ENUMERATION OF GRAPH COVERINGS AND SURFACE BRANCHED COVERINGS

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Preface

Lots of graphs having a symmetry property can be described as coverings of simpler graphs. In this manuscript, we examine several enumeration problems for various types of nonisomorphic graph coverings of a graph and some of their applications to a group theory or to a surface theory. This manuscript is organized as follows. In chapter 1, we introduce basic concepts. In chapter 2, by using covering graph construction, we count the positive isomorphism classes of cycle permutation graphs, which is equal to the number of double cosets of the dihedral group $\mathbb{D}_n$ in the symmetric group S_n on n elements. In chapter 3, we count nonisomorphic (connected) coverings of a graph and, as its application, we have another recursive formula for the number of conjugacy classes of subgroups of given index of a finitely generated free group. In chapter 4, we count nonisomorphic regular coverings of a graph whose covering transformation groups are abelian and, as its application, we count subgroups of given index of free abelian groups. The same work is done in chapter 5 for regular coverings having dihedral voltage groups. In chapter 6, we discuss a general counting formula for regular coverings having any finite voltage group. In chapter 7, after discussing a combinatorial proof of Hurwitz theorem for surface branched coverings, we consider the number of subgroups of surface groups. Finally, in chapter 8, we discuss a distribution of branched surface coverings of surfaces and some related topological properties including a generalization of the classical Alexander theorem.

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Chapter 1

Definitions and Notations

Let G be a connected finite simple graph with vertex set $V(G)$ and edge set $E(G)$. The *neighborhood* of a vertex $v \in V(G)$, denoted by $N(v)$, is the set of vertices adjacent to v. We use $|X|$ for the cardinality of a set X. The number $\beta(G) = |E(G)| - |V(G)| + 1$ is equal to the number of independent cycles in G and it is referred to as the *Betti number* of G.

Two graphs G and H are *isomorphic* if there exists a one-to-one correspondence between their vertex sets which preserves adjacency, and such a correspondence is called an *isomorphism* between G and H. An *automorphism* of a graph G is an isomorphism of G onto itself. Thus, an automorphism of G is a permutation of the vertex set $V(G)$ which preserves adjacency. Obviously, a composition of two automorphisms is also an automorphism. Hence the automorphisms of G form a permutation group, $\mathrm{Aut}\,(G)$, which acts on the vertex set $V(G)$.

A graph $\widetilde{G}$ is called a *covering* of G with projection $p : \widetilde{G} \to G$ if there is a surjection $p : V(\widetilde{G}) \to V(G)$ such that $p|_{N(\tilde{v})} : N(\tilde{v}) \to N(v)$ is a bijection for any vertex $v \in V(G)$ and $\tilde{v} \in p^{-1}(v)$. We also say that the projection $p : \widetilde{G} \to G$ is an *n-fold covering* of G if p is n-to-one. A covering $p : \widetilde{G} \to G$ is said to be *regular*(simply, *$\mathcal{A}$-covering*) if there is a subgroup $\mathcal{A}$ of the automorphism group $\mathrm{Aut}\,(\widetilde{G})$ of $\widetilde{G}$ acting freely on $\widetilde{G}$ so that the graph G is isomorphic to the quotient graph $\widetilde{G}/\mathcal{A}$, say by h, and the quotient map $\widetilde{G} \to \widetilde{G}/\mathcal{A}$ is the composition $h \circ p$ of p and h. The fibre of an edge or a vertex is its preimage under p.

Two coverings $p_i : \widetilde{G}_i \to G$, $i = 1, 2$, are said to be *isomorphic* (or, *equivalent*) if there exists a graph isomorphism $\Phi : \widetilde{G}_1 \to \widetilde{G}_2$ such that the diagram

$$\begin{tikzcd} \widetilde{G}_1 \arrow[rr, "\Phi"] \arrow[dr, "p_1"'] & & \widetilde{G}_2 \arrow[dl, "p_2"] \\ & G & \end{tikzcd}$$

commutes. Such a Φ is called a *covering isomorphism*. In particular, when $p_1 = p_2$ (say, $= p$) with $\widetilde{G}_1 = \widetilde{G}_2$ (say, $= \widetilde{G}$), it is called a *covering tansformation* of p, and the set of all covering transformations forms a group under the composition, called the *covering transformation group* of the covering $p : \widetilde{G} \to G$.

Every edge of a graph G gives rise to a pair of oppositely directed edges. By $e^{-1} = vu$, we mean the reverse edge to a directed edge $e = uv$. We denote the set of directed edges of G by $D(G)$. Each directed edge e has an initial vertex i_e and a terminal vertex t_e. Following [4], a *permutation voltage assignment* ϕ on a graph G is a map $\phi : D(G) \to S_n$ with the property that $\phi(e^{-1}) = \phi(e)^{-1}$ for each $e \in D(G)$, where S_n is the symmetric group on n elements $\{1, \ldots, n\}$. The *permutation derived graph* G^ϕ is defined as follows: $V(G^\phi) = V(G) \times \{1, \ldots, n\}$, and for each edge $e \in D(G)$ and $j \in \{1, \ldots, n\}$ let there be an edge (e, j) in $D(G^\phi)$ with $i_{(e,j)} = (i_e, j)$ and $t_{(e,j)} = (t_e, \phi(e)j)$. The natural projection $p : G^\phi \to G$ is a covering. Let $\mathcal{A}$ be a finite group. An *ordinary voltage assignment* (or, $\mathcal{A}$*-voltage assignment*) of G is a function $\phi : D(G) \to \mathcal{A}$ with the property that $\phi(e^{-1}) = \phi(e)^{-1}$ for each $e \in D(G)$. The values of ϕ are called *voltages*, and $\mathcal{A}$ is called the *voltage group*. The *ordinary derived graph* $G \times_\phi \mathcal{A}$ derived from an ordinary voltage assignment $\phi : D(G) \to \mathcal{A}$ has as its vertex set $V(G) \times \mathcal{A}$ and as its edge set $E(G) \times \mathcal{A}$, so that an edge (e, g) of $G \times_\phi \mathcal{A}$ joins a vertex (u, g) to $(v, \phi(e)g)$ for $e = uv \in D(G)$ and $g \in \mathcal{A}$. In the (ordinary) derived graph $G \times_\phi \mathcal{A}$, a vertex (u, g) is denoted by u_g, and an edge (e, g) by e_g. The first coordinate projection $p : G \times_\phi \mathcal{A} \to G$, called the natural projection, commutes with the left multiplication action of the $\phi(e)$ and the right action of $\mathcal{A}$ on the fibers, which is free and transitive, so that p is a regular $|\mathcal{A}|$-fold covering, called simply an $\mathcal{A}$*-covering*. Gross and Tucker ([4]) showed that every covering (resp. regular covering) of a graph G can be derived from a permutation (resp. ordinary) voltage assignment ϕ which assigns the identity voltage to the edges of an arbitrary fixed spanning tree T of G.

The following lemma is useful to count nonisomorphic objectives in enumerative combinatorics and will be repeatedly used in this manuscript.

Lemma 1 (Burnside's Lemma) *Let a finite group $\mathcal{A}$ act on a finite set X, and let $X/\mathcal{A}$ denote the set of orbits of the action. Then,*

$$|X/\mathcal{A}| = \frac{1}{|\mathcal{A}|}\sum_{g\in\mathcal{A}}|\mathrm{Fix}\,(g)| = \frac{1}{|\mathcal{A}|}\left(|X| + \sum_{g\in\mathcal{A},\, g\neq 1}|\mathrm{Fix}\,(g)|\right),$$

where $\mathrm{Fix}\,(g) = \{x \in X \mid gx = x\}$, *the set of fixed elements by* g.

Consider another group $\mathcal{A}$-action on a set Y. Two $\mathcal{A}$-actions are called *mutually orthogonal* if each non-identity element g of $\mathcal{A}$ has a fixed element in at most one action, that is, g cannot have a fixed element in both X and Y. Let $\mathcal{A}_x = \{g \in \mathcal{A} \,|\, gx = x\}$ denote the stabilizer of $x \in X$. Then, it follows from Burnside's Lemma that

$$|Y/\mathcal{A}_x| = \frac{|Y|}{|\mathcal{A}_x|} \qquad \text{and} \qquad |X/\mathcal{A}_y| = \frac{|X|}{|\mathcal{A}_y|}$$

for any $x \in X$ and any $y \in Y$.

Chapter 2

Cycle permutation graphs and the double cosets of $\mathbb{D}_n$ in S_n

Throughout this chapter, let S_n denote the symmetric group on n elements $\{1, 2, \ldots, n\}$ and let $\mathbb{D}_n$ denote the dihedral subgroup of S_n containing the n-cycle $\rho = (1\,2\,\cdots\,n)$, so that $|\mathbb{D}_n| = 2n$.

An *n-cycle permutation graph* $P_\alpha(C_n)$ consists of two copies of an n-cycle C_n , say C_x and C_y , with vertex sets $V(C_x) = \{x_1, x_2, \ldots, x_n\}$ and $V(C_y) = \{y_1, y_2, \ldots, y_n\}$, along with edges $x_i y_{\alpha(i)}$ for some $\alpha \in S_n$. The edges $x_i y_{\alpha(i)}$ are called the permutation edges of a cycle permutation graph $P_\alpha(C_n)$.

Let G denote the dumbbell graph with two vertices x, y, an edges $e = xy$ and two loops $e_x = xx, e_y = yy$ as illustrated in Figure 2.1. The permutation derived graph G^ϕ with the voltage assignment ϕ defined by $\phi(e_x) = \phi(e_y) = \rho$ and $\phi(e) = \alpha$, $\alpha \in S_n$, is clearly the cycle

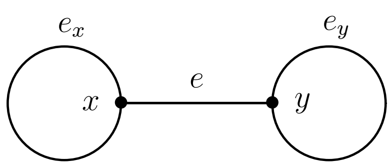

Figure 2.1: The dumbbell graph

permutation graph $P_\alpha(C_n)$. Moreover, with a suitable relabelling of the vertices of the inner cycle C_y of $P_\alpha(C_n)$, we can assume that the permutation edges are x_iy_i, $i = 1, 2, \ldots, n$. It is not difficult to show the following theorem.

Theorem 1 *A cycle permutation graph $P_\alpha(C_n)$ is isomorphic to the permutation derived graph G^ψ with voltage assignment ψ defined by $\psi(e_x) = \rho$, $\psi(e) =$ the identity in S_n and $\psi(e_y) = \alpha^{-1}\rho\alpha$ (or $\psi(e_y) = \alpha^{-1}\rho^{-1}\alpha$) over the dumbbell graph G.*

Note that the permutations $\alpha^{-1}\rho\alpha$ and $\alpha^{-1}\rho^{-1}\alpha$, $\alpha \in S_n$, have the same cycle type as the cycle ρ. Let Σ_n denote the conjugacy class of $\rho = (1\,2\,\cdots\,n)$ in S_n, *i.e.*, Σ_n is the set of all n-cycles in S_n. From the isomorphic identification in Theorem 1, it is enough to consider a permutation derived graph with a permutation voltage assignment which assigns the identity on the edge e, $\rho = (1\,2\,\cdots\,n)$ on the loop e_x and $\sigma \in \Sigma_n$ on the loop e_y of the dumbbell graph G for a cycle permutation graph. Hence, the set Σ_n can be identified with the set of all n-cyclic permutation graphs.

Two n-cycle permutation graphs $P_\alpha(C_n)$ and $P_\beta(C_n)$ are said to be isomorphic by a *positive natural isomorphism* Θ if $\Theta : P_\alpha(C_n) \to P_\beta(C_n)$ is an isomorphism satisfying $\Theta(C_x) = C_x$ and $\Theta(C_y) = C_y$.

The following theorem gives a group-theoretic characterization of two cyclic permutation graphs to be positively natural isomorphic.

Theorem 2 *Let α and β be two permutations in S_n Then the cyclic permutation graphs $P_\alpha(C_n)$ and $P_\beta(C_n)$ are isomorphic by a positive natural isomorphism if and only if there exists $d \in \mathbb{D}_n$ such that*

$$\beta^{-1}\rho\beta = d(\alpha^{-1}\rho\alpha)d^{-1} \qquad \text{or} \qquad \beta^{-1}\rho\beta = d(\alpha^{-1}\rho\alpha)^{-1}d^{-1}\,.$$

It is also equivalent to say that $\beta \in \mathbb{D}_n\alpha\mathbb{D}_n$, that is, the permutations α and β belong to the same double cosets of $\mathbb{D}_n$ in S_n.

Proof: Use the identification $P_\alpha(C_n) = G^\phi$ and $P_\beta(C_n) = G^\psi$ given in Theorem 1. If G^ϕ and G^ψ are isomorphic by a positive natural isomorphism, say Θ, then Θ maps the outer cycle of G^ϕ to the outer cycle of G^ψ isomorphically, which induces an element d in $\mathbb{D}_n$. (Note

Aut$(C_n) = \mathbb{D}_n$.) Then it follows that the path $x_i y_i y_{\alpha^{-1}\rho\alpha(i)} x_{\alpha^{-1}\rho\alpha(i)}$ (or $x_i y_i y_{\alpha^{-1}\rho^{-1}\alpha(i)} x_{\alpha^{-1}\rho^{-1}\alpha(i)}$ depending on the orientation of e_y) in G^ϕ is mapped to the path

$$x_{d(i)} y_{d(i)} y_{\beta^{-1}\rho\beta d(i)} x_{\beta^{-1}\rho\beta d(i)} \quad \text{or} \quad x_{d(i)} y_{d(i)} y_{\beta^{-1}\rho^{-1}\beta d(i)} x_{\beta^{-1}\rho^{-1}\beta d(i)}$$

depending on the orientation of e_y in G^ψ. In either case, we have $\beta^{-1}\rho\beta = d(\alpha^{-1}\rho\alpha)d^{-1}$ or $\beta^{-1}\rho\beta = d(\alpha^{-1}\rho\alpha)^{-1}d^{-1}$ from the construction of the derived covering G^ψ. Also, it gives

$$\begin{aligned} \rho &= (\beta d\alpha^{-1})\rho(\alpha d^{-1}\beta^{-1}) \\ &= (\beta d\alpha^{-1})\rho(\beta d\alpha^{-1})^{-1}, \end{aligned}$$

or

$$\rho^{-1} = (\beta d\alpha^{-1})\rho(\beta d\alpha^{-1})^{-1}$$

for some $d \in \mathbb{D}_n$. Hence, $\beta d\alpha^{-1}$ is contained in the normalizer $N(\rho, \rho^{-1})$ of $\{\rho, \rho^{-1}\}$ in $\mathbb{D}_n$. But $N(\rho, \rho^{-1}) = \mathbb{D}_n$. Therefore, $\beta \in \mathbb{D}_n \alpha \mathbb{D}_n$.

Conversely, if $\beta = d_1 \alpha d_2$ for some $d_1, d_2 \in \mathbb{D}_n$ then $\beta^{-1}\rho\beta = d_2^{-1}\alpha^{-1}d_1^{-1}\rho d_1 \alpha d_2$. Then the element d_2 in $\mathbb{D}_n$ induces an automorphism in the n-cycle C_n, and hence an isomorphism from the outer cycle of G^ϕ to the outer cycle of G^ψ. It is also easily extended to a positive natural isomorphism from G_n^α to G_n^β by the condition. $\square$

So far, we show that the number of double cosets of the dihedral group $\mathbb{D}_n$ in the symmetric group S_n is equal to the number $\mathbf{Iso}_P(C_n)$of positive natural isomorphism classes of n-cyclic permutation graphs. Also, every n-cyclic permutation graph can be constructed as an n-fold covering graph of the dumbbell.

To count the number $\mathbf{Iso}_P(C_n)$, let $\mathcal{I} : S_n \to S_n$ be the map defined by $\mathcal{I}(\sigma) = \sigma^{-1}$ for all $\sigma \in S_n$ and denote $\Gamma = \mathbb{D}_n \times \{1, \mathcal{I}\}$. Define a group action $\Gamma \times \Sigma_n \to \Sigma_n$ by $(d,1)(\sigma) = d\sigma d^{-1}$ and $(d,\mathcal{I})(\sigma) = d\sigma^{-1}d^{-1}$. Then, by Theorem 2 and Burnside's Lemma, we get

$$\mathbf{Iso}_P(C_n) = |\Sigma_n/\Gamma| = \frac{1}{4n} \sum_{\gamma\in\Gamma} |\mathrm{Fix}\,(\gamma)|,$$

where $\mathrm{Fix}\,(\gamma) = \{\sigma \in \Sigma_n \,:\, \gamma\sigma = \sigma\}$.

The authors computed $|\mathrm{Fix}\,(\gamma)|$ for each $\gamma \in \Gamma$ to get the following theorem.

n	3	4	5	6	7	8	9	10	11
$\mathbf{Iso}_P(C_n)$	1	2	4	12	39	202	1219	9468	83435

Table 2.1: The number $\mathbf{Iso}_P(C_n)$ for small n

Theorem 3 ([26]) *Let phi denote the Euler phi-function, that is, phi(n) is the number of natural numbers relatively prime to n. Then*

$\mathbf{Iso}_P(C_2) = 1$,

$\mathbf{Iso}_P(C_n)$

$$= \begin{cases} \dfrac{1}{4n}\left[\displaystyle\sum_{d|n} phi(d)^2 \left(\frac{n}{d}-1\right)!\, d^{\frac{n}{d}-1} + \frac{n}{2}!\left(3+\frac{n}{2}\right)2^{\frac{n}{2}-1}\right] & \text{if } n \text{ is even,} \\ \dfrac{1}{4n}\left[\displaystyle\sum_{d|n} phi(d)^2 \left(\frac{n}{d}-1\right)!\, d^{\frac{n}{d}-1} + n2^{\frac{n-1}{2}}\left(\frac{n-1}{2}\right)!\right] & \text{if } n \text{ is odd.} \end{cases}$$

In particular, for odd prime q,

$$\mathbf{Iso}_P(C_q) = \frac{1}{4q}\left[(q-1)! + (q-1)^2 + q2^{\frac{q-1}{2}}\left(\frac{q-1}{2}\right)!\right].$$

A short calculation gives the table 2.1 for $\mathbf{Iso}_P(C_n)$.

Question: Find an algorithm to list the representatives σ's in Σ_n of the double cosets of $\mathbb{D}_n$ in S_n. It gives how to draw all positively nonisomorphic cycle permutation graphs. Compute the size of each double coset of $\mathbb{D}_n$ in S_n. It gives how many permutations in S_n present the same cycle permutation graph.

Remark According to J.M. Montesinos [41], any closed orientable 3-dimensional manifold can be obtained as a finite sheeted covering of the 3-dimensional sphere $\mathbb{S}^3$ branched over the dumbell graph (*i.e.,* over the Hopf link with a bridge). Hence, the number $\mathbf{Iso}_P(C_n)$ of positive natural isomorphism classes of n-cyclic permutation graphs is equal to the number of closed orientable n-fold coverings of the sphere $\mathbb{S}^3$ branched over the dumbell graph.

Chapter 3

Graph coverings and subgroups of free groups

Let G be a connected graph and let T be a fixed spanning tree of G. A permutation voltage assignment ϕ is said to be *normalized* (with respect to T) if ϕ assigns the identity voltage to the edges of the fixed spanning tree T. Let $C^1_T(G; n)$ denote the set of all normalized permutation voltage assignments. In order to find an algebraic characterization of two n-fold coverings $p : G^\phi \to G$ and $q : G^\psi \to G$ to be isomorphic, we assume that they are isomorphic by a covering isomorphism $\Phi : G^\phi \to G^\psi$. Then $\Phi|_{p^{-1}(v)} : p^{-1}(v) \to q^{-1}(v)$ is a bijection between the n vertices $\{v_1, v_2, \ldots, v_n\}$ for all $v \in V(G)$. Now, we define $f : V(G) \to S_n$ by $f(v) = \Phi|_{p^{-1}(v)}$ for all $v \in V(G)$. For an edge $uv \in D(G)$, if (u, h) is joined to (v, k) in G^ϕ, then $\phi(uv)(h) = k$ and $(u, f(u)(h))$ is joined to $(v, f(v)(k))$ in G^ψ for any h. Thus, we have $\psi(uv)f(u) = f(v)\phi(uv)$, or $\psi(uv) = f(v)\phi(uv)f(u)^{-1}$ for all $uv \in D(G)$. The authors showed that the converse is also true.

Theorem 4 ([24]) *Two n-fold coverings $p : G^\phi \to G$ and $q : G^\psi \to G$ are isomorphic if and only if there exists a function $f : V(G) \to S_n$ such that $\psi(uv) = f(v)\phi(uv)f(u)^{-1}$ for each $uv \in D(G)$. Moreover, if $\phi, \psi \in C^1_T(G; n)$, then it is equivalent to say that there exists a permutation $\sigma \in S_n$ such that $\psi(uv) = \sigma\phi(uv)\sigma^{-1}$ for each $uv \in D(G) - D(T)$.*

By labeling the positively directed edges in $D(G) - D(T)$ as e_1, e_2, $\ldots$, $e_{\beta(G)}$, a normalized permutation voltage assignment can be identified as a $\beta(G)$-tuple of permutations in S_n, and the set $C^1_T(G;n)$ can be identified as

$$C^1_T(G;n) = S_n \times S_n \times \cdots \times S_n, \qquad (\beta(G) \quad \text{times}).$$

With an S_n-action on the set $C^1_T(G;n)$ defined by simultaneous coordinatewise conjugacy: for any $g \in S_n$ and any $(\sigma_1, \ldots, \sigma_{\beta(G)}) \in C^1_T(G;n)$,

$$g(\sigma_1, \sigma_2, \ldots, \sigma_{\beta(G)}) = (g\sigma_1 g^{-1}, g\sigma_2 g^{-1}, \ldots, g\sigma_{\beta(G)} g^{-1}),$$

it follows from Theorem 4 that two normalized permutation voltage assignments ϕ, ψ in $C^1_T(G;n)$ derive isomorphic coverings of G if and only if they belong to the same orbit under the S_n-action. That is, each $\beta(G)$-tuple of permutations $(\sigma_1, \ldots, \sigma_{\beta(G)})$, $\sigma_i \in S_n$ is identified with a normalized permutation voltage assignment ϕ in $C^1_T(G;n)$, and such two tuples derive isomorphic coverings of G if and only if they are simultaneous coordinatewise conjugate.

Two β-tuples of permutations $(\sigma_1, \sigma_2, \ldots, \sigma_\beta)$ and $(\tau_1, \tau_2, \ldots, \tau_\beta)$ in S_n are said to be *similar* by g, or simply *similar*, if they are simultaneous coordinatewise conjugate by g, that is,

$$\tau_i = g\sigma_i g^{-1} \;\text{ for }\; i = 1, 2, \ldots, \beta.$$

If we can find $g \in S_n$ that leaves fixed some k in $\{1, 2, \ldots, n\}$, then the tuples are said to be *k-similar*.

By Theorem 4, there is a one-to-one correspondence between the similarity classes of $\beta(G)$-tuples of permutations in S_n and the isomorphism classes of n-fold coverings of the graph G. We denote by $\mathbf{Iso}\,(G;n)$ the number of such isomorphism classes of n-fold coverings of G.

To count $\mathbf{Iso}\,(G;n)$ by Burnside's Lemma, we first count $\mathrm{Fix}\,(g)$ for each $g \in S_n$. Let $C(g)$ and $Z(g)$ denote the conjugacy class containing g and the center of g in the symmetric group S_n, respectively.

Lemma 2 *Under the S_n-action on $C^1_T(G;n) = S_n \times S_n \times \cdots \times S_n$, we have*

(1) *if g_1 and g_2 are conjugate, then* $|\mathrm{Fix}\,(g_1)| = |\mathrm{Fix}\,(g_2)|$,

(2) *for each $g \in S_n$,* $\mathrm{Fix}\,(g) = Z(g) \times Z(g) \times \cdots \times Z(g)$, $\quad \beta(G)$ *times,*

$\beta(G)$	1	2	3	4	5	6
$n=1$	1	1	1	1	1	1
$n=2$	2	4	8	16	32	64
$n=3$	3	11	49	251	1393	8051
$n=4$	5	43	681	14491	336465	7997683
$n=5$	7	161	14721	1730861	207388305	24883501301

Table 3.1: The number $\mathbf{Iso}\,(G;n)$ for small n and small $\beta(G)$

(3) $|C(g)||Z(g)| = n!$ for any $g \in S_n$.

By using Lemma 2 and Burnside's Lemma, we have

Theorem 5 ([24]) *The number of isomorphism classes of n-fold coverings of G is*

$$\mathbf{Iso}\,(G;n) = \sum_{\ell_1+2\ell_2+\cdots+n\ell_n=n} \left(\ell_1!\, 2^{\ell_2}\ell_2!\, \cdots n^{\ell_n}\ell_n!\right)^{\beta(G)-1}.$$

Next, we aim to compute the number $\mathbf{Isoc}\,(G;n)$ of isomorphism classes of *connected* n-fold coverings of G. Let $p : \tilde{G} \to G$ be an n-fold covering and let $\tilde{G}_1, \tilde{G}_2, \ldots, \tilde{G}_\ell$ be the components of $\tilde{G}$. Then $p_i = p|_{\tilde{G}_i} : \tilde{G}_i \to G$ is a connected covering of G for each $i = 1, 2, \ldots, \ell$. Let n_i be the fold number of the connected covering $p_i : \tilde{G}_i \to G$. Then $n_i \geq 1$ and $n_1 + \cdots + n_\ell = n$. In this case, the ordered sequence $[n_1 n_2 \cdots n_\ell]$ with $n_1 \leq n_2 \leq \cdots \leq n_\ell$ is just a partition of n, denoted by $\mathfrak{p}[n]$ or simply by $\mathfrak{p}$. Also, we say that a covering $p : \tilde{G} \to G$ has the *component type* of partition $\mathfrak{p}[n] = [n_1 n_2 \cdots n_\ell]$. Clearly, any two isomorphic n-fold coverings have the same component type. A partition $\mathfrak{p}$ of n is denoted by $[[k; \frac{n}{k}]]$ if every term of $\mathfrak{p}$ is k. Note that $[[k;m]]$ denotes the partition of the natural number km each of whose terms is k. For a partition $\mathfrak{p}$ of n, let $j_k(\mathfrak{p})$ denote the multiplicity of k in the partition $\mathfrak{p}$, so that $j_1(\mathfrak{p}) + 2j_2(\mathfrak{p}) + \cdots + nj_n(\mathfrak{p}) = n$. For convenience, let $\mathfrak{P}(n)$ denote the set of all partitions of a natural number n. For a partition $\mathfrak{p}$ of n, let $\mathbf{Iso}\,(G;\mathfrak{p})$ denote the number of nonisomorphic n-fold coverings of G having the component type $\mathfrak{p}$. Clearly,

$$\mathbf{Iso}\,(G;[[n;1]]) = \mathbf{Isoc}\,(G;n), \qquad \mathbf{Iso}\,(G;[[1;n]]) = 1,$$

and

$$\mathbf{Iso}\,(G;n) = \sum_{\mathfrak{p}\in\mathfrak{P}(n)} \mathbf{Iso}\,(G;\mathfrak{p}).$$

It gives a recursive formula for calculation the number $\mathbf{Isoc}\,(G;n)$ as follows.

Theorem 6 ([28]) For $n \geq 2$, the number of nonisomorphic connected n-fold coverings of G is

$$\begin{aligned}
&\mathbf{Isoc}\,(G;n) \\
&= \sum_{\ell_1+2\ell_2\cdots+(n-1)\ell_{n-1}=n-1} \big((\ell_1+1)^{\beta(G)-1}-1\big) \\
&\qquad\qquad \times \big(\ell_1!\,2^{\ell_2}\ell_2!\,\cdots\,(n-1)^{\ell_{n-1}}\ell_{n-1}!\big)^{\beta(G)-1} \\
&\quad + \sum_{2\ell_2+3\ell_3+\cdots+n\ell_n=n} \big(2^{\ell_2}\ell_2!\,3^{\ell_3}\ell_3!\,\cdots n^{\ell_n}\ell_n!\big)^{\beta(G)-1} \\
&\quad - \sum_{\substack{\mathfrak{p}\,\in\,\mathfrak{P}(n)\,-\,\{[[n;\,1]]\}\\ j_1(\mathfrak{p})\,=\,0}} \prod_{j_k(\mathfrak{p})\neq 0} \left(\frac{1}{j_k(\mathfrak{p})!}\prod_{\ell=0}^{j_k(\mathfrak{p})-1}(\mathbf{Isoc}\,(G;k)+\ell)\right),
\end{aligned}$$

where the summation over the empty index set is defined to be 0.

Proof: Since an n-fold covering of G having the component type $[[n;1]]$ is connected, we have

$$\mathbf{Iso}\,(G;[[n;1]]) = \mathbf{Isoc}\,(G;n)$$

and

$$\mathbf{Isoc}\,(G;n) = \mathbf{Iso}\,(G;n) - \sum_{\mathfrak{p}\in\mathfrak{P}(n)-\{[[n;1]]\}} \mathbf{Iso}\,(G;\mathfrak{p}),$$

where the summation over the empty index set is defined to be 0. Let $\mathfrak{p} \in \mathfrak{P}(n)$ with $j_1(\mathfrak{p}) \neq 0$ and let $p : \tilde{G} \to G$ be a covering having the component type $\mathfrak{p}$. Then $\tilde{G}$ has $j_1(\mathfrak{p})$ components which are isomorphic to G, and the restriction of $p : \tilde{G} \to G$ on the complement of one of such components in $\tilde{G}$ is an $(n-1)$-fold covering of G. Hence, we get

$$\sum_{\mathfrak{p}\in\mathfrak{P}(n),\,j_1(\mathfrak{p})\neq 0} \mathbf{Iso}\,(G;\mathfrak{p}) = \mathbf{Iso}\,(G;n-1).$$

It implies that

$$
\begin{aligned}
&\mathbf{Isoc}\,(G;n)\\
&\quad= \mathbf{Iso}\,(G;n) - \mathbf{Iso}\,(G;n-1) - \sum_{\substack{\mathfrak{p}\,\in\,\mathfrak{P}(n)\,-\,\{[[n;\,1]]\}\\ j_1(\mathfrak{p})\,=\,0}} \mathbf{Iso}\,(G;\mathfrak{p})\\
&\quad= \sum_{\ell_1+2\ell_2\cdots+(n-1)\ell_{n-1}=n-1} \big((\ell_1+1)^{\beta(G)-1}-1\big)\\
&\qquad\qquad\qquad\qquad \times \big(\ell_1!\,2^{\ell_2}\ell_2!\,\cdots\,(n-1)^{\ell_{n-1}}\ell_{n-1}!\big)^{\beta(G)-1}\\
&\qquad\quad + \sum_{2\ell_2+3\ell_3+\cdots+n\ell_n=n} \big(2^{\ell_2}\ell_2!\,3^{\ell_3}\ell_3!\,\cdots n^{\ell_n}\ell_n!\big)^{\beta(G)-1}\\
&\qquad\qquad\quad - \sum_{\substack{\mathfrak{p}\,\in\,\mathfrak{P}(n)\,-\,\{[[n;\,1]]\}\\ j_1(\mathfrak{p})\,=\,0}} \mathbf{Iso}\,(G;\mathfrak{p}),
\end{aligned}
$$

where $\mathbf{Iso}\,(G;0)=0$ by definition and the summation over the empty index set is defined to be 0. Since

$$\mathbf{Iso}\,(G;1) = \mathbf{Isoc}\,(G;1) = \mathbf{Iso}\,(G;[[1;m]]) = 1$$

for any natural number m, we have

$$\mathbf{Iso}\,(G;\mathfrak{p}) = \prod_{j_k(\mathfrak{p})\neq 0,k\neq 1} \mathbf{Iso}\,(G;[[k;j_k(\mathfrak{p})]])$$

for any partition $\mathfrak{p}\in\mathfrak{P}(n)-\{[[n;1]]\}$ with $j_1(\mathfrak{p})=0$. Now, to complete the proof, we need to estimate the number $\mathbf{Iso}\,(G;[[s;t]])$ for any natural numbers s and t. Let $p:\tilde{G}\to G$ be a covering having the component type $[[s;t]]$. Then $\tilde{G}$ has exactly t components and the restriction of $p:\tilde{G}\to G$ on each of such t components is a connected s-fold covering of G. Hence, a covering automorphism on a covering $p:\tilde{G}\to G$ having a component type $[[s;t]]$ must permute its t components so that each component maps onto its isomorphic copy. It implies that the number $\mathbf{Iso}\,(G;[[s;t]])$ is equal to the number of selections with repetition of t objects chosen from $\mathbf{Isoc}\,(G;s)$ types of objects, *i.e.,*

$$\mathbf{Iso}\,(G;[[s;t]]) = \binom{\mathbf{Isoc}\,(G;s)+t-1}{t} = \frac{1}{t\,!}\left(\prod_{i=0}^{t-1}(\mathbf{Isoc}\,(G;s)+i)\right).$$

This completes the proof. $\square$

Corollary 1 *Let $\mathfrak{p}$ be a partition of a natural number n. Then*

$$\mathbf{Iso}\,(G;\mathfrak{p}) = \prod_{j_k(\mathfrak{p})\neq 0} \frac{1}{j_k(\mathfrak{p})!}\left(\prod_{i=0}^{j_k(\mathfrak{p})-1} (\mathbf{Isoc}\,(G;k)+i)\right).$$

In particular, if $j_k(\mathfrak{p}) = 0$ or 1 for each $k = 1, 2, \ldots, n$, then

$$\mathbf{Iso}\,(G;\mathfrak{p}) = \prod_{j_k(\mathfrak{p})=1} \mathbf{Isoc}\,(G;k).$$

In fact, Liskovets [31] computed the number $\mathbf{Isoc}\,(G;n)$ in terms of the Möbius function and the number $\mathcal{S}_{\mathcal{F}_{\beta(G)}}(m)$:

$$\mathbf{Isoc}\,(G;n) = \frac{1}{n}\sum_{m|n} \mathcal{S}_{\mathcal{F}_{\beta(G)}}(m) \sum_{d|\frac{n}{m}} \mu\left(\frac{n}{md}\right) d^{(\beta(G)-1)m+1},$$

where $\mu(k)$ is the number-theoretic Möbius function and $\mathcal{S}_{\mathcal{F}_{\beta(G)}}(m)$ denotes the number of subgroups of index m in the free group $\mathcal{F}_{\beta(G)}$ generated by $\beta(G)$ elements.

EXAMPLE 1 By applying Theorem 6, we have

$$\mathbf{Isoc}\,(G;2) = (2^{\beta(G)-1}-1) + 2^{\beta(G)-1} = 2^{\beta(G)} - 1,$$

$$\mathbf{Isoc}\,(G;3) = (3^{\beta(G)-1}-1)2^{\beta(G)-1} + 3^{\beta(G)-1} = 6^{\beta(G)-1} + 3^{\beta(G)-1} - 2^{\beta(G)-1},$$

and

$$\begin{aligned}
&\mathbf{Isoc}\,(G;4)\\
&\quad = (4^{\beta(G)-1}-1)6^{\beta(G)-1} + (2^{\beta(G)-1}-1)2^{\beta(G)-1} + 8^{\beta(G)-1} + 4^{\beta(G)-1}\\
&\qquad\qquad - \mathbf{Iso}\,(G;[[2;2]])\\
&\quad = 24^{\beta(G)-1} + 8^{\beta(G)-1} + (2^{\beta(G)}-1)2^{\beta(G)-1} - 6^{\beta(G)-1}\\
&\qquad\qquad - (2^{\beta(G)}-1)2^{\beta(G)-1}\\
&\quad = 24^{\beta(G)-1} + 8^{\beta(G)-1} - 6^{\beta(G)-1}.
\end{aligned}$$

It is well-known (e.g., see [35]) in topology that the fundamental group of a graph G is a free group of rank $\beta(G)$, and there exists a one-to-one correspondence between the isomorphism classes of connected n-fold coverings of G and the conjugacy classes of subgroups of index n of the fundamental group of G. Thus, by using the enumerating formula for $\mathbf{Isoc}\,(G;n)$ in Theorem 6, we can compute the number of conjugacy classes of subgroups of index n of any finitely generated free group.

Notice that the number $\mathbf{Iso}\,(G;n)$ of nonisomorphic n-fold coverings of G can be expresses (in terms of $\mathbf{Isoc}\,(G;n)$) as follows.

$$\mathbf{Iso}\,(G;n) = \sum_{\mathfrak{p}\in\mathfrak{P}(n)} \prod_{j_k(\mathfrak{p})\neq 0} \left(\frac{1}{j_k(\mathfrak{p})!} \prod_{\ell=0}^{j_k(\mathfrak{p})-1} (\mathbf{Isoc}\,(G;k)+\ell) \right).$$

Remark An enumeration of the number of nonisomorphic n-fold coverings or n-fold connected coverings of a graph was also independently done by Hofmeister ([10, 14]). Liskovets ([31]) also enumerated those connected coverings by counting the conjugacy classes of subgroups of a finitely generated free group in terms of Möbius function.

Comparing with the combinatorial computation of the number $\mathbf{Isoc}$ $(G;n)$ of nonisomorphic *connected* n-fold coverings of G in Theorem 6, there is another group-theoretic computation of it with Burnside's Lemma. A $\beta(G)$-tuple of permutations $(\sigma_1,\ldots,\sigma_{\beta(G)})$, $\sigma_i \in S_n$ is called *transitive* if the permutation group $<\sigma_1,\ldots,\sigma_{\beta(G)}>$ generated by them acts transitively on the set $\{1,2,\ldots,n\}$. Let $\mathfrak{G}(n;\beta)$ denote the set of all transitive β-tuples of permutations in S_n.

Lemma 3 *The following are equivalent for a voltage assignment* $\phi = (\sigma_1,\ldots,\sigma_{\beta(G)})$ *in* $C^1_T(G;n)$.

(1) *It is transitive, i.e.,* $\phi \in \mathfrak{G}(n;\beta(G))$.

(2) *The associated transition graph with* $\{1,2,\ldots,n\}$ *as its vertex set and with pairs* $\{i,\sigma_j(i)\}_{i,j}$ *as its edges is connected.*

(3) *The permutation derived graph* G^ϕ *is connected.*

The following is a direct consequence of Lemma 3 and Theorem 4.

Lemma 4 ([32]) *There is a one-to-one correspondence among the following sets:*

(1) *The set of similarity classes of transitive $\beta(G)$-tuples of permutations in S_n.*

(2) *The set of nonisomorphic connected n-fold coverings of G.*

(3) *The set of conjugacy classes of subgroups of index n in the free group generated by $\beta(G)$ elements.*

Liskovets ([31]) used Burnside's Lemma to compute the number $\mathbf{Isoc}\,(G;n)$ of the conjugacy classes of subgroups of index n in the free group generated by $\beta = \beta(G)$ elements:

$$\begin{aligned} \mathbf{Isoc}\,(G;n) &= |\mathfrak{G}(n;\beta)/S_n| = \frac{1}{n!}\sum_{g\in S_n} |\mathrm{Fix}\,(g)| \\ &= \frac{1}{n}\sum_{m|n} \mathcal{S}_{\mathcal{F}}(m) \sum_{d|\frac{n}{m}} \mu\left(\frac{n}{md}\right) d^{(\beta-1)m+1}, \end{aligned}$$

where $\beta = \beta(G)$, $\mu(n)$ is the number-theoretic Möbius function and $\mathcal{S}_{\mathcal{F}}(m)$ denotes the number of subgroups of index m in the free group $\mathcal{F}$ generated by β elements.

In advance of stating Liskovets' method for computing the number $\mathbf{Isoc}\,(G;n)$, we introduce Hall's formula to count the number of subgroups of index n in a finitely generated free group. Let $\mathcal{F}$ be the free group of rank β generated by $Y = \{s_1, s_2, \ldots, s_\beta\}$. Let $\mathcal{U}$ be a subgroup of index n in $\mathcal{F}$ with a left coset representation:

$$\mathcal{F} = \mathcal{U}\,1 + \mathcal{U}g_2 + \cdots + \mathcal{U}g_n = \mathcal{U} + \mathcal{U}g_2 + \cdots + \mathcal{U}g_n.$$

Here, we can assume that the representatives g_i's with $g_1 = 1$, the identity, are selected to be a *Schreier system,*[1] even it is not unique in general. Define a function ϕ on the set $\{gs^\varepsilon \,|\, g \in \{g_i\}, s \in Y, \varepsilon = \pm 1\}$ so that $\phi(gs^\varepsilon)$ is the representative of the coset containing gs^ε, *i.e.,* $\phi(gs^\varepsilon) = g_i$ if $gs^\varepsilon \in \mathcal{U}g_i$. Then the function ϕ satisfies the following three conditions:

[1] A set W of reduced words in a free group $\mathcal{F}$ is a *Schreier system* if any $g = a_1a_2\cdots a_t$, in a reduced form, belongs to W, then the element $a_1a_2\cdots a_{t-1}$ is also belonging to W.

(i) $\phi(g_i s^\varepsilon) \in \{g_i\}$,

(ii) if $g_i s^\varepsilon \in \{g_i\}$, then $\phi(g_i s^\varepsilon) = g_i s^\varepsilon$,

(iii) $\phi[\phi(g_i s^\varepsilon) s^{-\varepsilon}] = g_i$.

For a subgroup $\mathcal{U}$ of a free group $\mathcal{F}$ generated by Y, the pair $\mathcal{U} = \mathcal{U}[\{g_i\}, \phi]$ of a Schreier system $\{g_i\}$ of coset representatives and a function ϕ on the set $\{gs^\varepsilon\}$ satisfying the three conditions (i)-(iii) listed above is called the *standard representation* for the subgroup $\mathcal{U}$. For a standard representation $\mathcal{U} = \mathcal{U}[\{g_i\}, \phi]$ for $\mathcal{U}$, it is known that the elements $gs\phi(gs)^{-1}$'s, where g runs over the representatives g_i's and s over the generating set Y, generate the subgroup $\mathcal{U}$. In particular, the subgroup $\mathcal{U}$ is also finitely generated.

The following lemma gives a criterion for recognizing different representations of the same subgroup.

Lemma 5 ([7]) *Let* $\mathcal{U}_1 = \mathcal{U}_1[\{g_i^{(1)}\}, \phi_1]$ *and* $\mathcal{U}_2 = \mathcal{U}_2[\{g_j^{(2)}\}, \phi_2]$ *be standard representations for the subgroups* $\mathcal{U}_1$ *and* $\mathcal{U}_2$*, respectively. Then* $\mathcal{U}_1 = \mathcal{U}_2$ *if and only if there is a one-to-one correspondence* $\{g_i^{(1)}\} \leftrightarrow \{g_j^{(2)}\}$ *between the representative sets mapping the identity onto itself such that if* $g_i^{(1)} \leftrightarrow g_j^{(2)}$ *including* $1 = g_1^{(1)} \leftrightarrow g_1^{(2)} = 1$*, then* $\phi_1(g_i^{(1)} s^\varepsilon) \leftrightarrow \phi_2(g_j^{(2)} s^\varepsilon)$ *for any* $s \in Y$.

Proof: The necessity is clear: Suppose $\mathcal{U}_1 = \mathcal{U}_2 = \mathcal{U}$. If $\mathcal{U} g_i^{(1)} = \mathcal{U} g_j^{(2)}$ is the same left coset of $\mathcal{U}$ in the two coset representatives, then the correspondence $g_i^{(1)} \leftrightarrow g_j^{(2)}$ gives a desired one-to-one correspondence between $\{g_i^{(1)}\}$ and $\{g_j^{(2)}\}$.

For the sufficiency, suppose that a 1-1 correspondence $\{g_i^{(1)}\} \leftrightarrow \{g_j^{(2)}\}$ is given with $1 = g_1^{(1)} \leftrightarrow g_1^{(2)} = 1$ such that if $g_i^{(1)} \leftrightarrow g_j^{(2)}$, then $\phi_1(g_i^{(1)} s^\varepsilon) \leftrightarrow \phi_2(g_j^{(2)} s^\varepsilon)$ for any $s \in Y$. By induction on length $\ell(f)$ of an element $f \in \mathcal{F}$, (*i.e.,* the length of the reduced word for f) we can show that an element belonging to a coset of $\mathcal{U}_1$ belongs to the corresponding coset of $\mathcal{U}_2$. This is true for $\ell(f) = 0$, since $f = 1$ and $1 \leftrightarrow 1$. And if f is in the cosets $\mathcal{U}_1 g_i^{(1)} = \mathcal{U}_2 g_j^{(2)}$, then fs^ε is in the corresponding cosets $\mathcal{U}_1 \phi_1(g_i^{(1)} s^\varepsilon) \leftrightarrow \mathcal{U}_2 \phi_2(g_j^{(2)} s^\varepsilon)$. Hence, the corresponding cosets are the same, and in particular $\mathcal{U}_1 = \mathcal{U}_2$. □

For the next lemma, let $\mathcal{U}$ be a subgroup of a (free or not) group $\mathcal{F}$ generated by $Y = \{s_1, \ldots, s_i, \ldots\}$. For an $s_i \in Y$, by multiplying s_i on the right of each (left) coset of $\mathcal{U}$ in $\mathcal{F}$, we have a permutation $\pi_i = \pi(s_i)$ on the left cosets. Since $Y = \{s_i\}$ generates $\mathcal{F}$, the π_i's generate a group which is transitive on the set of cosets. The following is a kind of converse for free groups.

Lemma 6 *Given a free group $\mathcal{F}$ generated by $Y = \{s_1, \ldots, s_i, \ldots\}$, and a set of indices $I = \{1, \ldots, i, \ldots\}$. With each generator s_i, associate a permutation π_i on the indices I. Suppose $J = \{1, \ldots, j, \ldots\}$ is the transitive constituent of I containing* 1. *Then in $\mathcal{F}$, there is a Schreier system $\{g_1 = 1, g_2, \ldots, g_j, \ldots\}$ indexed by J and a function ϕ on the set $\{g_j s_i^\varepsilon\}$ such that*

$$\phi(g_j s_i^\varepsilon) = g_k \quad \textit{if and only if} \quad \pi_i^\varepsilon(j) = k\,.$$

Proof: The permutations π_i generate a permutation group P of indices. Let E be the subgroup of P consisting of permutations π which fix the index 1. The mapping $s_i \mapsto \pi_i$ determines an epimorphism of $\mathcal{F} \to P$, and let U be the subgroup of $\mathcal{F}$ mapped onto E; $U \to E$. Now, choose a coset representative $\{g\}$ of U in $\mathcal{F}$ as a Schreier system:

$$\mathcal{F} = \mathcal{U}\,1 + \mathcal{U}g_2 + \cdots + \mathcal{U}g_j + \cdots,$$

and let $\phi(g_j s^\varepsilon)$ is the representative of the coset containing $g_j s^\varepsilon$, as before. If $g \mapsto \pi$, then $Ug \mapsto E\pi$, *i.e.,* the epimorphism $\mathcal{F} \to P$ preserves the left coset representative. If π maps the index 1 to j, we write $(1)\pi = j$, and assign the index j to g, putting $g = g_j$. Hence, the Schreier system $\{g\}$ is indexed by J, in which if $g_j \mapsto \pi$ and $s_i^\varepsilon \mapsto \pi_i^\varepsilon$ then $g_j s_i^\varepsilon \mapsto \pi\pi_i^\varepsilon$. Now, if $(j)\pi_i^\varepsilon = k$, then $(1)\pi\pi_i^\varepsilon = k$ and $\pi\pi_i^\varepsilon$ belongs to the left coset $E\eta$ of E consisting of those permutations τ of E which maps 1 to k, $(1)\tau = k$. Here, $Ug_k \to E\eta$. Hence, $g_j s_i^\varepsilon$ belongs to Ug_k, or $\phi(g_j s_i^\varepsilon) = g_k$. □

With each element f of $\mathcal{F}$ generated by $\{s_1, \ldots, s_\beta\}$, say $f = s_{i_1} \cdots s_{i_t}$, the associated permutation $\pi(f) = \pi(s_{i_1}) \cdots \pi(s_{i_t})$ defines a transitive n-degree permutation representation of the group $\mathcal{F}$, and those elements f such that $\pi(f)$ fixes 1 will form the subgroup $\mathcal{U}$. Conversely, any transitive n-degree permutation representation of the group $\mathcal{F}$ determines a subgroup of index n in the group $\mathcal{F}$, by Lemma 6. And, by Lemma 5, such kinds of two representations derived by a Schreier system $\{g_i\}$ and functions $\{\phi(g_i s)\}$ determines the same subgroup $\mathcal{U}$ if and only if they are equivalent via a permutation σ on $\{g_i\}$ such that $\sigma(1) = 1$. By identifying an n-degree permutation representation of $\mathcal{F}$ with its restriction on the generating set $Y = \{s_1, s_2, \ldots, s_\beta\}$, *i.e.,* with a transitive β-tuples of permutations on S_n, Lemma 5 can be rephrased as follows.

Lemma 7 *For $n > 1$, there exists a one-to-one correspondence between the subgroups of index n of the free group $\mathcal{F}$ and the 1-similarity classes of transitive β-tuples of permutations in S_n.*

So far, we used some group theory terminologies like Schreier systems or the standard representations for subgroups to have Lemma 7. But, we can give simpler proof by using graph coverings and a fundamental group theory as follows: Consider the free group $\mathcal{F}$ as the fundamental group $\pi_1(G, v)$ of a connected graph G with base vertex v. We assume that G has $\beta = \beta(G)$ cotree edges. It is well-known that every subgroup of $\mathcal{F} = \pi_1(G, v)$ is expressed as the image $p_\sharp(\pi_1(G^\phi, v_i))$ of the fundamental group of a connected covering $p : G^\phi \to G$, where $\phi = (\sigma_1, \ldots, \sigma_{\beta(G)})$ is a transitive permutation voltage assignment in $C^1_T(G; n)$, and v_i is a vertex in the fibre of v. Furthermore, for any two transitive permutation voltage assignments ϕ, ψ in $C^1_T(G; n)$,

$$p_\sharp(\pi_1(G^\phi, v_i)) = p_\sharp(\pi_1(G^\psi, v_j))$$

if and only if the two coverings are isomorphic by a covering isomorphism Φ which preserves the base point. Hence, we can say that in Theorem 4, the permutation σ leaves fixed 1, by relabeling of vertices in the fibre $p^{-1}(v)$ if necessary.

Remark As a generalization of Lemma 7, the connection between subgroups of any finitely presented group and its transitive permutational representations (see [8] Ch5 or [33]) can be formulated as follows:

Given a finitely presented group

$$\mathcal{A} = \{x_1, x_2, \ldots, x_r \,:\, f_1 = 1, f_2 = 1, \ldots\}$$

there is a one-to-one correspondence between the subgroups of index $n \geq 1$ in $\mathcal{A}$ and the root-similarity classes of transitive r-tuples $(x_1, x_2, \ldots, x_r)$ in S_n that satisfy the defining relations $\{f_j = 1\}, j = 1, 2, \ldots$.

Now, we may enumerate recursively the number of subgroups of index n in the free group $\mathcal{F}$.

Theorem 7 ([7]) *The number $\mathcal{S}_{\mathcal{F}}(n)$ of subgroups of index n in the free group $\mathcal{F}$ generated by β elements is given as*

$$\mathcal{S}_{\mathcal{F}}(n) = n(n!)^{\beta-1} - \sum_{t=1}^{n-1} (n-t)!^{\beta-1}\mathcal{S}_{\mathcal{F}}(t) \qquad \textit{with} \qquad \mathcal{S}_{\mathcal{F}}(1) = 1.$$

Proof: Clear for $n = 1$. Choose β permutations $P_1, \ldots, P_\beta$ on symbols $\{1, g_2, \ldots, g_n\}$. In general, $P_1, \ldots, P_\beta$ need not generate a group transitive on all of $1, g_2, \ldots, g_n$. Let the transitive constituent including 1 be $1, b_2, \ldots, b_t$. Disregarding the remaining letters, we may take as $\pi(s_1), \ldots, \pi(s_\beta)$ the permutations on $1, b_2, \ldots, b_t$, and these will determine a unique subgroup of index t. The remaining $n - t$ letters could occur in $P_1, \ldots, P_\beta$ in $[(n-t)!]^\beta$ ways. In addition, by Lemma 5, the same group will be determined if we replace $1, b_2, \ldots, b_t$ by any other combination $1, c_2, \ldots, c_t$ in the symbols $\{1, g_2, \ldots, g_n\}$, and the remaining $n - t$ letters in an arbitrary way. Also, the symbols $b_2, \ldots, b_t$ can be replaced by $c_2, \ldots, c_t$ from $g_2, \ldots, g_n$ in $(n-1)(n-2)\cdots(n-t+1)$ different ways. Thus a total of

$$(n-1)(n-2)\cdots(n-t+1)[(n-t)!]^\beta = (n-1)!\,[(n-t)!]^{\beta-1}$$

different permutations $P_1, \ldots, P_\beta$ may be associated with the same subgroup of index t, and $(n-1)!\,[(n-t)!]^{\beta-1}\mathcal{S}_{\mathcal{F}}(t)$ permutations are associated with the subgroups of index t. Hence, we get

$$\sum_{t=1}^{n} (n-1)!\,[(n-t)!]^{\beta-1}\mathcal{S}_{\mathcal{F}}(t) = (n!)^\beta\,.$$

Dividing by $(n-1)!$, we can get the desired formula. □

The symmetric group S_n acts naturally on the set $\{1, 2, \ldots, n\}$, and also acts on the set $\mathfrak{G}(n;\beta)$ by the simultaneous coordinatewise conjugacy. But, these two actions are mutually orthogonal, because any β-tuple in the set $\mathfrak{G}(n;\beta)$ is transitive. Hence, the group S_{n-1}, as the subgroup of S_n consisting of permutations σ fixing 1, *i.e.,* $\sigma(1) = 1$, acts freely on the set $\mathfrak{G}(n;\beta)$ and it follows by Lemma 7 and Burnside's Lemma

$$|\mathfrak{G}(n;\beta)| = (n-1)!\, \mathcal{S}_{\mathcal{F}}(n)\,,$$

where $\mathcal{F}$ is the free group generated by β elements.

Now, we go back to Liskovets' method for computing the number $\mathbf{Isoc}\,(G;n)$. It is already known that

$$\mathbf{Isoc}\,(G;n) = |\mathfrak{G}(n;\beta)/S_n| = \frac{1}{n!}\sum_{g\in S_n} |\mathrm{Fix}\,(g)|$$

and $\mathrm{Fix}\,(g) = \mathfrak{G}(n;\beta) \cap (Z(g)\times \ldots \times Z(g))$, where $\beta = \beta(G)$. If $\mathrm{Fix}\,(g) \neq \emptyset$ and $\phi = (\sigma_1, \ldots, \sigma_\beta)$ belongs to $\mathrm{Fix}\,(g)$, then g commutes with the the group $< \sigma_1, \ldots, \sigma_\beta >$, which is transitive on the set $\{1, 2, \ldots, n\}$. Hence, g must be a *regular* permutation, *i.e.,* it consists of independent cycles of the same length ℓ. For each $\ell m = n$, there exist $n!/(m!\,\ell^m)$ regular permutations g in S_n consisting of m cycles of length ℓ, and $|\mathrm{Fix}\,(g)|$ are equal for all such regular g. We denote this value by $|\mathrm{Fix}\,((\ell^m))|$, and call such g a permutation of type (ℓ^m). Hence, we get

$$\mathbf{Isoc}\,(G;n) = \frac{1}{n!}\sum_{g\in S_n} |\mathrm{Fix}\,(g)| = \sum_{\ell|n,\ell m=n} \frac{|\mathrm{Fix}\,((\ell^m))|}{m!\,\ell^m}\,.$$

The following lemma is well-known and an elementary exercise in group theory.

Lemma 8 *Let g_0 be the permutation in S_n of type (ℓ^m):*

$$g_0 = (12\cdots\ell)\,(\ell+1\cdots 2\ell)\,\cdots\,((m-1)\ell+1\cdots n)\,.$$

Then, the centralizer $Z(g_0)$ of g_0 is a wreath product $\mathbb{Z}_\ell \wr S_m$, where $\mathbb{Z}_\ell$ is the cyclic group generated by the ℓ-cycle $(12\cdots\ell)$. An element of the *wreath product* $\mathbb{Z}_\ell \wr S_m$ *is of the form* $a = (c_1, \ldots, c_m; \tilde{a})$*, where* $c_i \in \mathbb{Z}_\ell$ *and* $\tilde{a} \in S_m$*. The element* $a = (c_1, \ldots, c_m; \tilde{a})$ *represents a permutation in* S_n *acting on the set* $\{1, \ldots, n\}$ *as follows. Notice that each element in* $\{1, \ldots, n\}$ *is of the form* $k = (s-1)\ell + t = t_s (1 \le s \le m,\ 1 \le t \le \ell)$*. Then*

$$(t_s)a = (k)a = (k)(c_1, \ldots, c_m; \tilde{a}) = ((s)\tilde{a} - 1)\ell + (t)c_s = ((t)c_s)_{(s)\tilde{a}},$$

that is, first perform a cyclic transposition by the s*-th cycle* c_s *of the permutation* g_0 *for all* $s = 1, \ldots, m$*, and then shift through the action of the permutation* $\tilde{a}$ *in* S_m*. If* $b = (d_1, \ldots, d_m; \tilde{b}) \in \mathbb{Z}_\ell \wr S_m$*, then*

$$a \cdot b = (c_1 + d_{(1)\tilde{a}}, \ldots, c_m + d_{(m)\tilde{a}}; \tilde{a}\tilde{b}) = (c_1 d_{(1)\tilde{a}}, \ldots, c_m d_{(m)\tilde{a}}; \tilde{a}\tilde{b}),$$

where $(s)ab = ((s)a)b$ *for all* $s \in \{1, \ldots, m\}$*.*

Proof: Let $g_0 = (12 \cdots \ell)\,(\ell + 1 \cdots 2\ell) \cdots ((m-1)\ell + 1 \cdots n)$. Then g_0 can be identified with the element $(1, \ldots, 1; 1)$ in $\mathbb{Z}_\ell \wr S_m$. Then, for each $g = (c_1, \ldots, c_m; \tilde{a})$ in $\mathbb{Z}_\ell \wr S_m$, we have $gg_0 = (1 + c_1, \ldots, 1 + c_m; \tilde{a}) = g_0 g$. It implies that $\mathbb{Z}_\ell \wr S_m$ is a subgroup of the centralizer $Z(g_0)$ of g_0. Let $C(g_0)$ be the conjugacy class of g_0. Then $|C(g_0)| = n!/(m!\ell^m)$. Since $|S_n| = |Z(g_0)|\,|C(g_0)|$, $|Z(g_0)| = m!\ell^m = |\mathbb{Z}_\ell \wr S_m|$ and hence $\mathbb{Z}_\ell \wr S_m$ is the centralizer $Z(g_0)$ of g_0. □

From the notations, we have $|\mathrm{Fix}\,((\ell^m))| = |\mathrm{Fix}\,(g_0)|$ and

$$\mathrm{Fix}\,(g_0) = \big\{(a_1, \ldots, a_\beta) \in (\mathbb{Z}_\ell \wr S_m)^\beta \mid \\ < a_1, \ldots, a_\beta > \text{ is transitive in } \{1, \ldots, n = \ell m\}\big\}\,.$$

We set

$$F(\ell^m) = \big\{(a_1, \ldots, a_\beta) \in (\mathbb{Z}_\ell \wr S_m)^\beta \mid \\ < \tilde{a}_1, \ldots, \tilde{a}_\beta > \text{ is transitive in } \{1, \ldots, m\}\big\}\,.$$

The following lemma is due to Liskovets [31].

Lemma 9 *For any* $n = \ell m$ *and any* β*, we have*

(1) $|F(\ell^m)| = \displaystyle\sum_{k|\ell, kd=\ell} k^{m-1} |\mathrm{Fix}\,((d^m))|\,.$

(2) $|F(\ell^m)| = \ell^{\beta m} |\mathfrak{G}(m;\beta)|$.

Proof: Set $S = \{1, 1+k, \ldots, 1+(d-1)k\}$. For each $2 \le s \le m$ and $0 \le h_s \le k-1$, let

$$B(0, h_2, \ldots, h_m)_0 = S \cup \left(\bigcup_{s=2}^{m} S + (s-1)\,\ell + h_s \right)$$

and

$$B(h, s)_t = B(0, h_2, \ldots, h_m)_0 + t$$

for each $t = 1, 2, \ldots, k-1$, where all arithmetic is done by modulo ℓ. It is not hard to show that every element in $F(\ell^m)$ is transitive on each of the following sets $B(0, h_2, \ldots, h_m)_0, \ldots, B(0, h_2, \ldots, h_m)_{k-1}$ for some $s = 2, \ldots, m$ and $h_s = 0, 1, \ldots, k-1$. Notice that $\mathrm{Fix}\,((d^m))$ can be identified with the set of all elements in $(\mathbb{Z}_\ell \wr S_m)^\beta$ which is transitive on each of the sets $B(0, 0, \ldots, 0)_0, \ldots, B(0, 0, \ldots, 0)_{k-1}$. Moreover, $(0, h_2, \ldots, h_m; 1)\mathrm{Fix}\,((d^m))(0, h_2, \ldots, h_m; 1)^{-1}$ is the set of all elements in $(\mathbb{Z}_\ell \wr S_m)^\beta$ which is transitive on each of the following sets $B(0, h_2, \ldots, h_m)_0, \ldots, B(0, h_2, \ldots, h_m)_{k-1}$. Hence, we have (1).

Now, we aim to show (2). For each β-tuple $(\tilde{a}_1, \ldots, \tilde{a}_\beta)$ which is transitive in $\{1, \ldots, m\}$, there exists $\ell^{\beta m}$ elements $(b_1, \ldots, b_\beta)$ in $F(\ell^m)$ such that $(\tilde{b}_1, \ldots, \tilde{b}_\beta) = (\tilde{a}_1, \ldots, \tilde{a}_\beta)$. It implies (2). $\square$

Theorem 8 ([31]) *The number* $\mathbf{Isoc}\,(G; n)$ *of the conjugacy classes of subgroups of index* n *in the free group generated by* β *elements is given by the formula*

$$\mathbf{Isoc}\,(G; n) = \frac{1}{n} \sum_{m|n} \mathcal{S}_{\mathcal{F}}(m) \sum_{d|\frac{n}{m}} \mu\left(\frac{n}{md}\right) d^{(\beta-1)m+1},$$

where $\beta = \beta(G)$ *and* $\mu(n)$ *is the number-theoretic Möbius function.*

Proof:

Let $A(n) = \dfrac{|F(n^m)|}{n^{m-1}}$ and $a(n) = \dfrac{|\mathrm{Fix}\,((n)^m)|}{n^{m-1}}$. Then, by Lemma 9 (1),

$$\begin{aligned} A(n) &= \frac{1}{n^{m-1}}\left(\sum_{d|n}\left(\frac{n}{d}\right)^{m-1}|\mathrm{Fix}\,((d^m))|\right) = \sum_{d|n}\frac{1}{d^{m-1}}|\mathrm{Fix}\,((d^m))| \\ &= \sum_{d|n} a(d). \end{aligned}$$

By the Möbius inversion formula, we can see that

$$a(\ell) = \sum_{d|\ell}\mu(d)\,A\left(\frac{\ell}{d}\right) = \sum_{d|\ell}\mu(d)\left(\frac{d}{\ell}\right)^{m-1}\left|F\left(\frac{\ell^m}{d^m}\right)\right|.$$

By the definition of the function $a(n)$, we can get

$$|\mathrm{Fix}\,((\ell)^m)| = \sum_{d|\ell}\mu(d)\,d^{m-1}\left|F\left(\frac{\ell^m}{d^m}\right)\right|.$$

Recall that

$$\mathbf{Isoc}\,(G;n) = \frac{1}{n!}\sum_{g\in S_n}|\mathrm{Fix}\,(g)| = \sum_{\ell|n,\ell m=n}\frac{|\mathrm{Fix}\,((\ell^m))|}{m!\ell^m}$$

and

$$|\mathfrak{G}(n;\beta)| = (n-1)!\,\mathcal{S}_{\mathcal{F}}(n).$$

Now, by using these facts and Lemma 9 (2) together with an elementary computation, we have the theorem. □

The number **Isoc** $(G;n)$ for small n and $\beta(G)$ is listed in table 3.2.

Question: What are the relations between two different formulas for **Isoc** $(G;n)$?

$\beta(G)$	1	2	3	4	5	6
$n = 1$	1	1	1	1	1	1
$n = 2$	1	3	7	15	31	63
$n = 3$	1	7	41	235	1361	7987
$n = 4$	1	26	604	14120	334576	7987616
$n = 5$	1	97	13753	1712845	207009649	24875000437
$n = 6$	1	624	504243	371515454	268530771271	193466859054994

Table 3.2: The number $\mathbf{Isoc}\,(G; n)$ for small n and small $\beta(G)$

Remark The fundamental group of any (connected) graph G is a free group generated by $\beta(G)$ elements, and the conjugacy classes of its subgroups of index n are in one-to-one correspondence with the nonisomorphic connected n-fold coverings of G. Such a correspondence is established via the monomorphic image of the fundamental group of a connected covering of G. Since any covering of a graph is also a graph, *every subgroup of a free group is also a free group.* Moreover, any subgroup of index n in the free group generated by $\beta = \beta(G)$ elements is a monomorphic image of the fundamental group of an n-fold connected covering of G. Hence, it must be a free group generated by $1 + n(\beta(G) - 1)$ elements, because it must be equal to the Betti number of a connected n-fold covering of G.

Remark Related to the construction problem for all nonisomorphic connected n-fold coverings of a graph, one can ask the following two questions: (1) find a (minimal) generating set for each subgroup of a given index of a finitely generated free group $\mathcal{F}$ and (2) find all possible lists of a (minimal) generating set for each of those subgroups. The first question can be answered by Reidemeister-Schreier method. The second one can be done by the description of Aut $(\mathcal{F})$ (See [34]).

Chapter 4

Regular coverings with abelian voltage groups and subgroups of free abelian groups

Let $\mathcal{A}$ be a finite group and let $S_{\mathcal{A}}$ denote the symmetric group on the group elements of $\mathcal{A}$. It gives the (left) regular representation of $\mathcal{A} \to S_{\mathcal{A}}$ via $g \to L(g)$, the left multiplication by g on $\mathcal{A}$. Clearly, this representation is monomial and the group $\mathcal{A}$ can be identified with the group of left transformations $L(g)$'s: $\mathcal{A} \equiv \{L(g) \mid g \in \mathcal{A}\}$ (Cayley Theorem). Notice that a permutation voltage assignment $\phi : D(G) \to S_{\mathcal{A}}$ having its images in $\mathcal{A}$ is nothing but an $\mathcal{A}$-voltage assignment of G, and for such a voltage assignment ϕ, the permutation derived graph G^{ϕ} is just the ordinary derived graph $G \times_{\phi} \mathcal{A}$.

Let $C^1_T(G;\mathcal{A})$ denote the set of all normalized $\mathcal{A}$-voltage assignments of G. Recall ([4]) that any regular n-fold covering of G is isomorphic to an ordinary derived graph $G \times_{\phi} \mathcal{A}$ for a group $\mathcal{A}$ of order n and for a $\phi \in C^1_T(G;\mathcal{A})$. From the construction of an ordinary derived graph $G \times_{\phi} \mathcal{A}$, it is clear that if the regular graph covering $p : G \times_{\phi} \mathcal{A} \to G$ is connected, then the group $\mathcal{A}$ becomes the covering transformation group.

Let $\mathbf{Iso}\,(G;\mathcal{A})$ (resp. $\mathbf{Isoc}\,(G;\mathcal{A})$) denote the number of nonisomorphic (resp. connected) regular $\mathcal{A}$-coverings. We use $\mathbf{Iso}^R(G;n)$ to denote the number of nonisomorphic regular n-fold coverings regardless of the group $\mathcal{A}$ involved. Similarly, we define $\mathbf{Isoc}^R(G;n)$.

The algebraic characterization of two isomorphic graph coverings given in Theorem 4 can be rephrased for regular coverings as follows.

Theorem 9 ([16]) *Let $\phi \in C_T^1(G;\mathcal{A})$ and $\psi \in C_T^1(G;\mathcal{B})$ be any two ordinary voltage assignments in G. If their derived (regular) coverings $p_\phi : G \times_\phi \mathcal{A} \to G$ and $p_\psi : G \times_\psi \mathcal{B} \to G$ are connected, then they are isomorphic if and only if there exists a group isomorphism $\sigma : \mathcal{A} \to \mathcal{B}$ such that $\psi(uv) = \sigma(\phi(uv))$ for all $uv \in D(G) - D(T)$.*

In particular, if two voltages ϕ and ψ in $C_T^1(G;\mathcal{A})$ derive connected coverings, then their derived coverings are isomorphic if and only if there exists a group automorphism $\sigma \in \mathrm{Aut}(\mathcal{A})$ such that

$$\psi(uv) = \sigma(\phi(uv))$$

for all $uv \in D(G) - D(T)$.

As the case of the set $C_T^1(G;n)$, the set $C_T^1(G;\mathcal{A})$ of $\mathcal{A}$-voltage assignments of G can be identified as

$$C_T^1(G;\mathcal{A}) = \mathcal{A} \times \mathcal{A} \times \cdots \times \mathcal{A}, \qquad (\beta(G) \quad \text{times}),$$

that is, an $\mathcal{A}$-voltage assignment ϕ of G can be identified as a $\beta(G)$-tuple $(g_1, \ldots, g_{\beta(G)})$ of group elements $g_i \in \mathcal{A}$. Moreover, such a $\beta(G)$-tuple of g's derives a connected covering if and only if it is *transitive*. It means by definition that the subgroup $< g_1, \ldots, g_{\beta(G)} >$ generated by them acts transitively on the group $\mathcal{A}$ (under the left translation on $\mathcal{A}$), or equivalently $\{g_1, g_2, \ldots, g_{\beta(G)}\}$ generates $\mathcal{A}$.

Under the coordinatewise $\mathrm{Aut}\,(\mathcal{A})$-action on the set of transitive $\beta(G)$-tuples of group elements $g_i \in \mathcal{A}$, any two transitive $\beta(G)$-tuples of elements in $\mathcal{A}$ belong to the same orbit if and only if they derive (connected) isomorphic $\mathcal{A}$-coverings, by Theorem 9.

Clearly, the $\mathrm{Aut}\,(\mathcal{A})$-action on the set of transitive $\beta(G)$-tuples of group elements $g_i \in \mathcal{A}$ is free (having no fixed element), from which Burnside's Lemma gives an enumeration formula for $\mathbf{Isoc}\,(G;\mathcal{A})$ as follows.

Theorem 10 *For any finite group $\mathcal{A}$,*

$$\mathbf{Isoc}\,(G;\mathcal{A}) = \frac{|\mathfrak{G}(\mathcal{A};\beta(G))|}{|\mathrm{Aut}\,(\mathcal{A})|},$$

where $\mathfrak{G}(\mathcal{A};\beta) = \{\, (g_1, g_2, \ldots, g_\beta) \in \mathcal{A}^\beta \mid \{g_1, g_2, \ldots, g_\beta\}$ *generates* $\mathcal{A}\}$.

Note that the set $\mathfrak{G}(\mathcal{A};\beta(G))$ can be identified as the set of epimorphisms from the free group generated by $\beta(G)$ elements onto the group $\mathcal{A}$. Such kind identification will be reviewed again in chapter 6.

It is not difficult to show that the components of any regular covering $G \times_\phi \mathcal{A} \to G$ are isomorphic each other as coverings of G, and any two connected isomorphic regular coverings of G must have isomorphic covering transformation groups. To describe a component of the covering graph $G \times_\phi \mathcal{A}$ for $\phi \in C^1_T(G;\mathcal{A})$, let $\mathcal{A}_\phi(v)$ denote the *local voltage group* of ϕ at v which is, by definition, the subgroup of $\mathcal{A}$ consisting of all net ϕ-voltages of the closed walks based at $v \in V(G)$. The net ϕ-voltage of a closed walk is the product of the forward voltages (written from right to left) along the edges of the walk. Clearly, the local voltage groups $\mathcal{A}_\phi(v)$ of $\phi \in C^1_T(G;\mathcal{A})$, $v \in V(G)$, are independent of the choice of the base vertex v, and we simply denote it by $\mathcal{A}_\phi$. It is clear from the construction of the derived graph $G \times_\phi \mathcal{A}$ that for any voltage assignment $\phi \in C^1_T(G;\mathcal{A})$, the derived graph $G \times_\phi \mathcal{A}$ is connected if and only if the local voltage group $\mathcal{A}_\phi$ is just the full group $\mathcal{A}$. In fact, each component of $G \times_\phi \mathcal{A}$ is isomorphic to the component of $G \times_\phi \mathcal{A}$ containing the vertices $\{\, v_{id} \,|\, v \in V(G)\,\}$, called the *identity component* of $G \times_\phi \mathcal{A}$, where *id* denotes the identity element of the group $\mathcal{A}$. In fact, the identity component of an $\mathcal{A}$-covering $G \times_\phi \mathcal{A}$ is just the $\mathcal{A}_\phi$-covering $G \times_\phi \mathcal{A}_\phi$, by the construction of the derived graph. Now, it comes from Theorem 9 that two regular coverings of the same fold number of a graph are isomorphic if and only if their identity components are isomorphic as coverings. Notice that the order of any subgroup of a finite group $\mathcal{A}$ is a divisor of the order $|\mathcal{A}|$ of the group $\mathcal{A}$. The following theorem lists some basic formulas to count nonisomorphic regular coverings.

Theorem 11 ([22])

(1) *For any natural number n,* $\mathbf{Iso}^R(G;n) = \sum_{d|n} \mathbf{Isoc}^R(G;d)$.

(2) *For any natural number n,* $\mathbf{Isoc}^R(G;n) = \sum_{\mathcal{A}} \mathbf{Isoc}\,(G;\mathcal{A})$, *where $\mathcal{A}$ runs over all nonisomorphic groups of order n.*

(3) *For any finite group $\mathcal{A}$,* $\mathbf{Iso}\,(G;\mathcal{A}) = \sum_{\mathcal{S}} \mathbf{Isoc}\,(G;\mathcal{S})$, *where $\mathcal{S}$ runs over all nonisomorphic subgroups of $\mathcal{A}$.*

(4) *For any finite groups $\mathcal{A}$ and $\mathcal{B}$ with* $(|\mathcal{A}|, |\mathcal{B}|) = 1$,

$$\mathbf{Iso}\,(G;\mathcal{A}\oplus\mathcal{B}) = \mathbf{Iso}\,(G;\mathcal{A})\,\mathbf{Iso}\,(G;\mathcal{B})$$

and

$$\mathbf{Isoc}\,(G;\mathcal{A}\oplus\mathcal{B}) = \mathbf{Isoc}\,(G;\mathcal{A})\,\mathbf{Isoc}\,(G;\mathcal{B}).$$

(5) *For any two relatively prime numbers m and n,*

$$\mathbf{Iso}^R(G;mn) \geq \mathbf{Iso}^R(G;m)\,\mathbf{Iso}^R(G;n).$$

Note The number $\mathbf{Iso}^R(G;mn)$ can be strictly greater than the number $\mathbf{Iso}^R(G;m)\,\mathbf{Iso}^R(G;n)$, even if m and n are distinct primes. For example, if $\beta(G) \geq 2$, $m = 2$ and $n = 3$, then

$$\mathbf{Iso}^R(G;6) > \mathbf{Iso}^R(G;2)\,\mathbf{Iso}^R(G;3),$$

because

$$\begin{aligned}\mathbf{Iso}^R(G;6) &= \mathbf{Isoc}\,(G;\mathbb{Z}_6) + \mathbf{Isoc}\,(G;\mathbb{D}_3) \\ &\qquad + \mathbf{Isoc}\,(G;\mathbb{Z}_3) + \mathbf{Isoc}\,(G;\mathbb{Z}_2) + 1 \\ &= \mathbf{Iso}\,(G;\mathbb{Z}_6) + \mathbf{Isoc}\,(G;\mathbb{D}_3),\end{aligned}$$

and $\quad \mathbf{Iso}\,(G;\mathbb{Z}_6) = \mathbf{Iso}\,(G;\mathbb{Z}_2)\,\mathbf{Iso}\,(G;\mathbb{Z}_3) = \mathbf{Iso}^R(G;2)\,\mathbf{Iso}^R(G;3).$

Example 2 Let $\mathbb{Z}_{p^m}$ be the cyclic group of order p^m, p prime. Then $\mathrm{Aut}\,(\mathbb{Z}_{p^m})$ can be identified with the set of all elements of $\mathbb{Z}_{p^m}$ which are relatively prime to p^m, that is, the set $\{\lambda \in \mathbb{Z}_{p^m} : (\lambda, p^m) = 1\}$, and

$$\mathfrak{G}(\mathbb{Z}_{p^m};\beta(G)) = \{\, (g_1, g_2, \ldots, g_{\beta(G)}) \in (\mathbb{Z}_{p^m})^{\beta(G)} \mid \text{at least one of } g_i\text{'s generates } \mathbb{Z}_{p^m}\}.$$

It implies that

$|\mathrm{Aut}\,(\mathbb{Z}_{p^m})| = p^{m-1}(p-1)$ and $|\mathfrak{G}(\mathbb{Z}_{p^m};\beta(G))| = p^{\beta(G)m} - p^{\beta(G)(m-1)}$.

Then, by Theorem 10,

$$\mathbf{Isoc}\,(G;\mathbb{Z}_{p^m}) = \frac{p^{\beta(G)m} - p^{\beta(G)(m-1)}}{p^{m-1}(p-1)} = p^{(\beta(G)-1)(m-1)}\frac{p^{\beta(G)}-1}{p-1}$$

for $m > 0$. Now, by Theorem 11(3) and the lattice structure of subgroups of $\mathbb{Z}_{p^m}$, we have

$$\begin{aligned}&\mathbf{Iso}\,(G;\mathbb{Z}_{p^m})\\ &= 1 + \sum_{h=1}^{m} p^{(\beta(G)-1)(h-1)}\frac{p^{\beta(G)}-1}{p-1} = 1 + \frac{p^{\beta(G)}-1}{p-1}\,\frac{p^{m(\beta(G)-1)}-1}{p^{\beta(G)-1}-1}.\end{aligned}$$

From Example 2 and Theorem 11(4), we can get

Theorem 12 ([16],[22]) *For any $n = p_1^{s_1}p_2^{s_2}\cdots p_\ell^{s_\ell} > 1$ (a prime factorization), the number of isomorphism classes of connected $\mathbb{Z}_n$-coverings of G is*

$$\mathbf{Isoc}\,(G;\mathbb{Z}_n) = \begin{cases} 0 & \text{if } \beta(G) = 0, \\ \displaystyle\prod_{i=1}^{\ell} p_i^{(\beta(G)-1)(s_i-1)}\frac{p_i^{\beta(G)}-1}{p_i-1} & \text{if } \beta(G) \geq 1 \end{cases}$$

And, the number of nonisomorphic $\mathbb{Z}_n$-coverings of G is

$$\mathbf{Iso}\,(G;\mathbb{Z}_n) = \begin{cases} 1 & \text{if } \beta(G) = 0, \\ \displaystyle\prod_{i=1}^{\ell}(s_i+1) & \text{if } \beta(G) = 1, \\ \displaystyle\prod_{i=1}^{\ell}\left(1 + \frac{(p_i^{\beta(G)}-1)(p_i^{s_i(\beta(G)-1)}-1)}{(p_i-1)(p_i^{\beta(G)-1}-1)}\right) & \text{if } \beta(G) \geq 2\,. \end{cases}$$

For the remain of this chapter, we aim to describe the enumeration of nonisomorphic regular coverings having a finite *abelian* voltage group. By the classification of finite abelian groups, any finite abelian group $\mathcal{A}$ is isomorphic to a direct sum of finite cyclic groups of order powers of prime numbers. In order to compute the number $\mathbf{Iso}\,(G;\mathcal{A})$, it suffices, by Theorem 11((3),(4)), to compute the number $\mathbf{Iso}\,(G;\oplus_{h=1}^{\ell} m_h\mathbb{Z}_{p^{s_h}})$ or the number $\mathbf{Isoc}\,(G;\oplus_{h=1}^{\ell} m_h\mathbb{Z}_{p^{s_h}})$ for a prime p. To do this, we first introduce the following lemma.

Lemma 10 ([22])

(1) *For any natural numbers m and n with $m \leq n$, and a prime p, we have*

$$|\mathfrak{G}(m\mathbb{Z}_p;n)| = p^{\frac{m(m-1)}{2}}(p^n-1)(p^{n-1}-1)\cdots(p^{n-m+1}-1),$$

and

$$|\mathrm{Aut}\,(m\mathbb{Z}_p)| = |\mathfrak{G}(m\mathbb{Z}_p;m)| = p^{\frac{m(m-1)}{2}}(p^m-1)(p^{m-1}-1)\cdots(p-1).$$

(2) *For any natural number $s \geq 1$, we have*

$$|\mathfrak{G}(m\mathbb{Z}_{p^s};n)| = p^{(s-1)mn}|\mathfrak{G}(m\mathbb{Z}_p;n)|,$$

and

$$|\mathrm{Aut}\,(m\mathbb{Z}_{p^s})| = p^{(s-1)m^2}|\mathrm{Aut}\,(m\mathbb{Z}_p)|.$$

By Theorems 11(3), 10 and Lemma 10, we have

Corollary 2 ([22]) *For any m, the number of nonisomorphic connected $m\mathbb{Z}_p$-coverings of G is*

$$\mathbf{Isoc}\,(G;m\mathbb{Z}_p) = \frac{(p^{\beta(G)}-1)(p^{\beta(G)-1}-1)\cdots(p^{\beta(G)-m+1}-1)}{(p^m-1)(p^{m-1}-1)\cdots(p-1)}.$$

The number of nonisomorphic $m\mathbb{Z}_p$-coverings of G is

$$\mathbf{Iso}\,(G;m\mathbb{Z}_p) = 1 + \sum_{h=1}^{m}\frac{(p^{\beta(G)}-1)(p^{\beta(G)-1}-1)\cdots(p^{\beta(G)-h+1}-1)}{(p^h-1)(p^{h-1}-1)\cdots(p-1)}.$$

This formula for the number $\mathbf{Iso}\,(G; m\mathbb{Z}_p)$ in Corollary 2 is much more explicit than that of Hofmeister's in [12].

Remark It is well-known (see [46]) that the number of the m-dimensional subspaces of the n-dimensional vector space $n\mathbb{Z}_p$ over the field $\mathbb{Z}_p$ is equal to the Gaussian coefficient

$$\left[\begin{array}{c} n \\ m \end{array}\right]_p = \frac{\displaystyle\prod_{i=n-m+1}^{n} (p^i - 1)}{\displaystyle\prod_{i=1}^{m} (p^i - 1)}.$$

Hence, we can say that the number of nonisomorphic connected $m\mathbb{Z}_p$-coverings of a connected graph G is equal to the number of the m-dimensional subspaces of the $\beta(G)$-dimensional vector space $\beta(G)\mathbb{Z}_p$.

Let $m_1\mathbb{Z}_{p^{s_1}} \oplus m_2\mathbb{Z}_{p^{s_2}}$ be the direct sum of two abelian groups $m_1\mathbb{Z}_{p^{s_1}}$ and $m_2\mathbb{Z}_{p^{s_2}}$ (say, $s_2 < s_1$) and let $g_1 = (g_{11}, g_{12}), \ldots, g_n = (g_{n1}, g_{n2}) \in m_1\mathbb{Z}_{p^{s_1}} \oplus m_2\mathbb{Z}_{p^{s_2}}$. Then $\{g_1, \ldots, g_n\}$ generates $m_1\mathbb{Z}_{p^{s_1}} \oplus m_2\mathbb{Z}_{p^{s_2}}$ if and only if $\{(p^{s_1-1}g_{11}, p^{s_2-1}g_{12}), \ldots, (p^{s_1-1}g_{n1}, p^{s_2-1}g_{n2})\}$ generates $(m_1 + m_2)\mathbb{Z}_p$. An analogous argument to the proof of Lemma 10 gives

$$|\mathfrak{G}(m_1\mathbb{Z}_{p^{s_1}} \oplus m_2\mathbb{Z}_{p^{s_2}}; n)| = p^{n\left(m_1(s_1-1)+m_2(s_2-1)\right)}|\mathfrak{G}((m_1 + m_2)\mathbb{Z}_p; n)|.$$

But, in general,

$$|\mathrm{Aut}\,(m_1\mathbb{Z}_{p^{s_1}} \oplus m_2\mathbb{Z}_{p^{s_2}})| \neq |\mathfrak{G}(m_1\mathbb{Z}_{p^{s_1}} \oplus m_2\mathbb{Z}_{p^{s_2}}; m_1 + m_2)|.$$

Note that the group $m_1\mathbb{Z}_{p^{s_1}} \oplus m_2\mathbb{Z}_{p^{s_2}}$ is an elementary abelian p-group, so that its automorphism group is isomorphic to the group of nonsingular linear transformations of the vector space $m_1\mathbb{Z}_{p^{s_1}} \oplus m_2\mathbb{Z}_{p^{s_2}}$. Now, an elementary exercise gives

$$|\mathrm{Aut}\,(m_1\mathbb{Z}_{p^{s_1}} \oplus m_2\mathbb{Z}_{p^{s_2}})| = p^{g(m_i,s_i)} \prod_{i=1}^{2} \prod_{h=1}^{m_i} \left(p^{m_i-h+1} - 1\right),$$

where

$$g(m_i, s_i) = m\left(\sum_{i=1}^{2} m_i(s_i - 1)\right) - m_1m_2(s_1 - s_2 - 1) + \frac{m(m-1)}{2}$$

with $m = m_1 + m_2$ and $s_2 < s_1$. In general, we have the following.

Lemma 11 *Let $m_1, \ldots, m_\ell$ and $s_1, \ldots, s_\ell$ be natural numbers with $s_\ell < \ldots < s_1$. Let p be a prime number. Then we have*

(1) $$\left|\mathfrak{G}\left(\oplus_{h=1}^{\ell} m_h \mathbb{Z}_{p^{s_h}}; n\right)\right| = p^{n\left(m_1(s_1-1)+\cdots+m_\ell(s_\ell-1)\right)} |\mathfrak{G}((m_1+\cdots+m_\ell)\mathbb{Z}_p; n)|.$$

(2) $$\left|\mathrm{Aut}\left(\oplus_{h=1}^{\ell} m_h \mathbb{Z}_{p^{s_h}}\right)\right| = p^{g(m_i,s_i)} \prod_{i=1}^{\ell}\prod_{h=1}^{m_i}\left(p^{m_i-h+1}-1\right),$$

where

$$g(m_i,s_i) = m\left(\sum_{i=1}^{\ell} m_i(s_i-1)\right) - \sum_{i=1}^{\ell-1} m_i\left(\sum_{j=i+1}^{\ell} m_j(s_i-s_j-1)\right) + \frac{m(m-1)}{2}$$

with $m = m_1+\cdots+m_\ell$.

Now, the following comes from Theorem 10 and Lemma 11.

Theorem 13 ([22]) *Let $m_1, \ldots, m_\ell$ and $s_1, \ldots, s_\ell$ be natural numbers with $s_\ell < \cdots < s_1$. Then the number of nonisomorphic connected $\oplus_{h=1}^{\ell} m_h \mathbb{Z}_{p^{s_h}}$-coverings of G is*

$$\mathbf{Isoc}\left(G; \oplus_{h=1}^{\ell} m_h \mathbb{Z}_{p^{s_h}}\right) = p^{f(\beta(G),m_i,s_i)} \frac{\prod_{i=1}^{m} p^{\beta(G)-i+1}-1}{\prod_{j=1}^{\ell}\prod_{h=1}^{m_j} p^{m_j-h+1}-1},$$

where $m = m_1+\cdots+m_\ell$, p is prime and

$$f(\beta(G),m_i,s_i) = (\beta(G)-m)\left(\sum_{i=1}^{\ell} m_i(s_i-1)\right) + \sum_{i=1}^{\ell-1} m_i\left(\sum_{j=i+1}^{\ell} m_j(s_i-s_j-1)\right).$$

		Isoc			**Iso**		
β	(p,q)	$\mathbb{Z}_{p^3}\oplus\mathbb{Z}_p$	$\mathbb{Z}_{q^2}$	$\mathbb{Z}_{p^3}\oplus\mathbb{Z}_p\oplus\mathbb{Z}_{q^2}$	$\mathbb{Z}_{p^3}\oplus\mathbb{Z}_p$	$\mathbb{Z}_{q^2}$	$\mathbb{Z}_{p^3}\oplus\mathbb{Z}_p\oplus\mathbb{Z}_{q^2}$
1	$(2,3)$	0	1	0	4	3	12
2	$(2,5)$	6	30	180	32	37	1184
3	$(3,5)$	1404	775	1088100	2757	807	2224899
4	$(3,7)$	126360	137200	1695792000	161451	137601	22215819051

Table 4.1: The number **Isoc** $(G;\mathcal{A})$ and **Iso** $(G;\mathcal{A})$ for some $\mathcal{A}$ and small $\beta(G)$

Now, we can compute the number **Iso** $(G;\mathcal{A})$ for any finite abelian group $\mathcal{A}$ by using Theorems 11((3),(4)) and 13 repeatedly if necessary. For example, if p and q are two distinct prime numbers, then

$$\begin{aligned}
&\mathbf{Iso}\ (G;\mathbb{Z}_{p^3}\oplus\mathbb{Z}_p\oplus\mathbb{Z}_{q^2})\\
&=\mathbf{Iso}\ (G;\mathbb{Z}_{p^3}\oplus\mathbb{Z}_p)\ \mathbf{Iso}\ (G;\mathbb{Z}_{q^2})\\
&=\left(1+\sum_{i=1}^{3}\mathbf{Isoc}\ \left(G;\mathbb{Z}_{p^i}\right)+\sum_{i=1}^{3}\mathbf{Isoc}\ \left(G;\mathbb{Z}_{p^i}\oplus\mathbb{Z}_p\right)\right)\\
&\qquad\qquad\times\left(1+\sum_{i=1}^{2}\mathbf{Isoc}\ \left(G;\mathbb{Z}_{q^i}\right)\right)\\
&=\left(1+\frac{p^{\beta(G)}-1}{p-1}\left(1+p^{\beta(G)-1}\left(1+p^{\beta(G)-1}\right)\right)\right.\\
&\qquad\left.+\frac{\left(p^{\beta(G)}-1\right)\left(p^{\beta(G)-1}-1\right)}{(p-1)\left(p^2-1\right)}\left(1+p^{\beta(G)-2}(p+1)\left(1+p^{\beta(G)-1}\right)\right)\right)\\
&\qquad\qquad\times\left(1+\frac{\left(q^{\beta(G)}-1\right)\left(q^{\beta(G)-1}+1\right)}{q-1}\right).
\end{aligned}$$

For some abelian groups $\mathcal{A}$ and small $\beta(G)$, the numbers **Isoc** $(G;\mathcal{A})$ and **Iso** $(G;\mathcal{A})$ are listed in table 4.1.

Remark For a connected $\mathcal{A}$-covering $p:\tilde{G}\to G$, the image $p_*(\pi_1(\tilde{G}))$ of the fundamental group of the covering graph $\tilde{G}$ is a normal subgroup of the fundamental group $\pi_1(G)$ of the base graph G, and the quotient group $\pi_1(G)/p_*(\pi_1(\tilde{G}))$ is isomorphic to $\mathcal{A}$. If $\mathcal{A}$ is abelian,

β	1	2	3	4	5	6	7	8	9	10
$n = 1$	1	1	1	1	1	1	1	1	1	1
$n = 2$	1	3	7	15	31	63	127	255	511	1023
$n = 3$	1	4	13	40	121	364	1093	3280	9841	29524
$n = 4$	1	7	35	155	651	2667	10795	43435	174251	698027
$n = 5$	1	6	31	156	781	3906	19531	97656	488281	2441406
$n = 6$	1	12	91	600	3751	22932	138811	836400	5028751	30203052
$n = 7$	1	8	57	400	2801	19608	137257	960800	6725601	47079208
$n = 8$	1	15	155	1395	11811	97155	788035	6347715	50955971	408345795

Table 4.2: The number of subgroups of index n in $\oplus_1^\beta \mathbb{Z}$

then $p_*(\pi_1(\tilde{G}))$ contains the commutator subgroup $[\pi_1(G), \pi_1(G)]$ of the free group $\pi_1(G)$. Since $[\pi_1(G), \pi_1(G)]$ is a normal subgroup of $\pi_1(G)$, the natural homomorphism $q : \pi_1(G) \to \pi_1(G)/[\pi_1(G), \pi_1(G)]$ induces a one-to-one correspondence between the set of all subgroups of $\pi_1(G)$ containing $[\pi_1(G), \pi_1(G)]$ and the set of all subgroups of the quotient group $\pi_1(G)/[\pi_1(G), \pi_1(G)]$. Notice that $\pi_1(G)/[\pi_1(G), \pi_1(G)]$ is the free abelian group generated by $\beta(G)$ elements. Now, from a well-known classification theorem for regular coverings of a topological space, it follows that the number $\sum_{\mathcal{A}} \mathbf{Isoc}\,(G; \mathcal{A}) = \sum_{\mathcal{A}} \frac{|\mathfrak{G}(\mathcal{A}; \beta(G))|}{|\mathrm{Aut}\,(\mathcal{A})|}$, where $\mathcal{A}$ runs over all nonisomorphic *abelian* groups of order n, is equal to the number of subgroups of index n of the free abelian group $\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$ generated by $\beta(G)$ elements. For small n and small β, these numbers are listed in table 4.2.

Chapter 5

Regular coverings having dihedral voltage groups

In this chapter, we consider a dihedral group as a nonabelian voltage group, and aim to compute the number of nonisomorphic regular coverings having a dihedral voltage group. Recall that the dihedral group of order $2n$ can be presented as follows:

$$\mathbb{D}_n = \left\langle a, b \,:\, a^2 = 1 = b^n, aba = b^{-1} \right\rangle.$$

Note that $\mathbb{D}_1 = \mathbb{Z}_2$, $\mathbb{D}_2 = \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $\mathbb{D}_n$ is not abelian for $n \geq 3$ with $\langle a \rangle = \mathbb{Z}_2$ and $\langle b \rangle = \mathbb{Z}_n$, and an element of $\mathbb{D}_n$ can be of the form b^i or ab^i for $i = 0, 1, \ldots, n-1$.

Notice that any subgroup of the dihedral group $\mathbb{D}_n$ is isomorphic to one of $\mathbb{D}_i$ (i is a divisor of n) or $\mathbb{Z}_j$ (j is a divisor of n), where

$\mathbb{Z}_1 = \{identity\}$. It follows from Theorem 11(3) that for any $n \geq 3$

$$\begin{aligned}
&\mathbf{Iso}\,(G;\mathbb{D}_n)\\
&= \begin{cases} \displaystyle\sum_{m|n} \mathbf{Isoc}\,(G;\mathbb{Z}_m) + \sum_{m|n} \mathbf{Isoc}\,(G;\mathbb{D}_m) & \text{if } n \text{ is odd} \\ \displaystyle\sum_{m|n} \mathbf{Isoc}\,(G;\mathbb{Z}_m) + \sum_{m|n,\, m\neq 1} \mathbf{Isoc}\,(G;\mathbb{D}_m) & \text{if } n \text{ is even} \end{cases}\\
&= \begin{cases} \displaystyle\mathbf{Iso}\,(G;\mathbb{Z}_n) + \sum_{m|n} \mathbf{Isoc}\,(G;\mathbb{D}_m) & \text{if } n \text{ is odd} \\ \displaystyle\mathbf{Iso}\,(G;\mathbb{Z}_n) + \sum_{m|n,\, m\neq 1} \mathbf{Isoc}\,(G;\mathbb{D}_m) & \text{if } n \text{ is even.} \end{cases}
\end{aligned}$$

To compute the number $\mathbf{Isoc}\,(G;\mathbb{D}_n)$, we first compute $|\mathrm{Aut}(\mathbb{D}_n)|$ and $|\mathfrak{G}(\mathbb{D}_n;r)|$.

Lemma 12 *Let n be a natural number with prime decomposition $p_1^{m_1} \cdots p_\ell^{m_\ell}$. If $n \geq 3$, then*

(1) $|\mathrm{Aut}(\mathbb{D}_n)| = n \cdot phi(n) = n\, p_1^{m_1-1}(p_1-1)\cdots p_\ell^{m_\ell-1}(p_\ell-1)$.

(2) *For any natural number r,*

$$|\mathfrak{G}(\mathbb{D}_n;r)| = (2^r-1)\prod_{i=1}^{\ell} p_i^{(m_i-1)r+1}\left(p_i^{r-1}-1\right).$$

Proof: It is not hard to show that

$$\mathrm{Aut}(\mathbb{D}_n) = \left\{\sigma_j^i \,:\, \sigma_j^i(a) = ab^i, \sigma_j^i(b) = b^j,\, 0 \leq i,j \leq n-1,\, (n,j)=1\right\}.$$

It implies that $|\mathrm{Aut}(\mathbb{D}_n)| = n\cdot phi(n) = n\, p_1^{m_1-1}(p_1-1)\cdots p_\ell^{m_\ell-1}(p_\ell-1)$.

Next, we compute the number $|\mathfrak{G}(\mathbb{D}_n;r)|$. Since the prime decomposition of n is $p_1^{m_1}\cdots p_\ell^{m_\ell}$, $\mathbb{Z}_n =< b >$ is isomorphic to $\oplus_{i=1}^{\ell}\mathbb{Z}_{p_i^{m_i}}$, where $\mathbb{Z}_{p_i^{m_i}} =< b_i >$ with $b = b_1\cdots b_\ell$. Note that $\mathbb{D}_n = \mathbb{Z}_n \cup a\mathbb{Z}_n$, disjoint union. It is clear that if $(g_1,\ldots,g_r) \in \mathfrak{G}(\mathbb{D}_n;r)$ then there exists at least one j $(1 \leq j \leq r)$ such that $g_j \in a\mathbb{Z}_n = \{ab^i \,|\, i = 1,\ldots,n\}$. Given any nonempty subset S of $\{1,2,\ldots,r\}$, let $\mathfrak{G}[S]$ denote the set

$$\{(g_1,\ldots,g_r) \in \mathfrak{G}(\mathbb{D}_n;r) \,:\, g_j \in a\mathbb{Z}_n \text{ for } j \in S, \text{ and } g_j \in \mathbb{Z}_n \text{ for } j \notin S\}.$$

Then

$$\bigcup_{S(\neq\emptyset)\subset\{1,2,\ldots,r\}} \mathfrak{G}[S] = \mathfrak{G}(\mathbb{D}_n; r).$$

Moreover, $\mathfrak{G}[S]$ and $\mathfrak{G}[T]$ are disjoint for any two distinct nonempty subsets S and T of $\{1, 2, \ldots, r\}$. It implies that

$$|\mathfrak{G}(\mathbb{D}_n; r)| = \left| \bigcup_{S(\neq\emptyset)\subset\{1,2,\ldots,r\}} \mathfrak{G}[S] \right| = \sum_{S(\neq\emptyset)\subset\{1,2,\ldots,r\}} |\mathfrak{G}[S]|.$$

For convenience, for each $g \in \mathbb{D}_n$, let

$$g = \begin{cases} (g_1', \ldots, g_\ell') & \text{if } g \in \mathbb{Z}_n = \oplus_{i=1}^{\ell} \mathbb{Z}_{p_i^{m_i}} \\ a(g_1', \ldots, g_\ell') & \text{if } g \in a\mathbb{Z}_n = a \oplus_{i=1}^{\ell} \mathbb{Z}_{p_i^{m_i}}. \end{cases}$$

Let S be a nonempty subset of $\{1, \ldots, r\}$ and $(g_1, \ldots, g_r) \in (\mathbb{D}_n)^r \equiv \prod_{i=1}^{r} \mathbb{D}_n$. Then $(g_1, \ldots, g_r) \in \mathfrak{G}[S]$ if and only if for each $i = 1, \ldots, \ell$,

$$(g_{1_i}', \ldots, g_{r_i}') \in \prod_{i=1}^{r} \mathbb{Z}_{p_i^{m_i}} - \bigcup_{k=0}^{p_i-1} \left(\prod_{j\notin S} \mathbb{Z}_{p_i^{m_i-1}} \times \prod_{j\in S} b_i^k \mathbb{Z}_{p_i^{m_i-1}} \right),$$

where $\mathbb{Z}_{p_i^{m_i-1}}$ is the subgroup of $\mathbb{Z}_{p_i^{m_i}}$ generated by $b_i^{p_i}$. It implies that for any nonempty subset S of $\{1, 2, \ldots, r\}$,

$$\begin{aligned} |\mathfrak{G}[S]| &= \prod_{i=1}^{\ell} \left(p_i^{m_i r} - p_i^{(m_i-1)|S|} \cdot p_i \cdot p_i^{(m_i-1)(r-|S|)} \right) \\ &= \prod_{i=1}^{\ell} p_i^{(m_i-1)r+1} \left(p_i^{r-1} - 1 \right), \end{aligned}$$

which does not depend on the set S. Now, the cardinality $|\mathfrak{G}(\mathbb{D}_n; r)|$ of the set $\mathfrak{G}(\mathbb{D}_n; r)$ is

$$\sum_{S(\neq\emptyset)\subset\{1,2,\ldots,r\}} |\mathfrak{G}[S]| = (2^r - 1) \prod_{i=1}^{\ell} p_i^{(m_i-1)r+1} \left(p_i^{r-1} - 1 \right).$$

□

Now, the next theorem follows from Theorem 10 and Lemma 12.

Theorem 14 ([22]) *For any $n \geq 3$, the number of nonisomorphic connected $\mathbb{D}_n$-coverings of G is*

$$\mathbf{Isoc}\,(G; \mathbb{D}_n) = \left(2^{\beta(G)} - 1\right) \prod_{i=1}^{\ell} p_i^{(m_i-1)(\beta(G)-2)} \frac{p_i^{\beta(G)-1} - 1}{p_i - 1},$$

where $p_1^{m_1} \cdots p_\ell^{m_\ell}$ is the prime decomposition of n.

For any edge e in the cotree $G - T$, we have $\beta(G - e) = \beta(G) - 1$. By Example 2, Theorems 13 and 14, we have

$$\mathbf{Isoc}\,(G; \mathbb{D}_n) = (2^{\beta(G)} - 1)\mathbf{Isoc}\,(G - e; \mathbb{Z}_n)$$

for any $n \geq 3$. Thus, if n is odd, then

$$\begin{aligned} \sum_{m|n} \mathbf{Isoc}\,(G; \mathbb{D}_m) &= \left(2^{\beta(G)} - 1\right) \sum_{m|n} \mathbf{Isoc}\,(G - e; \mathbb{Z}_m) \\ &= \left(2^{\beta(G)} - 1\right) \mathbf{Iso}\,(G - e; \mathbb{Z}_n). \end{aligned}$$

If n is even, then

$$\begin{aligned} &\sum_{m|n,\, m\neq 1} \mathbf{Isoc}\,(G; \mathbb{D}_m) \\ &\quad = \sum_{m|n,\, m\geq 3} \mathbf{Isoc}\,(G; \mathbb{D}_m) + \mathbf{Isoc}\,(G; \mathbb{D}_2) \\ &\quad = \left(2^{\beta(G)} - 1\right) \left(\sum_{m|n} \mathbf{Isoc}\,(G - e; \mathbb{Z}_m) - [1 + \mathbf{Isoc}\,(G - e; \mathbb{Z}_2)] \right) \\ &\qquad\qquad + \mathbf{Isoc}\,(G; \mathbb{D}_2) \\ &\quad = \left(2^{\beta(G)} - 1\right) \mathbf{Iso}\,(G - e; \mathbb{Z}_n) - \left(2^{\beta(G)} - 1\right) 2^{\beta(G)-1} \\ &\qquad\qquad + \frac{1}{3} \left(2^{\beta(G)} - 1\right) \left(2^{\beta(G)-1} - 1\right) \\ &\quad = \left(2^{\beta(G)} - 1\right) \mathbf{Iso}\,(G - e; \mathbb{Z}_n) - \frac{1}{3} \left(4^{\beta(G)} - 1\right). \end{aligned}$$

We summarize our discussion as follows.

β	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$	$n=11$
1	0	0	0	0	0	0	0	0	0
2	3	3	3	3	3	3	3	3	3
3	28	42	42	84	56	84	84	126	84
4	195	420	465	1365	855	1680	1755	3255	1995
5	1240	3720	4836	18600	12400	29760	33480	72540	45384

Table 5.1: The number $\mathbf{Isoc}\,(G;\mathbb{D}_n)$ for small n and small $\beta(G)$

β	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$	$n=11$
1	3	3	3	4	3	4	4	4	3
2	11	14	13	27	15	29	26	35	19
3	49	85	81	231	121	281	250	431	225
4	251	591	637	2251	1271	3231	3086	6267	3475
5	1393	4403	5649	23899	15233	42099	44674	102555	61521

Table 5.2: The number $\mathbf{Iso}\,(G;\mathbb{D}_n)$ for small n and small $\beta(G)$

Theorem 15 ([16], [22]) *For any $n \geq 3$, the number of nonisomorphic $\mathbb{D}_n$-coverings of G is*

$$\mathbf{Iso}\,(G;\mathbb{D}_n)$$
$$= \begin{cases} \mathbf{Iso}\,(G;\mathbb{Z}_n) + \left(2^{\beta(G)}-1\right)\mathbf{Iso}\,(G-e;\mathbb{Z}_n) & \text{if } n \text{ is odd,} \\[2ex] \mathbf{Iso}\,(G;\mathbb{Z}_n) + \left(2^{\beta(G)}-1\right)\mathbf{Iso}\,(G-e;\mathbb{Z}_n) & \\ \qquad\qquad\qquad\qquad - \dfrac{4^{\beta(G)}-1}{3} & \text{if } n \text{ is even,} \end{cases}$$

where e is an edge in the cotree $G-T$.

Recall that the number $\mathbf{Iso}\,(G;\mathbb{Z}_n)$ was computed in Theorem 12. The numbers $\mathbf{Isoc}\,(G;\mathbb{D}_n)$ and $\mathbf{Iso}\,(G;\mathbb{D}_n)$ for small n and $\beta(G)$ are listed in tables 5.1 and 5.2.

Let p be a prime number. Then every group of order p or p^2 is abelian. Hence, there is only one group of order p up to isomorphism; it is the cyclic group $\mathbb{Z}_p$, and there are only two groups of order p^2 up to isomorphism; they are $\mathbb{Z}_{p^2}$ and $\mathbb{Z}_p \oplus \mathbb{Z}_p$. Let p and q are distinct primes. If $p < q$, $p \not| \, (q-1)$, then there are only one nonisomorphic group of order pq; it is the cyclic group $\mathbb{Z}_{pq}$ which is isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_q$. If

$p < q$, $p \,|\, (q-1)$, then there are only two nonisomorphic groups of order pq; one of them is the cyclic group $\mathbb{Z}_{pq}$ and the other is a nonabelian group $\mathcal{K}$ generated by two elements a and b such that

$$|<a>| = p; \qquad |<b>| = q; \qquad ab = b^s a,$$

where $s \neq 1$ and $s^p \equiv 1 \pmod q$. More on the classification of finite groups that needed in this manuscript can be found in [44, 45].

The following come from the classification of finite groups and Theorem 11 (2). For a prime p, the numbers of p-, p^2-, pq- or p^3-fold nonisomorphic connected regular coverings of G are

$$\mathbf{Isoc}^R(G;p) = \frac{p^{\beta(G)}-1}{p-1},$$

$$\mathbf{Isoc}^R(G;p^2) = \frac{(p^{\beta(G)}-1)(p^{\beta(G)-1}-1)}{(p^2-1)(p-1)} + p^{(\beta(G)-1)}\frac{p^{\beta(G)}-1}{p-1},$$

$$\mathbf{Isoc}^R(G;pq) = \begin{cases} \dfrac{p^{\beta(G)}-1}{p-1}\,\dfrac{q^{\beta(G)}-1}{q-1} & \text{if } p<q,\ p \nmid (q-1), \\ \dfrac{\left(p^{\beta(G)}-1\right)\left(q^{\beta(G)-1}-1\right)}{q-1} & \\ \qquad + \dfrac{p^{\beta(G)}-1}{p-1}\,\dfrac{q^{\beta(G)}-1}{q-1} & \text{if } p<q,\ p \,|\, (q-1), \end{cases}$$

$$\begin{aligned} \mathbf{Isoc}^R(G;p^3) \;=\; & \frac{(p^{\beta(G)}-1)(p^{\beta(G)-1}-1)(p^{\beta(G)-2}-1)}{(p^3-1)(p^2-1)(p-1)} \\ & + p^{\beta(G)-2}\frac{(p^{\beta(G)}-1)(p^{\beta(G)-1}-1)}{(p^2-1)(p-1)}(p+2) \\ & + p^{\beta(G)-1}\frac{p^{\beta(G)}-1}{2(p-1)}\left(3p^{\beta(G)-1}-1\right). \end{aligned}$$

Now, by using Theorem 11 (1), we have

$$\mathbf{Iso}^R(G;p) = \frac{p^{\beta(G)}-1}{p-1} + 1,$$

$$\mathbf{Iso}^R(G;p^2) = \frac{(p^{\beta(G)}-1)(p^{\beta(G)-1}-1)}{(p^2-1)(p-1)} + \frac{p^{\beta(G)}-1}{p-1}(p^{\beta(G)-1}+1) + 1,$$

β	$n=1$	2	3	4	5	6	7	8	9	10	11
1	1	1	1	1	1	1	1	1	1	1	1
2	1	3	4	7	6	15	8	19	13	21	12
3	1	7	13	35	31	119	57	211	130	259	133
4	1	15	40	155	156	795	400	1955	1210	2805	1464
5	1	31	121	651	781	4991	2801	16771	11011	29047	16105

Table 5.3: The number $\mathbf{Isoc}^R(G;n)$ for small n and small $\beta(G)$

β	$n=1$	2	3	4	5	6	7	8	9	10	11
1	1	2	2	3	2	4	2	4	3	4	2
2	1	4	5	11	7	23	9	30	18	31	13
3	1	8	14	43	32	140	58	254	144	298	134
4	1	16	41	171	157	851	401	2126	1251	2977	1465
5	1	32	122	683	782	5144	2802	17452	11133	29860	16106

Table 5.4: The number $\mathbf{Iso}^R(G;n)$ for small n and small $\beta(G)$

$$\mathbf{Iso}^R(G;pq) = \begin{cases} \dfrac{p^{\beta(G)}+p-2}{p-1}\,\dfrac{q^{\beta(G)}+q-2}{q-1} & \text{if } p<q,\ p\nmid(q-1), \\ \dfrac{p^{\beta(G)}+p-2}{p-1}\,\dfrac{q^{\beta(G)}+q-2}{q-1} & \\ \quad +\dfrac{\left(p^{\beta(G)}-1\right)\left(q^{\beta(G)-1}-1\right)}{q-1} & \text{if } p<q,\ p\,|\,(q-1), \end{cases}$$

$$\begin{aligned}
&\mathbf{Iso}^R(G;p^3) \\
&= \frac{(p^{\beta(G)}-1)(p^{\beta(G)-1}-1)(p^{\beta(G)-2}-1)}{(p^3-1)(p^2-1)(p-1)} \\
&\quad + \frac{(p^{\beta(G)}-1)(p^{\beta(G)-1}-1)}{2(p^2-1)(p-1)}\left(p^{\beta(G)+1}+p^{\beta(G)-1}+4\cdot p^{\beta(G)-2}+2\right) \\
&\quad + \frac{p^{\beta(G)}-1}{p-1}\left(p^{2(\beta(G)-1)}+p^{\beta(G)-1}+1\right)+1.
\end{aligned}$$

Remark More enumerations of graph coverings satisfying some properties like concrete or bipartite coverings were studied in the sequel.

Hofmeister ([11], [13]) introduced the notion of a concrete (resp. concrete regular) covering of a graph G and gave formulas for enumerating the isomorphism classes of concrete (resp. concrete regular) coverings of G. An n-fold covering $p : \widetilde{G} \to G$ is said to be *concrete* if it is accompanied by an explicit partition $\mathcal{P} = \{P_1, \ldots, P_n\}$ of $V(\widetilde{G})$ such that every partition set P_i meets every vertex fiber exactly once. The partition sets P_i are the *sheets* of the covering p. A *concrete regular* covering is a concrete covering $p : \widetilde{G} \to G$ which is regular and every covering transformation of $\widetilde{G}$ preserves the sheets. Later, R. Feng et al [3] showed that the number of nonisomorphic n-fold concrete (resp. concrete regular) coverings of G is equal to that of nonisomorphic n-fold (resp. regular) coverings of the join $G + \infty$ of G and an extra vertex ∞. As a consequence, the isomorphism classes of concrete (resp. concrete regular) coverings of a graph can be enumerated by using known formulas for enumerating the isomorphism classes of coverings (resp. regular coverings) of a graph. It also gives a new formula to compute the number of the isomorphism classes of graphs with n vertices because the number of nonisomorphic concrete double coverings of the complete graph on n vertices is equal to the number of nonisomorphic graphs with n vertices. For enumeration of bipartite coverings, see [2] and [17].

Chapter 6

Regular coverings; A general case

In this chapter, we introduce a general formula to enumerate $\mathcal{A}$-coverings of a graph G for any finite group $\mathcal{A}$ in terms of the Möbius function defined on the subgroup lattice of $\mathcal{A}$ by P. Hall in [6]. G. Jones [20] [21] used such Möbius function to find a method for counting normal subgroups of a surface group or a crystallographic group, and applied it to count some covering surfaces. To apply the Jones' method to a graph covering case, first recall that the set $C^1_T(G;\mathcal{A})$ of $\mathcal{A}$-voltage assignments of G can be identified as

$$C^1_T(G;\mathcal{A}) = \mathcal{A} \times \mathcal{A} \times \cdots \times \mathcal{A}, \qquad (\beta(G) \quad \text{times}),$$

from which every $\mathcal{A}$-covering of the graph G can be derived. Let $\mathcal{F}_\beta$ denote the free group generated by β elements, where $\beta = \beta(G)$. Then, the $\mathcal{A}$-voltage assignments in $C^1_T(G;\mathcal{A})$ correspond bijectively to homomorphisms from the free group $\mathcal{F}_\beta$ to the voltage group $\mathcal{A}$, thus $|C^1_T(G;\mathcal{A})| = |\mathrm{Hom}(\mathcal{F}_\beta, \mathcal{A})| = |\mathcal{A}|^\beta$. Also, it can be written as

$$|C^1_T(G;\mathcal{A})| = |\mathrm{Hom}(\mathcal{F}_\beta, \mathcal{A})| = \sum_{K \le \mathcal{A}} |\mathrm{Epi}(\mathcal{F}_\beta, K)|$$

the sum of the numbers of epimorphisms from the free group $\mathcal{F}_\beta$ onto subgroups K of the group $\mathcal{A}$, and such epimorphisms correspond bijectively to transitive K-voltage assignments in $\mathfrak{G}(K;\beta)$. It follows that

$|\mathrm{Epi}(\mathcal{F}_\beta, K)| = |\mathfrak{G}(K;\beta)|$. Now, one can invert the equation

$$|\mathrm{Hom}(\mathcal{F}_\beta, \mathcal{A})| = \sum_{K \le \mathcal{A}} |\mathrm{Epi}(\mathcal{F}_\beta, K)|,$$

to count epimorphisms in terms of homomorphisms, by introducing the *Möbius funtion* for $\mathcal{A}$. This assigns an integer $\mu(K)$ to each subgroup K of $\mathcal{A}$ by the recursive formula

$$\sum_{H \ge K} \mu(H) = \delta_{K,\mathcal{A}} = \begin{cases} 1 \text{ if } K = \mathcal{A}, \\ 0 \text{ if } K < \mathcal{A}. \end{cases}$$

The equation

$$|\mathrm{Epi}(\mathcal{F}_\beta, \mathcal{A})| = \sum_{K \le \mathcal{A}} \mu(K) |\mathrm{Hom}(\mathcal{F}_\beta, K)|$$

is then easily deduced, and Theorem 10 gives

$$\begin{aligned} \mathbf{Isoc}\,(G;\mathcal{A}) &= \frac{1}{|\mathrm{Aut}\,(\mathcal{A})|} \sum_{K \le \mathcal{A}} \mu(K) |\mathrm{Hom}(\mathcal{F}_\beta, K)| \\ &= \frac{1}{|\mathrm{Aut}\,(\mathcal{A})|} \sum_{K \le \mathcal{A}} \mu(K) |K|^\beta. \end{aligned}$$

Example 3 (1) The cyclic group $\mathcal{A} = \mathbb{Z}_n$ has a unique subgroup $\mathbb{Z}_m$ for each m dividing n, and has no other subgroups. The Möbius function on the subgroup is $\mu(\mathbb{Z}_m) = \mu(n/m)$ (the Möbius function of the elementary number theory) and $|\mathrm{Aut}\,(\mathbb{Z}_n)| = phi(n)$ (Euler *phi*-function), so it implies that

$$\mathbf{Isoc}\,(G;\mathbb{Z}_n) = \frac{1}{phi(n)} \sum_{m|n} \mu\left(\frac{n}{m}\right) m^\beta.$$

This coincides with the formula given in Theorem 12.
(2) Let $\mathcal{A} = \mathbb{D}_n = \langle a, b \,:\, a^2 = 1 = b^n, aba = b^{-1} \rangle$ be the dihedral group of order $2n$. For convenience, let $\mathbb{Z}_m =< b^{\frac{n}{m}} >$ and let $\mathbb{D}_m^{(i)} = \mathbb{Z}_m \cup a(\mathbb{Z}_m b^i)$ for $i = 0, \ldots, \frac{n}{m} - 1$. Then each subgroup of $\mathbb{D}_n$ is one of $\mathbb{Z}_m$

or $\mathbb{D}_m^{(i)}$ for each m dividing n. Now, consider the lattice induced by the subgroups of $\mathbb{D}_n$. Then, for each subgroup $\mathcal{S}$ of $\mathbb{D}_n$, we have

$$\mu(\mathcal{S}) = \begin{cases} \mu\Big(\dfrac{n}{m}\Big) & \text{if } \mathcal{S} = \mathbb{D}_m^{(i)} \text{ for each } i = 0, \ldots, \frac{n}{m} - 1, \\ -\dfrac{n}{m}\,\mu\Big(\dfrac{n}{m}\Big) & \text{if } \mathcal{S} = \mathbb{Z}_m. \end{cases}$$

Since $|\mathrm{Aut}\,(\mathbb{D}_n)| = n \cdot phi(n)$ for $n \geq 3$, we have

$$\begin{aligned} \mathbf{Isoc}\,(G; \mathbb{D}_n) &= \frac{1}{n \cdot phi(n)} \left(\sum_{m|n} \frac{n}{m}\,\mu\Big(\frac{n}{m}\Big)\,(2m)^\beta - \sum_{m|n} \frac{n}{m}\,\mu\Big(\frac{n}{m}\Big)\,m^\beta \right) \\ &= \frac{2^\beta - 1}{phi(n)} \sum_{m|n} \mu\left(\frac{n}{m}\right) m^{\beta-1} \end{aligned}$$

for $n \geq 3$. This coincides with the formula given in Theorem 14.

Chapter 7

New classifications of branched coverings and the number of subgroups of a surface group

A *surface* $\mathbb{S}$ is a compact connected 2-manifold without boundary. By the classification of surfaces, a surface $\mathbb{S}$ is homeomorphic to one of the following:

$$\mathbb{S}_k = \begin{cases} \text{the orientable surface with } k \text{ handles} & \text{if } k > 0, \\ \text{the sphere } \mathbb{S}^2 & \text{if } k = 0, \\ \text{the nonorientable surface with } -k \text{ crosscaps} & \text{if } k < 0. \end{cases}$$

A continuous surjective map $p : \tilde{\mathbb{S}} \to \mathbb{S}$ is a *branched covering* if $p|_{\tilde{\mathbb{S}}-p^{-1}(B)} : \tilde{\mathbb{S}} - p^{-1}(B) \to \mathbb{S} - B$ is a covering for a finite subset B of $\mathbb{S}$. The *branch set* B of a branched covering $p : \tilde{\mathbb{S}} \to \mathbb{S}$ is the collection of points $x \in \mathbb{S}$ which have the property that x has no neighborhood N_x such that each component of $p^{-1}(N_x)$ is mapped homeomorphically onto N_x by p. A branched covering $p : \tilde{\mathbb{S}} \to \mathbb{S}$ is regular (or $\mathcal{A}$-covering) if $p|_{\tilde{\mathbb{S}}-p^{-1}(B)} : \tilde{\mathbb{S}} - p^{-1}(B) \to \mathbb{S} - B$ is a regular covering (with the covering transformation group $\mathcal{A}$). Two branched coverings $p_i : \tilde{\mathbb{S}}_i \to \mathbb{S}$ $(i = 1, 2)$ are *isomorphic* (or *equivalent*) if there exists a homeomorphism $\tilde{h} : \tilde{\mathbb{S}}_1 \to \tilde{\mathbb{S}}_2$ such that $p_2 \circ \tilde{h} = p_1$.

A (branched) covering of a surface is closely related to a graph covering which is embeddable into it. To see such a kind of relation, we first review a graph emdedding.

An *embedding* of a graph G into a surface $\mathbb{S}$ is a homeomorphism $\imath : G \to \mathbb{S}$ of G into $\mathbb{S}$. If every component of $\mathbb{S} - \imath(G)$, called a *region*, is homeomorphic to an open disk, then the embedding $\imath : G \to \mathbb{S}$ is called a 2-*cell* embedding, and the regions are called *faces* of the embedding. When a graph G is 2-cell embedded into a surface, every boundary walk of a face induces a walk in the graph G of the same length. A face of a 2-cell embedding of a graph G into a surface is said to be n-*sided* if its boundary walk is of length n. Note that if G is disconnected, no embedding of G into a surface $\mathbb{S}$ will be a 2-cell embedding.

An *embedding scheme* (ρ, λ) for a graph G consists of a *rotation scheme* ρ which assigns a cyclic permutation ρ_v on $N(v) = \{\, e \in D(G) : i_e = v \,\}$ to each $v \in V(G)$ and a *voltage assignment* λ which assigns a value $\lambda(e)$ in $\mathbb{Z}_2 = \{1, -1\}$ to each $e \in E(G)$.

Stahl [43] showed that every embedding scheme for a graph G determines a 2-cell embedding of G into a surface $\mathbb{S}$, and every 2-cell embedding of G into a surface $\mathbb{S}$ is determined by such a scheme. To see the relation between an embedding scheme for a graph and its 2-cell embedding to a surface, we give the following example.

Example 4 Let G be a figure eight having a vertex v and two loops ℓ_1 and ℓ_2, and let (ρ, λ) be an embedding scheme defined by $\rho_v = (\ell_1 \ell_2 \ell_1^{-1} \ell_2^{-1})$, $\lambda(\ell_1) = 1$ and $\lambda(\ell_2) = -1$. In a geometric presentation of G in $\mathbb{R}^3$ with directed loops initiating at v in counterclockwise order according to the rotation scheme ρ_v as in Figure 7.1 (b), we attach a closed disk at the vertex v and 1-bands along loops ℓ_1 and ℓ_2, where a 1-band is twisted if $\lambda(\ell_i) = -1$ and untwisted if $\lambda(\ell_i) = 1$ as in Figure 7.1 (c). Finally, we attach a closed disk along each boundary of the graph with 1-bands. Note that there exists only one component of the boundary of the graph with 1-bands in this example, and we get a 2-cell embedding of the figure eight into the Klein bottle with only one face as in Figure 7.1 (d). Conversely, if there exists such an embedding as in Figure 7.1 (d), it induces an embedding scheme (ρ, λ) as described above.

The orientability of the surface $\mathbb{S}$ can be detected by looking at

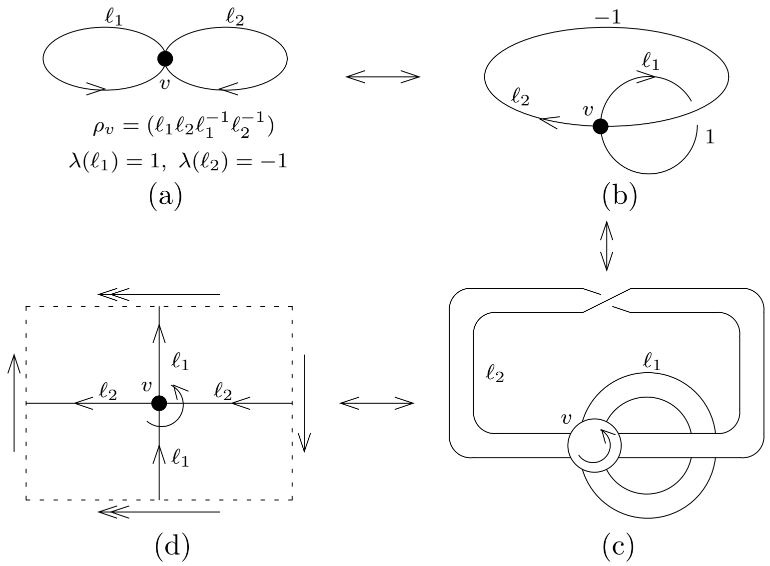

Figure 7.1: An embedding scheme for a figure eight embedded to the Klein bottle

the voltage assignments of cycles of G. In fact, $\mathbb{S}$ is orientable if and only if every cycle of G is λ-*trivial,* that is, the number of edges e with $\lambda(e) = -1$ is even in every cycle of G. In particular, every 2-cell embedding of G into an orientable surface can be determined by an embedding scheme (ρ, λ) with $\lambda(e) = 1$ for each $e \in E(G)$.

Let $\imath : G \to \mathbb{S}$ be a 2-cell embedding and (ρ, λ) the associated embedding scheme. Let ϕ be either an ordinary or a permutation voltage assignment. The derived graph G^ϕ has the *derived embedding scheme* $(\rho^\phi, \lambda^\phi)$, which is defined by $(\rho^\phi)_{v_g}(e_g) = (\rho_v(e))_g$ and $\lambda^\phi(e_g) = \lambda(e)$ for each $e_g \in D(G^\phi)$. Then $(\rho^\phi, \lambda^\phi)$ induces a 2-cell embedding of G^ϕ into a surface, say $\tilde{\imath} : G^\phi \to \mathbb{S}^\phi$, such that the following diagram

$$\begin{array}{ccc} G^\phi & \xrightarrow{\tilde{\imath}} & \mathbb{S}^\phi \\ {\scriptstyle p_\phi}\downarrow & & \downarrow{\scriptstyle \tilde{p}_\phi} \\ G & \xrightarrow{\imath} & \mathbb{S} \end{array}$$

commutes. Moreover, if G^ϕ is connected, then $\mathbb{S}^\phi$ is also connected.

Gross and Tucker [4] showed the following relation between branched coverings of a surface and coverings of a graph.

Theorem 16 *Let (ρ, λ) be an embedding scheme for a graph G which induces a 2-cell embedding $\imath : G \to \mathbb{S}$.*

(1) *Let $\phi : D(G) \to S_n$ be a permutation voltage assignment. Then the natural covering projection $p_\phi : G^\phi \to G$ can be extended to a branched n-fold covering $\tilde{p}_\phi : \mathbb{S}^\phi \to \mathbb{S}$ that has at most one branch point inside each face. If the net voltage on a face R has cycle structure $(c_1, c_2, \ldots, c_n)$, then the projection $\tilde{p}_\phi$ has a branch point inside face R with exactly c_j prebranch points of order j (i.e., locally j to 1 map) for $j = 1, 2, \ldots, n$.*

Conversely, if $p : \tilde{\mathbb{S}} \to \mathbb{S}$ is an n-fold branched covering such that each face of the embedding $\imath : G \to \mathbb{S}$ has at most one branch point interior of it, and no branch points in G, then there exists a permutation voltage assignment $\phi : D(G) \to S_n$ such that the branched covering $\tilde{p}_\phi : \mathbb{S}^\phi \to \mathbb{S}$ is isomorphic to the given branched covering $p : \tilde{\mathbb{S}} \to \mathbb{S}$.

(2) *Let $\mathcal{A}$ be a finite group and let $\phi : D(G) \to \mathcal{A}$ be an $\mathcal{A}$-voltage assignment. Then the natural covering projection $p_\phi : G \times_\phi \mathcal{A} \to G$ can be extended to a branched $\mathcal{A}$-covering $\tilde{p}_\phi : \mathbb{S}^\phi \to \mathbb{S}$ that has at most one branch point inside each face. If the net voltage on an n-sided face R is g in the voltage group $\mathcal{A}$, then $(\tilde{p}_\phi)^{-1}(R)$ consists of $|\mathcal{A}|/o(g)$ numbers of $n \cdot o(g)$-sided faces, where $o(g)$ is the order of g in $\mathcal{A}$.*

Conversely, if $p : \tilde{\mathbb{S}} \to \mathbb{S}$ is a branched $\mathcal{A}$-covering such that each face of the embedding $\imath : G \to \mathbb{S}$ has at most one branch point interior of it, and no branch points in G, then there exists an $\mathcal{A}$-voltage assignment $\phi : D(G) \to \mathcal{A}$ such that the branched $\mathcal{A}$-covering $\tilde{p}_\phi : \mathbb{S}^\phi \to \mathbb{S}$ is isomorphic to the given branched $\mathcal{A}$-covering $p : \tilde{\mathbb{S}} \to \mathbb{S}$. $\square$

It follows from Theorem 16 that any (regular) branched surface covering of a surface $\mathbb{S}$ can be derived from a suitable 2-cell embedding $\imath : G \to \mathbb{S}$ of a graph G and a voltage assignment ϕ. We call $\mathbb{S}^\phi$ the branched covering surface of $\mathbb{S}$ *induced by a* 2*-cell embedding* $\imath : G \to \mathbb{S}$ and a *voltage assignment* ϕ.

Notice that a 2-cell embedding of a graph G into a surface $\mathbb{S}$ determines a cell decomposition of the surface having the graph G as its 1-skeleton. Let ϕ be a voltage assignment of G and let G be 2-cell embedded in a surface $\mathbb{S}$. Then the lifted embedding scheme determines the cell decomposition of the surface $\mathbb{S}^\phi$ having the covering graph G^ϕ as its 1-skeleton. Moreover, the branched covering map $\tilde{p}_\phi : \mathbb{S}^\phi \to \mathbb{S}$ preserves cells, that is, it assigns i-cell to i-cell for each $i = 0, 1, 2$ and the restriction of $\tilde{p}_\phi$ to its 1-skeleton is just the covering $p_\phi : G^\phi \to G$. It implies that if two branched coverings $\tilde{p}_\phi : \mathbb{S}^\phi \to \mathbb{S}$ and $\tilde{p}_\psi : \mathbb{S}^\psi \to \mathbb{S}$ are isomorphic, then the two coverings $p_\phi : G^\phi \to G$ and $p_\psi : G^\psi \to G$ are isomorphic as graph coverings. Conversely, if two coverings $p_\phi : G^\phi \to G$ and $p_\psi : G^\psi \to G$ are isomorphic, then, by Theorem 4, there exists a function $f : V(G) \to S_n$ such that $\psi(uv) = f(v)\phi(uv)f(u)^{-1}$ for each uv in $D(G)$. Notice that the map $\Phi : G^\phi \to G^\psi$ defined by $\Phi(u_g) = u_{f(u)(g)}$ is a covering isomorphism. Let $(uv)_g = u_g v_{\phi(uv)(g)}$ maps to $(uw)_g = u_g w_{\phi(uw)(g)}$ by the induced rotation system $(\rho^\phi)_{u_g}$. By the definition of Φ, $\Phi((uv)_g) = u_{f(u)(g)} v_{f(v)\phi(uv)(g)} = u_{f(u)(g)} v_{\psi(uv)f(u)(g)}$ and $\Phi((uw)_g) = u_{f(u)(g)} w_{\psi(uw)f(u)(g)}$. So, $\Phi\,\rho^\phi = \rho^\psi\,\Phi$. Now, by combining this fact with $\lambda^\psi(\Phi(e_g)) = \lambda(e) = \lambda^\phi(e_g)$, we can show that Φ is extended to a cell preserving homeomorphism $\tilde{h}$ from $\mathbb{S}^\phi$ to $\mathbb{S}^\psi$ such that $\tilde{p}_\psi \circ \tilde{h} = \tilde{p}_\phi$, that is, two branched coverings $\tilde{p}_\phi : \mathbb{S}^\phi \to \mathbb{S}$ and $\tilde{p}_\psi : \mathbb{S}^\psi \to \mathbb{S}$ are isomorphic. We proved the following.

Theorem 17 *Let G be a graph* 2*-cell embedded in $\mathbb{S}$ and let ϕ and ψ be two voltage assignments of G. Then two branched coverings $\tilde{p}_\phi : \mathbb{S}^\phi \to \mathbb{S}$ $\tilde{p}_\psi : \mathbb{S}^\psi \to \mathbb{S}$ are isomorphic as surface branched coverings if and only if the two coverings $p_\phi : G^\phi \to G$ and $p_\psi : G^\psi \to G$ are isomorphic as graph coverings.*

Let $\mathfrak{B}_m$ be the graph consisting of one vertex and m self loops, say $\ell_1, \ldots, \ell_m$. We call it the *bouquet of m circles* or simply, a *bouquet*. Clearly, $\mathfrak{B}_m$ is irreducible (*i.e.,* having no vertices of degree 2) if $m \geq 2$. A surface $\mathbb{S}_k$ can be represented by a $4k$-gon with identification data $\prod_{s=1}^{k} a_s b_s a_s^{-1} b_s^{-1}$ on its boundary if $k > 0$; $-2k$-gon with identification data $\prod_{s=1}^{-k} a_s a_s$ on its boundary if $k < 0$; and bigon with identification data aa^{-1} on its boundary if $k = 0$.

Let B be a finite set of points in $\mathbb{S}_k$. For our purpose, we assume that $|B| > 0$ when $k = 0$. If $* \in \mathbb{S}_k - B$, then the fundamental group $\pi_1(\mathbb{S}_k - B, *)$ of the punctured surface $\mathbb{S}_k - B$ with base point $*$ can be presented as follows:

$$\left\langle a_1, \dots, a_k, b_1, \dots, b_k, c_1, \dots, c_{|B|} \,;\, \prod_{s=1}^{k} a_s b_s a_s^{-1} b_s^{-1} \prod_{t=1}^{|B|} c_t = 1 \right\rangle \quad \text{if } k > 0;$$

$$\left\langle a_1, \dots, a_{-k}, c_1, \dots, c_{|B|} \,;\, \prod_{s=1}^{-k} a_s a_s \prod_{t=1}^{|B|} c_t = 1 \right\rangle \quad \text{if } k < 0;$$

$$\left\langle c_1, \dots, c_{|B|} \,;\, \prod_{t=1}^{|B|} c_t = 1 \right\rangle \quad \text{if } k = 0.$$

We call this the *standard presentation* of the fundamental group $\pi_1(\mathbb{S}_k - B, *)$. For each $t = 1, 2, \dots, |B|$, we take a simple closed curve based at $*$ lying in the face determined by the polygonal representation of the surface $\mathbb{S}_k$ so that it represents the homotopy class of the generator c_t. Then, it induces a 2-cell embedding of a bouquet of m circles into the surface $\mathbb{S}_k$ such that the embedding has $|B|$ 1-sided regions and one $(|B|+4k)$-sided region if $k > 0$; $|B|$ 1-sided regions and one $(|B|-2k)$-sided region if $k < 0$; and $|B|$ 1-sided regions and one $|B|$-sided region if $k = 0$, where m is the number of the generators of the corresponding fundamental group. We call this embedding $\imath : \mathfrak{B}_m \to \mathbb{S}_k$ the *standard embedding*, denoted by $\mathfrak{B}_m \hookrightarrow \mathbb{S}_k - B$.

For example, Figure 7.2 illustrates the standard embeddings of bouquets with $|B| = 3$. Figure 7.2 (a) represents the standard embedding $\mathfrak{B}_7 \hookrightarrow \mathbb{S}_2 - B$ and (b) does the standard embedding $\mathfrak{B}_6 \hookrightarrow \mathbb{S}_{-3} - B$.

For convenience, let $a_k = 2k$ if $k \geq 0$, and $a_k = -k$ if $k < 0$. Let $C^1(\mathfrak{B}_{a_k+|B|} \hookrightarrow \mathbb{S}_k - B; n)$ (resp. $C^1(\mathfrak{B}_{a_k+|B|} \hookrightarrow \mathbb{S}_k - B; \mathcal{A})$) denote the subset of $(S_n)^{a_k+|B|}$ (resp. of $(\mathcal{A})^{a_k+|B|}$) consisting of all $(a_k+|B|)$-tuples $(\sigma_1, \dots, \sigma_{a_k+|B|})$ which satisfy the following three conditions:

(C1) The subgroup $< \sigma_1, \dots, \sigma_{a_k+|B|} >$ generated by $\{\sigma_1, \dots, \sigma_{a_k+|B|}\}$ is transitive on $\{1, 2, \dots, n\}$ (resp. is the full group $\mathcal{A}$), and

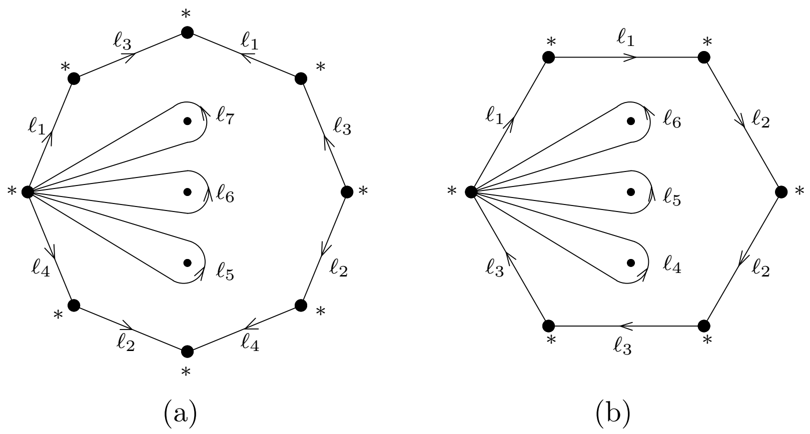

Figure 7.2: Two examples of standard embeddings

(C2) (i) if $k \geq 0$, then

$$\prod_{i=1}^{k} \sigma_i \sigma_{k+i} \sigma_i^{-1} \sigma_{k+i}^{-1} \prod_{i=1}^{|B|} \sigma_{2k+i} = 1,$$

(ii) if $k < 0$, then

$$\prod_{i=1}^{-k} \sigma_i \sigma_i \prod_{i=1}^{|B|} \sigma_{-k+i} = 1,$$

(C3) $\sigma_i \neq 1$ for each $i = a_k + 1, \ldots, a_k + |B|$.

Note that condition (C1) guarantees that the surface $\mathbb{S}^\phi$ is connected, and conditions (C2) and (C3) do that the set B is the same as the branch set of the branched covering $\tilde{p}_\phi : \mathbb{S}^\phi \to \mathbb{S}$. By using a similar method as in [23], we can obtain the following theorem.

Theorem 18 (Existence and classification of branched coverings) *Every permutation voltage assignment in $C^1(\mathfrak{B}_{a_k+|B|} \hookrightarrow \mathbb{S}_k - B; n)$ induces a connected branched n-fold covering of $\mathbb{S}_k$ with branch set B. Conversely, every connected branched n-fold covering of $\mathbb{S}_k$ with branch set B can be derived from a voltage assignment in $C^1(\mathfrak{B}_{a_k+|B|} \hookrightarrow \mathbb{S}_k - B; n)$. Moreover, for any given two permutation voltage assignments $\phi, \psi \in C^1(\mathfrak{B}_{a_k+|B|} \hookrightarrow \mathbb{S} - B; n)$, two branched n-fold surface coverings*

$\tilde{p}_\phi : \mathbb{S}^\phi \to \mathbb{S}$ and $\tilde{p}_\psi : \mathbb{S}^\psi \to \mathbb{S}$ are isomorphic if and only if two graph coverings $p_\phi : \mathfrak{B}^\phi_{a_k+|B|} \to \mathfrak{B}_{a_k+|B|}$ and $p_\psi : \mathfrak{B}^\psi_{a_k+|B|} \to \mathfrak{B}_{a_k+|B|}$ are isomorphic. It is also equivalent to say that there exists a permutation $\sigma \in S_n$ such that

$$\psi(\ell_i) = \sigma\phi(\ell_i)\sigma^{-1}$$

for all $\ell_i \in D(\mathfrak{B}_{a_k+|B|})$, where $a_k = 2k$ if $k \geq 0$, and $a_k = -k$ if $k < 0$. □

For a finite group $\mathcal{A}$, let $S_\mathcal{A}$ denote the symmetric group on the group elements of $\mathcal{A}$. It gives the (left) regular representation $\mathcal{A} \to \mathcal{S}_\mathcal{A}$ of $\mathcal{A}$ via $g \to L(g)$, the left translation by g on $\mathcal{A}$. Clearly, this representation is faithful and the group $\mathcal{A}$ can be identified with the group of left transformations $L(g)$'s: $\mathcal{A} \equiv \{L(g) \mid g \in \mathcal{A}\}$ (Cayley Theorem). Notice that a permutation voltage assignment $\phi : D(G) \to S_\mathcal{A}$ having its images in $\mathcal{A}$ can be considered as an $\mathcal{A}$-voltage assignment of G, and for such a voltage assignment ϕ, the permutation derived graph G^ϕ is nothing but the ordinary derived graph $G \times_\phi \mathcal{A}$. By using this fact, Kwak et al. showed the following.

Theorem 19 [23](Existence and classification of regular branched coverings) *Every ordinary voltage assignment in $C^1(\mathfrak{B}_{a_k+|B|} \hookrightarrow \mathbb{S}_k - B; \mathcal{A})$ induces a connected branched $\mathcal{A}$-covering of $\mathbb{S}_k$ with branch set B. Conversely, every connected branched $\mathcal{A}$-covering of $\mathbb{S}_k$ with branch set B can be derived from a voltage assignment in $C^1(\mathfrak{B}_{a_k+|B|} \hookrightarrow \mathbb{S}_k - B; \mathcal{A})$. Moreover, for any given two voltage assignments $\phi, \psi \in C^1(\mathfrak{B}_{a_k+|B|} \hookrightarrow \mathbb{S} - B; \mathcal{A})$, two branched $\mathcal{A}$-coverings $\tilde{p}_\phi : \mathbb{S}^\phi \to \mathbb{S}$ and $\tilde{p}_\psi : \mathbb{S}^\psi \to \mathbb{S}$ are isomorphic if and only if two graph coverings $p_\phi : \mathfrak{B}_{a_k+|B|} \times_\phi \mathcal{A} \to \mathfrak{B}_{a_k+|B|}$ and $p_\psi : \mathfrak{B}_{a_k+|B|} \times_\psi \mathcal{A} \to \mathfrak{B}_{a_k+|B|}$ are isomorphic. It is also equivalent to say that there exists a group automorphism σ of $\mathcal{A}$ such that*

$$\psi(\ell_i) = \sigma\left(\phi(\ell_i)\right)$$

for all $\ell_i \in D(\mathfrak{B}_{a_k+|B|})$, where $a_k = 2k$ if $k \geq 0$, and $a_k = -k$ if $k < 0$. □

There are two classical Hurwitz theorems: the existence and the classification theorems of surface branched coverings. Let $p : \tilde{\mathbb{S}} \to \mathbb{S}$ be an n-fold surface branched covering, where $\tilde{\mathbb{S}}$ is possibly disconnected.

Hurwitz [18] introduced a system, called *Hurwitz system*, for p as follows: Consider the associated covering $p|_{\tilde{\mathbb{S}}-p^{-1}(B)} : \tilde{\mathbb{S}} - p^{-1}(B) \to \mathbb{S} - B$ of p. A Hurwitz system is a representation $H_p : \pi_1(\mathbb{S} - B, *) \to S_n$, which is determined by choosing a one-to-one correspondence $p^{-1}(*) \leftrightarrow \{1, 2, \ldots, n\}$ and assigning to a loop α in $\mathbb{S} - B$ based at $*$ the permutation of $\{1, 2, \ldots, n\}$ induced by the liftings of α. For any finite set B of points in $\mathbb{S}$ and a representation $H : \pi_1(\mathbb{S} - B, *) \to S_n$, there exists an n-fold branched covering $p : \tilde{\mathbb{S}} \to \mathbb{S}$, where $\tilde{\mathbb{S}}$ is perhaps not connected, with branch set contained in B and $H_p = H$ (*Hurwitz existence theorem*). Two n-fold branched coverings $p_i : \tilde{\mathbb{S}}_i \to \mathbb{S},\ i = 1, 2$, are isomorphic if and only if $H_{p_2} = H_{p_1}$ modulo inner automorphisms of S_n. (*Hurwitz classification theorem*).

Every group homomorphism from $\pi_1(\mathbb{S} - B, *)$ to S_n is uniquely determined by its values on the generator set $\{a_s,\ b_s,\ c_t\}$ of $\pi_1(\mathbb{S}-B, *)$ which preserves the corresponding relation in the standard presentation of $\pi_1(\mathbb{S} - B, *)$. Hence, a Hurwitz system $H_p : \pi_1(\mathbb{S} - B, *) \to S_n$ for a branched n-fold covering $p : \tilde{\mathbb{S}} \to \mathbb{S}$ is nothing but a voltage assignment in $C^1(\mathfrak{B}_m; n)$ which satisfies the conditions (C2) and (C3), and that of a connected branched n-fold covering $p : \tilde{\mathbb{S}} \to \mathbb{S}$ is nothing but a voltage assignment in $C^1(\mathfrak{B}_m \hookrightarrow \mathbb{S} - B; n)$. So, Theorems 18 and 19 are new combinatorial statements of the Hurwitz existence and classification theorems for branched coverings and for branched regular coverings, respectively.

Remark Let $p : \widetilde{\mathbb{S}} \to \mathbb{S}$ be an n-fold connected unbranched covering and let $* \in \mathbb{S}$. The *monodromy representation* of $\pi_1(\mathbb{S}, *)$ is a homomorphism $H_p : \pi_1(\mathbb{S}, *) \to S_n$ determined by choosing a one-to-one correspondence $p^{-1}(*) \leftrightarrow \{1, 2, \ldots, n\}$ and assigning to a loop α in $\pi_1(\mathbb{S}, *)$ the permutation of $\{1, 2, \ldots, n\}$ induced by the liftings of α. This permutation maps $\tilde{*} \in p^{-1}(*)$ to the terminal point of the lifting α having $\tilde{*}$ as an initial point, *i.e.,* $H_p(\alpha)(\tilde{*}) = \tilde{\alpha}(1)$. The image of the monodromy representation is a subgroup of S_n and is called the *monodromy group*. Its element is called a *monodromy map*. Notice that the monodromy representation for a surface covering is equal to the Hurwitz system for a surface covering and hence it can be identified with a voltage assignment ϕ in $C^1(\mathfrak{B}_m \hookrightarrow \mathbb{S}_k; n)$. For a voltage assignment ϕ in $C^1(\mathfrak{B}_m \hookrightarrow \mathbb{S}_k; n)$, the monodromy group of the corresponding covering is nothing but the subgroup $< \phi >$ of S_n generated by the image of ϕ. Obviously, we have $\{\alpha \in \pi_1(\mathbb{S}, *) \mid H_p(\alpha)(\tilde{*}) = \tilde{*}\} = p_\sharp \pi_1(\tilde{\mathbb{S}}, \tilde{*})$. It implies that the index $[\pi_1(\mathbb{S}, x) : p_\sharp \pi_1(\tilde{\mathbb{S}}, \tilde{*})] = n$ equals the fold number of the covering.

To derive some formulas for enumerating the isomorphism classes of surface branched coverings, we define an S_n-action on the set $C^1(\mathfrak{B}_m; n)$ by a simultaneously coordinatewise conjugation, that is, for any $g \in S_n$ and any $(\sigma_1, \ldots, \sigma_m) \in C^1(\mathfrak{B}_m; n)$,

$$g \cdot (\sigma_1, \ldots, \sigma_m) = (g\sigma_1 g^{-1}, \ldots, g\sigma_m g^{-1}).$$

It follows from Theorem 18 that two voltage assignments in $C^1(\mathfrak{B}_{a_k+|B|} \hookrightarrow \mathbb{S}_k - B; n)$ derive isomorphic branched coverings of $\mathbb{S}_k$ if and only if they belong to the same orbit under the S_n-action. Hence we have the following.

Lemma 13 *Let k be any integer and let B be a finite subset of the surface $\mathbb{S}_k$. Then the number of isomorphism classes of connected n-fold branched coverings of the surface $\mathbb{S}_k$ with branch set B is*

$$\mathbf{Isoc}\,(\mathbb{S}_k, B; n) = \left| C^1(\mathfrak{B}_{a_k+|B|} \hookrightarrow \mathbb{S}_k - B; n)/S_n \right|.$$

□

Now, we aim to express the number $\mathbf{Isoc}\,(\mathbb{S}_k, B; n)$ in terms of known parameters.

Let $\mathfrak{C}(\mathfrak{B}_m; n)$ denote the set of all m-tuples $(\sigma_1, \ldots, \sigma_m)$ in $(S_n)^m$ such that the group $<\sigma_1, \ldots, \sigma_m>$ generated by $\{\sigma_1, \ldots, \sigma_m\}$ is transitive on $\{1, 2, \ldots, n\}$, that is,

$$\mathfrak{C}(\mathfrak{B}_m; n) = \{(\sigma_1, \sigma_2, \ldots, \sigma_m) \in (S_n)^m : <\sigma_1, \sigma_2, \ldots, \sigma_m> \text{ is transitive on } \{1, 2, \ldots, n\}\}.$$

Then $\mathfrak{C}(\mathfrak{B}_m; n)$ contains all representatives of connected n-fold coverings of the bouquet of m-circles $\mathfrak{B}_m$ and the number $\mathbf{Isoc}\,(\mathfrak{B}_m; n)$ of isomorphism classes of connected n-fold coverings of $\mathfrak{B}_m$ is equal to $|\mathfrak{C}(\mathfrak{B}_m; n)/S_n|$, where the S_n-action on $\mathfrak{C}(\mathfrak{B}_m; n)$ is also defined by the simultaneously coordinatewise conjugation.

Lemma 14 *Let k be an integer and b a nonnegative integer. For each $0 \leq t \leq b$, let*

$$S(k,b,t) = \big\{ \phi \in (S_n)^{a_k+b} \ : \ \phi \text{ satisfies (C1), (C2)} \\ \qquad\qquad \textit{and } \sigma_i = 1, \ ^\forall i = a_k+1, \ldots, a_k+t \ \big\},$$

where $\phi = (\sigma_1, \sigma_2, \ldots, \sigma_{a_k+b})$. If $t = b$, then the set $S(k,b,b)$ is equal to the set $C^1(\mathfrak{B}_{a_k} \hookrightarrow \mathbb{S}_k; n)$, and if $t \neq b$, then there is a one-to-one correspondence between the sets $S(k,b,t)$ and $\mathfrak{C}(\mathfrak{B}_{a_k+b-t-1}; n)$. Moreover, the correspondence preserves the S_n-action on the both sets which are defined by simultaneously coordinatewise conjugacy.

Proof: The case of $t = b$ is clear. Assume that $t \neq b$. Then every element in $S(k,b,t)$ is of the form $(\sigma_1, \ldots, \sigma_{a_k}, 1, \ldots, 1, \sigma_{a_k+t+1}, \ldots, \sigma_{a_k+b})$. It comes from conditions (C1)-(C2) that the function $f : S(k,b,t) \to \mathfrak{C}(\mathfrak{B}_{a_k+b-t-1}; n)$ defined by

$$f(\sigma_1, \ldots, \sigma_{a_k}, 1, \ldots, 1, \sigma_{a_k+t+1}, \ldots, \sigma_{a_k+b}) \\ = (\sigma_1, \ldots, \sigma_{a_k}, \sigma_{a_k+t+1}, \ldots, \sigma_{a_k+b-1})$$

is well-defined and bijective (Note that the function f is defined by deleting 1's and the last coordinate). This completes the proof. □

Theorem 20 *Let k be any integer and let B be a b-subset of the surface $\mathbb{S}_k$. Then the number of connected n-fold branched coverings of the surface $\mathbb{S}_k$ with branch set B is*

$$\mathbf{Isoc}\,(\mathbb{S}_k, B; n) = (-1)^b \mathbf{Isoc}\,(\mathbb{S}_k, \emptyset; n) + \sum_{t=0}^{b-1} (-1)^t \binom{b}{t} \mathbf{Isoc}\,(\mathfrak{B}_{a_k+b-t-1}; n),$$

where $\mathfrak{B}_m$ is a bouquet of m circles, $a_k = 2k$ if $k \geq 0$, and $a_k = -k$ if $k < 0$.

Proof: For each $i = a_k+1, \ldots, a_k+b$, let $\mathcal{P}_i$ be the property that the i-th coordinate of an element of $(S_n)^{a_k+b}$ is the identity. For each subset S of $\{a_k+1, \ldots, a_k+b\}$, let $N(\mathcal{P}_S)$ be the number of elements in the product $(S_n)^{a_k+b}$ which satisfy conditions (C1), (C2) and the properties

$\mathcal{P}_i$ for all $i \in S$. Notice that $N(\mathcal{P}_\emptyset)$ is the number of all elements in the product $(S_n)^{a_k+b}$ which satisfy conditions (C1) and (C2), and that the set $C^1(\mathfrak{B}_{a_k+b} \hookrightarrow \mathbb{S}_k - B; n)$ is equal to the set of elements of $(S_n)^{a_k+b}$ which satisfy conditions (C1) and (C2), but not any other property $\mathcal{P}_i$ for $i = a_k+1, \ldots, a_k+b$. It comes from the principle of inclusion and exclusion that

$$\left|C^1(\mathfrak{B}_{a_k+b} \hookrightarrow \mathbb{S}_k - B; n)\right| = \sum_{t=0}^{b} (-1)^t \left(\sum_{\substack{S \subset \{a_k+1,\ldots,a_k+b\} \\ |S|=t}} N(\mathcal{P}_S) \right).$$

Since $N(\mathcal{P}_S) = N(\mathcal{P}_{S'})$ for any two subsets S, S' of $\{a_k+1, \ldots, a_k+b\}$ with the same cardinality, we have

$$\begin{aligned}
&\sum_{\substack{S \subset \{a_k+1,\ldots,a_k+b\} \\ |S|=t}} N(\mathcal{P}_S) \\
&= \binom{b}{t} \Big| \{ \phi \in (S_n)^{a_k+b} \ : \ \phi \text{ satisfies (C1), (C2)} \\
&\qquad\qquad \text{and } \sigma_i = 1, \ {}^\forall i = a_k+1, \ldots, a_k+t \ \} \Big| .
\end{aligned}$$

Now, it comes from Lemma 14 that

$$\begin{aligned}
&|C^1(\mathfrak{B}_{a_k+b} \hookrightarrow \mathbb{S}_k - B; n)| \\
&\quad = \sum_{t=0}^{b-1} (-1)^t \binom{b}{t} |\mathfrak{C}(\mathfrak{B}_{a_k+b-t-1}; n)| + (-1)^b \left|C^1(\mathfrak{B}_{a_k} \hookrightarrow \mathbb{S}_k; n)\right| .
\end{aligned}$$

By taking the S_n-action on the underlying sets of the both sides of this equation, we have

$$\mathbf{Isoc}\,(\mathbb{S}_k, B; n) = (-1)^b \mathbf{Isoc}\,(\mathbb{S}_k, \emptyset; n) + \sum_{t=0}^{b-1} (-1)^t \binom{b}{t} \mathbf{Isoc}\,(\mathfrak{B}_{a_k+b-t-1}; n).$$

$\square$

By using Burnside's Lemma, Mednykh ([37], [38]) counted the number of subgroups in the fundamental group $\pi_1(\mathbb{S}_k, *)$ of an orientable surface $\mathbb{S}_k$ and the number of conjugacy classes of subgroups in $\pi_1(\mathbb{S}_k, *)$. The same problem for a nonorientable surface was done by A. Mednykh and G. Pozdnyakova in [40].

Theorem 21 ([37], [38], [40])

(1) *The number of subgroups of index n in the fundamental group $\pi_1(\mathbb{S}_k, *)$ of a (orientable or nonorientable) surface $\mathbb{S}_k$ of genus k is*

$$\mathcal{S}_k(n) \equiv \mathcal{S}_{\pi_1(\mathbb{S}_k,*)}(n) = n \sum_{s=1}^{n} \frac{(-1)^{s+1}}{s} \sum_{\substack{i_1+i_2+\cdots+i_s=n \\ i_1,i_2,\ldots,i_s \geq 1}} \beta_{i_1}\beta_{i_2}\cdots\beta_{i_s},$$

where

$$\beta_h = \sum_{\lambda \in D_h} \left(\frac{h!}{f^{(\lambda)}}\right)^t, \qquad t = \begin{cases} 2k-2 & \text{if } k \geq 0, \\ k-2 & \text{if } k < 0, \end{cases}$$

D_h is the set of all irreducible representation of the group S_h, and $f^{(\lambda)}$ is the degree of the representation λ.

(2) *The number of nonisomorphic connected n-fold unbranched coverings of a surface $\mathbb{S}_k$ of genus k is*

$$\mathbf{Isoc}\,(\mathbb{S}_k, \emptyset; n)$$
$$= \begin{cases} \dfrac{1}{n} \displaystyle\sum_{m|n} \mathcal{S}_k(m) \sum_{d|\frac{n}{m}} \mu\left(\frac{n}{md}\right) d^{(2k-2)m+2} & \text{if } k \geq 0, \\ \dfrac{1}{n} \displaystyle\sum_{m|n} \sum_{d|\frac{n}{m}} \mu\left(\frac{n}{md}\right) d^{(-k-2)m+1}[(2,d)\mathcal{S}_k^-(m) + d\,\mathcal{S}_k^+(m)] & \\ & \text{if } k < 0, \end{cases}$$

where $\mu(m)$ is the Möbius function, $\mathcal{S}_k^+(m) = 0$ if m is odd, and $\mathcal{S}_k^+(m) = \mathcal{S}_k(\frac{m}{2})$ if m is even, $\mathcal{S}_k^-(m) = \mathcal{S}_k(m) - \mathcal{S}_k^+(m)$, and $(2,d)$ denotes the greatest common divisor of 2 and d.

Next, we aim to compute the number $\mathbf{Isoc}^R(\mathbb{S}_k, B; n)$ of nonisomorphic connected regular n-fold branched coverings of the surface $\mathbb{S}_k$ with branch set B. Clearly, any two connected regular branched coverings are not isomorphic if their covering transformation group (or voltage groups) are not isomorphic. Since every connected regular n-fold

$(\lvert B\rvert, n)$	$k=-4$	$k=-3$	$k=-2$	$k=-1$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$(0,2)$	15	7	3	1	0	3	15	63	255
$(0,3)$	90	18	4	1	0	4	100	2884	96104
$(1,2)$	0	0	0	0	0	0	0	0	0
$(1,3)$	145	23	3	0	0	3	135	5103	185895
$(2,2)$	16	8	4	2	1	4	16	64	256
$(2,3)$	981	171	31	6	1	31	991	34231	1218031

Table 7.1: The number $\mathbf{Isoc}\,(\mathbb{S}_k, B; n)$ for small k, n and small $|B|$

branched covering is isomorphic to a connected branched $\mathcal{A}$-covering for some group $\mathcal{A}$ of order n, we have

$$\mathbf{Isoc}^R(\mathbb{S}_k, B; n) = \sum_{\mathcal{A}} \frac{\left|C^1(\mathfrak{B}_{a_k+|B|} \hookrightarrow \mathbb{S}_k - B; \mathcal{A})\right|}{|\mathrm{Aut}\,(\mathcal{A})|} = \sum_{\mathcal{A}} \mathbf{Isoc}\,(\mathbb{S}_k, B; \mathcal{A}),$$

where $\mathcal{A}$ runs over all representatives of isomorphism classes of groups of order n. Recall that the number $\mathbf{Isoc}\,(\mathbb{S}_k, B; \mathcal{A})$ of nonisomorphic connected $\mathcal{A}$-coverings of the surface $\mathbb{S}_k$ with branch set B is equal to number of the orbits of the coordinatewise $\mathrm{Aut}\,(\mathcal{A})$-action on the set $\mathfrak{G}(k, B; \mathcal{A})$. Note that this $\mathrm{Aut}\,(\mathcal{A})$-action on the set $\mathfrak{G}(k, B; \mathcal{A})$ is free because $\sigma_1, \ldots, \sigma_m$ generates $\mathcal{A}$. Now, by the Burnside Lemma, we have

$$\mathbf{Isoc}\,(\mathbb{S}_k, B; \mathcal{A}) = \frac{|\mathfrak{G}(k, B; \mathcal{A})|}{|\mathrm{Aut}\,(\mathcal{A})|} = \frac{|C^1(\mathfrak{B}_m \hookrightarrow \mathbb{S}_k - B; \mathcal{A})|}{|\mathrm{Aut}\,(\mathcal{A})|},$$

where $m = 2k + |B|$ if $k \geq 0$; and $m = -k + |B|$ if $k < 0$.

We summarize our discussion as follows.

Theorem 22 *Let k be any integer and let B be a b-subset of the surface $\mathbb{S}_k$. Then we have*

(1) *the number of nonisomorphic connected regular n-fold branched coverings of the surface $\mathbb{S}_k$ with branch set B is*

$$\mathbf{Isoc}^R(\mathbb{S}_k, B; n) = \sum_{\mathcal{A}} \mathbf{Isoc}\,(\mathbb{S}_k, B; \mathcal{A}),$$

where $\mathcal{A}$ runs over all representatives of isomorphism classes of groups of order n, and

(2) *the number of nonisomorphic connected regular $\mathcal{A}$-coverings of the surface $\mathbb{S}_k$ with branch set B is*

$$\mathbf{Isoc}\,(\mathbb{S}_k, B; \mathcal{A}) = \frac{|\mathfrak{G}(k, B; \mathcal{A})|}{|\mathrm{Aut}\,(\mathcal{A})|} = \frac{|C^1(\mathfrak{B}_m \hookrightarrow \mathbb{S}_k - B; n)|}{|\mathrm{Aut}\,(\mathcal{A})|},$$

where $m = 2k + |B|$ if $k \geq 0$; and $m = -k + |B|$ if $k < 0$. $\square$

By Theorem 22, we now need to compute the number $\mathbf{Isoc}\,(\mathbb{S}_k, B; \mathcal{A})$ for each finite group $\mathcal{A}$ of order n. By using a method similar to the proof of Theorem 20, we can have the following theorem.

Theorem 23 *Let k be any integer and let B be a b-subset of the surface $\mathbb{S}_k$. Then, for any finite group $\mathcal{A}$, the number of branched connected $\mathcal{A}$-coverings of the surface $\mathbb{S}_k$ with branch set B is*

$$\mathbf{Isoc}\,(\mathbb{S}_k, B; \mathcal{A}) = (-1)^b \mathbf{Isoc}\,(\mathbb{S}_k, \emptyset; \mathcal{A}) + \sum_{t=0}^{b-1} (-1)^t \binom{b}{t} \mathbf{Isoc}\,(\mathfrak{B}_{a_k+b-t-1}; \mathcal{A}),$$

where $\mathfrak{B}_m$ is bouquet of m circles, $a_k = 2k$ if $k \geq 0$, and $a_k = -k$ if $k < 0$. $\square$

Recall that an explicit computing of the number $\mathbf{Isoc}\,(\mathfrak{B}_m; \mathcal{A})$ was done for any m and any finite abelian group $\mathcal{A}$ or dihedral groups $\mathbb{D}_n$ of order $2n$ (See chapters 4-6). But the number $\mathbf{Isoc}\,(\mathbb{S}_k, \emptyset; \mathcal{A})$ is known if $\mathcal{A}$ is $\mathbb{Z}_p$ or $\mathbb{D}_p$ (see [23, 29] or see next chapter 8).

As a final discussion of this chapter, we aim to introduce a formula for computing the number $\mathbf{Isoc}\,(\mathbb{S}_k, \emptyset; \mathcal{A})$ when $\mathcal{A}$ is abelian. If $\mathcal{A}$ is an abelian group and $\mathbb{S}_k$ is an orientable surface, then the number $\mathbf{Isoc}\,(\mathbb{S}_k, \emptyset; \mathcal{A})$ of connected $\mathcal{A}$-coverings of the surface $\mathbb{S}_k$ is equal to the number $\mathbf{Isoc}\,(\mathfrak{B}_{2k}; \mathcal{A})$ of connected $\mathcal{A}$-coverings of the bouquet of $2k$ circles $\mathfrak{B}_{2k}$. In this case, we computed this number in chapter 4. By the classification theorem of finite abelian groups, we can express a finite abelian group $\mathcal{A}$ as follows.

$$\mathcal{A} = \mathcal{A}_o \oplus \mathcal{A}_e = \left(\oplus_{i=1}^{s} \oplus_{j=1}^{t_i} m_{i_j} \mathbb{Z}_{p_i^{\ell_{i_j}}} \right) \bigoplus \left(\oplus_{\mu=1}^{\ell} m_\mu \mathbb{Z}_{2^{\gamma_\mu}} \right),$$

where p_i are odd primes and $p_i \neq p_{i'}$ if $i \neq i'$. Let $\theta(\mathcal{A})$ denote the number of direct summands of $\mathcal{A}$ whose order is a multiple of 4 and $\omega(\mathcal{A})$ denote the number of direct summands of $\mathcal{A}$ whose order is 2. For example, $\mathbb{Z}_6 \oplus \mathbb{Z}_8 = \mathbb{Z}_3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_8$, $\theta(\mathbb{Z}_6 \oplus \mathbb{Z}_8) = 1$ and $\omega(\mathbb{Z}_6 \oplus \mathbb{Z}_8) = 1$.

$(\lvert B\rvert,p)$	$k=-4$	$k=-3$	$k=-2$	$k=-1$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$(0,2)$	15	7	3	1	0	3	15	63	255
$(0,3)$	13	4	1	0	0	4	40	364	3280
$(1,2)$	0	0	0	0	0	0	0	0	0
$(1,3)$	27	9	3	1	0	0	0	0	0
$(2,2)$	16	8	4	2	1	4	16	64	256
$(2,3)$	54	18	6	2	1	9	81	729	6561

Table 7.2: The number $\mathbf{Isoc}\,(\mathbb{S}_k, B; \mathbb{Z}_p)$ for small k, p and small $|B|$

Lemma 15 *Let k be any integer and let B be a b-subset of the surface $\mathbb{S}_k$. Let $\mathcal{A}$ be an abelian group. Then we have the following.*

(1) *If $k \geq 0$, then* $\mathbf{Isoc}\,(\mathbb{S}_k, \emptyset; \mathcal{A}) = \mathbf{Isoc}\,(\mathfrak{B}_{2k}; \mathcal{A})$,

(2) *If $k < 0$, then*

$$\mathbf{Isoc}\,(\mathbb{S}_k, \emptyset; \mathcal{A})$$
$$= \begin{cases} \dfrac{2^{\theta(\mathcal{A})}\left(2^{-k-\theta(\mathcal{A})}-1\right)}{2^{-k-(\theta(\mathcal{A})+\omega(\mathcal{A}))}-1}\mathbf{Isoc}\,(\mathfrak{B}_{-k-1};\mathcal{A}) & \text{if } \theta(\mathcal{A})+\omega(\mathcal{A}) < -k, \\ \dfrac{2^{\theta(\mathcal{A})}\left(2^{-k-\theta(\mathcal{A})}-1\right)}{(2^{-k}-1)2^{\sum_{\mu=1}^{\ell} m_\mu(\gamma_\mu-1)}}\mathbf{Isoc}\,(\mathfrak{B}_{-k-1};\mathcal{A}_o)\mathbf{Isoc}\,(\mathfrak{B}_{-k};\mathcal{A}_e) & \text{if } \theta(\mathcal{A})+\omega(\mathcal{A}) = -k \text{ and } \theta(\mathcal{A}) \neq -k, \\ 0 & \text{otherwise,} \end{cases}$$

where $\mathcal{A} = \mathcal{A}_o \oplus \mathcal{A}_e = \left(\oplus_{i=1}^{s} \oplus_{j=1}^{t_i} m_{i_j}\mathbb{Z}_{p_i^{\ell_{i_j}}}\right) \bigoplus \left(\oplus_{\mu=1}^{\ell} m_\mu \mathbb{Z}_{2^{\gamma_\mu}}\right)$.

□

As an illustration, we compute $\mathbf{Isoc}^R(\mathbb{S}_k, B; p)$ for any prime p. Recall that $\mathbf{Isoc}\,(\mathfrak{B}_m; \mathbb{Z}_p) = \frac{p^m-1}{p-1}$ for a prime number p. Since every group of order p is isomorphic to the cyclic group $\mathbb{Z}_p$, it comes from Theorem 22 that $\mathbf{Isoc}^R(\mathbb{S}_k, B; p) = \mathbf{Isoc}\,(\mathbb{S}_k, B; \mathbb{Z}_p)$ for any k and any $B \subset \mathbb{S}_k$. Now, by applying Theorem 23 and Lemma 15, we have the following.

Theorem 24 *Let B be a b-subset of the surface $\mathbb{S}_k$ and let p a prime. Then the number $\mathbf{Isoc}^R(\mathbb{S}_k, B; p)$ of nonisomorphic regular connected branched p-fold coverings of $\mathbb{S}_k$ with branch set B is*

$$\mathbf{Isoc}^R(\mathbb{S}_k, B; p) = \begin{cases} \dfrac{p^{2k}-1}{p-1} & \text{if } k \geq 0 \text{ and } b = 0, \\ p^{2k-1}\left((p-1)^{b-1} + (-1)^b\right) & \text{if } k \geq 0 \text{ and } b \neq 0, \\ 2^{-k} - 1 & \text{if } k < 0,\ b = 0 \text{ and } p = 2, \\ 2^{-k-1}\left(1 + (-1)^b\right) & \text{if } k < 0,\ b \neq 0 \text{ and } p = 2, \\ \dfrac{p^{-k-1}-1}{p-1} & \text{if } k < 0,\ b = 0 \text{ and } p \neq 2, \\ p^{-k-1}(p-1)^{b-1} & \text{if } k < 0,\ b \neq 0 \text{ and } p \neq 2. \end{cases}$$

In fact, $\mathbf{Isoc}^R(\mathbb{S}_k, B; p)$ for $k \geq 0$ was computed by Mednykh in [36], [37]. In [30], we can also found an explicit formula for computing the numbers $\mathbf{Isoc}^R(\mathbb{S}_k, B; 2p)$ and $\mathbf{Isoc}^R(\mathbb{S}_k, B; p^2)$ when p is a prime number.

This kind of enumeration of regular coverings will be continued in the next chapter.

Chapter 8

Distributions of branched surface coverings

A well-known theorem of Alexander ([1]) says that every orientable surface is a branched covering of the sphere $\mathbb{S}^2$, and every nonorientable surface is a branched covering of the projective plane. In the study of surface branched coverings, we can ask naturally as a generalization of Alexander's theorem: *In how many different ways can a given surface be a branched covering of another given surface?* To give a systematic answer of this question, we define two polynomials, called *branched covering distribution polynomials.*

(i) For each $i \in \mathbb{Z}$, let $a_i(\mathbb{S}, B; n)$ denote the number of equivalence classes of branched n-fold coverings $p : \mathbb{S}_i \to \mathbb{S}$ with branch set B, and let

$$R_{(\mathbb{S},B;n)}(x) = \sum_{i=-\infty}^{\infty} a_i(\mathbb{S}, B; n)x^i.$$

(ii) For each $i \in \mathbb{Z}$, let $a_i(\mathbb{S}, B; \mathcal{A})$ denote the number of equivalence classes of branched $\mathcal{A}$-coverings $p : \mathbb{S}_i \to \mathbb{S}$ with branch set B, and let

$$R_{(\mathbb{S},B;\mathcal{A})}(x) = \sum_{i=-\infty}^{\infty} a_i(\mathbb{S}, B; \mathcal{A})x^i.$$

These two polynomials can have at most finitely many nonzero terms by the Riemann-Hurwitz equation: $\chi(\tilde{\mathbb{S}}) = n\chi(\mathbb{S}) - \sum_{b\in B} \mathrm{def}(b)$, where

$\mathrm{def}(b) = n - |p^{-1}(b)|$ and χ denotes the Euler characteristic. (Here, $n = |\mathcal{A}|$ for an $\mathcal{A}$ covering.)

Remark From the covering distribution polynomials, we see that the number

$$R_{(\mathbb{S},B;n)}(1) = \sum_{i=-\infty}^{\infty} a_i(\mathbb{S}, B; n)$$

is equal to the total number **Isoc** $(\mathbb{S}, B; n)$ of nonequivalent branched n-fold coverings of the (orientable or nonorientable) surface $\mathbb{S}$ with branch set B. In particular, the total number $R_{(\mathbb{S},\emptyset;n)}(1)$ of nonequivalent unbranched n-fold coverings of the surface $\mathbb{S}$ is equal to the number of the conjugacy classes of the subgroups of index n of the fundamental group $\pi_1(\mathbb{S}, *)$. Also, for the regular coverings, the number

$$R_{(\mathbb{S},B;\mathcal{A})}(1) = \sum_{i=-\infty}^{\infty} a_i(\mathbb{S}, B; \mathcal{A})$$

is equal to the total number **Isoc** $(\mathbb{S}, B; \mathcal{A})$ of nonequivalent branched $\mathcal{A}$-coverings of the surface $\mathbb{S}$ with branch set B. The total number **Isoc** $(\mathbb{S}, \emptyset; \mathcal{A}) = R_{(\mathbb{S},\emptyset;\mathcal{A})}(1)$ of nonequivalent unbranched $\mathcal{A}$-coverings of the surface $\mathbb{S}$ is equal to the number of the normal subgroups $\mathcal{H}$ of the fundamental group $\pi_1(\mathbb{S}, *)$ such that the quotient group $\pi_1(\mathbb{S}, *)/\mathcal{H}$ is isomorphic to $\mathcal{A}$.

Now, we are interest in the number $R_{(\mathbb{S},B;\mathcal{A})}(1)$ and in the polynomial $R_{(\mathbb{S},B;\mathcal{A})}(x)$. In chapter 7, the number $R_{(\mathbb{S},B;\mathcal{A})}(1)$ was discussed and computed when $\mathcal{A}$ is an abelian group. Notice that the computation of the polynomial $R_{(\mathbb{S},B;\mathcal{A})}(x)$ is harder than the computation of the number $R_{(\mathbb{S},B;\mathcal{A})}(1)$. The polynomial $R_{(\mathbb{S},B;\mathcal{A})}(x)$ is known for the case when $\mathcal{A}$ is the cyclic group $\mathbb{Z}_p$ of prime order p or the dihedral group $\mathbb{D}_p$ of order $2p$. (See [23, 29].)

By Theorem 24 and the Riemann-Hurwitz equation, we can obtain the following which also can be found in [23] .

Theorem 25 ([23]) *Let $\mathcal{A} = \mathbb{Z}_p$ and p be a prime.*

$\|B\|$	$p=2$	3	5	7	11	13	17
0	0	0	0	0	0	0	0
1	0	0	0	0	0	0	0
2	1	1	1	1	1	1	1
3	0	x	$3x^2$	$5x^3$	$9x^5$	$11x^6$	$15x^8$
4	x	$3x^2$	$13x^4$	$31x^6$	$91x^{10}$	$133x^{12}$	$241x^{16}$
5	0	$5x^3$	$51x^6$	$185x^9$	$909x^{15}$	$1595x^{18}$	$3855x^{24}$
6	x^2	$11x^4$	$205x^8$	$1111x^{12}$	$9091x^{20}$	$19141x^{24}$	$61681x^{32}$
7	0	$21x^5$	$819x^{10}$	$6665x^{15}$	$90909x^{25}$	$229691x^{30}$	$986895x^{40}$
8	x^3	$43x^6$	$3277x^{12}$	$39991x^{18}$	$909091x^{30}$	$2756293x^{36}$	$15790321x^{48}$
9	0	$85x^7$	$13107x^{14}$	$239945x^{21}$	$9090909x^{35}$	$33075515x^{42}$	$252645135x^{56}$

Table 8.1: The polynomial $R_{(\mathbb{S},B,\mathbb{Z}_p)}(x)$ for the sphere $\mathbb{S}_0$

(1) *Let B be a finite set of points in an orientable surface $\mathbb{S}_k$ $(k \geq 0)$ and let $b = |B|$. Then we have*

$$a_i(\mathbb{S}_k, B; \mathbb{Z}_p) = \begin{cases} \dfrac{p^{2k}-1}{p-1} & \text{if } i = 1 + p(k-1),\ b = 0, \\ p^{2k-1}\left((p-1)^{b-1} + (-1)^b\right) & \text{if } i = pk + \dfrac{p-1}{2}(b-2) \text{ and } b \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(2) *Let B be a finite set of points in a nonorientable surface $\mathbb{S}_k$ $(k < 0)$ and let $b = |B|$. Then we have*

$$a_i(\mathbb{S}_k, B; \mathbb{Z}_2) = \begin{cases} 1 & \text{if } i = -k-1,\ b = 0, \\ 2^{-k} - 2 & \text{if } i = 2(k+1),\ k \neq -1,\ b = 0, \\ 2^{-k} & \text{if } i = 2(k+1) - b,\ b \neq 0,\ b = \text{even}, \\ 0 & \text{otherwise.} \end{cases}$$

(3) *Let B be a finite set of points in a nonorientable surface $\mathbb{S}_k$ $(k < 0)$ and let $b = |B|$. Then, for each odd prime p, we have*

$\lvert B\rvert$	$p=2$	$p=3$	$p=5$	$p=7$	$p=11$	$p=13$
0	$3x$	$4x$	$6x$	$8x$	$12x$	$14x$
1	0	0	0	0	0	0
2	$4x^2$	$9x^3$	$25x^5$	$49x^7$	$121x^{11}$	$169x^{13}$
3	0	$9x^4$	$75x^7$	$245x^{10}$	$1089x^{16}$	$1859x^{19}$
4	$4x^3$	$27x^5$	$325x^9$	$1519x^{13}$	$11011x^{21}$	$22477x^{25}$
5	0	$45x^6$	$1275x^{11}$	$9065x^{16}$	$109989x^{26}$	$269555x^{31}$
6	$4x^4$	$99x^7$	$5125x^{13}$	$54439x^{19}$	$1100011x^{31}$	$3234829x^{37}$
7	0	$189x^8$	$20475x^{15}$	$326585x^{22}$	$10999989x^{36}$	$38817779x^{43}$
8	$4x^5$	$387x^9$	$81925x^{17}$	$1959559x^{25}$	$110000011x^{41}$	$465813517x^{49}$
9	0	$765x^{10}$	$327675x^{19}$	$11757305x^{28}$	$1099999989x^{46}$	$2147483647x^{55}$

Table 8.2: The polynomial $R_{(\mathbb{S},B,\mathbb{Z}_p)}(x)$ for the torus $\mathbb{S}_1$

$\lvert B\rvert$	$p=2$	$p=3$	$p=5$	$p=7$	$p=11$	$p=13$
0	1	0	0	0	0	0
1	0	x^{-1}	x^{-1}	x^{-1}	x^{-1}	x^{-1}
2	$2x^{-2}$	$2x^{-3}$	$4x^{-5}$	$6x^{-7}$	$10x^{-11}$	$12x^{-13}$
3	0	$4x^{-5}$	$16x^{-9}$	$36x^{-13}$	$100x^{-21}$	$144x^{-25}$
4	$2x^{-4}$	$8x^{-7}$	$64x^{-13}$	$216x^{-19}$	$1000x^{-31}$	$1728x^{-37}$
5	0	$16x^{-9}$	$256x^{-17}$	$1296x^{-25}$	$10000x^{-41}$	$20736x^{-49}$
6	$2x^{-6}$	$32x^{-11}$	$1024x^{-21}$	$7776x^{-31}$	$100000x^{-51}$	$248832x^{-61}$
7	0	$64x^{-13}$	$4096x^{-25}$	$46656x^{-37}$	$1000000x^{-61}$	$2985984x^{-73}$
8	$2x^{-8}$	$128x^{-15}$	$16384x^{-29}$	$279936x^{-43}$	$10000000x^{-71}$	$35831808x^{-85}$
9	0	$256x^{-17}$	$65536x^{-33}$	$1679616x^{-49}$	$100000000x^{-81}$	$429981696x^{-97}$

Table 8.3: The polynomial $R_{(\mathbb{S},B,\mathbb{Z}_p)}(x)$ for the projective plane $\mathbb{S}_{-1}$

$$a_i(\mathbb{S}_k, B; \mathbb{Z}_p) = \begin{cases} \dfrac{p^{-k-1}-1}{p-1} & \text{if } i = p(k+2)-2,\ b=0, \\ p^{-k-1}(p-1)^{b-1} & \text{if } i = p(k+2)-b(p-1)-2,\ b\neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The following can be found in [29].

Theorem 26 ([29]) *Let $\mathcal{A} = \mathbb{D}_p$ and p be an odd prime.*

(1) *Let B be a finite subset of the sphere $\mathbb{S}_0$ and let $b = |B|$. Then*

we have

$$a_i(\mathbb{S}_0, B; \mathbb{D}_p) = \begin{cases} \binom{b}{2s} p^{2s-2}(p-1)^{b-2s-1} & \text{if } i = b(p-1)+1-s(p-2)-2p \\ & \qquad \text{for } 1 \le s \le \left\lfloor \frac{b-1}{2} \right\rfloor,\ b \ge 3, \\ \dfrac{p^{b-2}-1}{p-1} & \text{if } i = p\left(\frac{b-4}{2}\right)+1,\ b(\ge 3) \text{ is even}, \\ 0 & \text{otherwise.} \end{cases}$$

(2) *Let* B *be a finite subset of an orientable surface* $\mathbb{S}_k$ $(k > 0)$ *and let* $b = |B|$*. Then we have*

$$a_i(\mathbb{S}_k, B; \mathbb{D}_p) = \begin{cases} \left(4^k - 1\right) \dfrac{p^{2k-2}-1}{p-1} & \text{if } i = 2p(k-1)+1,\ b = 0, \\ \left(4^k - 1\right) p^{2k-2}(p-1)^{b-1} & \text{if } i = 2p(k-1)+b(p-1)+1, \\ & \qquad b \ne 0, \\ \binom{b}{2s} 4^k p^{2k+2s-2} & \text{if } i = 2p(k-1)+b(p-1) \\ \qquad \times (p-1)^{b-2s-1} & \qquad +1-s(p-2) \\ & \qquad \text{for } 1 \le s \le \left\lfloor \frac{b-1}{2} \right\rfloor,\ b \ne 0, \\ 4^k \left(\dfrac{p^{2k+b-2}-1}{p-1} \right) & \text{if } i = p\left(2(k-1)+\frac{b}{2}\right)+1, \\ & \qquad b(\ne 0) \text{ is even}, \\ 0 & \text{otherwise,} \end{cases}$$

(3) *Let* B *be a finite set of points in a nonorientable surface* $\mathbb{S}_k$ $(k < 0)$

and let $b = |B|$. Then we have

$$a_i(\mathbb{S}_k, B; \mathbb{D}_p) = \begin{cases} \dfrac{p^{-k-1}-1}{p-1} & \text{if } i = 1 + p(-k-2),\ b = 0, \\ \left(2^{-k}-2\right)\dfrac{p^{-k-2}-1}{p-1} & \text{if } i = 2p(k+2) - 2,\ b = 0, \\ p^{-k-2}\left((p-1)^{b-1} + (-1)^{b}\right) & \text{if } i = p(-k-2) + b(p-1) + 1, \\ & \qquad b \neq 0, \\ (2^{-k}-2)p^{-k-2}(p-1)^{b-1} & \text{if } i = 2p(k+2) - 2b(p-1) - 2, \\ & \qquad b \neq 0, \\ \dbinom{b}{2s} 2^{-k} p^{-k+2s-2} & \text{if } i = 2p(k+2) - 2b(p-1) \\ \qquad \times (p-1)^{b-2s-1} & \qquad +2s(p-2) - 2, \\ & \text{for } 1 \leq s \leq \left\lfloor \frac{\beta-1}{2} \right\rfloor,\ b \neq 0, \\ 2^{-k}\left(\dfrac{p^{-k+b-2}-1}{p-1}\right) & \text{if } i = p(2k+4-b) - 2, \\ & \qquad b(\neq 0) \text{ is even}, \\ 0 & \text{otherwise}, \end{cases}$$

From Theorem 26, we have the following.

Corollary 3 *Let $\mathcal{A} = \mathbb{D}_p$ and p be an odd prime.*

(1) *Let B be a finite set of points in the sphere $\mathbb{S}_0$ and let $b = |B|$. Then*

$$\mathbf{Isoc}\,(\mathbb{S}_0, B; \mathbb{D}_p) = \begin{cases} \displaystyle\sum_{s=1}^{\left\lfloor \frac{b-1}{2} \right\rfloor} \binom{b}{2s} p^{2s-2}(p-1)^{b-2s-1} & \text{if } b \geq 3, \\ \qquad + \dfrac{p^{b-2}-1}{p-1}\,\dfrac{\left(1+(-1)^{b}\right)}{2} & \\ 0 & \text{otherwise.} \end{cases}$$

$\|B\|$	$p=3$	$p=5$
0	0	0
1	0	0
2	0	0
3	3	3
4	$4x+12x^2$	$6x+24x^4$
5	$45x^3+40x^4$	$125x^5+160x^8$
6	$40x^4+270x^5+120x^6$	$156x^6+1500x^9+960x^{12}$
7	$567x^6+1260x^7+336x^8$	$4375x^{10}+14000x^{13}+5376x^{16}$
8	$364x^7+4536x^8+5040x^9+896x^{10}$	$3906x^{11}+70000x^{14}+112000x^{17}+28672x^{20}$

$\|B\|$	$p=7$	$p=11$
0	0	0
1	0	0
2	0	0
3	3	3
4	$8x+36x^6$	$12x+60x^{10}$
5	$245x^7+360x^{12}$	$605x^{11}+1000x^{20}$
6	$400x^8+4410x^{13}+3240x^{18}$	$1464x^{12}+18150x^{21}+15000x^{30}$
7	$16807x^{14}+61740x^{19}+27216x^{24}$	$102487x^{22}+423500x^{31}+210000x^{40}$
8	$19608x^{15}+403368x^{20}+740880x^{25}+217728x^{30}$	

Table 8.4: The polynomial $R_{(\mathbb{S},B,\mathbb{D}_p)}(x)$ for the sphere $\mathbb{S}_0$

$\|B\|$	$p=3$	$p=5$
0	0	0
1	$3x^3$	$3x^5$
2	$16x^4+6x^5$	$24x^6+12x^9$
3	$108x^6+12x^7$	$300x^{10}+48x^{13}$
4	$160x^7+432x^8+24x^9$	$624x^{11}+2400x^{14}+192x^{17}$
5	$1620x^9+1440x^{10}+48x^{11}$	$12500x^{15}+16000x^{18}+768x^{21}$
6	$1456x^{10}+9720x^{11}+4320x^{12}+96x^{13}$	$15624x^{16}+150000x^{19}+96000x^{22}+3072x^{25}$

$\|B\|$	$p=7$	$p=11$
0	0	0
1	$3x^7$	$3x^{11}$
2	$32x^8+18x^{13}$	$48x^{12}+30x^{21}$
3	$588x^{14}+108x^{19}$	$1452x^{22}+300x^{31}$
4	$1600x^{15}+7056x^{20}+648x^{25}$	$5856x^{23}+29040x^{32}+3000x^{41}$
5	$48020x^{21}+70560x^{26}+3888x^{31}$	$292820x^{33}+484000x^{42}+30000x^{51}$
6	$78432x^{22}+864360x^{27}+635040x^{32}+23328x^{37}$	

Table 8.5: The polynomial $R_{(\mathbb{S},B,\mathbb{D}_p)}(x)$ for the torus $\mathbb{S}_1$

$\lvert B\rvert$	$p=3$	$p=5$
0	0	0
1	0	0
2	$2x^{-2}+x^{2}$	$2x^{-2}+x^{4}$
3	$18x^{-6}+x^{4}$	$30x^{-10}+3x^{8}$
4	$72x^{-10}+26x^{-8}+3x^{6}$	$240x^{-18}+62x^{-12}+13x^{12}$
5	$240x^{-14}+270x^{-12}+5x^{8}$	$1600x^{-26}+1250x^{-20}+51x^{16}$
6	$720x^{-18}+1620x^{-16}+242x^{-14}+11x^{10}$	$9600x^{-34}+15000x^{-28}+1562x^{-22}+205x^{20}$

$\lvert B\rvert$	$p=7$	$p=11$
0	0	0
1	0	0
2	$2x^{-2}+x^{6}$	$2x^{-2}+x^{10}$
3	$42x^{-14}+5x^{12}$	$66x^{-22}+9x^{20}$
4	$504x^{-26}+114x^{-16}+31x^{18}$	$1320x^{-42}+266x^{-24}+91x^{30}$
5	$5040x^{-38}+3430x^{-28}+185x^{24}$	$22000x^{-62}+13310x^{-44}+909x^{40}$
6	$45360x^{-50}+61740x^{-40}+5602x^{-30}+1111x^{30}$	

Table 8.6: The polynomial $R_{(\mathbb{S},B,\mathbb{D}_p)}(x)$ for the projective plane $\mathbb{S}_{-1}$

(2) *Let B be a finite set of points in an orientable surface $\mathbb{S}_k$ $(k>0)$ and let $b=|B|$. Then we have*

$$
\mathbf{Isoc}\,(\mathbb{S}_k, B; \mathbb{D}_p)
= \begin{cases}
\left(4^k-1\right)\dfrac{p^{2k-2}-1}{p-1} & \text{if } b=0,\\[2ex]
\left(4^k-1\right)p^{2k-2}(p-1)^{b-1} & \text{if } b\neq 0.\\
\qquad +\displaystyle\sum_{s=1}^{\left\lfloor\frac{b-1}{2}\right\rfloor}\binom{b}{2s}4^k p^{2k+2s-2}(p-1)^{b-2s-1} & \\
\qquad\qquad +4^k\dfrac{p^{2k+b-2}-1}{p-1}\dfrac{\left(1+(-1)^b\right)}{2} &
\end{cases}
$$

(3) *Let B be a finite set of points in a nonorientable surface $\mathbb{S}_k$ $(k<0)$*

$(\|B\|,p)$	$k=-4$	$k=-3$	$k=-2$	$k=-1$	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
$(0,2)$	69	10	1	0	0	0	60	2520	92820
$(0,3)$	115	12	1	0	0	0	90	9828	996030
$(1,2)$	0	0	0	0	0	3	135	5103	185898
$(1,3)$	0	0	0	0	0	3	375	39375	3984375
$(2,2)$	919	149	23	3	0	22	910	33502	1211470
$(2,3)$	4021	393	37	3	0	36	3996	407484	40937436

Table 8.7: The number $\mathbf{Isoc}\,(\mathbb{S}_k, B; \mathbb{D}_p)$ for small k, p and small $|B|$

and let $b = |B|$. Then we have

$$\mathbf{Isoc}\,(\mathbb{S}_k, B; \mathbb{D}_p)$$
$$= \begin{cases} \dfrac{p^{-k-1}-1}{p-1} + (2^{-k}-2)\dfrac{p^{-k-2}-1}{p-1} & \text{if } b = 0, \\[2ex] p^{-k-2}\left((p-1)^{b-1} + (-1)^b\right) & \text{if } b \neq 0. \\ \quad + p^{-k-2}(2^{-k}-2)(p-1)^{b-1} & \\ \qquad + \displaystyle\sum_{s=1}^{\lfloor \frac{b-1}{2} \rfloor} \binom{b}{2s} 2^{-k} p^{-k+2s-2}(p-1)^{b-2s-1} & \\ \qquad\quad + 2^{-k}\,\dfrac{p^{-k+b-2}-1}{p-1}\,\dfrac{\left(1+(-1)^b\right)}{2} & \end{cases}$$

From Theorems 25 and 26, we can make tables 8.1-8.6, and derive many interesting topological properties of branched regular surface coverings. We list some of them in the following.

A group $\mathcal{A}$ action on a surface $\mathbb{S}$ is *pseudofree* if the number of fixed points of the action is finite, *i.e.,* the cardinality of the set $\{x \in \mathbb{S} \mid gx = x \text{ for some } g \neq id \text{ in } \mathcal{A}\}$ is finite. A group action on a surface is *spherical* if the quotient surface of the action is homeomorphic to the sphere.

1. For any $k \geq 0$, there are exactly $4^k - 1$ nonequivalent connected unbranched double coverings of $\mathbb{S}_k$, and all of their covering surfaces are $\mathbb{S}_{2k-1}$.

2. For any surface $\mathbb{S}$, there does not exist a connected branched double covering of $\mathbb{S}$ with odd number of branch points.

3. For any $k \geq 0$ and even number $2b$, $b \geq 1$, there are exactly 4^k nonequivalent connected branched double coverings of $\mathbb{S}_k$ having given $2b$ branch points, and all of their covering surfaces are $\mathbb{S}_{2k+b-1}$.
4. There exists a unique connected unbranched double covering of the projective plain $\mathbb{S}_{-1}$ up to equivalence, and its covering surface is the sphere. For any $k \leq -2$, there exist $2^{-k} - 1$ connected unbranched double coverings of $\mathbb{S}_k$ up to equivalence, and one of their covering surfaces is the orientable surface $\mathbb{S}_{-k-1}$ and all others are the nonorientable surface $\mathbb{S}_{2(k+1)}$.
5. For any $k \leq -1$ and even number $2b$, $b \geq 1$, there are exactly 2^{-k} nonequivalent connected branched double coverings of $\mathbb{S}_k$ having given $2b$ branch points, and all of their covering surfaces are the nonorientable surface $\mathbb{S}_{2(k-b+1)}$.
6. Every orientable surface is a branched double covering of the sphere $\mathbb{S}^2$. Every nonorientable surface is a branched double or triple covering of the projective plane $\mathbb{S}_{-1}$ (This is Alexander's theorem).
7. Let p be prime ≥ 2. Then the dihedral group $\mathbb{D}_p$ can act freely on the surface $\mathbb{S}_k$ if and only if either $k \geq 1$ and $k - 1 \equiv 0 \pmod{p}$ or $k \leq -3$ and $k + 2 \equiv 0 \pmod{2p}$. Moreover,

i. if $k \geq 1$, $k - 1 \equiv 0 \pmod{p}$ and $k - 1 \not\equiv 0 \pmod{2p}$, then $\mathbb{S}_k/\mathbb{D}_p$ is the nonorientable surface $\mathbb{S}_{\frac{1-k}{p}-2}$;
ii. if $k \geq 2$ and $k - 1 \equiv 0 \pmod{2p}$, then $\mathbb{S}_k/\mathbb{D}_p$ is either the orientable surface $\mathbb{S}_{\frac{k-1}{2p}+1}$ or the nonorientable surface $\mathbb{S}_{\frac{1-k}{p}-2}$;
iii. if $k \leq -3$ and $k + 2 \equiv 0 \pmod{2p}$, then $\mathbb{S}_k/\mathbb{D}_p$ is the nonorientable surface $\mathbb{S}_{\frac{k+2}{2p}-2}$.

8. For any prime $p \geq 2$, a surface $\mathbb{S}_k$ has a spherical pseudofree $\mathbb{D}_p$-action if and only if $k = (p-1)m + n$, where $m, n \geq 0$ and $m + 1 \geq n$. Moreover, for such a $k = (p-1)m + n$, the number of branch points of the $\mathbb{D}_p$-covering $p : \mathbb{S}_k \to \mathbb{S}_k/\mathbb{D}_p = \mathbb{S}_0$ is $m + n + 3$.

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