Theory of 1D Vlasov-Maxwell Equilibria

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Abstract. Current sheets are magnetic structures which are important for activity processes in the coronae of the Sun and of other stars. To lowest order current sheets can be described as one-dimensional equilibria of a collisionless plasma described by the Vlasov-Maxwell equations. In the present paper, a general mathematical framework for one-dimensional equilibrium solutions of the Vlasov-Maxwell equations is presented. The equilibrium equations for the magnetic vector potential are equivalent to the motion of a particle in a two-dimensional potential.The theory is applied to some examples for one-dimensional current sheets.

INTRODUCTION

Magnetic activity processes in the coronae of the Sun and of other stars belong to the most fascinating phenomena in plasma astrophysics. Astrophysical plasmas usually satisfy the conditions for ideal magnetohydrodynamics (MHD) being valid to describe them extremely well, but nevertheless non-ideal processes such as, for example, magnetic reconnection are known to play an important role in most coronal activity processes.

It is well-known that one way of overcoming this apparent contradiction is the formation of strongly localized regions of strong electric current, i.e. magnetic current sheets. In these regions ideal MHD can break down, allowing for non-ideal process such as magnetic reconnection to occur. Although these non-ideal processes occur only on very small length scales, they still have a global effect in the release of stored magnetic energy.

To understand the physics behind these non-ideal processes better, kinetic theory instead of MHD has to be used and because the time and length scales on which theses processes occur are usually much shorter than typical collision times and mean free paths, it is appropriate to use collisionless theory.

Plasma equilibria are suitable starting points for investigating these processes. For collisionless plasmas, the relevant equilibria are self-consistent solutions of the Vlasov-Maxwell (VM) equations[1, 2]. Due to the generic structure of current sheets they can be modelled by 1D equilibria as a first approximation.

The Vlasov equation is formally solved by making the equilibrium distribution functions of the particle species depend on the one-particle constants of motion only. Using the resulting charge and current densities in the steady state Maxwell equations leads to a system of partial differential equations for the electric potential and the magnetic vector potential.

GENERAL THEORY

We assume all quantities depend only on z and that the magnetic field has components B_x and B_y. The magnetic field components are written in terms of a vector potential $\mathbf{A}$ where

$$B_x = -\frac{dA_y}{dz} \tag{1}$$

$$B_y = \frac{dA_x}{dz}. \tag{2}$$

Due to the symmetries of the system (time independence and spatial independence of x and z) the three obvious constants of motion for each particle species are the Hamiltonian or particle energy for each species s,

$$H_s = \frac{1}{2m_s}[(p_{xs} - q_s A_x)^2 + (p_{ys} - q_s A_y)^2 + p_{zs}^2] + q_s \phi, \tag{3}$$

where ϕ is the electric potential, the canonical momentum in the x-direction, p_{xs},

$$p_{xs} = m_s v_x + q_s A_x, \tag{4}$$

and the canonical momentum in the y-direction, p_{ys},

$$p_{ys} = m_s v_y + q_s A_y, \tag{5}$$

where m_s and q_s are the mass and charge of each species.

All positive functions f_s satisfying the appropriate conditions for existence of velocity moments etc and depending only on the constants of motion,

$$f_s = f_s(H_s, p_{xs}, p_{ys}) \tag{6}$$

solve the steady-state Vlasov equation

$$\mathbf{v} \cdot \frac{\partial f_s}{\partial \mathbf{r}} + \frac{q_s}{m_s}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f_s}{\partial \mathbf{v}} = 0. \tag{7}$$

To calculate 1D VM equilibria we have to solve the steady-state Maxwell equations

$$-\frac{d^2\phi}{dz^2} = \frac{1}{\varepsilon_0}\sigma(A_x, A_y, \phi) \tag{8}$$

$$-\frac{d^2 A_x}{dz^2} = \mu_0 j_x(A_x, A_y, \phi) \tag{9}$$

$$-\frac{d^2 A_y}{dz^2} = \mu_0 j_y(A_x, A_y, \phi) \tag{10}$$

where the source terms are the electric charge density and the current densities in the x and y directions, which are defined in the usual way as velocity moments of the

equilibrium distribution functions, f_s. We assume that the plasma is quasineutral, which can be expressed to lowest order as

$$\sigma(A_x, A_y, \phi) = 0. \tag{11}$$

This equation implicitly defines the quasineutral electric potential ϕ_{qn} where

$$\phi_{qn} = \phi_{qn}(A_x, A_y). \tag{12}$$

An interesting aspect of the 1D Vlasov-Maxwell equilibrium problem is that is completely equivalent to the classical mechanics problem of a (pseudo-)particle in a 2D potential[3]. To start with, the force balance condition for the equilibrium requires that

$$\frac{B^2}{2\mu_0} + P = P_T = \text{constant}, \tag{13}$$

where P is the P_{zz} component of the plasma pressure tensor. It turns out that Ampère's law can be written as

$$\frac{d^2 A_x}{dz^2} = -\mu_0 \frac{\partial P}{\partial A_x} \tag{14}$$

$$\frac{d^2 A_y}{dz^2} = -\mu_0 \frac{\partial P}{\partial A_y}. \tag{15}$$

These two equations are the equations of motion for a pseudo-particle with position A_x, A_y in the potential $P(A_x, A_y)$ and z as time. The Hamiltonian of this pseudo-particle is the total pressure defined in Eq. (13) (modulo a factor μ_0)

$$H_A = \frac{1}{2}\left(\frac{dA_x}{dz}\right)^2 + \frac{1}{2}\left(\frac{dA_y}{dz}\right)^2 + \mu_0 P(A_x, A_y). \tag{16}$$

As we will see this equivalence allows us to study some properties of the solutions without having to solve the equations in the same way as discussing particle motion in a given potential.

EXAMPLES

As an example we consider the case of an anisotropi drifting Maxwellian distribution function given by

$$f_s = c_s \exp\left(-\frac{H_s}{k_B T_{s\perp}} + \frac{\Delta T_s}{2m_s k_B T_{s\perp} T_{s\parallel}} b_s p_{xs}^2 + \frac{\Delta T_s}{2m_s k_B T_{s\perp} T_{s\parallel}} p_{ys}^2\right), \tag{17}$$

where $\Delta T_s = T_{s\parallel} - T_{s\perp} \geq 0$ is assumed. The resulting pressure $P(A_x, A_y)$ is a Gaussian function of A_x and A_y with the parameter b_s determining the difference in the width of the Gaussian in the A_x- and A_y-directions. The resulting equations cannot be solved
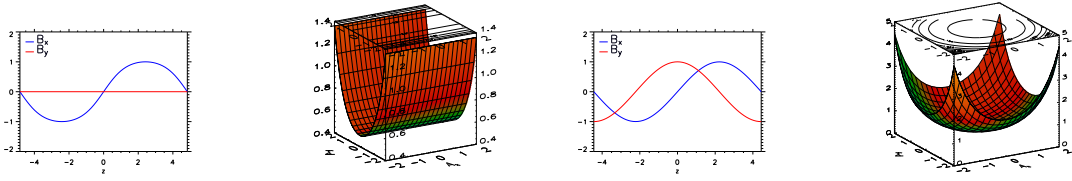
FIGURE 1. The two examples with the $b_s = 0.0$ case shown in the two plots on the left and $b_s = 1.0$ case shown in the two plots on the right.

analytically (except for one special case), but by looking at the shape of the pressure function one can immediately see that both A_x and A_y will be periodic functions of z, although usually with different periods.

We show plots for the two special cases $b_s = 0.0$ and $b_s = 1.0$ in Figures 1. The case $b_s = 0.0$ gives rise to purely anti-parallel magnetic field configurations ($B_x = 0$), whereas the case $b_s = 1.0$ allows for analytical solutions, which are linear force-free [4]. In the linear force-free case the pressure function P is isotropic in A_x and A_y, i.e. corresponding to a central potential allowing for circular orbits. Varying the parameter b_s from 0 to 1 allows for a transition from the anti-parallel to linear force-free configurations.

DISCUSSION AND CONCLUSIONS

A general theory of 1D Vlasov-Maxwell equilibria has been presented. It has been shown that the general properties of the equilibria can be deduced from the pressure function $P(A_x, A_y)$, which can be regarded as the potential for the motion of a pseudo-particle in two dimensions. For the future it is planned to carry out 2D Particle-in-Cell simulations of magnetic reconnection in these equilibria and to study how the characteristics of magnetic reconnection change during the transition from anti–parallel to force-free configurations.

ACKNOWLEDGMENTS

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