^{-i\omega\theta}\bar{u}_0] \\ \dot{h}_{11}(0) - d\Delta h_{11}(0) - a_0 h_{11}(0) - b_0 h_{11}(-1) = 4cRe\,(e^{i\omega}) b_0 |u_0|^2. \end{array}\right. \tag{5.34}$$

Theorem 5.3. *Consider Eq. (5.23) with $n = 1$ and boundary conditions (5.2) or (5.3), and suppose that (5.15), (5.16) and (5.22) hold. Then a Hopf bifurcation occurs at $\alpha = 0$ on a locally 2-dimensinal center manifold of the origin. On this manifold, the flow is given in polar coordinates by equation (5.19), with*

$$K_1 = Re\Big(\frac{< u_0, (a_1 + b_1 e^{-i\omega}) u_0 >}{< u_0, (1 + b_0 e^{-i\omega}) u_0 >}\Big), \ K_2 = \frac{1}{4} Re\,(C_1 + C_2),$$

where $Re\, C_1$ is given by (5.30) and C_2 is determined by (5.32), (5.33) and (5.34).

Example 5.4. Consider the problem:

$$\begin{array}{l} \dfrac{\partial u(t,x)}{\partial t} = d\dfrac{\partial^2 u(t,x)}{\partial x^2} + b(x,\alpha) u(t-1,x)[1 + u(t,x)], \quad t > 0, \ x \in (0,\pi) \\ \dfrac{\partial u}{\partial x}(t,0) = \dfrac{\partial u}{\partial x}(t,\pi) = 0. \end{array} \tag{5.35}$$

We note that the linear part of (5.35) is as in (5.26). Let $b(x,\alpha) = -\frac{\pi}{2} + \alpha b_1(x) + O(\alpha^2)$, with $\int_0^\pi b_1(x)dx \neq 0$. The coefficient K_1 in (5.19) is still given by $K_1 = -\frac{2}{4+\pi^2}\int_0^\pi b_1(x)dx$.

From (5.30), and since $\omega = -\frac{\pi}{2}$, we have $C_1 = 0$. From (5.33)–(5.34), we get $h_{11} = 0$ and

$$\left\{\begin{array}{l} \dot{h}_{20}(\theta) - i\pi h_{20}(\theta) = i\pi^2[\beta e^{\frac{i\pi\theta}{2}} + \bar{\beta} e^{\frac{-i\pi\theta}{2}}] \\ \dot{h}_{20}(0) - d\Delta h_{20}(0) + \frac{\pi}{2} h_{20}(-1) = i\pi, \end{array}\right. \tag{5.36}$$

where β is as in Example 5.2. Define $z = h_{20}(0)$. This system gives $h_{20}(-1) = -z + \frac{8}{3(4+\pi^2)}(-4 + 2\pi + 2i + i\pi)$, for $z \in D = \{v \in W^{2,2}[0,\pi] : v'(0) = v'(\pi) = 0\}$ such that

$$dz'' + (\frac{\pi}{2} - i\pi)z = \frac{\pi}{3(4+\pi^2)}(-16 + 8\pi + 20i + 4i\pi - 3i\pi^2).$$

Solving this equation, we obtain $h_{20}(-1) + ih_{20}(0) = \frac{2}{15(4+\pi^2)}(-12 + 20\pi - 3\pi^2 - 4i + 9i\pi^2)$. Using (5.32), we finally get

$$K_2 = \frac{1}{4} Re\, C_2 = \frac{\pi(2 - 3\pi)}{5(4+\pi^2)} < 0$$

(see also [6]). We then conclude that the Hopf bifurcation gives raise to nontrivial stable periodic orbits on the center manifold. The direction of the Hopf bifurcation is super-, respectively subcritical, if $\int_0^\pi b_1(x)dx < 0$, respectively > 0.

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