Drift Time Detection and Adjustment Procedures for Processes Subject to Linear Trend

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In this paper, we consider production processes when the mean of a quality characteristic is drifted linearly with time. First, we introduce a procedure to detect the drift time of the process mean as early as possible. Then, based on the estimate of the drift time, a new adjustment procedure based on the maximum likelihood estimate of the drift time is developed to keep the process mean on target. We analyze and compare the performance of the proposed estimator with cumulative sum (CUSUM) and exponentially moving average (EWMA) change point estimation procedures. It is observed that the proposed procedure indeed estimates the drift time effectively for moderate and large trend rates. However, there is a noticeable decrease in its ability to detect drift time at small trend rates. Furthermore, the performance of the new adjustment procedure is compared with EWMA controllers. It is shown that the new procedure is more stable through a wide range of trend rates and its performance does not depend on any parameters of the process.

Keywords: Drift time; Linear trend; Maximum likelihood estimate; Process adjustment

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1. Introduction

Statistical control charts are widely used to monitor process characteristics in industry. Shewhart developed the control chart in 1924 (see, Shewhart 1931) for differentiating between the inevitable random causes and the assignable causes in a process. When a control chart signals an out-of-control alarm, process engineers initiate a search for the assignable cause of the process disturbance. The experience and the knowledge of the process engineers allow them to identify the combination of process variables responsible for the process change.

The output characteristics of many manufacturing processes exhibit trend patterns. The trend can be positive or negative and linear or nonlinear. An example for a positive trend in process mean includes tool-wear where a tool wear out results in linearly increasing product dimension, while a continuous clogging of a spray nozzle represents a negative trend in process mean. Identifying when the process changes would simplify the search for assignable causes and minimize the production of defective units. When the change time is accurately estimated, it would enable the process engineers to quickly identify the assignable causes and make the proper adjustments. The sooner assignable causes are detected the sooner corrective actions are activated which leads to reduction in losses due to deviations from the mean.

The maximum likelihood estimation technique has been used to identify the point of step-change in the process mean. Samuel *et al.* (1998) and Pignatiello and Samuel (2001) base their approach on Hinkley (1970). They consider an estimator based on the maximum likelihood estimator (MLE) for step-change point in process mean once the Shewhart

$\overline{X}$ control chart issues a signal. Hinkley discusses the asymptotic properties of the estimator.

The problem of two-phase regression; a change from one straight line regression to another; has been studied by a number of authors. Hudson (1966) develops the maximum likelihood estimate of the intersection between two straight lines and Hinkley (1969, 1971) derives the asymptotic distribution for the maximum likelihood estimate of the intersection. Bacon *et al.* (1971) use Bayesian analysis to estimate the transition between two intersecting straight lines. Maronna *et al.* (1978) and Beckman *et al.* (1979) study the same problem using likelihood ratio test and show the critical values for the test by simulation.

The two-phase regression is different from the problem of identifying the drift time in a process monitored by a control chart since a) drift time depends on the run length distribution of the control chart, and b) the small data sampling in quality control makes asymptotic distributions impractical in most of quality control applications. Hence, the two-phase problem can be considered as a drift time identification problem without the implications of the quality control system.

In this paper, we consider Shewhart, CUSUM, and EWMA control charts for individual observations and the MLE procedure to identify process drift time. The procedure is applied once the control chart issues an out-of-control signal. The MLE of the drift time is compared with CUSUM procedure in Page (1954) and EWMA estimator procedure proposed in Nishina (1992).

This paper is organized as follows. The process model for a process subject to a linear trend in the process mean is introduced in section 2 followed by the derivation and confidence interval of the drift time estimator in section 3. A numerical example is given in section 4 to illustrate the approach. In section 5, we analyze and compare the performance of the proposed estimator with CUSUM and EWMA estimator procedures. A new adjustment procedure for linearly trended processes is presented in section 6. We compare the performance of the proposed procedure with EWMA controllers in section 7.

2. Process Model

The process under study considers monitoring the characteristic of a product using a control chart. The process is subject to linear trend. The objective is to determine the earliest possible time to detect the trend occurrence.

We assume that the process is initially in-control; i.e., observations are assumed to be normally and independently distributed from a normal distribution with a known mean μ_0 and known standard deviation σ_0. After an unknown point in time τ, the process starts to drift away from the standard mean μ_0 by $\beta\sigma_0 / \textit{unit time}$ where β is unknown. In other words, the quality characteristic at time *t*, y_t, can be represented as

$$y_t \sim N\left(\mu_0, \sigma_0^2\right), \qquad t \le \tau$$

$$y_t \sim N\left(\mu_0 + \beta\sigma_0(t-\tau), \sigma_0^2\right), \qquad t > \tau$$

We assume that y_T is the first observation to exceed the control limit of the control chart and that this signal is not a false alarm. Thus, $y_1, y_2, ..., y_\tau$ are observations from an in-control process, while $y_{\tau+1}, y_{\tau+2}, ..., y_T$ are observations from out-of-control process.

3. Derivation of the Drift Time Estimator

The log likelihood function for the linear trend process model can be derived as:

The *pdf* of the trend is

$$f(y_t) = \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-(y_t-\mu_t)^2/2\sigma_0^2}$$

and the corresponding likelihood function is

$$L(\beta,\tau) = \prod_{t=1}^{T} \frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-(y_t-\mu_t)^2/2\sigma_0^2}$$

The log likelihood function is

$$l(\beta,\tau) = \log L(\beta,\tau) = \sum_{t=1}^{T} \log\left[\frac{1}{\sqrt{2\pi\sigma_0^2}} e^{-(y_t-\mu_t)^2/2\sigma_0^2}\right]$$

$$= -\frac{T}{2}\log(2\pi) - \frac{T}{2}\log\sigma_0^2 - \frac{1}{2\sigma_0^2}\left\{\sum_{t=1}^{\tau}(y_t-\mu_0)^2 + \sum_{t=\tau+1}^{T}(y_t-(t-\tau)\beta\sigma_0-\mu_0)^2\right\}$$

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$$\hat{\beta} = \frac{\sum_{t=\hat{\tau}+1}^{T} (y_t - \mu_0)(t - \hat{\tau})}{\sigma_0 \sum_{t=\hat{\tau}+1}^{T} (t - \hat{\tau})^2} \qquad (2)$$

Substituting the value of $\hat{\beta}$ from equation (2) in equation (1), we obtain $\hat{\tau}$ that maximizes the log likelihood function

$$\hat{\tau} = \arg\min_{\tau} \left[-2 \sum_{t=\tau+1}^{T} y_t (t - \tau)\hat{\beta} + 2\mu_0 \sum_{t=\tau+1}^{T} (t - \tau)\hat{\beta} + \sum_{t=\tau+1}^{T} (t - \tau)^2 \hat{\beta}^2 \sigma_0 \right] = \arg\min_{\tau} [D_\tau] \qquad (3)$$

where $\hat{\tau}$ is the value of τ in the range $1 \le \tau < T$ which minimizes D_τ in equation (3).

3.1. Confidence Interval Estimation

For any given value of $\hat{\tau}$, it can be shown from equation (2) that the asymptotic distribution of $\hat{\beta}(\hat{\tau})$ is

$$\hat{\beta}(\hat{\tau}) \sim N\left(\beta, \frac{4(T - \hat{\tau})}{[(T - \hat{\tau})(T - \hat{\tau} + 1)]^2} \right)$$

Therefore a $100(1-\alpha)$ confidence interval on β is given by

$$\hat{\beta}(\hat{\tau}) \pm Z_{\alpha/2} \frac{2\sqrt{(T - \hat{\tau})}}{[(T - \hat{\tau})(T - \hat{\tau} + 1)]}$$

where $Z_{\alpha/2}$ is the percentage point of the standard normal distribution such that $P\left(z \geq Z_{\alpha/2}\right) = \alpha/2$.

In order to derive the confidence interval for a drift time τ, we present an approach based on the likelihood ratio (LR) statistic. According to the process model in section (2), the standard mean μ_0 of the process is assumed to start changing at time τ with an unknown trend rate β. The control chart issues an out-of-control signal at time *T* when the process reaches the control limit of the control chart. The least squares estimator of β is the value which minimizes the sum of squares function

$$S = \sum_{t=1}^{\tau} \left(y_t - \mu_0\right)^2 + \sum_{t=\tau+1}^{T} \left(y_t - \mu_0 - (t-\tau)\beta\sigma_0\right)^2$$

The minimum value of *S* is denoted by $\hat{S}$ which is achieved at $\hat{\tau}$ and $\hat{\beta}\left(\hat{\tau}\right)$. Since the least squares estimators are the MLE under Gaussian errors for linear and nonlinear models, the MLE of β is used for the confidence interval estimation procedure.

We use the grid search procedure described by Lerman (1980) to estimate the confidence interval for τ. This procedure can be used for linear and nonlinear regression models. The procedure is based on carrying out a grid search over τ to map $S(\tau)$, the minimum value of the residual sum of squares at τ. $S(\tau)$ is defined for the process model as

$$S(\tau) = \sum_{t=1}^{\tau} (y_t - \mu_0)^2 + \sum_{t=\tau+1}^{T} \left(y_t - \mu_0 - (t-\tau)\hat{\beta}(\tau)\sigma_0\right)^2, \qquad 2 \le \tau \le T-2 \tag{4}$$

where $\hat{\beta}(\tau)$ is the MLE of β at τ.

In the two-phase model literature, Hinkley (1971) and Feder (1975) suggest using the two-sided LR test for the null hypothesis $H_0 : \gamma = \gamma_0$ for finite samples*,* where γ represents the intersection point of two regression lines

$$\frac{\left(S^0 - \hat{S}\right)}{s^2} \le F_{1,m-p}^{\alpha} \tag{5}$$

and

$$s^2 = \hat{S}/(m-p)$$

where *m* is the number of data points in the regression model. S^0 is the minimum value of residual sum of squares under hypothesis H_0, and *p* (=1) denotes the dimension of the parameter space, s^2 is the residual mean square, and $F_{v,w}^{\alpha}$ denotes the upper α per cent value of the central *F* distribution with *v* and *w* degrees of freedom.

Based on (5) the values of $S(\tau)$ can be used to estimate an approximate confidence interval for the drift time τ such as

$$S(\tau) \le \hat{S} + s^2 F_{1,m-p}^{\alpha} \tag{6}$$

Lerman (1980) shows that confidence intervals based on (6) give reliable results for quite small samples in linear regression models.

Following the procedure above, we summarize the steps required to draw the confidence intervals based on the drift time τ. First, we sequentially split the observations into two parts using different values of τ and fit the process model, then calculate $S(\tau)$ at the different values of τ from (4). Find $\hat{S}$ (the minimum value of $S(\tau)$), then estimate the confidence interval from (6).

3.2. Confidence Interval Analysis

The confidence interval procedure is studied for different values of linear trend rates. A process is simulated with the first 50 observations normally distributed with mean 10.0 and standard deviation 1.0. Starting from observation 51, observations are randomly generated from a normal distribution with mean $10.0+(t-50)\beta$ (where $t>50$) and standard deviation 1.0 until a Shewhart (3σ) control chart issues an out-of-control signal at time *T*. This scenario is repeated for $\beta = 0.1, 0.5,$ and 1.0. Figure 1 shows the relation between $S(\tau)$ and τ. It can be seen that the confidence intervals decrease and the estimate for the change point is enhanced with the increase of the trend rate β. The estimate of the drift time τ and its 95% confidence interval for the above example are shown in table 1.

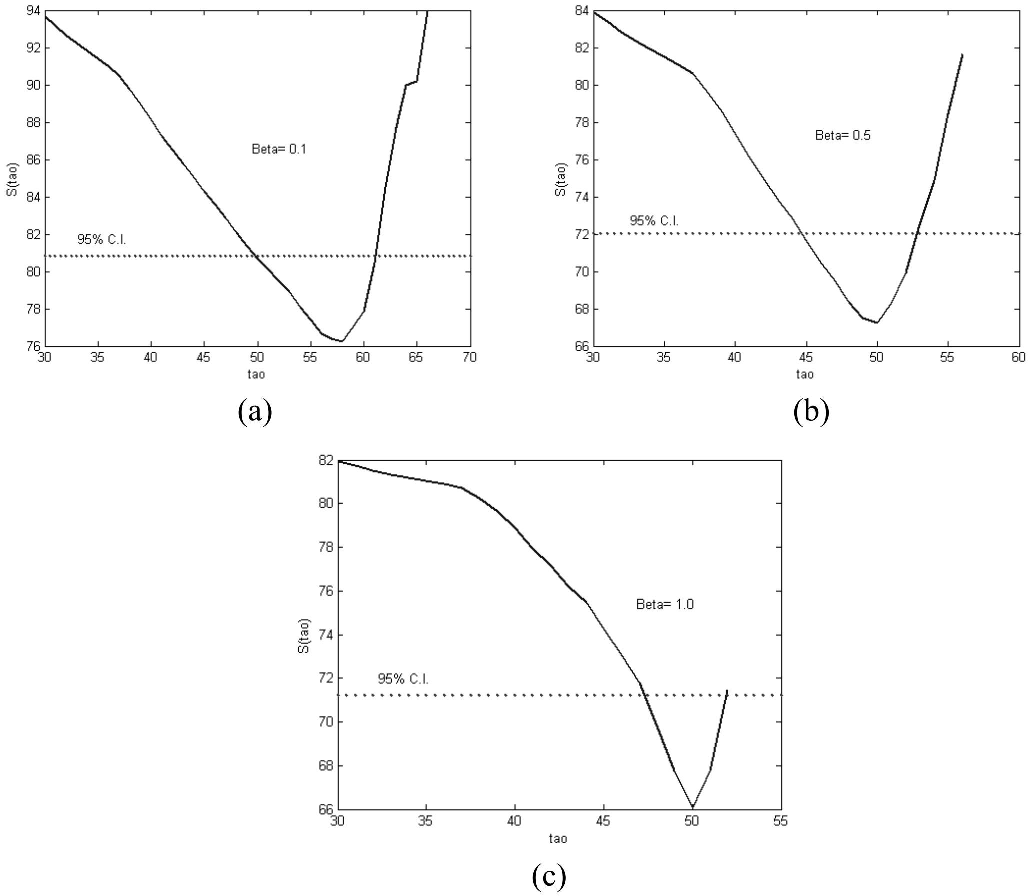

Figure 1. Relation between $S(\tau)$ and τ (a) at $\beta = 0.1$; (b) at $\beta = 0.5$; (c) at $\beta = 1.0$

Table 1. Simulated process results

β	T	$\hat{\tau}$	95% C.I. of τ
0.1	67	58	(49.73, 61.09)
0.5	57	50	(44.66, 52.83)
1.0	53	50	(47.30, 51.93)

4. Numerical Example

Consider the turning process of the outside diameter of steel rods. The nominal outer diameter (OD) of the steel rods is 10.0 mm and the in-control production output from the turning operation is considered independent and normally distributed with a mean $\mu_0 = 10.0$ and standard deviation $\sigma_0 = 1.0$.

Although, we intend to use individual observations collected at fixed intervals of time, the same procedure can be applied for subgroups of size *M*.

A Shewhart control chart for individuals X-chart is used for the detection of out-of-control process and the corresponding upper control limit UCL and lower control limit LCL are 13.0 and 7.0 respectively.

The steel rods OD measurements are shown in table 2. The process starts to drift from the in-control mean ($\mu_0 = 10.0$) after the 10^{th} observation with a trend rate equals to 0.1. From table 2, it can be seen that Shewhart X-chart issues an out-of-control signal at the 30^{th} observation since $y_{30} > UCL$, hence $T = 30$. Figure 2, illustrates the observations from the numerical example on a Shewhart chart.

Table 2. Steel rods OD measurements

t	y_t	t	y_t
1	9.87	16	9.99
2	9.03	17	11.96
3	10.89	18	11.22
4	9.64	19	11.05
5	8.39	20	11.56
6	9.57	21	12.20
7	9.93	22	12.09
8	10.36	23	11.99
9	9.46	24	10.35
10	8.91	25	11.24
11	9.27	26	11.08
12	10.95	27	12.44
13	11.66	28	10.46
14	9.76	29	12.72
15	10.81	30	14.10

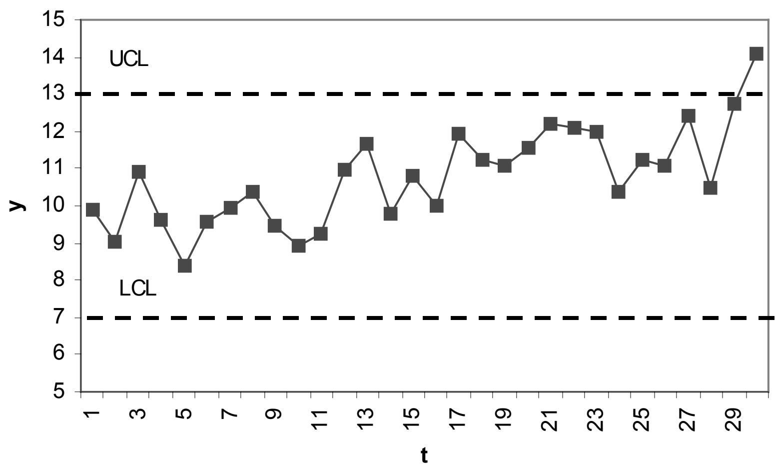

Figure 2. Shewhart chart for the numerical example

To estimate the drift time of the process using the proposed approach, we determine the value of τ in the range $1 \leq \tau < T$ which minimizes D_τ.

The values of $\hat{\beta}$ and D_τ for the numerical example are shown in table 3. It can be seen that the minimum value of D_τ is associated with the 10th observation. Thus, we conclude that the 10th observation is the last observation from in-control process. This coincides with the exact observation from the actual generated data. Thus, process engineers would search for the assignable causes that might have occurred between observations 10 and 11. Also, process engineers can use the estimate of the linear trend $\hat{\beta}$ in the process to propose corrective actions. The relation between D_τ and τ for the numerical example is shown in figure 3.

Table 3. Example results

τ	$\hat{\beta}$	D_τ	τ	$\hat{\beta}$	D_τ
29	4.10	-16.84	14	0.17	-42.04
28	2.18	-23.87	13	0.15	-42.61
27	1.30	-23.67	12	0.14	-43.37
26	0.93	-25.99	11	0.13	-44.11
25	0.70	-27.25	10	0.12	-44.40
24	0.56	-28.30	9	0.12	-44.25
23	0.45	-28.47	8	0.11	-43.87
22	0.38	-29.46	7	0.10	-43.48
21	0.33	-30.99	6	0.09	-43.03
20	0.29	-32.96	5	0.09	-42.45
19	0.26	-34.90	4	0.08	-41.59
18	0.24	-36.56	3	0.08	-40.68
17	0.22	-38.10	2	0.07	-39.93
16	0.20	-39.85	1	0.07	-39.06
15	0.18	-41.02			

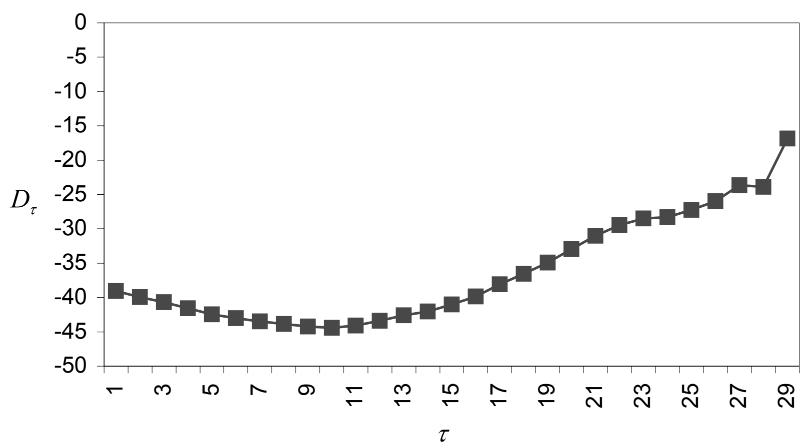

Figure 3. Estimates of D_τ at different values of τ for the numerical example

5. Performance Analysis

The performance of the proposed estimator is studied by simulation. In the simulation study, the drift time of process is designed to occur at $\tau = 50$. Therefore, observations 1 to 50 are independently and randomly generated from a normal distribution with mean 10.0 and standard deviation 1.0. Starting from observation 51, observations are independently and randomly generated from a normal distribution with mean $10.0+(t-50)\beta$ (where $t>50$) and standard deviation 1.0 until the control chart issues a signal. This procedure is repeated 10,000 times for each value of β. This scenario is repeated with same randomly generated data to analyze the performance of the MLE procedure when it is accompanied with Shewhart X-chart, CUSUM, and EWMA charts. The parameters of these charts are set such that the control charts have almost equivalent performances based on in-control average run length value 370.

During the simulation study, when a false alarm is encountered at time t; before the actual designed drift time; the control chart is restarted at time $t+1$ without changing the actual designed drift time. For example, when a false alarm occurs at $t=10$, the control chart statistic will be restarted at $t=11$. Therefore, the individual observation at $t=11$ is treated as if it is the first observation during the in-control period. This procedure deals with false alarms in the same way they have been treated in practice. A similar procedure is used in Pignatiello and Samuel (2001), Fahmy and Elsayed (2005), and Perry and Pignatiello (2005).

5.1. CUSUM Change Point Estimation Procedure

Page (1954) proposes to use this procedure to identify the change point using CUSUM control charts. The CUSUM estimator procedure is summarized for an CUSUM chart with parameters *K* and *H* as follows

a) CUSUM statistics are

$$C_i^+ = \max\left[0, y_t - (\mu_0 + K) + C_{i-1}^+\right] \tag{7}$$

$$C_i^- = \max\left[0, \left(\mu_0 - K\right) - y_t + C_{i-1}^-\right] \tag{8}$$

where the starting values are $C_0^+ = C_0^- = 0$

b) CUSUM chart issues an out-of-control signal when C_i^+ or C_i^- exceeds the control limit H

b) If $C_i^+ \geq H$, the time when shift occurs is identified by counting the number of consecutive periods since C_i^+ is greater than zero. The same procedure is used when $C_i^- \geq H$. The change point estimate using CUSUM chart to identify the change point τ can be expressed as

$$\hat{\tau}_{CUSUM} = \left\{t : C_t^{\pm} \leq 0, 0 < C_i^{\pm} < H \left(i = t+1, \ldots. T-1\right), C_T^{\pm} \geq H\right\} \tag{9}$$

5.2. EWMA Change Point Estimation Procedure

A similar approach to the CUSUM estimator is proposed by Nishina (1992) to use EWMA control charts to identify the change point in a process. The EWMA estimator is summarized for an EWMA chart with parameters λ and *L* as follows

a) EWMA statistics E_t at time *t* is defined as

$$E_t = \lambda\, y_t + \left(1 - \lambda\right) E_{t-1} \tag{10}$$

where $0 < \lambda \leq 1$ and $E_0 = \mu_0$

b) EWMA chart issues an out-of-control signal when E_t exceeds the control limits

$$UCL_t = \mu_0 + L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}\left[1-(1-\lambda)^{2t}\right]} \qquad (11)$$

$$LCL_t = \mu_0 - L\sigma\sqrt{\frac{\lambda}{(2-\lambda)}\left[1-(1-\lambda)^{2t}\right]} \qquad (12)$$

c) If $E_t \geq UCL_t$, the time when shift occurred will be identified by counting the number of consecutive periods since E_t exceeds the standard mean μ_0. The EWMA estimator for the change point τ can be expressed as

$$\hat{\tau}_{EWMA} = \{t : E_t \leq \mu_0, \mu_0 < E_i < UCL_i \ (i = t+1, \ldots, T-1), E_T \geq UCL_T\} \qquad (13)$$

5.3. MLE with Shewhart Chart

The simulation results for the MLE when it is accompanied with Shewhart X-chart are shown in tables 4 and 5. Table 4 shows the average run length (ARL) and its standard error for Shewhart (3σ) at different trend rates, the expected value of the deviation d and its standard error. The deviation d is the difference between the estimate of drift time $\hat{\tau}$ and the actual drift time $(\tau = 50)$; i.e., $d = \hat{\tau} - \tau$. The estimate of $E(d)$ and its standard error are

$$E(d) = \frac{1}{10{,}000}\sum_{i=1}^{10{,}000} d_i$$

$$SE(d) = \frac{s}{\sqrt{10{,}000}}$$

where

$$s = \sqrt{\frac{\sum_{i=1}^{10,000}(d_i - E(d))}{(10,000-1)}}$$

The estimated probability of drift time detection within n observations from the actual drift time is shown in table 5. Where $\hat{p}(|\hat{\tau} - \tau| \leq n)$ is the estimated probability that the absolute difference between $\hat{\tau}$ and τ is less than or equal n.

Table 4. Simulation results for the estimates of drift time (Shewhart chart)

β	Shewhart (3σ)			
	Run Length		MLE	
	ARL	$SE(RL)$	$E(d)$	$SE(d)$
0.05	30.69	0.100	3.51	0.106
0.1	18.54	0.056	1.97	0.071
0.2	11.04	0.032	1.02	0.048
0.3	8.13	0.023	0.64	0.038
0.4	6.55	0.018	0.43	0.033
0.5	5.53	0.015	0.28	0.030
0.6	4.84	0.013	0.18	0.027
0.7	4.29	0.012	0.12	0.025
0.8	3.87	0.010	0.02	0.024
0.9	3.55	0.010	-0.02	0.023
1.0	3.29	0.009	-0.05	0.022

As shown in table 5, the proposed estimator detects the actual drift time within ± 1 observation in 50% or more of the 10,000 simulation trials for trend rates greater than 0.4. These results indicate the usefulness of the proposed estimator in detecting the actual drift time of a process and the decreasing the size of search window.

Table 5. Estimated probability of detection for MLE with Shewhart chart based on 10,000 trials

β	0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$\hat{p}(\hat{\tau} = \tau)$	0.047	0.072	0.114	0.154	0.189	0.221	0.259	0.291	0.324	0.351	0.383
$\hat{p}(\lvert\hat{\tau} - \tau\rvert \le 1)$	0.134	0.207	0.309	0.394	0.477	0.547	0.613	0.668	0.716	0.752	0.792
$\hat{p}(\lvert\hat{\tau} - \tau\rvert \le 2)$	0.218	0.328	0.480	0.596	0.695	0.773	0.833	0.873	0.905	0.923	0.936
$\hat{p}(\lvert\hat{\tau} - \tau\rvert \le 3)$	0.297	0.432	0.617	0.746	0.836	0.895	0.926	0.945	0.955	0.960	0.965
$\hat{p}(\lvert\hat{\tau} - \tau\rvert \le 4)$	0.369	0.526	0.727	0.853	0.920	0.953	0.965	0.971	0.973	0.976	0.978
$\hat{p}(\lvert\hat{\tau} - \tau\rvert \le 5)$	0.431	0.603	0.812	0.915	0.957	0.970	0.975	0.978	0.980	0.982	0.984
$\hat{p}(\lvert\hat{\tau} - \tau\rvert \le 6)$	0.485	0.671	0.876	0.952	0.973						
$\hat{p}(\lvert\hat{\tau} - \tau\rvert \le 7)$	0.537	0.731	0.919	0.970	0.979						
$\hat{p}(\lvert\hat{\tau} - \tau\rvert \le 8)$	0.582	0.780	0.948	0.977	0.984						
$\hat{p}(\lvert\hat{\tau} - \tau\rvert \le 9)$	0.629	0.828	0.967	0.982	0.986						
$\hat{p}(\lvert\hat{\tau} - \tau\rvert \le 10)$	0.673	0.869	0.976	0.985	0.988						
$\hat{p}(\lvert\hat{\tau} - \tau\rvert \le 11)$	0.712	0.902									
$\hat{p}(\lvert\hat{\tau} - \tau\rvert \le 12)$	0.748	0.925									
$\hat{p}(\lvert\hat{\tau} - \tau\rvert \le 13)$	0.778	0.944									
$\hat{p}(\lvert\hat{\tau} - \tau\rvert \le 14)$	0.810	0.960									
$\hat{p}(\lvert\hat{\tau} - \tau\rvert \le 15)$	0.835	0.968									

5.4. MLE with CUSUM Chart

In this section, we compare the performance of the MLE using equation (3) with the CUSUM estimator using equation (9). Both estimators are used to identify the drifting point when an CUSUM chart issues an out-of-control signal. We consider an CUSUM chart with parameters *K*= 0.25 and *H*= 8. The CUSUM chart with theses parameters has the same in-control ARL as the Shewhart (3σ) control chart.

Table 6 shows the ARL of the CUSUM (0.25,8) and the expected value of the deviation d for the MLE and CUSUM estimators along with their standard errors under different trend rates.

The comparison between the MLE and CUSUM estimator shows that the CUSUM estimator performs slightly better than MLE for $\beta = 0.05$ but for other trend rates the MLE performs much better than CUSUM estimator. Also, it should be noted that the standard error of the MLE estimator is significantly reduced with the increase of the trend rate.

Table 6. Simulation results for the estimates of drift time (CUSUM chart)

β	CUSUM (0.25,8)					
	Run Length		MLE		CUSUM Estimator	
	ARL	$SE(RL)$	$E(d)$	$SE(d)$	$E(d)$	$SE(d)$
0.05	19.71	0.052	2.24	0.129	1.72	0.103
0.1	13.38	0.032	0.05	0.098	-1.64	0.093
0.2	9.14	0.020	-0.83	0.075	-3.47	0.089
0.3	7.33	0.015	-0.87	0.064	-4.21	0.087
0.4	6.29	0.013	-0.87	0.057	-4.62	0.087
0.5	5.59	0.011	-0.88	0.053	-4.89	0.086
0.6	5.09	0.010	-0.80	0.049	-5.05	0.086
0.7	4.69	0.009	-0.74	0.045	-5.16	0.086
0.8	4.37	0.008	-0.67	0.041	-5.26	0.086
0.9	4.11	0.008	-0.65	0.040	-5.35	0.086
1.0	3.89	0.007	-0.61	0.038	-5.41	0.086

The estimated probability for MLE and CUSUM estimators in detecting the drift time within n observations from the actual drift time is shown in table 7.

The results in table 7 indicate that the MLE significantly outperforms the CUSUM estimator especially in the range of trend magnitude from 0.2 to 1.0. For $\beta = 0.05$. Although on average the CUSUM estimator slightly outperforms the proposed MLE, the results indicate that the MLE correctly identifies the drift time similar, if not more effectively than the CUSUM estimator. While for $\beta = 0.1$, both estimators have almost the same performance in detecting 0.1 linear trend rate.

Table 7. Estimated probability of detecting the actual drift time for MLE and CUSUM estimators based on 10,000 trials

β		0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$\hat{p}(\hat{\tau} = \tau)$	MLE	0.044	0.067	0.116	0.163	0.201	0.239	0.276	0.307	0.343	0.371	0.403
	CUSUM	0.040	0.069	0.106	0.135	0.158	0.177	0.197	0.212	0.227	0.238	0.250
$\hat{p}(\lvert\hat{\tau} - \tau\rvert \le 1)$	MLE	0.116	0.193	0.320	0.423	0.504	0.573	0.635	0.686	0.728	0.762	0.792
	CUSUM	0.114	0.192	0.284	0.349	0.392	0.423	0.441	0.452	0.457	0.461	0.462
$\hat{p}(\lvert\hat{\tau} - \tau\rvert \le 2)$	MLE	0.190	0.311	0.494	0.617	0.705	0.774	0.823	0.853	0.882	0.901	0.912
	CUSUM	0.194	0.317	0.447	0.519	0.550	0.561	0.564	0.562	0.558	0.554	0.551
$\hat{p}(\lvert\hat{\tau} - \tau\rvert \le 3)$	MLE	0.259	0.414	0.623	0.749	0.819	0.865	0.896	0.910	0.926	0.936	0.942
	CUSUM	0.274	0.434	0.573	0.622	0.630	0.627	0.623	0.618	0.614	0.610	0.607
$\hat{p}(\lvert\hat{\tau} - \tau\rvert \le 4)$	MLE	0.325	0.506	0.724	0.835	0.882	0.910	0.929	0.937	0.948	0.953	0.958
	CUSUM	0.360	0.542	0.669	0.690	0.683	0.675	0.671	0.667	0.664	0.661	0.658
$\hat{p}(\lvert\hat{\tau} - \tau\rvert \le 5)$	MLE	0.385	0.587	0.797	0.881	0.912	0.929	0.942	0.948	0.956	0.960	0.965
	CUSUM	0.432	0.632	0.729	0.729	0.719	0.711	0.708	0.704	0.701	0.698	0.696
$\hat{p}(\lvert\hat{\tau} - \tau\rvert \le 6)$	MLE	0.439	0.659	0.848	0.910	0.929						
	CUSUM	0.502	0.703	0.766	0.756	0.747						
$\hat{p}(\lvert\hat{\tau} - \tau\rvert \le 7)$	MLE	0.490	0.713	0.881	0.927	0.940						
	CUSUM	0.567	0.757	0.793	0.781	0.772						
$\hat{p}(\lvert\hat{\tau} - \tau\rvert \le 8)$	MLE	0.540	0.761	0.905	0.937	0.947						
	CUSUM	0.628	0.800	0.815	0.804	0.797						
$\hat{p}(\lvert\hat{\tau} - \tau\rvert \le 9)$	MLE	0.592	0.805	0.921	0.944	0.953						
	CUSUM	0.687	0.833	0.832	0.822	0.815						
$\hat{p}(\lvert\hat{\tau} - \tau\rvert \le 10)$	MLE	0.634	0.838	0.931	0.949	0.957						
	CUSUM	0.739	0.856	0.848	0.840	0.833						
$\hat{p}(\lvert\hat{\tau} - \tau\rvert \le 11)$	MLE	0.675	0.867									
	CUSUM	0.783	0.873									
$\hat{p}(\lvert\hat{\tau} - \tau\rvert \le 12)$	MLE	0.714	0.888									
	CUSUM	0.821	0.887									
$\hat{p}(\lvert\hat{\tau} - \tau\rvert \le 13)$	MLE	0.746	0.905									
	CUSUM	0.856	0.900									
$\hat{p}(\lvert\hat{\tau} - \tau\rvert \le 14)$	MLE	0.776	0.919									
	CUSUM	0.880	0.908									
$\hat{p}(\lvert\hat{\tau} - \tau\rvert \le 15)$	MLE	0.806	0.928									
	CUSUM	0.900	0.916									

5.5 MLE with EWMA Chart

The same simulation study is repeated to analyze the performance of the proposed MLE when it is combined with EWMA (0.1,2.7) control chart. We compare its performance with EWMA estimator using equation (13). The EWMA chart with these parameters has a similar 370 in-control ARL which is the same as that of the Shewhart (3σ) and the CUSUM (0.25,8) charts.

The ARL of EWMA (0.1,2.7) and the expected value of d, the deviation from the true drift time $\tau = 50$, for different trend rates along with their standard errors are shown in table 8. The results indicate that the performance of the MLE when it is combined with EWMA chart is similar to the previous case when it is combined with CUSUM chart. It is also observed that at $\beta = 0.05$ and $\beta = 0.1$, EWMA estimator is closer to the true drift time than MLE. However, for trend rates greater than 0.1, the MLE provides much better average estimate for the drift time.

Table 8. Simulation results for the estimates of drift time (EWMA chart)

β	EWMA (0.1,2.7)					
	Run Length		MLE		EWMA Estimator	
	ARL	$SE(RL)$	$E(d)$	$SE(d)$	$E(d)$	$SE(d)$
0.05	19.21	0.052	3.70	0.124	3.13	0.101
0.1	12.85	0.032	0.77	0.095	-0.07	0.092
0.2	8.68	0.020	-0.45	0.073	-1.94	0.087
0.3	6.93	0.016	-0.66	0.062	-2.72	0.086
0.4	5.94	0.013	-0.78	0.057	-3.11	0.085
0.5	5.26	0.011	-0.78	0.051	-3.38	0.084
0.6	4.77	0.010	-0.79	0.049	-3.57	0.084
0.7	4.40	0.009	-0.72	0.044	-3.72	0.083
0.8	4.11	0.009	-0.69	0.041	-3.81	0.083
0.9	3.86	0.008	-0.65	0.039	-3.91	0.083
1.0	3.64	0.008	-0.64	0.038	-3.98	0.082

The estimated probability for MLE and EWMA estimators in detecting the drift time within n observations from the actual drift time is shown in table 9.

Table 9. Estimated probability of detecting the actual drift time for MLE and EWMA estimators based on 10,000 trials

β		0.05	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$\hat{p}(\hat{\tau}=\tau)$	MLE	0.039	0.064	0.112	0.160	0.194	0.233	0.268	0.299	0.334	0.365	0.395
	EWMA	0.034	0.049	0.069	0.086	0.103	0.117	0.130	0.147	0.162	0.176	0.190
$\hat{p}(\|\hat{\tau}-\tau\|\le 1)$	MLE	0.108	0.184	0.307	0.409	0.493	0.561	0.621	0.670	0.716	0.749	0.780
	EWMA	0.094	0.139	0.203	0.255	0.303	0.343	0.380	0.414	0.442	0.470	0.496
$\hat{p}(\|\hat{\tau}-\tau\|\le 2)$	MLE	0.182	0.297	0.476	0.602	0.696	0.765	0.810	0.842	0.874	0.891	0.904
	EWMA	0.159	0.236	0.345	0.431	0.500	0.550	0.586	0.612	0.628	0.638	0.642
$\hat{p}(\|\hat{\tau}-\tau\|\le 3)$	MLE	0.245	0.397	0.608	0.738	0.812	0.860	0.888	0.905	0.920	0.929	0.936
	EWMA	0.222	0.337	0.493	0.594	0.647	0.675	0.687	0.691	0.691	0.690	0.689
$\hat{p}(\|\hat{\tau}-\tau\|\le 4)$	MLE	0.307	0.489	0.715	0.828	0.878	0.908	0.924	0.934	0.942	0.948	0.952
	EWMA	0.292	0.445	0.626	0.700	0.726	0.729	0.727	0.725	0.724	0.722	0.721
$\hat{p}(\|\hat{\tau}-\tau\|\le 5)$	MLE	0.364	0.569	0.793	0.878	0.910	0.928	0.937	0.946	0.952	0.957	0.959
	EWMA	0.362	0.546	0.716	0.760	0.761	0.757	0.754	0.751	0.750	0.749	0.748
$\hat{p}(\|\hat{\tau}-\tau\|\le 6)$	MLE	0.417	0.637	0.847	0.911	0.928						
	EWMA	0.435	0.639	0.780	0.790	0.785						
$\hat{p}(\|\hat{\tau}-\tau\|\le 7)$	MLE	0.469	0.697	0.884	0.927	0.939						
	EWMA	0.506	0.718	0.817	0.813	0.808						
$\hat{p}(\|\hat{\tau}-\tau\|\le 8)$	MLE	0.515	0.748	0.911	0.938	0.946						
	EWMA	0.573	0.779	0.839	0.831	0.826						
$\hat{p}(\|\hat{\tau}-\tau\|\le 9)$	MLE	0.564	0.797	0.927	0.944	0.952						
	EWMA	0.638	0.827	0.854	0.845	0.841						
$\hat{p}(\|\hat{\tau}-\tau\|\le 10)$	MLE	0.609	0.835	0.936	0.949	0.957						
	EWMA	0.699	0.861	0.867	0.859	0.855						
$\hat{p}(\|\hat{\tau}-\tau\|\le 11)$	MLE	0.653	0.869									
	EWMA	0.756	0.885									
$\hat{p}(\|\hat{\tau}-\tau\|\le 12)$	MLE	0.695	0.892									
	EWMA	0.801	0.902									
$\hat{p}(\|\hat{\tau}-\tau\|\le 13)$	MLE	0.732	0.909									
	EWMA	0.843	0.914									
$\hat{p}(\|\hat{\tau}-\tau\|\le 14)$	MLE	0.766	0.925									
	EWMA	0.874	0.921									
$\hat{p}(\|\hat{\tau}-\tau\|\le 15)$	MLE	0.795	0.934									
	EWMA	0.900	0.928									

From table 9, we conclude that the MLE significantly outperforms the EWMA estimator especially in the range of trend magnitudes from 0.2 to 1.0. While on average the EWMA estimator slightly outperforms the MLE for $\beta = 0.05$. These results are similar to the results from the comparison between MLE and CUSUM estimator.

5.6. Effect of Change Point Location on the Performance of MLE

We now investigate the sensitivity of the proposed MLE to the location /time of the change point τ. The same simulation procedure is repeated at several instants of drift times $\tau = 150, 250, \text{and } 350$ to compare the performance of the different estimators at different locations of the change point τ.

The results for the average performance of MLE when it is used with Shewhart (3σ) chart at different locations of τ are presented in table 10. This comparison shows the robustness of the MLE against the change point location τ.

In table 11, we show the average performance of the MLE and CUSUM estimator for CUSUM (0.25,8) control chart at different locations of τ. This comparison between the performance between MLE and CUSUM estimators shows that both estimators have the same performance regardless of change point location τ.

Table 10. Simulation results for the estimates of drift time $\tau = 150$, 250, and 350

(Shewhart chart)

β	τ	Shewhart (3σ)			
		Run Length		MLE Estimator	
		ARL	$SE(RL)$	$E(d)$	$SE(d)$
0.05	150	30.73	0.100	3.42	0.108
	250	30.64	0.100	3.52	0.107
	350	30.61	0.101	3.59	0.107
0.1	150	18.47	0.056	1.96	0.072
	250	18.43	0.056	1.97	0.072
	350	18.47	0.056	2.08	0.071
0.2	150	11.08	0.032	0.98	0.048
	250	11.05	0.032	1.06	0.046
	350	11.03	0.031	0.98	0.050
0.3	150	8.09	0.023	0.62	0.040
	250	8.12	0.023	0.64	0.039
	350	8.12	0.023	0.60	0.040
0.4	150	6.52	0.018	0.40	0.034
	250	6.51	0.018	0.43	0.034
	350	6.54	0.018	0.37	0.034
0.5	150	5.50	0.015	0.26	0.030
	250	5.50	0.015	0.30	0.029
	350	5.51	0.015	0.21	0.031
0.6	150	4.79	0.013	0.14	0.029
	250	4.79	0.013	0.19	0.027
	350	4.80	0.013	0.11	0.030
0.7	150	4.26	0.012	0.08	0.026
	250	4.27	0.012	0.11	0.025
	350	4.27	0.012	0.01	0.029
0.8	150	3.86	0.011	0.02	0.025
	250	3.86	0.010	0.07	0.022
	350	3.88	0.010	-0.02	0.026
0.9	150	3.54	0.010	0.00	0.023
	250	3.54	0.010	-0.01	0.023
	350	3.55	0.009	-0.04	0.023
1.0	150	3.28	0.009	-0.05	0.023
	250	3.26	0.009	-0.09	0.024
	350	3.28	0.009	-0.06	0.022

Table 11. Simulation results for the estimates of drift time $\tau = 150$, 250, and 350 (CUSUM chart)

β	τ	CUSUM (0.25, 8)					
		Run Length		MLE Estimator		CUSUM Estimator	
		ARL	$SE(RL)$	$E(d)$	$SE(d)$	$E(d)$	$SE(d)$
0.05	150	19.68	0.051	2.34	0.127	1.56	0.108
	250	19.66	0.052	2.39	0.126	1.67	0.107
	350	19.72	0.051	2.32	0.127	1.71	0.106
0.1	150	13.38	0.032	-0.01	0.098	-1.77	0.099
	250	13.36	0.032	0.18	0.096	-1.76	0.097
	350	13.38	0.032	0.05	0.097	-1.68	0.096
0.2	150	9.13	0.020	-0.73	0.073	-3.70	0.096
	250	9.14	0.020	-0.63	0.071	-3.60	0.093
	350	9.14	0.020	-0.83	0.075	-3.57	0.092
0.3	150	7.34	0.015	-0.81	0.062	-4.43	0.094
	250	7.34	0.015	-0.85	0.061	-4.30	0.091
	350	7.35	0.015	-0.90	0.063	-4.30	0.091
0.4	150	6.29	0.013	-0.84	0.055	-4.82	0.094
	250	6.29	0.013	-0.76	0.052	-4.69	0.091
	350	6.29	0.013	-0.89	0.056	-4.69	0.090
0.5	150	5.58	0.011	-0.81	0.051	-5.07	0.093
	250	5.58	0.011	-0.73	0.048	-4.95	0.091
	350	5.58	0.011	-0.77	0.049	-4.97	0.090
0.6	150	5.08	0.010	-0.74	0.046	-5.23	0.093
	250	5.07	0.010	-0.68	0.044	-5.08	0.090
	350	5.07	0.010	-0.79	0.046	-5.13	0.090
0.7	150	4.69	0.009	-0.68	0.042	-5.36	0.093
	250	4.68	0.009	-0.66	0.041	-5.20	0.090
	350	4.68	0.009	-0.77	0.044	-5.23	0.090
0.8	150	4.37	0.008	-0.63	0.039	-5.43	0.092
	250	4.37	0.008	-0.60	0.038	-5.30	0.090
	350	4.37	0.008	-0.71	0.041	-5.32	0.090
0.9	150	4.11	0.008	-0.58	0.036	-5.51	0.092
	250	4.10	0.008	-0.56	0.036	-5.38	0.090
	350	4.10	0.008	-0.66	0.039	-5.40	0.090
1.0	150	3.89	0.007	-0.58	0.035	-5.56	0.092
	250	3.89	0.007	-0.53	0.034	-5.45	0.090
	350	3.88	0.007	-0.62	0.037	-5.45	0.090

Finally, we show the average performance of the MLE and EWMA estimator for EWMA (0.1,2.7) control chart when $\tau = 150, 250, \text{and } 350$ in table12. The comparison between these two estimators reveals that both estimators are robust to change point location τ.

Table 12. Simulation results for the estimates of drift time $\tau = 150$, 250, and 350

(EWMA chart)

β	τ	EWMA (0.1, 2.7)					
		Run Length		MLE Estimator		EWMA Estimator	
		ARL	$SE(RL)$	$E(d)$	$SE(d)$	$E(d)$	$SE(d)$
0.05	150	19.23	0.052	3.65	0.125	2.92	0.104
	250	19.15	0.053	3.67	0.123	3.05	0.103
	350	19.25	0.052	3.69	0.124	3.10	0.103
0.1	150	12.86	0.032	0.63	0.096	-0.24	0.095
	250	12.84	0.032	0.86	0.092	-0.18	0.095
	350	12.89	0.032	0.72	0.095	-0.23	0.096
0.2	150	8.66	0.020	-0.41	0.073	-2.19	0.091
	250	8.69	0.020	-0.41	0.071	-2.11	0.091
	350	8.69	0.020	-0.52	0.075	-2.16	0.092
0.3	150	6.92	0.016	-0.65	0.062	-2.90	0.089
	250	6.94	0.016	-0.65	0.061	-2.82	0.090
	350	6.94	0.016	-0.74	0.063	-2.89	0.091
0.4	150	5.92	0.013	-0.78	0.056	-3.29	0.088
	250	5.93	0.013	-0.68	0.053	-3.22	0.089
	350	5.93	0.013	-0.78	0.056	-3.30	0.090
0.5	150	5.25	0.011	-0.80	0.051	-3.56	0.088
	250	5.25	0.011	-0.68	0.048	-3.49	0.088
	350	5.25	0.011	-0.75	0.050	-3.53	0.089
0.6	150	4.76	0.010	-0.80	0.049	-3.75	0.087
	250	4.76	0.010	-0.68	0.046	-3.66	0.088
	350	4.77	0.010	-0.81	0.048	-3.72	0.089
0.7	150	4.39	0.010	-0.76	0.045	-3.87	0.087
	250	4.39	0.009	-0.67	0.043	-3.79	0.087
	350	4.39	0.009	-0.81	0.046	-3.87	0.089
0.8	150	4.09	0.009	-0.66	0.040	-3.99	0.087
	250	4.09	0.009	-0.63	0.040	-3.91	0.087
	350	4.10	0.009	-0.76	0.043	-3.97	0.088
0.9	150	3.85	0.008	-0.63	0.038	-4.07	0.087
	250	3.85	0.008	-0.61	0.038	-3.99	0.087
	350	3.85	0.008	-0.70	0.040	-4.06	0.088
1.0	150	3.63	0.008	-0.65	0.038	-4.14	0.086
	250	3.65	0.008	-0.57	0.035	-4.07	0.087
	350	3.64	0.008	-0.66	0.038	-4.12	0.088

6. Derivation of the Adjustment Procedure

In this section, we present a new feedback adjustment procedure for linearly trended processes. The procedure is based on the maximum likelihood estimators for the process linear trend β and the drift time τ presented in section 3.

It is assumed that the changes in properties of the quality characteristic / process output y_t are only due to a linear trend disturbance according to the model

$$\begin{cases} y_t = \mu_0 + x_{t-1} + \beta\sigma_0(t-\tau) + \varepsilon_t, & t > \tau \\ y_t = \mu_0 + \varepsilon_t, & t \le \tau \end{cases} \tag{14}$$

where x_{t-1} is the level of the controllable factor at time t-1and $\varepsilon_t \sim N(0, \sigma_0^2)$.

The adjustment procedure works under a hierarchical three-phase methodology. During the first phase, process output is monitored using a statistical process control chart. In the second phase estimates β and tW

at $t=T$:

$$\hat{\beta}_T = \frac{\sum_{t=\hat{\tau}+1}^{T} y_t(t-\hat{\tau}) - \mu_0 \sum_{t=\hat{\tau}+1}^{T}(t-\hat{\tau})}{\sigma_0 \sum_{t=\hat{\tau}+1}^{T}(t-\hat{\tau})^2}$$

at $t=T+1$:

$$\hat{\beta}_{T+1} = \frac{\sum_{t=\hat{\tau}+1}^{T} y_t(t-\hat{\tau}) + w_{T+1}(T+1-\hat{\tau}) - \mu_0 \sum_{t=\hat{\tau}+1}^{T+1}(t-\hat{\tau})}{\sigma_0 \sum_{t=\hat{\tau}+1}^{T+1}(t-\hat{\tau})^2}$$

$$\hat{\beta}_{T+1} = \frac{\hat{\beta}_T \sigma_0 \sum_{t=\hat{\tau}+1}^{T}(t-\hat{\tau})^2 + w_{T+1}(T+1-\hat{\tau}) - \mu_0(T+1-\hat{\tau})}{\sigma_0 \sum_{t=\hat{\tau}+1}^{T+1}(t-\hat{\tau})^2}$$

where w_t is the unadjusted process output. w_t is defined as

$$w_t = y_t - x_{t-1}$$

Generally,

$$\hat{\beta}_{T+i} = \frac{\hat{\beta}_{T+i-1} \sigma_0 \sum_{t=\hat{\tau}+1}^{T+i-1}(t-\hat{\tau})^2 + w_{T+i}(T+i-\hat{\tau}) - \mu_0(T+i-\hat{\tau})}{\sigma_0 \sum_{t=\hat{\tau}+1}^{T+i}(t-\hat{\tau})^2} \tag{16}$$

It is clear from equation (16) that the updating process for the estimate of linear trend rate depends on the previous estimate value of $\hat{\beta}$, the current unadjusted process value and $\hat{\tau}$. This recursive nature of the updating process makes it more practical.

7. Process Adjustment Simulations

Simulation is used to study the performance of the proposed adjustment procedure and compare it with the Exponentially Weighted Moving Average (EWMA) controllers at different levels of linear trend disturbance according to the model described in equation (14).

EWMA feedback controllers have been used for years in semiconductor industry and many authors study their performance especially under linear trend disturbance (Ingolfsson and Sachs (1993), and Del Castillo (1999, 2001)).

The so-called double EWMA controller developed by Butler and Stefani (1994) can be written according to equation (14) as

$$x_t = \mu_0 - a_t - R_t \tag{17}$$

$$a_t = \lambda_1 \left(y_t - x_{t-1} \right) + \left(1 - \lambda_1 \right) a_{t-1}, \quad 0 \le \lambda_1 \le 1 \tag{18}$$

$$R_t = \lambda_2 \left(y_t - x_{t-1} - a_{t-1} \right) + \left(1 - \lambda_2 \right) R_{t-1}, \quad 0 \le \lambda_2 \le 1 \tag{19}$$

A single EWMA is obtained when $\lambda_2 = 0$ and $R_0 = 0$ in Equation (19).

The performance of the proposed adjustment procedure is compared with single EWMA and double EWMA controllers. As for the adjustment performance characterization, we use the normalized mean square error as the performance index. The smaller the value, the better the performance. The normalized mean square error is defined as

$$MSE / \sigma_0^2 = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{y_{T+i} - \mu_0}{\sigma_0}\right)^2 \qquad (20)$$

where n is the number of adjustments in each simulation run.

In the simulation study we assume that the process starts to drift away from the target value at time $\tau = 50$. Namely, the first 50 observations are independently and randomly generated from a normal distribution with mean 10.0 and standard deviation 1.0. Starting from observation 51, observations are independently and randomly generated from a normal distribution with mean $10.0 + (t - 50)\beta$ (where $t > 50$) and standard deviation 1.0. The adjustment starts after a Shewhart (3σ) chart issues an out-of-control signal at time T. The performa optimal trade-off between the long-run performance and transient performance at lower and higher trend rate respectively (Del Castillo 1999).

Tables 13 shows the mean and standard deviation (in parentheses) of the normalized mean square error for the different controllers under linear trend rates from 0.1 to 1.0 when production ends at $t = 100$.

It is shown that the proposed adjustment procedure has a more stable performance over a wide range of trend rates while the performance of the EWMA controllers depends on the controller parameters and their performances gradually deteriorate especially at higher trend rates $\beta > 0.40$.

Table 13. Mean and standard deviation (in parentheses) of the normalized mean square error for the different controllers based on 1000 trials (production ends at $t = 100$)

β	Proposed Procedure	Single EWMA		Double EWMA	
		$\lambda_1 = 0.30$	$\lambda_1 = 0.90$	$\lambda_1 = 0.03, \lambda_2 = 0.29$	$\lambda_1 = 0.20, \lambda_2 = 0.85$
0.1	2.037 (0.640)	1.353 (0.345)	1.830 (0.562)	1.266 (0.345)	2.165 (0.652)
0.2	2.010 (0.570)	1.727 (0.319)	1.848 (0.501)	1.398 (0.313)	2.125 (0.581)
0.3	2.005 (0.544)	2.303 (0.320)	1.904 (0.479)	1.593 (0.304)	2.112 (0.553)
0.4	2.005 (0.539)	3.090 (0.334)	1.989 (0.475)	1.861 (0.305)	2.107 (0.548)
0.5	2.008 (0.534)	4.098 (0.363)	2.102 (0.472)	2.203 (0.316)	2.104 (0.546)
0.6	2.012 (0.528)	5.297 (0.387)	2.236 (0.466)	2.607 (0.322)	2.102 (0.537)
0.7	2.017 (0.522)	6.711 (0.413)	2.394 (0.461)	3.082 (0.332)	2.101 (0.530)
0.8	2.020 (0.519)	8.328 (0.435)	2.576 (0.457)	3.625 (0.337)	2.099 (0.525)
0.9	2.026 (0.520)	10.160 (0.461)	2.785 (0.458)	4.241 (0.345)	2.100 (0.524)
1.0	2.032 (0.520)	12.206 (0.491)	3.019 (0.457)	4.930 (0.358)	2.103 (0.522)

The behaviour of the proposed adjustment procedure is investigated further when a production ends at $t = 150$; i.e. adjustment period is extended by 50 observations. As shown in table 14, the results indicate that the standard deviation of the normalized mean square error is decreased for all controllers. The performance of the proposed procedure is consistent throughout the entire simulation study. In general, a better performance of the EWMA controllers is achieved when the adjustment period is extended due to the long run property of the EWMA controllers. However, the proposed procedure still outperforms EWMA controllers when $\beta > 0.6$.

Table 14. Mean and standard deviation (in parentheses) of the normalized mean square error for the different controllers based on 1000 trials (production ends at $t = 150$)

β	Proposed Procedure	Single EWMA		Double EWMA	
		$\lambda_1 = 0.30$	$\lambda_1 = 0.90$	$\lambda_1 = 0.03, \lambda_2 = 0.29$	$\lambda_1 = 0.20, \lambda_2 = 0.85$
0.1	2.007 (0.377)	1.307 (0.190)	1.822 (0.331)	1.223 (0.192)	2.126 (0.383)
0.2	1.998 (0.360)	1.664 (0.186)	1.851 (0.316)	1.290 (0.185)	2.113 (0.365)
0.3	1.996 (0.354)	2.230 (0.188)	1.910 (0.311)	1.388 (0.184)	2.107 (0.360)
0.4	1.997 (0.350)	3.013 (0.195)	1.994 (0.308)	1.523 (0.184)	2.103 (0.357)
0.5	1.997 (0.345)	4.015 (0.208)	2.103 (0.305)	1.696 (0.188)	2.099 (0.353)
0.6	1.999 (0.345)	5.228 (0.215)	2.238 (0.304)	1.904 (0.189)	2.098 (0.351)
0.7	2.002 (0.344)	6.658 (0.225)	2.397 (0.303)	2.149 (0.191)	2.097 (0.350)
0.8	2.005 (0.344)	8.299 (0.232)	2.581 (0.302)	2.430 (0.190)	2.097 (0.350)
0.9	2.009 (0.344)	10.161 (0.249)	2.791 (0.302)	2.750 (0.194)	2.098 (0.349)
1.0	2.014 (0.346)	12.238 (0.265)	3.024 (0.302)	3.106 (0.198)	2.099 (0.349)

From an operating point of view, the proposed procedure can be applied using computer-controlled system especially with the increasing computational power of today's computers which allow for efficient real time use of this procedure in practice. Also, it should be noted that the proposed procedure is not intended for a continuous adjustment scheme. Rather, it is triggered when the accompanied control chart issues an out-of-control signal. Based on the results of this research, it can be seen that the proposed adjustment procedure has better performance and consistency than EWMA controllers when a short adjustment period is considered.

8. Conclusions

In this paper, we propose an estimator for identifying the change point in processes subject to a linear trend in process mean. The performance of the proposed estimator is analyzed when it is used with a Shewhart X-chart, CUSUM and EWMA charts. The results show that the proposed estimator outperforms the CUSUM and the EWMA estimators in estimating the actual drift time for almost all investigated trend rates. The study also shows that the proposed estimator maintains same performance regardless of the location of the change point.

We also introduce a new adjustment procedure for linearly trended processes. Simulation study shows that the proposed adjustment procedure is more stable than EWMA controllers over a wide range of linear trend rates and its performance does not depend on the selection of any parameters. These characteristics make it also an ideal controller for processes subject to random trend rates. The proposed adjustment procedure can be easily implemented in practice especially for computer-controlled processes.

A possible extension of this research could be the detection of the drift time and the adjustment procedure for nonlinear drifts as well as considering a drift in the process variance.

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References

BECKMAN, R. J., and COOK, R. D., 1979, Testing for two-phase regressions. *Technometrics*, 21, 65-69.

BACON, D. W., and WATTS, D. G., 1971, Estimating the transition between two intersecting straight lines. *Biometrika*, 58, 525-534.

BUTLER, S. W. and STEFANI, J. A., 1994, Supervisory run-to-run control of a polysilicon gate etch using in situ ellipsometry. *IEEE Transactions on Semiconductor Manufacturing*, 7, 193-201.

DEL CASTELLO, E., 1999, Long-run and transient analysis of a double EWMA feedback controller. *IIE Transactions*, 31, 1157-1169.

DEL CASTELLO, E., 2001, Some properties of EWMA feedback quality adjustment schemes for drifting disturbances. *Journal of Quality Technology*, 33, 153-166.

FAHMY, H. M., and ELSAYED, E. A., 2005, Drift time detection and adjustment procedures for processes subject to linear trend. Report No. 014, Department of Industrial and Systems Engineering, Rutgers University, USA.

FEDER, P. I., 1975, The log likelihood ratio in segmented regression. *The Annals of Statistics*, 3, 84-97.

HINKLEY, D. V., 1969, Inference about the intersection in two-phase regression. *Biometrika*, 56, 495-504.

HINKLEY, D. V., 1970, Inference about the change-point in a sequence of random variable. *Biometrika*, 57, 1-17.

HINKLEY, D. V., 1971, Inference in two-phase regression. *Journal of the American Statistical Association*, 66, 736-743.

HUDSON, D. J., 1966, Fitting segmented curves whose join points have to be estimated. *Journal of the American Statistical Association*, 61, 1097-1124.

INGOLFSSON, A. and SACHS, E., 1993, Stability and sensitivity of an EWMA controller. *Journal of Quality Technology*, 25, 271-287.

LERMAN, P. M., 1980, Fitting segmented regression models by grid search. *Applied Statistics*, 29, 77-84.

MARONNA, R., and YOHAI, V. J., 1978, A bivariate test for the detection of a systematic change in mean. *Journal of the American Statistical Association*, 73, 640-645.

NISHINA, K., 1992, A comparison of control charts from the viewpoint of change-point estimation. *Quality and Reliability Engineering International*, 8, 537-541.

PAGE, E. S., 1954, Continuous inspection schemes. *Biometrics*, 41, 100-115.

PERRY, M. B., and PIGNATIELLO, J. J., JR., 2005, Estimation of the change point of the fraction nonconforming in SPC applications. *International Journal of Reliability, Quality and Safety Engineering,* 12, 2, 95-110.

PIGNATIELLO, J. J., JR., and SAMUEL, T. R., 2001, Estimation of the change point of a normal process mean in SPC applications. *Journal of Quality Technology*, 33, 82-95.

SAMUEL, T. R., PIGNATIELLO, J. J., JR., and CALVIN, J. A., 1998, Identifying the time of a step change with $\overline{X}$ control charts. *Quality Engineering*, 10, 521-527.

SHEWHART, W. A., 1931, *Economic Control of Quality of Manufactured Product* (New York, NY: D. Van Nostrand).