Quantiser design for AWGN and Rayleigh fading BPSK channels with soft output decoding

J. Bakus and A.K. Khandani

Abstract: A new method of combined source-channel coding for the scalar quantisation of a discrete memoryless source is presented that takes advantage of the channel soft output values produced by the demodulator. The case of transmitting a memoryless Gaussian source using binary phase shift keying (BPSK) modulation over over an additive white Gaussian noise (AWGN) channel with and without Rayleigh fading is studied. Numerical results using some closed form expressions are presented, showing up to 1 dB improvement in the end-to-end distortion with respect to a traditional channel optimised scalar quantiser.

1 Introduction

This work studies the transmission of a discrete time continuous amplitude signal over a noisy channel using a scalar quantiser in conjunction with a soft output channel decoder. An improvement in the end-to-end quantisation distortion is achieved by providing a soft reconstruction rule using the channel output value produced by the demodulator. The corresponding quantisation and reconstruction rules are iteratively optimised using a procedure similar to the Lloyd–Max algorithm [1, 2]. We study the case of transmitting a memoryless Gaussian source using binary phase shift keying (BPSK) modulation over over an additive white Gaussian noise (AWGN) channel with and without Rayleigh fading. Numerical results using some closed form expressions are presented, showing up to 1 dB improvement in the end-to-end distortion with respect to a traditional channel optimised scalar quantiser.

Optimum fixed-rate scalar quantisers, introduced by Max [1] and Lloyd [2], minimise the average distortion for a given number of threshold points. This approach was later extended by Linde, Buzo and Gray to a vector quantiser [3]. The original Lloyd–Max algorithm does not consider the effect of channel noise. Kurtenbach and Wintz were among the first researchers to investigate the effect of channel noise in a quantisation system [4]. Later, Rydbeck and Sundberg showed that the codeword assignment for a quantiser operating over a noisy channel also plays an important role in system performance [5].

In [6], Farvardin and Vaishampayan presented an algorithm based on the Lloyd–Max algorithm for quantiser design over a noisy channel resulting in the so-called channel optimised scalar quantiser (COSQ). The COSQ algorithm was further extended by the same researchers [7, 8] to vector sources, and the extended version is known as the channel optimised vector quantiser (COVQ).

One of the first attempts to design a quantiser with soft reconstruction decoding was made by Vaishampayan and Farvardin by extending the COVQ design algorithm to include the modulation signal set [9]. Another approach to soft reconstruction was proposed by Phamdo and Alajaji, who applied the COVQ algorithm to the case that the demodulator output is uniformly quantised to allow for a soft decision decoding [10, 11]. This increases the number of channel output symbols and results in a finer reconstruction than the classical COVQ. This approach has successfully been used by Zhu and Alajaji [12] to design a COVQ for a turbo-code channel.

In [13, 14] Skoglund and Hedelin study the problem of transmitting a source via vector quantisation over a noisy channel. They present a soft decoder that is optimal in the mean square sense and discuss rules to design the corresponding quantiser and reconstructor pair. They propose an efficient algorithm for optimum decoding, as well as a variety of suboptimum methods of reduced complexity, where the key idea is to use a Hadamard formulation of the optimum decoder. This idea is further applied to image transmission in [15]. In [16], Ottosson and Skoglund present methods for combined soft decoding and multiuser detection in a frequency selective Rayleigh fading CDMA system.

Ho in [17] uses the reliability information and the same soft reconstruction rule as in this work. The design algorithm is based on the iterative COVQ, and the effects of the channel noise are incorporated by transmitting the encoded training sequence through a turbo-code channel at every iteration.

The current work and previous related works [6–11, 14, 17, 18] are in some way or other based on the following two basic principles:

(i) Quantiser partitions are selected to minimise the distortion due to the individual input samples (refer to (15)).

(ii) Quantiser output is selected as a proper combination of a set of reconstruction values. In [14, 17–19], as well as in the current work, this is selected as the sum of the reconstruction values multiplied by their probabilities (refer to (11)).

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The main contribution of this paper is to formulate these two basic principles explicitly for the case of a memoryless Gaussian source that is scalar quantised and transmitted using BPSK modulation over an AWGN channel with and without Rayleigh fading. The main objective is to take the maximum advantage of the information provided in the continuous channel output to reduce the end-to-end source distortion. The main differences between the current paper and earlier related work [18, 19] are: (i) using closed form expressions to design the quantisers for the Gaussian distributed samples (instead of a training sequence), and (ii) extending the design to use channel output of uncoded BPSK over an AWGN channel with Raleigh fading (again, using closed form expressions).

2 System overview

The block diagram of the system is shown in Fig. 1. The scalar quantiser $\gamma(\cdot)$ maps the input sample x to a quantiser partition $S \in \{S_1, \ldots, S_N\}$, and each quantiser partition is mapped to a binary code word $\boldsymbol{u} \in \{\boldsymbol{u}_1, \ldots, \boldsymbol{u}_N\}$ composed of r bits, where $N \leq 2^r$. These bits are modulated using BPSK and transmitted over the noisy channel. At the receiver side, for each transmitted code word $\boldsymbol{u}$, the receiver produces a channel output vector $\boldsymbol{v}$ composed of r channel output values v. The reconstructor $g(\cdot)$ maps the channel output vector $\boldsymbol{v}$ to an output sample $\hat{x}$.

The transmitter, noisy channel, and receiver can be considered as an equivalent channel with input $\boldsymbol{u}$ and output $\boldsymbol{v}$, as shown in Fig. 1. To design the quantiser and the reconstructor, the relationship between the input $\boldsymbol{u}$ and the output $\boldsymbol{v}$ of this equivalent channel is required, and several different models to approximate this relationship are presented.

We assume that the equivalent channel between $\boldsymbol{u}$ and $\boldsymbol{v}$ is memoryless, i.e.

$$P(\boldsymbol{v}|\boldsymbol{u}) = \prod_{i=1}^{r} P(v_i|u_i) \tag{1}$$

where u_i, v_i, $i = 1, \ldots, r$ are the components of $\boldsymbol{u}$ and $\boldsymbol{v}$ respectively. We also assume that the components of $\boldsymbol{u}$ are independent of each other, i.e.

$$P(\boldsymbol{u}) = \prod_{i=1}^{r} P(u_i) \tag{2}$$

Using (1) and (2) we obtain

$$P(\boldsymbol{u}|\boldsymbol{v}) = \prod_{i=1}^{r} P(u_i|v_i) \tag{3}$$

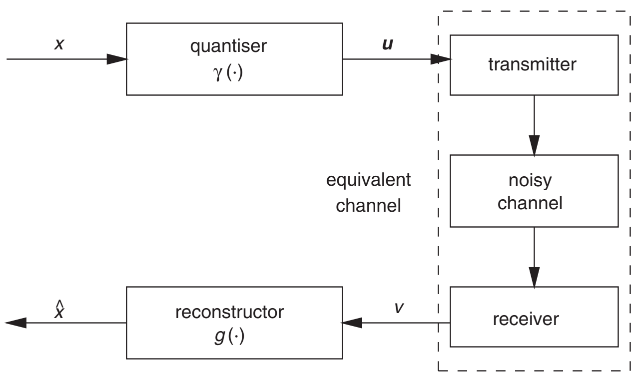

Fig. 1 *System block diagram*

In practice, the bit independence assumption in (3) is usually not entirely valid, however, the same assumptions are used in all the schemes that are compared. The validity of the independence assumption can be improved by interleaving the quantiser output bits before transmission.

An additive white Gaussian noise (AWGN) channel is investigated, both with and without fading. In both cases the transmitted bit u is modulated to one of two values $\{+E_b, -E_b\}$ and received as channel output v. The AWGN channel model adds a normal distributed noise value n with a variance σ_n^2 to the transmitted value as

$$v = u + n \tag{4}$$

Given the received value v the probability of the transmitted bit u is given by

$$\begin{aligned} P(v|u=0) &= \frac{1}{\sqrt{2\pi\sigma_n^2}} \exp\left(-\frac{(v+E_b)^2}{2\sigma_n^2}\right) \\ P(v|u=1) &= \frac{1}{\sqrt{2\pi\sigma_n^2}} \exp\left(-\frac{(v-E_b)^2}{2\sigma_n^2}\right) \end{aligned} \tag{5}$$

The Rayleigh channel model is defined as

$$v = gu + n \tag{6}$$

where g is Rayleigh distributed fading and n is the additive white Gaussian noise. The probability of channel output v given the transmitted bit u is computed in the Appendix as

$$\begin{aligned} P(v|u=0) =& D\left\{ +\frac{1}{2A}\exp(-C) - \frac{B\sqrt{\pi}}{4A\sqrt{A}} \right. \\ & \left. \times \exp\left(\frac{B^2}{4A} - C\right)\left[1 - \operatorname{erf}\left(\frac{B}{2\sqrt{A}}\right)\right]\right\} \\ P(v|u=1) =& D\left\{ +\frac{1}{2A}\exp(-C) + \frac{B\sqrt{\pi}}{4A\sqrt{A}} \right. \\ & \left. \times \exp\left(\frac{B^2}{4A} - C\right)\left[1 + \operatorname{erf}\left(\frac{B}{2\sqrt{A}}\right)\right]\right\} \end{aligned} \tag{7}$$

where

$$\begin{aligned} A &= \frac{\sigma_n^2 + s^2}{2\sigma_n^2 s^2} \\ B &= \frac{v}{\sigma_n^2} \\ C &= \frac{v^2}{2\sigma_n^2} \\ D &= \frac{1}{s^2\sqrt{2\pi\sigma_n^2}} \end{aligned} \tag{8}$$

Here, σ_n^2 is the variance of the noise n, and s is the parameter for the Rayleigh fading [Note 1].

Assuming that the source bits are equally likely, i.e. $P(u=0) = P(u=1) = 1/2$, the probability of received value v given the transmitted bit u is given by

$$\begin{aligned} P(u=1|v) &= \frac{P(v|u=1)}{P(v|u=0) + P(v|u=1)} \\ P(u=0|v) &= \frac{P(v|u=0)}{P(v|u=0) + P(v|u=1)} \end{aligned} \tag{9}$$

where the probabilities $P(v|u)$ are calculated using (5) and (7) for the respective channels. Using (3) the reconstructor $g(\cdot)$ calculates the corresponding code word probabilities $P(\boldsymbol{u} = \boldsymbol{u}_i|\boldsymbol{v})$, $i = 1, \ldots, N$, where $P(\boldsymbol{u} = \boldsymbol{u}_i|\boldsymbol{v})$ is the probability

Note 1: The Raleigh probability distribution is $P_g(x) = x/s \exp(x^2 2s^2)$, $x \geq 0$.

that code word $\boldsymbol{u}_i$ was transmitted conditioned on the channel output $\boldsymbol{v}$. Using these probabilities and a set of reconstruction levels $\{R_1,\ldots,R_N\}$, the reconstructor generates the output sample $\hat{x}$ using one of two reconstruction rules: hard reconstruction rule or soft reconstruction rule.

The hard reconstruction rule selects the codeword which is most likely transmitted and outputs the corresponding reconstruction level, i.e.

$$\hat{x} = g(\boldsymbol{v}) = R_i : P(\boldsymbol{u} = \boldsymbol{u}_i|\boldsymbol{v}) > P(\boldsymbol{u} = \boldsymbol{u}_j|\boldsymbol{v}),\ \forall j \neq i \quad (10)$$

Systems with this reconstruction rule are well established in the literature [1, 2, 4, 6–8]. The soft reconstruction rule generates the reconstructed sample as a sum of the codeword probabilities multiplied with their corresponding reconstruction levels, i.e.

$$\hat{x} = g(\boldsymbol{v}) = \sum_{i=1}^{N} R_i P(\boldsymbol{u} = \boldsymbol{u}_i|\boldsymbol{v}) \quad (11)$$

3 Quantiser design

To design the quantiser, consider the distortion of a sample x quantised to partition i as

$$D_i(x) = \int_{\boldsymbol{v}} P(\boldsymbol{v}|\boldsymbol{u} = \boldsymbol{u}_i)[x - g(\boldsymbol{v})]^2 d\boldsymbol{v} \quad (12)$$

where $P(\boldsymbol{v}|\boldsymbol{u}=\boldsymbol{u}_i) = P(\boldsymbol{v}|x \in S_i)$ is the probability of receiving a channel output $\boldsymbol{v}$ conditioned on transmitting $\boldsymbol{u}_i$, and $g(\cdot)$ is the reconstruction function. Equation (12) can be written as

$$D_i(x) = x^2 - 2xB_i + C_i \quad (13)$$

where

$$\begin{aligned} B_i &= \int_{\boldsymbol{v}} g(\boldsymbol{v}) P(\boldsymbol{v}|\boldsymbol{u} = \boldsymbol{u}_i) d\boldsymbol{v} \\ C_i &= \int_{\boldsymbol{v}} g(\boldsymbol{v})^2 P(\boldsymbol{v}|\boldsymbol{u} = \boldsymbol{u}_i) d\boldsymbol{v} \end{aligned} \quad (14)$$

The structure of the quantiser is captured in the quantities B_i and C_i given in (14).

The quantiser is designed by fixing the reconstruction function, and finding the best partitions S_i, $i = 1,\ldots, N$ where

$$S_i = \{x : D_i(x) \leq D_j(x), \forall j \neq i\} \quad (15)$$

with $D_i(x)$ given in (13). The reconstruction levels $\{R_1,\ldots, R_N\}$ of the decoder are calculated as the expected value of x conditioned on the corresponding partition:

$$R_i = E(x|x \in S_i) = \frac{\int_{S_i} xP(x)dx}{\int_{S_i} P(x)dx}, i = 1, \ldots, N \quad (16)$$

Averaging $D_i(x)$ over the samples in each partition and summing over all the partitions result in the total distortion

$$D = \sum_{i=1}^{N} \int_{S_i} P(x)(x^2 - 2xB_i + C_i)dx \quad (17)$$

where $P(x)$ is the source sample distribution. The quantiser is designed using an iterative method by optimising the encoder with (15) and the decoder with (16) until the total distortion in (17) reaches a local minimum.

In this work, we use a Gaussian memoryless source distribution, and as a result a closed form expression exists for the necessary integrals over the source sample distribution. In the case that the probability distribution of the input source is not known, we can use a training sequence to estimate these integrals. In this case, for each training sample x, the partition which gives the lowest distortion is found. As each sample is quantised, statistics are collected for the partitions to estimate $P(x \in S_i)$ and $E(x|x \in S_i)$, and then calculate the necessary integrals with

$$\begin{aligned} \int_{S_i} xP(x)dx &= P(x \in S_i)E(x|x \in S_i) \\ \int_{S_i} P(x)dx &= P(x \in S_i) \end{aligned} \quad (18)$$

3.1 Hard decision quantiser design

Using the hard reconstruction function in (10) and an appropriate equivalent channel, the presented formulation becomes either the classical Lloyd–Max quantiser [1, 2] or the channel optimised scalar quantiser [6]. These two are then compared with the proposed model based on a soft reconstruction rule.

The hard noiseless channel model ignores the effect of channel noise, where $\boldsymbol{v} = \boldsymbol{u}$, and the quantities B_i and C_i in (14) become

$$\begin{aligned} B_i &= R_i \\ C_i &= R_i^2 \end{aligned} \quad (19)$$

This results in the classical Lloyd–Max design algorithm [1, 2], where the reconstruction levels are given by

$$R_i = \frac{\int_{S_i} xP(x)dx}{\int_{S_i} P(x)dx}, i = 1, \ldots, N \quad (20)$$

The hard binary symmetric channel model replaces the equivalent channel model with a binary symmetric channel with a bit error probability of ε. In this case, the relationship between the transmitted codeword $\boldsymbol{u}$ and the received codeword $v = \hat{\boldsymbol{u}}$, is given by

$$\begin{aligned} P(j|i) &\equiv P(\hat{\boldsymbol{u}} = \boldsymbol{u}_j|\boldsymbol{u}_i) \\ &= (\varepsilon)^{h_{i,j}}(1 - \varepsilon)^{(r - h_{i,j})} \end{aligned} \quad (21)$$

where $\hat{\boldsymbol{u}}$ is the received binary code word, $h_{i,j}$ is the Hamming distance between codewords $\boldsymbol{u}_i$ and $\boldsymbol{u}_j$, and r is the number of bits in a word. With this model and using the hard reconstruction rule, the quantities B_i and C_i in (14) become:

$$\begin{aligned} B_i &= \sum_{j=1}^{N} P(j|i)R_j \\ C_i &= \sum_{j=1}^{N} P(j|i)R_j^2 \end{aligned} \quad (22)$$

This results in the channel optimised quantiser of [6] with the reconstruction levels given by

$$R_i = \frac{\sum_{j=1}^{N} P(i|j) \int_{S_j} xP(x)dx}{\sum_{j=1}^{N} P(i|j) \int_{S_j} P(x)dx}, i = 1, \ldots, N \quad (23)$$

3.2 Soft decision quantiser design

For a system with soft decision, the reconstruction function in (11) is substituted into the distortion expression in (14) resulting in:

$$\begin{aligned} B_i &= \sum_{k=1}^{N} R_k \Theta_i(k) \\ C_i &= \sum_{j=1}^{N} \sum_{k=1}^{N} R_j R_k \Psi_i(j, k) \end{aligned} \quad (24)$$

where

$$\Theta_i(k) = \int_{\boldsymbol{v}} P(\boldsymbol{v}|\boldsymbol{u} = \boldsymbol{u}_i)P(\boldsymbol{u} = \boldsymbol{u}_k|\boldsymbol{v})d\boldsymbol{v}$$
$$\Psi_i(j,k) = \int_{\boldsymbol{v}} P(\boldsymbol{v}|\boldsymbol{u} = \boldsymbol{u}_i)P(\boldsymbol{u} = \boldsymbol{u}_j|\boldsymbol{v})P(\boldsymbol{u} = \boldsymbol{u}_k|\boldsymbol{v})d\boldsymbol{v} \quad (25)$$

The two sets of parameters, $\Theta_i(k)$ and $\Psi_i(j, k)$, capture the characteristics of the channel with soft reconstruction rule. The indices i, j and k denote the individual codewords, and $\Theta_i(k)$, $\Psi_i(j, k)$ are computed for all possible codeword combinations.

To calculate these parameters, we first compute the single bit versions $\theta_i(k)$ and $\psi_i(j, k)$, where

$$\theta_i(k) = \int_v P(v|u = i)P(u = k|v)dv$$
$$\psi_i(j,k) = \int_v P(v|u = i)P(u = j|v)P(u = k|v)dv \quad (26)$$

The indices i, j and k in (26) can take the binary values 0 or 1. The two quantities are evaluated for all possible bit combinations, resulting in four different values for $\theta_i(k)$ and eight different values for $\psi_i(j, k)$. Using the independence assumptions in (1) and (3) the set of parameters $\Theta_i(k)$ and $\Psi_i(j, k)$ are calculated by multiplying their single bit components, i.e. $\theta_i(k)$ and $\psi_i(j, k)$.

The quantiser partitions are determined by substituting B_i and C_i in (15) and the reconstruction levels are computed as

$$R_i = E(x|x \in S_i) = \frac{\int_{S_i} xP(x)dx}{\int_{S_i} P(x)dx}, \text{if} \int_{S_i} P(x)dx > 0, \forall i \quad (27)$$

In order to estimate the channel parameters $\theta_i(j)$ and $\psi_i(j, k)$, (5) or (7) (depending on the channel) are substituted in (26) and the resulting expressions are numerically integrated. Alternatively, the parameters can be estimated by passing a test bit stream through the channel and collecting certain statistics. In this approach the equivalent channel in Fig. 1 is replaced by a reliability information channel model. This is based on the assumption that the equivalent channel model outputs a reliability information value $P(u=k|v)$ for every input bit. This value is calculated using (5) or (7) for the AWGN or Rayleigh channels, respectively.

To compute $\theta_i(k)$, we define the received reliability information $P(u|v)$ as a random variable X_k:

$$X_k = P(u = k|v), \quad k = 0, 1 \quad (28)$$

Note that X_k, as a function of v, is a random variable. The identities in (5), (7) and (9) provide a one-to-one relationship between v and X_k, and the value of $k=0,1$ determines which of the identities are used for this purpose. With this one-to-one mapping, the probability distribution of X_k is expressed in terms of the probability distribution of v by the transformation:

$$P(X_k)dX_k = P(v)dv \quad (29)$$

Conditioning both sides of the equation on $u=i$ results in

$$P(X_k|u = i)dX_k = P(v|u = i)dv \quad (30)$$

Substituting (28) and (30) in (26) results in

$$\begin{aligned}\theta_i(k) &= \int_v P(v|u = i)P(u = k|v)dv \\ &= \int_0^1 X_kP(X_k|u = i)dX_k \\ &= E(X_k|u = i) \end{aligned} \quad (31)$$

The quantity $E(X_k|u=i)$ is measured by passing a test bit stream through the channel and observing the channel output.

Similarly, the quantity $\psi_i(j, k)$ is calculated by defining a random variable $Y_{j,k}$, as

$$Y_{j,k} = P(u = j|v)P(u = k|v), \quad j,k = 0, 1 \quad (32)$$

Using the identities in (5), (7) and (9), we obtain

$$P(Y_{j,k})dY_{j,k} = P(v)dv \quad (33)$$

Conditioning both sides on $u=i$ results in

$$P(Y_{j,k}|u = i)dY_{j,k} = P(v|u = i)dv \quad (34)$$

Substituting (32) and (34) in (26) results in

$$\begin{aligned}\psi_i(j,k) &= \int_v P(v|u = i)P(u = j|v)P(u = k|v)dv \\ &= \int_0^1 Y_{j,k}P(Y_{j,k}|u = i)dY_{j,k} \\ &= E(Y_{j,k}|u = i) \end{aligned} \quad (35)$$

The quantity $E(Y_{j,k}|u=i)$ is also measured by passing a test bit stream through the channel and observing the output. Using the parameters, $\theta_i(j)$ and $\psi_i(j, k)$, we calculate the parameters, $\Theta_i(j)$ and $\Psi_i(j, k)$, and design the quantiser as outlined earlier.

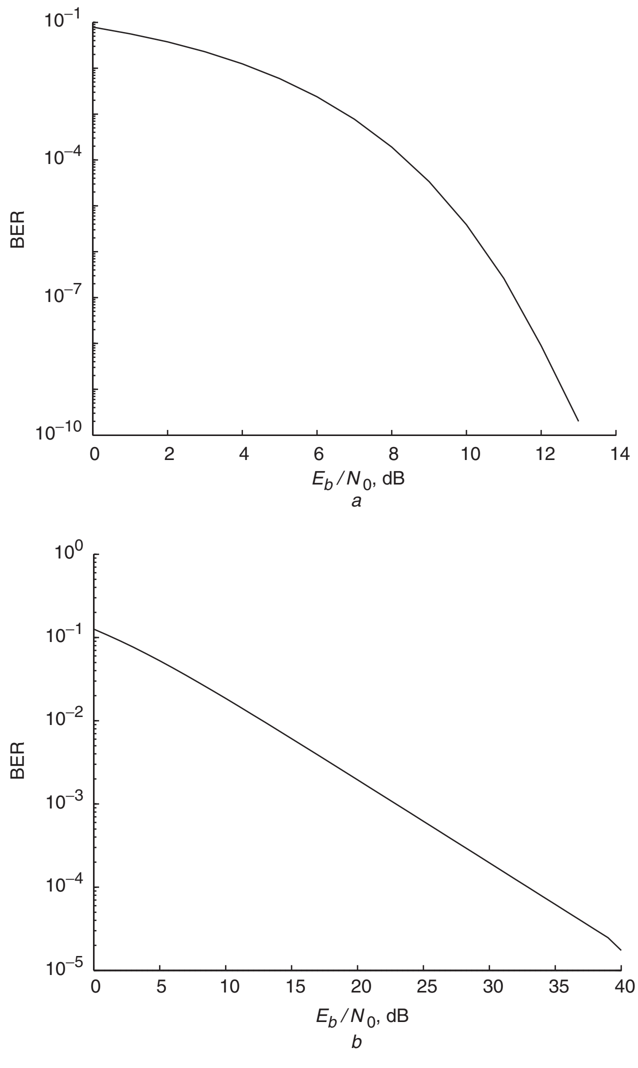

Fig. 2 *Bit-error-rate curves for AWGN and Rayleigh channels*
a AWGN
b Rayleigh

4 Numerical results

4.1 Simulation setup

The proposed quantisation scheme has been simulated for an independent identically distributed Gaussian input. The bits are modulated with binary phase shift keying (BPSK) and with additive white Gaussian noise (AWGN) both with and without Rayleigh fading. The bit-error-rate curves for both of these channels are shown in Fig. 2. A total of three different quantisers are designed and tested, as listed in Table 1. The first quantiser is the classical Lloyd–Max quantiser designed assuming a noiseless channel and tested over a noisy channel with hard decision rule. The channel optimised quantiser is designed taking into account the effect of the noise and tested with a hard decision rule. Finally, the soft decision rule quantiser is designed using the method developed in the current article.

To reduce the impact of the initialisation, the codewords are initialised randomly and the best of 100 designs is chosen. In each case, the quantiser and reconstructor design algorithms are iterated until the relative change in the mean square error is less that 10^{-3}. Since the source sample distribution is known to be Gaussian, the quantisers are designed by closed form evaluation of (18) without the use of a training sequence.

Table 1: Quantisers tested

Quantiser	Channel model
Lloyd–Max	Noiseless
Channel optimised	Binary symmetric
Soft decision	Reliability information

4.2 Quantiser results

The results for 4- and 5-bit quantisers are shown in Figs. 3 and 4, respectively, for both the AWGN and Rayleigh channels. As expected, the Lloyd–Max quantiser operating with hard reconstruction rule results in the worst performance. The channel optimised quantiser also uses a hard reconstruction rule, however, the design takes the effect of the channel noise into account and the performance is substantially improved. The soft reconstruction quantisers offer an improvement over both of the hard reconstruction quantisers. Of the two soft decision quantisers, the system of equations design performs better than the expected value design.

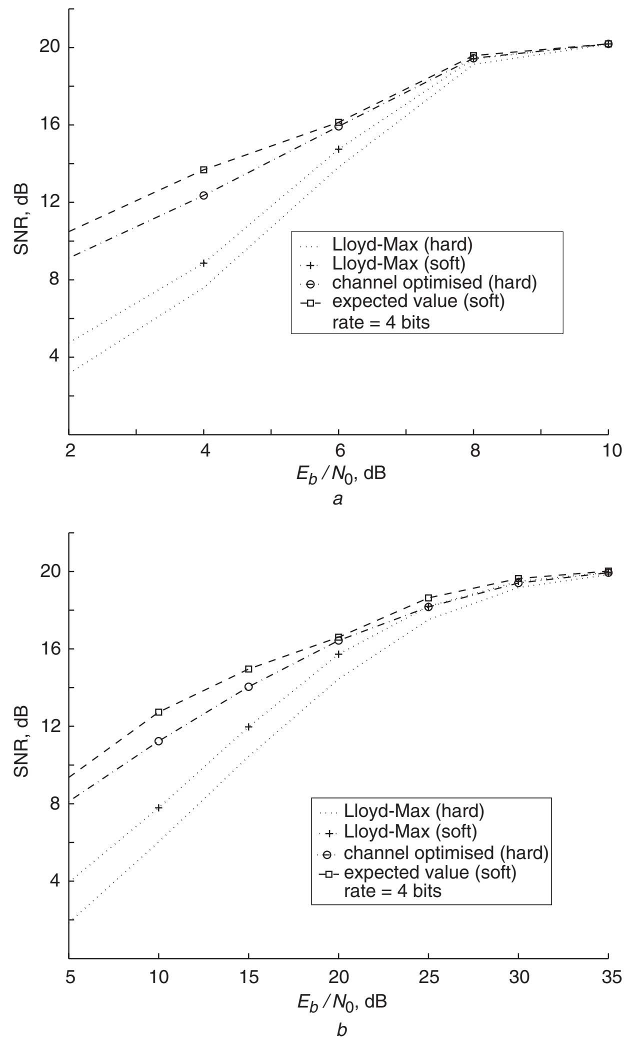

Fig. 3 *Distortion for 4-bit Lloyd–Max and channel optimised quantisers using hard and soft reconstruction rules*
a AWGN
b Rayleigh

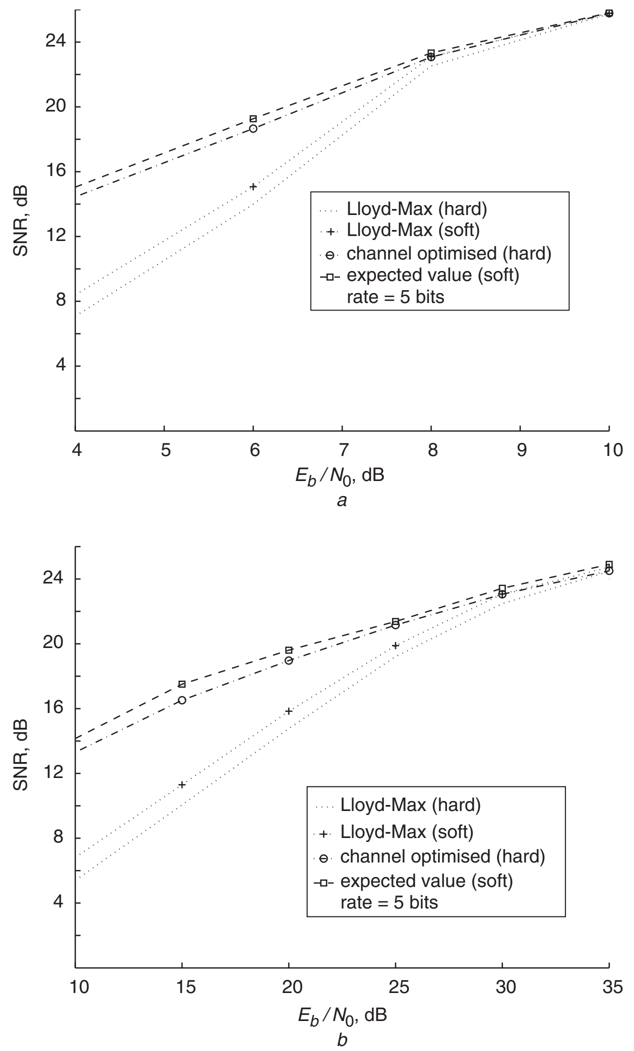

Fig. 4 *Distortion for 5-bit Lloyd–Max and channel optimised quantisers using hard and soft reconstruction rules*
a AWGN
b Rayleigh

5 Summary

We have discussed a new method of quantiser design with soft output decoding. A system is presented that transmits a discrete time, continuous amplitude signal over AWGN channel with and without Rayleigh fading. The transmitted samples are encoded using a fixed rate scalar quantiser, and modulated using BPSK. The system performance is improved by using a soft reconstruction of samples using the reliability information available at the channel decoder. The quantiser and the reconstructor are jointly optimised using an iterative algorithm to minimise the end-to-end distortion. The performance of the presented quantisers has been shown to be up to 1.0 dB better than that of the conventional channel optimised scalar quantiser.

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7 Appendix: Rayleigh channel probability derivation

This Appendix presents the derivation for the probability distribution for the Rayleigh channel given in (7). Considering the case $u = 1$, the random variable v in (6) is the sum of two random variables g and n with the following distributions:

$$\begin{aligned} P_n(x) &= \frac{1}{\sqrt{2\pi\sigma_n^2}} \exp\left(-\frac{x^2}{2\sigma_n^2}\right) \\ P_g(x) &= \frac{x}{s}\exp(x^2 2s^2), \quad x \geq 0 \end{aligned} \tag{36}$$

The probability distribution of v is given as a convolution of the above distribution given by

$$\begin{aligned} P_v &= P_n(x) * P_g(x) \\ P_v &= \int_0^{\infty} \frac{1}{\sqrt{2\pi\sigma_n^2}} \exp\left(-\frac{(v-\tau)^2}{2\sigma_n^2}\right) \frac{\tau}{s^2}\exp(\tau^2 2s^2)d\tau \end{aligned} \tag{37}$$

Using

$$\begin{aligned} A &= \frac{\sigma_n^2 + s^2}{2\sigma_n^2 s^2} \\ B &= \frac{v}{\sigma_n^2} \\ C &= \frac{v^2}{2\sigma_n^2} \\ D &= \frac{1}{s^2\sqrt{2\pi\sigma_n^2}} \end{aligned} \tag{38}$$

Equation (37) becomes

$$P_v = D\int_0^{\infty} \tau \exp(-A\tau^2 + B\tau - C)d\tau \tag{39}$$

Evaluating the integral results in

$$P_v(x) = I(\infty) - I(0) \tag{40}$$

where $I(\tau)$ is the indefinite integral of (39) given by

$$\begin{aligned} I(\tau) = D\Bigg[&-\frac{1}{2A}\exp(-A\tau^2 + B\tau - C) \\ &+ \frac{B\sqrt{\pi}}{4A\sqrt{A}}\exp\left(\frac{B^2}{4A} - C\right)\mathrm{erf}\left(\sqrt{A}\tau - \frac{B}{2\sqrt{A}}\right)\Bigg] \end{aligned} \tag{41}$$

and

$$\begin{aligned} I(0) &= D\Bigg[-\frac{1}{2A}\exp(-C) + \frac{B\sqrt{\pi}}{4A\sqrt{A}}\exp\left(\frac{B^2}{4A} - C\right) \\ &\quad \times \mathrm{erf}\left(-\frac{B}{2\sqrt{A}}\right)\Bigg] \\ I(\infty) &= D\left[\frac{B\sqrt{\pi}}{4A\sqrt{A}}\exp\left(\frac{B^2}{4A} - C\right)\right] \end{aligned} \tag{42}$$

Evaluation of the above integrals, results in

$$\begin{aligned} P_v = D\Bigg\{ &+\frac{1}{2A}\exp(-C) + \frac{B\sqrt{\pi}}{4A\sqrt{A}}\exp\left(\frac{B^2}{4A} - C\right) \\ &\times \left[1 + \mathrm{erf}\left(\frac{B}{2\sqrt{A}}\right)\right]\Bigg\} \end{aligned} \tag{43}$$

The case for $u = -1$ follows similarly.