SEQUENTIAL DESIGN OF EXPERIMENTS FOR A RAYLEIGH INVERSE SCATTERING PROBLEM.

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ABSTRACT

We consider the problem of imaging a Rayleigh scattering medium using an array of sensors. We study the effect of noise on the estimation of Rayleigh scatterer reflection powers by setting up a sequential design of experiments. We derive an expression for the mean squared error for estimating the scatterer reflection powers. We keep the transmitted spatial waveforms fixed and find the optimal energy allocation strategy that minimizes this error under a fixed energy constraint. We show that this sequential design approach performs better than a single step experiment. Closed-form expressions for the optimal transmission scheme and the minimum mean squared error are provided.

1. INTRODUCTION

In this paper, we address the problem of imaging a medium of multiple scatterers using an array of sensors by optimally designing a sequence of measurements. The probing method uses an array of transducers, e.g., antennas that both illuminate and measure the backscattered signal field. The method consists of the following four signal processing steps at the transducer array: (i) transmission of time varying signals into the medium; (ii) recording of the backscattered field from the medium; (iii) retransmission into the medium of a spatially filtered version of the recorded backscatter signals; (iv) measurement and spatial filtering of the backscattered signals.

Over the past decade, the problem of imaging has been widely studied in areas such as non-destructive testing [1], land mine detection, active audio, underwater acoustics [2] and ultrasonic medical imaging [3]. One recent approach to imaging which follows such a four step method uses the concept of time reversal [4,5]. Time reversal works by exploiting the reciprocity of the channel, i.e., a receiver can reflect back a time reversed signal, thereby focusing the signal at the transmitter source [6]. Furthermore, with suitable prefiltering and aperture, the signal energy can also be focused at an arbitrary spatial location [7].

Iterative time reversal techniques [8] have also been used to achieve selective focusing. In [9], the concept of adaptive beamforming [10] and interference cancellation is used to focus energy at a specified location. Though time reversal imaging methods have been studied in detail for both deterministic and random scattering environments, the performance of these methods in the presence of receiver noise has not been thoroughly studied.

Design of experiments is another area which has found wide range of applications in statistical decision making [11, 12]. Sequential design [13–15] uses the knowledge of the past measurements to improve upon the performance of an estimator. Applied to the problem of imaging a scattered medium, a carefully designed sequence of measurements sounding the channel could alter the statistics of the next measurement to yield an overall reduction in mean squared error (MSE).

In [16], experimental design was performed for imaging a scattering medium with a simple additive Gaussian measurement noise. In this paper, we consider the general problem of imaging a Rayleigh scattering medium and systematically study the effect of receiver noise on the imaging performance. We evaluate the imaging performance through the MSE of the least squared (LS) estimates of the scatterer reflection powers. In the single scatterer case, we obtain a closed-form expression for the MSE under the optimal transmission strategy. We assume that the spatial properties of the transmitted signals are fixed and find the optimal energy allocation scheme between the two transmissions involved in steps (i) and (iii) under the constraint that the total transmitted energy is fixed. We then show that we achieve a better performance than a single step strategy using this two-step design. We then extend the results to the case of constrained optimization.

In Section 2, we present the concept of imaging a medium using an iterative process of array measurements. In Section 3, we formulate the MSE criterion

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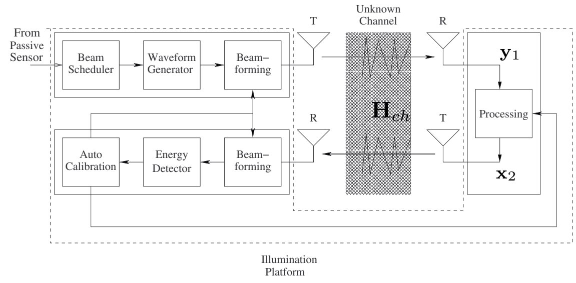


Fig. 1. Measurement setup

for LS estimation of the scatterer reflection powers. We offer an optimal two-step design that minimizes the MSE and show that this strategy outperforms the conventional beamformer. Furthermore, we provide simulation results to verify the optimal solution obtained analytically. We conclude this paper in Section 4.

2. MODEL AND MATHEMATICAL DESCRIPTION

The block diagram in Fig. 1 provides a high level description of the system. The signal flow in the block diagram is read clockwise from the upper left corner of the diagram. The three blocks surrounded by the box on the upper left of the diagram incorporate voxel selection (beam scheduling), spatio-temporal waveform selection and beamsteering followed by transmission into the medium, denoted as a dispersive spatio-temporal channel function $\mathbf{H}_{ch}$. The block on the right of the diagram processes the received backscattered signal and reinserts it into the medium $\mathbf{H}_{ch}$.

We have W transducers located at positions $\{\mathbf{r}_k^a\}_{k=1}^W$, that transmit narrowband signals with center frequency ω rad/sec. The channel denoted by $\mathbf{h}_i$, between any candidate voxel i at location $\mathbf{r}_i^v$ and the W transducers is given by,

$$\mathbf{h}_i = \left[\left(\frac{\exp(-j\omega/c\|\mathbf{r}_k^a - \mathbf{r}_i^v\|)}{\|\mathbf{r}_k^a - \mathbf{r}_i^v\|} \right)_{k=1\ldots W} \right]^T. \quad (1)$$

This channel model is a narrowband near-field approximation, which ignores the effect of multiple scattering and has been widely adopted in other scattering studies [17]. In our problem, we assume that the imaging area is divided into M voxels at locations $\{\mathbf{r}_i^v\}_{i=1}^M$. Then the channel between the transmitted field and the measured backscattered field at the transducer array is

$$\begin{aligned} \mathbf{H}_{ch} &= \mathbf{HDH}^T \\ \mathbf{H} &= [\mathbf{h}_1, \cdots, \mathbf{h}_M] \\ \mathbf{D} &= \mathrm{diag}(\mathbf{d}); \quad \mathbf{d} = [d_1, \ldots, d_M]^T, \end{aligned}$$

where each voxel location is characterized by its scatter coefficient $\{d_i\}_{i=1}^M$. Under the Rayleigh scattering model, these scatter coefficients are circularly symmetric complex normal random variables with $\mathrm{E}[d_i] = 0$, $\mathrm{E}\left[|d_i|^2\right] = r_d(i)$, $\mathrm{E}[d_i d_{i_1}] = 0$, $\mathrm{E}\left[d_i (d_{i_1})^*\right] = 0$; $i_1 \neq i$. This implies that each element of the channel matrix $\mathbf{H}_{ch}$ is a complex normal random variable and hence Rayleigh in magnitude. Note that matrix $\mathbf{H}$ is $W \times M$, $\mathbf{D}$ is $M \times M$, and $\mathbf{H}_{ch}$ is $W \times W$.

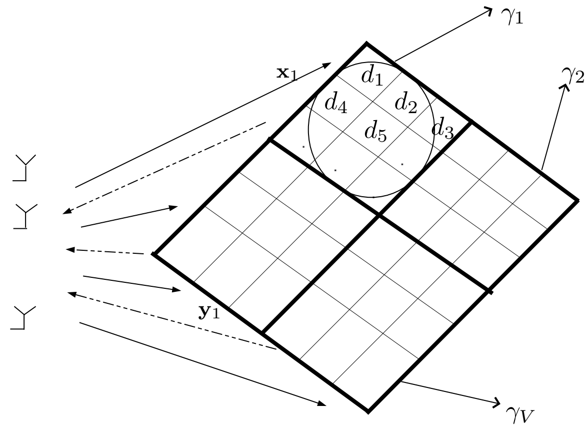


Fig. 2. Scattering medium

The two-step probing mechanism for estimating the power distribution of the scatter coefficients $\mathbf{d}$ involves four signal processing steps and generates the following sequence of noise contaminated signals.

Step 1(a): The transducer array transmits a complex amplitude vector, $\mathbf{x}_1$.

Step 1(b): The transducer array receives the backscattered signal $\mathbf{H}_{ch}\mathbf{x}_1$ plus noise $\mathbf{n}_1$,

$$\mathbf{y}_1 = \mathbf{H}_{ch}\ \mathbf{x}_1 + \mathbf{n}_1. \quad (2)$$

Step 2(a): The transducer array transmits $\mathbf{x}_2 = \mathbf{x}_2(\mathbf{y}_1)$ which, in general, is a function of $\mathbf{y}_1$ allowing the system to exploit information about $\mathbf{H}_{ch}$ in the signal $\mathbf{y}_1$.

Step 2(b): The transducer array receives the second backscattered signal,

$$\mathbf{y}_2 = \mathbf{H}_{ch}\ \mathbf{x}_2 + \mathbf{n}_2. \quad (3)$$

The noises $\mathbf{n}_1, \mathbf{n}_2$ are i.i.d complex normal random vectors with zero mean and covariance matrix $\sigma^2\mathbf{I}$. For a Rayleigh scattering medium, $\mathbf{y}_1$ is complex normal with a mean of $\mathrm{E}[\mathbf{y}_1] = 0$ and a covariance matrix given by

$$\begin{aligned} \mathbf{R}_{\mathbf{y}_1} &= \mathrm{E}\left[\mathbf{y}_1\mathbf{y}_1^H\right] \\ &= \sum_{l=1}^V \gamma_l \mathbf{R}_l(\mathbf{x}_1) + \sigma^2\mathbf{I}, \end{aligned} \quad (4)$$

where

$$\mathbf{R}_l(\mathbf{x}_1) = \frac{\int_{\mathbf{r}} \hat{r}_l(\mathbf{r})\mathbf{h}_l(\mathbf{r})\mathbf{h}_l(\mathbf{r})^H |\mathbf{h}_l^T(\mathbf{r})\mathbf{x}_1|^2 d\mathbf{r}}{\int_{\mathbf{r}} \hat{r}_l(\mathbf{r})d\mathbf{r}}$$

$$\gamma_l = \int_{\mathbf{r}} \hat{r}_l(\mathbf{r})d\mathbf{r}, \quad l = 1, \ldots, V, \tag{5}$$

and $\hat{r}_l(\mathbf{r})$ are the autocorrelation coefficients of the scatterer distribution. Our goal is to estimate $\boldsymbol{\gamma} = [\gamma_1, \ldots, \gamma_V]^T$, the non-negative, Rayleigh scattering reflection powers that are determined through the statistics of the scatter coefficients $\{d_i\}$. Figure 2 shows the transducer array and the scattering medium. The imaging area is divided into V cells and $\{\gamma_i\}_{i=1}^V$ denotes the average scatterer reflection power in the cells.

Step 2 (i.e., 2(a) and 2(b)) can be repeated to generate a sequential n-step procedure where the n transmitted signals would be $\mathbf{x}_j = \mathbf{x}_j(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_{j-1})$, $j = 1, 2, \ldots, n$. For the n-step procedure, the sequence of received signals are distributed as

$$\mathbf{y}_j|\mathbf{x}_j \sim \mathcal{CN}\left(0, \mathbf{R}_{\mathbf{y}_j}\right), \quad j = 1, 2, \ldots, n,$$

$$\mathbf{R}_{\mathbf{y}_j} = \sum_{i=1}^{V} \gamma_i \mathbf{R}_i\left(\mathbf{x}_j(\{\mathbf{y}_k\}_{k=1}^{j-1})\right) + \sigma^2\mathbf{I}. \tag{6}$$

Given the observations $\mathbf{y}_1, \ldots, \mathbf{y}_{j-1}$ at any step j, the objective is to design the next transmitted signal $\mathbf{x}_j(\mathbf{y}_1, \ldots, \mathbf{y}_{j-1})$ in order to improve the estimator performance.

3. MEAN SQUARED ERROR CALCULATION

We divide the analysis of the MSE into two parts: The one-stage estimator and two-stage estimator. In the general setup, our goal is *to design a sequence of experiments to improve upon the performance of a one-step estimator under the constraint that the total transmitted energy, E_0 is fixed.*

3.1. One-step estimator

Given N sample observations of $\mathbf{y}_1$ ($W \times 1$), an estimator $\widehat{\boldsymbol{\gamma}}_1(\{\mathbf{y}_{1k}\}_{k=1}^N)$ can be obtained by least squares fitting of $\gamma_1, \ldots, \gamma_V$ to the set of W^2 equations,

$$\widehat{\mathbf{R}}_{\mathbf{y}_1} = \frac{1}{N}\sum_{k=1}^{N} \mathbf{y}_{1k}\mathbf{y}_{1k}^H = \sum_{i=1}^{V} \gamma_i \mathbf{R}_i(\mathbf{x}_1) + \sigma^2\mathbf{I} \tag{7}$$

Equation (7) can be rewritten as

$$\mathbf{M}\widehat{\boldsymbol{\gamma}}_1 = \mathbf{vec}\left(\widehat{\mathbf{R}}_{\mathbf{y}_1} - \sigma^2\mathbf{I}\right),$$

where $\mathbf{M} = [\mathbf{vec}\left(\mathbf{R}_1(\mathbf{x}_1)\right), \ldots, \mathbf{vec}\left(\mathbf{R}_V(\mathbf{x}_1)\right)]$ and $\mathbf{vec}(\mathrm{X})$ returns a vector obtained by stacking the columns of the matrix X. Given (7), the LS estimate of $\boldsymbol{\gamma}$ is given by,

$$\widehat{\boldsymbol{\gamma}}_1 = \left(\mathbf{M}^H\mathbf{M}\right)^{-1}\mathbf{M}^H\mathbf{vec}\left(\widehat{\mathbf{R}}_{\mathbf{y}_1} - \sigma^2\mathbf{I}\right). \tag{8}$$

The MSE for the one-step estimator can be written as

$$\begin{aligned}\mathrm{MSE}_1 &= \mathrm{E}\left[\left(\widehat{\boldsymbol{\gamma}}_1 - \boldsymbol{\gamma}\right)\left(\widehat{\boldsymbol{\gamma}}_1 - \boldsymbol{\gamma}\right)^H\right] \\ &= \frac{1}{N}\left(\mathbf{M}^H\mathbf{M}\right)^{-1}\boldsymbol{\Pi}\left(\mathbf{M}^H\mathbf{M}\right)^{-1},\end{aligned} \tag{9}$$

where

$$\boldsymbol{\Pi} = N\,\mathbf{M}^H\mathrm{E}\left[\,\mathbf{vec}(\widehat{\mathbf{R}}_{\mathbf{y}_1} - \mathbf{R}_{\mathbf{y}_1})\mathbf{vec}(\widehat{\mathbf{R}}_{\mathbf{y}_1} - \mathbf{R}_{\mathbf{y}_1})^H\right]\mathbf{M}.$$

Using the fourth-order moment property for complex Gaussian vectors,

$$\boldsymbol{\Pi}_{i,j} = \mathrm{tr}\left[\mathbf{R}_i(\mathbf{x}_1)\mathbf{R}_{\mathbf{y}_1}\mathbf{R}_j(\mathbf{x}_1)\mathbf{R}_{\mathbf{y}_1}\right]. \tag{10}$$

Furthermore,

$$\begin{aligned}\left(\mathbf{M}^H\mathbf{M}\right)_{i,j} &= \mathbf{vec}\left(\mathbf{R}_i(\mathbf{x}_1)\right)^H\mathbf{vec}\left(\mathbf{R}_j(\mathbf{x}_1)\right) \\ &= \mathrm{tr}\left[\mathbf{R}_i(\mathbf{x}_1)\mathbf{R}_j(\mathbf{x}_1)\right].\end{aligned} \tag{11}$$

For a single unknown scatterer reflector power (γ), the LS one-step estimator from (8) and the corresponding MSE_1 from (9) become

$$\widehat{\gamma}_1(\mathbf{y}_1) = \frac{\mathrm{tr}\left(\mathbf{R}_1(\mathbf{x}_1)(\widehat{\mathbf{R}}_{\mathbf{y}_1} - \sigma^2\mathbf{I})\right)}{\mathrm{tr}\left(\mathbf{R}_1^2(\mathbf{x}_1)\right)},$$

$$\mathrm{MSE}_1(\mathbf{x}_1) = \frac{1}{N}\left(\frac{\gamma^2\mathrm{tr}\left(\mathbf{R}_1^4(\mathbf{x}_1)\right)}{\mathrm{tr}^2\left(\mathbf{R}_1^2(\mathbf{x}_1)\right)} + \frac{\sigma^4}{\mathrm{tr}\left(\mathbf{R}_1^2(\mathbf{x}_1)\right)} + \frac{2\sigma^2\gamma\mathrm{tr}\left(\mathbf{R}_1^3(\mathbf{x}_1)\right)}{\mathrm{tr}^2\left(\mathbf{R}_1^2(\mathbf{x}_1)\right)}\right). \tag{12}$$

Note that though the received signals are corrupted by complex normal noise, the estimate $\widehat{\gamma}_1$ is real as both matrices $\mathbf{R}_1(\mathbf{x}_1)$ and $\mathbf{R}_{\mathbf{y}_1}$ are Hermitian symmetric.

3.2. Two-step sequential design

For a two-step sequential design, we search for a waveform $\mathbf{x}_2(\mathbf{y}_1)$ which yields a lower MSE than that achievable using $\mathbf{x}_1$ under the constraint that $\mathrm{E}\left[E_1 + E_2\right] \leq E_0$ where E_1 and E_2 are the average energies used in the first and second transmissions respectively. We assume here that the spatial properties of $\mathbf{x}_1$ and $\mathbf{x}_2$ are fixed, and go after the energy allocation between the two steps that minimizes the MSE. The transmitted signal at the second step $\mathbf{x}_2(\mathbf{y}_1)$ can be written as

$$\mathbf{x}_2(\mathbf{y}_1) = \sqrt{E_2(\mathbf{y}_1)}\frac{\mathbf{x}_1}{\|\mathbf{x}_1\|} = \sqrt{E_2(\mathbf{y}_1)}\widetilde{\mathbf{x}}_1, \tag{13}$$

where $E_2 = \mathrm{E}\left[\|\mathbf{x}_2(\mathbf{y}_1)\|^2\right] = \mathrm{E}\left[E_2(\mathbf{y}_1)\right]$, $E_1 = \|\mathbf{x}_1\|^2$ and $\widetilde{\mathbf{x}}_1$ is the normalized version of $\mathbf{x}_1$. We first look at the two-step design for a single scatterer case. Let $\widehat{\gamma}_1(\mathbf{y}_1)$ and $\widehat{\gamma}_1(\mathbf{y}_2)$ be the LS estimates of γ obtained from the two steps by transmitting signals $\mathbf{x}_1$ and $\mathbf{x}_2(\mathbf{y}_1)$ respectively. The overall two-step estimate of γ is

$$\widehat{\gamma}_2 = \frac{w_1\widehat{\gamma}_1(\mathbf{y}_1) + w_2\widehat{\gamma}_1(\mathbf{y}_2)}{w_1 + w_2}, \tag{14}$$

where the weights w_1 and w_2 are chosen to minimize the MSE. The MSE of the two-step estimate is

$$\begin{aligned} \mathrm{MSE}_2 &= \mathrm{E}\left[(\widehat{\gamma}_2 - \gamma)^2\right] = \mathrm{E}_{\mathbf{y}_1}\left[\mathrm{E}_{\mathbf{y}_2|\mathbf{y}_1}\left[(\widehat{\gamma}_2 - \gamma)^2\right]\right] \\ &= \mathrm{E}_{\mathbf{y}_1}\left[\mathrm{MSE}_2|\mathbf{y}_1\right], \end{aligned} \tag{15}$$

where

$$\mathrm{MSE}_2|\mathbf{y}_1 = \frac{w_1^2(\widehat{\gamma}_1(\mathbf{y}_1) - \gamma)^2 + w_2^2(\mathrm{MSE}_1(\mathbf{x}_2(\mathbf{y}_1)))}{(w_1 + w_2)^2}.$$

Minimizing $\mathrm{MSE}_2|\mathbf{y}_1$ with respect to w_1 and w_2, we get $w_1\left(\widehat{\gamma}_1(\mathbf{y}_1) - \gamma\right)^2 = w_2\mathrm{MSE}_1(\mathbf{x}_2(\mathbf{y}_1))$. Substituting for the optimal weights in equation (15), the two-stage MSE is

$$\mathrm{MSE}_2 = \mathrm{E}_{\mathbf{y}_1}\left[\frac{1}{\left(\frac{1}{(\widehat{\gamma}_1 - \gamma)^2} + \frac{1}{\mathrm{MSE}_1(\mathbf{x}_2(\mathbf{y}_1))}\right)}\right]. \tag{16}$$

In [16], the design and the improvement in MSE using a sequential procedure for an additive gaussian channel model was studied and the optimal solution was found to be a thresholding strategy. A thresholding solution to energy at the second stage can be written as,

$$E_2(\mathbf{y}_1) = A\,\mathrm{I}\left(\left[\frac{\widehat{\gamma}_1(\mathbf{y}_1) - \gamma}{\sqrt{\mathrm{MSE}_1(\mathbf{x}_1)}}\right]^2 > \rho\right), \tag{17}$$

where $\mathrm{I}(\cdot)$ is the indicator function and A is chosen to satisfy the energy constraint. This solution implies that if the particular realization of $\widehat{\gamma}_1$ was closer than average to the true value, then it is fairly accurate and thus there is no need to retransmit energy.

We first look at the solution to this problem at low SNR ($\mathrm{SNR} = \frac{E_0}{\sigma^2}$). At low SNR, the MSE for the two stages can be approximated as

$$\mathrm{MSE}_1(\mathbf{x}_1) \approx \frac{\mathrm{H}}{NE_1^2}, \tag{18}$$

$$\mathrm{MSE}_1(\mathbf{x}_2(\mathbf{y}_1)) \approx \frac{\mathrm{H}}{NE_2^2(\mathbf{y}_1)}, \tag{19}$$

where

$$\mathrm{H} = \frac{\sigma^4}{\mathrm{tr}\left(\mathbf{R}_1^2(\widetilde{\mathbf{x}}_1)\right)}.$$

When N is large, the averaging associated with first estimate of γ drives the standardized MSE $\left(\frac{\widehat{\gamma}_1(\mathbf{y}_1) - \gamma}{\sqrt{\mathrm{MSE}_1(\mathbf{x}_1)}}\right)$ to asymptotically zero mean unit variance normal random variable, n_1. Substituting for n_1 and $\mathrm{MSE}_1(\mathbf{x}_1)$, $\mathrm{MSE}_1(\mathbf{x}_2(\mathbf{y}_1))$ from equations (18), (19) into equation (16), the MSE for the two-step design is

$$\begin{aligned} \mathrm{MSE}_2 &= \frac{\mathrm{H}}{N}\,\mathrm{E}_{\mathrm{n}_1}\left[\frac{\mathrm{n}_1^2\mathrm{I}(\mathrm{n}_1^2 > \rho)}{E_1^2 + \mathrm{n}_1^2E_2^2(\mathbf{y}_1)}\right] \\ &= \frac{\mathrm{H}}{N}\mathrm{E}_{\mathrm{n}_1}\left[\frac{\mathrm{n}_1^2\mathrm{I}(\mathrm{n}_1^2 > \rho)}{E_1^2 + \mathrm{n}_1^2A^2} + \frac{\mathrm{n}_1^2\mathrm{I}(\mathrm{n}_1^2 \le \rho)}{E_1^2}\right] \end{aligned} \tag{20}$$

So our goal now is to minimize this two-step MSE for the optimal energy allocation between the two steps subject to the energy constraint which can be written as

$$\mathrm{E}_{\mathbf{y}_1}\left[E_1 + E_2(\mathbf{x}_2(\mathbf{y}_1))\right] \le E_0 \tag{21}$$

Substituting the suboptimal energy solution from equation (17) into (21), we obtain

$$\begin{aligned} E_1 + A\,\mathrm{E}\left[\mathrm{I}(|\mathrm{n}_1|^2 > \rho)\right] &\le E_0 \\ A &\le E_0\frac{(1-\alpha)}{2\mathrm{Q}(\sqrt{\rho})}, \end{aligned} \tag{22}$$

where $\alpha = \frac{E_1}{E_0}$ is the fraction of energy allocated to the first step. Putting back the constraint into the MSE_2 expression in equation (20) we get

$$\begin{aligned} \mathrm{MSE}_2 &= \frac{\mathrm{H}}{NE_0^2}\mathrm{E}_{\mathrm{n}_1}\left[\frac{\mathrm{n}_1^2\mathrm{I}(\mathrm{n}_1^2 > \rho)}{\alpha^2 + \left(\frac{\mathrm{n}_1(1-\alpha)}{2\mathrm{Q}(\sqrt{\rho})}\right)^2} + \frac{\mathrm{n}_1^2\mathrm{I}(\mathrm{n}_1^2 \le \rho)}{\alpha^2}\right] \\ &= \frac{H}{NE_0^2}\left(2\int_{\sqrt{\rho}}^{\infty}\frac{\mathrm{n}_1^2}{\alpha^2 + \mathrm{n}_1^2\left(\frac{(1-\alpha)}{2\mathrm{Q}(\sqrt{\rho})}\right)^2}\mathrm{f}(\mathrm{n}_1)d\mathrm{n}_1 \right.\\ &\quad \left. + \frac{1}{\alpha^2}\left[-\sqrt{\frac{2\rho}{\pi}}e^{\frac{\rho}{2}} + 1 - 2\mathrm{Q}(\sqrt{\rho})\right]\right), \end{aligned} \tag{23}$$

where the integral is evaluated numerically. Minimizing MSE_2 in the above expression, the optimal solution to ρ and α and the corresponding MSE at low SNR is found to be

$$\rho_{\mathrm{opt}} \approx 0.8885, \quad \alpha_{\mathrm{opt}} = \frac{E_{1\mathrm{opt}}}{E_0} \approx 0.66, \tag{24}$$

$$\mathrm{MSE}_2(\gamma) \approx 0.6821\frac{H}{NE_0^2} = 0.6821\,\mathrm{MSE}_1(\gamma) \tag{25}$$

corresponding to a reduction in MSE by 68%. In [16], we fixed E_2 to be a constant for every $\mathbf{y}_1$ received rather than constraining the average energy used over all possible $\mathbf{y}_1$ and obtained a reduction in MSE of 92%. Figures 3 and 4 show the analytical (solid line) and simulation (dashed line) plots of the gain $= \frac{\mathrm{MSE}_2}{\mathrm{MSE}_1}$ as a function of ρ for α_{opt} and as a function of α for ρ_{opt} at SNR $= -10$dB. Since $\frac{\text{MSE}_2}{\text{MSE}_1} < 1$, we obtain a reduction in MSE using our sequential design approach. The plot of gain in MSE vs. SNR corresponding to α_{opt} and ρ_{opt} at low SNR is shown in Fig. 5 through simulation (dashed line) and analytically (solid line).

Our design procedure is more critical at a lower SNR for the following reason. It is important to note that the solution to $\mathbf{x}_2(\mathbf{y}_1)$ and the weights depend on the value of γ which is unknown. However, if we are given information of the form $\gamma \in [\gamma_a, \gamma_b]$ for any $-\infty < \gamma_a, \gamma_b < \infty$, then it is possible to incorporate this knowledge in making the optimal decision for $\mathbf{x}_2$ by replacing γ with γ_g in (17) :

$$E_2(\mathbf{y}_1) = A\,\mathbf{I}\left(\left[\frac{\widehat{\gamma}_1(\mathbf{y}_1) - \gamma}{\sqrt{\text{MSE}_1(\mathbf{x}_1)}} + \frac{\sqrt{N}E_1(\gamma - \gamma_g)}{\sqrt{\text{H}}}\right]^2 > \rho\right), \tag{26}$$

where γ_g is a guess of γ. Since γ is bounded, the guess term $|\frac{\sqrt{N}E_1(\gamma-\gamma_g)}{\sqrt{\text{H}}}|$ is also bounded. For a typical low SNR scenario, the energy transmitted tends to zero thereby making this term negligibly small. As a result, there is no loss of optimality due to the guess factor γ_g in the solution in (26). To demonstrate this concept, we plot the gain in MSE versus the error in the guess of γ for varying SNR in Fig. 6. The figure validates the fact that as SNR decreases, the error in the guess of γ plays a negligible role in the gain in MSE.

The LS solution $(\widehat{\boldsymbol{\gamma}}_1)$ allows for negative estimates of $\boldsymbol{\gamma}$. In practice, quadratic programming should be used to solve for $\boldsymbol{\gamma}$ when $\boldsymbol{\gamma} \geq 0$. In the single scatterer case, the constrained solution $(\gamma \geq 0)$ can be written as

$$\widetilde{\gamma}_1 = \widehat{\gamma}_1 \ \text{I}(\widehat{\gamma}_1 \geq 0). \tag{27}$$

Then the MSE of the constrained one-step estimator MSE_{1c} can be computed as

$$\begin{aligned} \widetilde{\gamma}_1 - \gamma &= (\widehat{\gamma}_1 - \gamma)\ \text{I}(\widehat{\gamma}_1 \geq 0) - \gamma\text{I}(\widehat{\gamma}_1 < 0) \\ \text{MSE}_{1c} &= \text{E}\left[(\widetilde{\gamma}_1 - \gamma)^2\right] \\ &= \text{E}\left[(\widehat{\gamma}_1 - \gamma)^2\ \text{I}(\widehat{\gamma}_1 \geq 0)\right] + \gamma^2\text{E}\left[\text{I}(\widehat{\gamma}_1 < 0)\right]. \end{aligned}$$

When the number of sample observations N is large enough, we get

$$\begin{aligned} \text{MSE}_{1c} &= \text{MSE}_1\text{E}\left[\text{n}_1^2\text{I}\left(\text{n}_1 > \frac{-\gamma}{\sqrt{\text{MSE}_1}}\right)\right] \\ &\quad + \gamma^2\text{E}\left[\text{I}\left(\text{n}_1 \leq \frac{\gamma}{\sqrt{\text{MSE}_1}}\right)\right] \\ &= \text{MSE}_1\left\{\text{Q}(-\sqrt{\text{s}}) - \frac{\sqrt{\text{s}}}{\sqrt{2\pi}}e^{-\frac{\text{s}}{2}} + \text{s}\ \text{Q}(\sqrt{\text{s}})\right\}, \end{aligned}$$

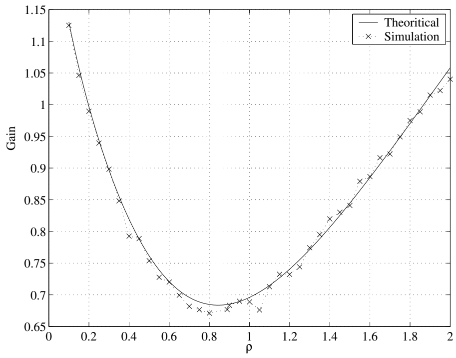
where $\text{s} = \frac{\gamma^2}{\text{MSE}_1}$. Using the same type of two-step design applied for the unconstrained case, we can show that the gain in this constrained case for low SNR is

$$\frac{\text{MSE}_{2c}(\widetilde{\gamma}_1)}{\text{MSE}_{1c}(\widetilde{\gamma}_1)} \approx 0.1263 \tag{28}$$

and all the above discussions regarding the optimal solution and the guess of γ approach can be directly extended to this constrained optimization.

4. CONCLUSIONS AND FUTURE WORK

The problem of imaging a Rayleigh scattering medium using an array of sensors rises in many applications. We obtained the MSE for the LS solution to the scatterer reflection powers. For a two-step sequential design, we found the optimal transmission scheme that minimizes the MSE and proved that we can gain over conventional one-step strategies. The gains in MSE obtained analytically are verified through simulations. We also extended the results to the constrained optimization case. Future work involves extending these results to multiple scatterers. We also intend to solve the problem of optimizing the transmitted spatial waveform rather than just looking at the energy allocation. In addition, we need to generalize this approach from a two-step method to an iterative sequence of measurements.



$(\frac{\text{MSE}_2}{\text{MSE}_1})$ vs. ρ for α_{opt}

Fig. 3. Gain(α_{opt}) vs. ρ. at SNR $= -10$dB.

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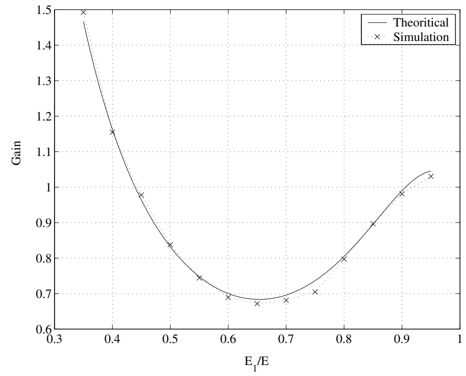
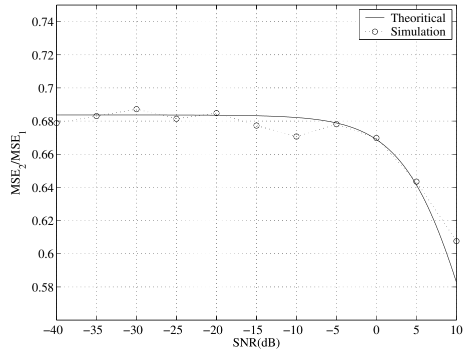


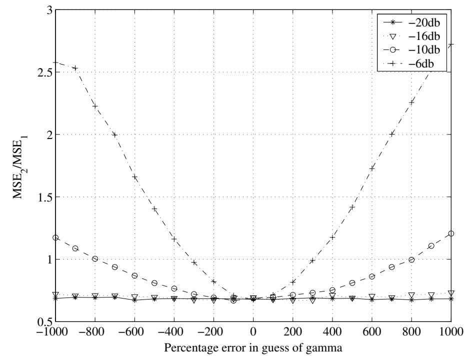
$\left(\frac{\mathrm{MSE}_2}{\mathrm{MSE}_1}\right)$ vs. α for ρ_{opt}

Fig. 4. Gain(ρ_{opt}) vs. α. at SNR $= -10$dB.



$\frac{\mathrm{MSE}_2}{\mathrm{MSE}_1}$ vs. SNR

Fig. 5. Gain vs. SNR for low SNR optimal $\alpha_{\mathrm{opt}} \approx 0.66$ and $\rho_{\mathrm{opt}} \approx 0.8885$.



$\frac{\mathrm{MSE}_2}{\mathrm{MSE}_1}$ vs. $\left(\frac{\gamma_g - \gamma}{\gamma}\right) 100$

Fig. 6. Gain vs. Percentage error in guess of γ_1 for varying SNR.

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