Face Enumeration Using Generalized Binomial Coefficients

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Abstract

We introduce a new encoding of the face numbers of a simplicial complex, the s-polynomial, obtained by multiplying each face number with an appropriate generalized binomial coefficient. We prove that the face numbers of the barycentric subdivision of the free join of two CW-complexes may be found by multiplying the s-polynomials of the barycentric subdivisions of the original complexes. We show that the s-polynomial of the order complex of any simplicial poset and of certain graded planar posets has non-negative coefficients. By calculating the s-polynomial of the barycentric subdivision of n-cube in two ways, we provide a combinatorial proof for the following identity of Bernoulli polynomials:

$$1+2^n\sum_{k=1}^{n}\frac{\binom{n}{k-1}}{k}\cdot\left(B_k\left(x+\frac{1}{2}\right)-B_k\left(\frac{1}{2}\right)\right)=(2x+1)^n.$$

Introduction

Relations between the face numbers of a cell complex are often shown using polynomial substitutions into the polynomial obtained by multiplying each face number with the appropriate power of a variable. An excellent example of this tradition is the use of the h-vector of a simplicial complex [16, Chapter II, Section 2]) which allows to express the famous Upper Bound Theorem [16, Chapter II, Section 3]), limiting the face numbers of a simplicial sphere of fixed dimension and with a fixed number of vertices, or the Dehn-Sommerville equations [16, Chapter II, Section 6]) in their most compact form.

In this paper we suggest to have a look at the *s-polynomial* $\sum_{k=0}^{d} s_k\cdot x^k := \sum_{j=1}^{d} f_{j-1}\binom{x}{j}$ (the choice of the notation reflects the implied role of the Stirling numbers) also encoding the face numbers

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f_{j-1} ($j = 0, \ldots, d$) of a simplicial complex, but in a different way. The original motivation to study the s-polynomial is that, in terms of this invariant, the effect of taking the diamond product of two graded posets on the face numbers of their order complexes may be expressed as a simple polynomial multiplication. (This is shown in Section 3.) It was already known by Kalai [14] that the diamond product of the face lattices of two polytopes is the face lattice of their Cartesian product. Ehrenborg and Readdy [8] were the first to describe the effect of taking a diamond product in terms of the cd-index. It was recently observed by the present author [12] that the dual of this operation describes the effect of taking the free join of two CW-spheres on their face posets. Since the free join of two spheres is a sphere and, by famous conjecture of Stanley [18, Conjecture 2.1], CW-spheres should have a positive cd-index, taking the dual diamond product should preserve the positivity of the cd-index in this case. It was shown by Ehrenborg and Fox [7] that the diamond product (and thus also its dual) of Eulerian posets *always* preserves the non-negativity of the cd-index of its arguments, independently of the original face structures. Unfortunately the explicit formulas they derive seem to be difficult to use in concrete calculations.

As a first step to a more explicit understanding of the effect of the diamond product on the flag numbers one may want to return to studying the face numbers of the order complexes involved, and that's where the s-polynomial or of the order complex (which we refer to as the s-polynomial or s-vector of the graded poset) may be helpful. Since the original context is a search for positivity results on flag invariants, the question naturally arises: which graded partially ordered sets have an non-negative s-vector? We find some surprising partial answers to this question.

The first surprise is that for an arbitrary graded simplicial poset (independently of the underlying topological structure), the s-polynomial has non-negative coefficients: in fact it simply encodes the face numbers of the underlying CW-complex. There is also a reasonably large class of graded planar posets whose s-polynomials have non-negative coefficients. It was first observed by the present author and L. Billera [4] that the most natural enumeration of the facets in the order complex of a graded planar poset is analogous in many ways to a shelling. Surprisingly, in the study of this invariant inspired by a truly geometric operation we find a familiar "shelling-like" argument in the world of planar analogues, while in the "truly geometric sense" the underlying topology seems to play little role. (Both non-negativity results are shown in Section 4). It turns out that the face lattice of every 3-dimensional polytope has an s-polynomial with non-negative coefficients, and it is not easy yet possible to find counterexamples among 4-dimensional polytopes. The fact that one needs to invoke some of the latest findings on the flag numbers of 4-polytopes to find a counterexample to s-positivity might indicate that the class of polytopes whose face lattice has a non-negative s-vector might be a large and interesting one. In fact, this class is shown to contain all simplicial and simple polytopes and it is closed under taking the Cartesian product of polytopes.

Although even some 4-dimensional cubical polytopes occur among the counterexamples to the non-negativity of all coefficients of the s-polynomial, further study of the s-polynomial of cubical posets in general seems to be worthwhile since they may be expressed in terms of their face numbers and the mysterious Bernoulli polynomials. A deeper understanding of the properties of the s-polynomial and the structure of cubical posets might lead to combinatorial proofs of identities for the Bernoulli polynomials. As a first example, in Section 5 we derive such an identity by calculating the s-polynomial of an n-cube in two different ways. We hope that many more examples connecting the study of s-polynomials to that of of Bernoulli polynomials will be found in the future, and in the concluding remarks we indicate the reason why this may be the case not only for order complexes of cubical posets.

1 Preliminaries

A *simplicial complex* $\triangle$ is a family of subsets of a finite vertex set V such that $\{v\} \in \triangle$ for all $v \in V$ and $\triangle$ is closed under taking subsets. The elements of $\triangle$ are called *faces*. The dimension of a face σ is $|\sigma| - 1$, the number of faces of dimension j is denoted by f_j. The dimension of the complex $\triangle$ is the maximum dimension of its faces. For a $(d-1)$-dimensional simplicial complex, the vector $(f_{-1}, f_0, \ldots, f_{d-1})$ is the *f-vector* of $\triangle$.

In this paper we focus on simplicial complexes that arise as order complexes of graded partially ordered sets. Given any poset P, its *order complex* $\triangle(P)$ is the simplicial complex whose vertices are the elements of P, and whose faces are the increasing chains. A poset is *graded* if it has a unique minimum element $\widehat{0}$, a unique maximum element $\widehat{1}$, and a rank function ρ. For a graded poset of rank $n+1$, the order complex $\triangle(P \setminus \{\widehat{0}, \widehat{1}\})$ is *completely balanced*, i.e., its vertices may be colored with $\dim(\triangle(P \setminus \{\widehat{0}, \widehat{1}\})) - 1 = n$ colors in such a way that no two vertices of the same color belong to the same face. We color each vertex with its rank for this purpose. The number of faces whose vertices are colored with the set $S \subseteq \{1, \ldots, n\}$ is denoted by f_S, the vector $(f_S : S \subseteq \{1, \ldots, n)$ is called the *flag f-vector* of the poset.

A graded poset is *Eulerian* if every interval $[x, y]$ of positive rank in it satisfies $\sum_{x \leq z \leq y} (-1)^{\rho(z)} = 0$. All linear relations holding for the flag f-vector of an arbitrary Eulerian poset of rank $n+1$ were determined by Bayer and Billera [2].

Theorem 1.1 (Bayer and Billera) *For every Eulerian poset of rank $n+1$, every subset $S \subseteq [1, n]$, and every maximal interval $[i, \ell]$ of $[1, n] \setminus S$,*

$$\left((-1)^{i-1} + (-1)^{\ell+1}\right) f_S(P) + \sum_{j=i}^{\ell} (-1)^j f_{S \cup \{j\}}(P) = 0.$$

Furthermore, every linear equality holding for the flag vector of all Eulerian posets of rank $n+1$ is a consequence of these equations.

These linear relations were rephrased by J. Fine (see Bayer and Klapper [3]) by stating the existence of the *cd*-index (which will be not defined, nor used in this paper). Fine also conjectured that the *cd*-index of the face poset of a polytope has non-negative coefficients, this result was shown by Stanley [18, Corollary 2.2]. Stanley's proves this result by introducing the notion of spherical shellings for spherical CW-complexes, and showing that the boundary of a polytope is spherically shellable. Stanley also has a conjecture [18, Conjecture 2.1], according to which all Gorenstein* posets have a non-negative *cd*-index.

The poset-operation that inspired the present work is the *diamond product* $P \diamond Q = (P \setminus \{\widehat{0}_P\}) \times (Q \setminus \{\widehat{0}_Q\}) \cup \{\widehat{0}\}$ of two graded posets P and Q. Here $P \times Q$ is the direct product $\{(p,q) : p \in P, q \in Q\}$ with the partial order $(p,q) \leq (p',q')$ iff $p \leq_P p'$ and $q \leq_Q q'$. If both P and Q are Eulerian posets, so is their diamond product. The effect of this product on the *cd*-index of Eulerian posets was first studied by Ehrenborg and Readdy [8]. Ehrenborg and Fox have shown [7] that $P \diamond Q$ has a non-negative *cd*-index, whenever the same holds for P and Q. As observed by the present author [12, Corollary 1.3], taking the *free join* of two CW-spheres Ω and Ω' induces taking the *dual diamond product* at the level of face posets (where all face posets are augmented in question with a maximum element $\widehat{1}$). For a definition of the free join of two CW-complexes see May [15, Chapter 10, Section 2]. The term "dual diamond product" to indicate $P \diamond^* Q = (P \setminus \{\widehat{1}_P\}) \times (Q \setminus \{\widehat{1}_Q\}) \cup \{\widehat{1}\}$ was first used by Ehrenborg and Readdy [10].

The proof of the Ehrenborg-Fox result (which is directly applicable to the dual diamond product as well) relies on presenting some explicit formulas connecting the *cd*-indices (and thus flag f-vectors) of the posets involved. It seems very hard to use these formulas in explicit computations, and it seems equally hard to find simpler formulas connecting the flag f-vectors. The present work focuses on an invariant that makes at least the calculation of the (non-flag) f-vector of a diamond product easier (see Section 3).

2 The s-vector of a simplicial complex

Definition 2.1 *We define the* s-polynomial *of a* $(d-1)$*-dimensional simplicial complex in terms of its* f*-vector* $(f_{-1}, \ldots, f_{d-1})$ *by the formula*

$$\sum_{k=0}^{d} s_k \cdot x^k = \sum_{j=0}^{d} f_{j-1} \cdot \binom{x}{j},$$

We refer to the vector $(s_0, \ldots, s_d)$ *as the* s-vector *of the simplicial complex.*

As usual, the generalized binomial coefficient $\binom{x}{j}$ stands for the polynomial

$$\binom{x}{j} = \frac{(x)_j}{j!} = \frac{x(x-1)\cdots(x-j+1)}{j!}.$$

The term s-vector is motivated by the fact that the f-vector may be expressed in terms of the s-vector using multiples of the Stirling numbers of the first kind. In fact, using the well-known formula

$$(x)_n = \sum_{k=0}^{n} s(n,k) \cdot x^k,$$

we may write

$$s_k = \sum_{j=k}^{d} f_{j-1} \cdot \frac{s(j,k)}{j!}. \tag{1}$$

As a consequence of the well-known equation

$$x^n = \sum_{k=0}^{n} S(n,k) \cdot (x)_k \quad \text{we may write} \quad x^n = \sum_{k=0}^{n} S(n,k) \cdot k! \cdot \binom{x}{k}.$$

Here the numbers $S(n,k)$ are the Stirling numbers of the second kind. Using this formula we may express the f-vector in terms of the s-vector as follows:

$$f_{j-1} = j! \sum_{k=j}^{d} S(k,j) \cdot s_k. \tag{2}$$

In other words, the s-vector, just like the h-vector (see Stanley [16, Chapter II, Section 2]), is an equivalent encoding of the f-vector.

3 Face numbers in a (dual) diamond product of graded posets

In this section we describe the effect of taking the diamond product of two graded posets on the face numbers of their order complexes. It should be noted that the order complex $\triangle(P \setminus \{\widehat{0}, \widehat{1}\})$ of a graded poset P is identifiable with the order complex $\triangle(P^* \setminus \{\widehat{0}, \widehat{1}\})$ associated to the dual poset P^*, obtained by reversing the partial order. Taking the dual changes the rank function, and thus the flag f-vector, but there is no change at the level of (non-flag) f-vectors. To simplify our notation and language we make the following definition.

Definition 3.1 *Given a graded poset P, we call the s-vector resp. s-polynomial of $\triangle\left(P \setminus \{\widehat{0}, \widehat{1}\}\right)$ the s-vector resp. s-polynomial of P. We use the notation $s_P(x)$ to denote the s-polynomial of P.*

Our main result is the following.

Theorem 3.2 *Any two graded posets P and Q satisfy*

$$s_{P \diamond Q}(x) = s_{P \diamond^* Q}(x) = s_P(x) \cdot s_Q(x).$$

Proof: Let us compute the f-vector of $\triangle(P \diamond^* Q \setminus \{0, \widehat{1}\})$ (the proof for the diamond product is identical). To obtain $f_{j-1} := f_{j-1}\left(\triangle(P \diamond^* Q \setminus \{0, \widehat{1}\})\right)$ we must enumerate all increasing chains of the form $\widehat{0} < (p_1, q_1) < \cdots < (p_j, q_j) < \widehat{1}$ such that $\{p_1, \ldots, p_j\} \subseteq P \setminus \{\widehat{1}_P\}$, $\{q_1, \ldots, q_j\} \subseteq Q \setminus \{\widehat{1}_Q\}$ and $(p_1, q_1) \neq (\widehat{0}_P, \widehat{0}_Q) = \widehat{0}$. Let us fix the sets $A := \{p_1, \ldots, p_j\} \setminus \{\widehat{0}_P\}$ and $B := \{q_1, \ldots, q_j\} \setminus \{\widehat{0}_Q\}$ and enumerate all chains for which A and B are the fixed sets. Starting from $(p_0, q_0) := (\widehat{0}_P, \widehat{0}_Q)$, each time we move one step up from $(p_{\alpha-1}, q_{\alpha-1})$ to (p_α, q_α) (where $\alpha = 1, 2, \ldots, j$) we must either change the first coordinate to the next larger element of A, or the second coordinate to the next larger element of B, or increase both coordinates to the next larger available element. Assuming $|A| = i$ and $|B| = k$, there must be exactly $j - k$ steps when only the first coordinate increases, $j - i$ steps when only the second coordinate increases, and $i + k - j$ steps when both coordinates increase. Thus we obtain

$$f_{j-1} = \sum_{\substack{i,k \leq j \\ i+k \geq j}} \binom{j}{j-i,\, j-k,\, i+k-j} f_{i-1}\left(\triangle\left(P \setminus \{\widehat{0}, \widehat{1}\}\right)\right) f_{k-1}\left(\triangle\left(Q \setminus \{\widehat{0}, \widehat{1}\}\right)\right).$$

Multiplying both sides with $\binom{x}{j}$ and summing over j yields

$$\sum_j f_{j-1}\binom{x}{j} = \sum_j \sum_{\substack{i,k \leq j \\ i+k \geq j}} \binom{j}{j-i,\, j-k,\, i+k-j} f_{i-1}\left(\triangle\left(P \setminus \{\widehat{0}, \widehat{1}\}\right)\right) f_{k-1}\left(\triangle\left(Q \setminus \{\widehat{0}, \widehat{1}\}\right)\right)\binom{x}{j}.$$

For a fixed pair (i, k) on the right hand side the product of the appropriate face numbers is multiplied with

$$\sum_{j=\max(i,k)}^{i+k} \binom{j}{j-i,\, j-k,\, i+k-j}\binom{x}{j} \tag{3}$$

which we claim to be equal to $\binom{x}{i} \cdot \binom{x}{k}$. In fact, w.l.o.g. we may assume that x is a sufficiently large positive integer, since two polynomials from $\mathbb{R}[x]$ are equal, if they agree for infinitely many substitutions into x. Then the sum (3) counts the number of pairs of subsets (Y, Z) of an x-element set X such that $|Y| = i$ and $|Z| = k$: introducing $j := |Y \cup Z|$ there are $\binom{x}{j}$ ways to choose $Y \cup Z$, which then needs to be partitioned into a $(j-k)$-element set $Y \setminus Z$, an $(i-k)$-element set $Z \setminus Y$ and a $(k+i-j)$-element set $Y \cap Z$. On the other hand we may choose Y and Z independently, giving $\binom{x}{i} \cdot \binom{x}{k}$ as stated. Therefore we may write

$$\begin{aligned}
\sum_j f_{j-1}\binom{x}{j} &= \sum_j \sum_{\substack{i,k \leq j \\ i+k \geq j}} f_{i-1}\left(\triangle\left(P \setminus \{\widehat{0}, \widehat{1}\}\right)\right) f_{k-1}\left(\triangle\left(Q \setminus \{\widehat{0}, \widehat{1}\}\right)\right)\binom{x}{i}\binom{x}{k} \\
&= \sum_i f_{i-1}\left(\triangle\left(P \setminus \{\widehat{0}, \widehat{1}\}\right)\right)\binom{x}{i} \cdot \sum_k f_{k-1}\left(\triangle\left(Q \setminus \{\widehat{0}, \widehat{1}\}\right)\right)\binom{x}{k},
\end{aligned}$$

and the statement of the theorem follows. $\square$

An important consequence of this theorem is that the class of graded posets with non-negative s-vectors is closed under taking the (dual) diamond product.

Corollary 3.3 *If the s-vectors of the graded posets P and Q are non-negative then the same holds for $P \diamond Q$ and $P \diamond^* Q$.*

4 Graded posets with non-negative s-vectors

In this section we present two large classes (and one "small" class) of graded posets consisting only of graded posets with non-negative s-vectors. We also provide examples of graded posets whose s-polynomial does have a negative coefficient. These examples help us avoid making false generalizations.

The first important class of graded posets with a non-negative s-vector is the class of *simplicial posets*, where a surprisingly simple connection exists.

Definition 4.1 *A graded poset P is* simplicial *if for every $x \in P \setminus \widehat{1}$, the interval $[\widehat{0}, x]$ is a Boolean algebra.*

The surprising coincidence is that for simplicial posets the entries of the s-vector are not only positive but also have a very natural meaning.

Theorem 4.2 *Every simplicial poset P of rank $n+1$ satisfies*

$$s_k\left(\triangle\left(P \setminus \{\widehat{0}, \widehat{1}\}\right)\right) = f_{\{k\}}(P) \quad \text{for } k = 1, 2, \ldots, n \text{ and } \quad s_0\left(\triangle\left(P \setminus \{\widehat{0}, \widehat{1}\}\right)\right) = 1 = f_{\emptyset}(P).$$

Here $f_S(P)$ is the appropriate entry from the flag f-vector.

Proof: For $j \geq 1$ we have

$$f_{j-1}\left(\triangle\left(P \setminus \{\widehat{0}, \widehat{1}\}\right)\right) = \sum_{k=j}^{n} f_{\{k\}}(P) \cdot S(k, j) \cdot j!$$

In fact, when we choose a j-element increasing chain $p_1 < \cdots < p_j$ in $P \setminus \{0, \widehat{1}\}$, there are $f_{\{k\}}(P)$ ways to choose its top element p_j at rank k ($k = j, j+1, \ldots, n$) and then there are $S(k, j) \cdot j!$ ways to choose a $(j-1)$-element increasing chain in the open interval $(\widehat{0}, p_j)$. (Since $[\widehat{0}, p_j]$ is a Boolean algebra of rank k, choosing a $(j-1)$-element chain in $(\widehat{0}, p_j)$ is equivalent to choosing an ordered set-partition of a k-element set into j-parts). Obviously, we also have

$$f_{-1}\left(\triangle\left(P \setminus \{\widehat{0}, \widehat{1}\}\right)\right) = 1 = f_{\emptyset}(P),$$

thus the invertible matrix that sends the vector $(f_\emptyset(P), f_{\{1\}}(P), \ldots, f_{\{n\}}(P))$ into the f-vector of $\triangle\left(P \setminus \{\widehat{0}, \widehat{1}\}\right)$ is the same as the matrix sending the s-vector of $\triangle\left(P \setminus \{\widehat{0}, \widehat{1}\}\right)$ into its f-vector. $\square$

The next large class of posets is a subclass of graded planar posets.

Definition 4.3 *A graded poset P is* planar *if its Hasse diagram may be drawn in the plane with non-crossing edges, such that whenever $q \in P$ covers $p \in P$, the vertex representing q is above the vertex representing P. We call a graded planar poset P* barely branching *if for all $p \in P \setminus \{\widehat{0}, \widehat{1}\}$ at least one of the intervals $[\widehat{0}, p]$ and $[p, \widehat{1}]$ is a chain.*

Example 4.4 A barely branching graded planar poset is shown in Fig. 1.

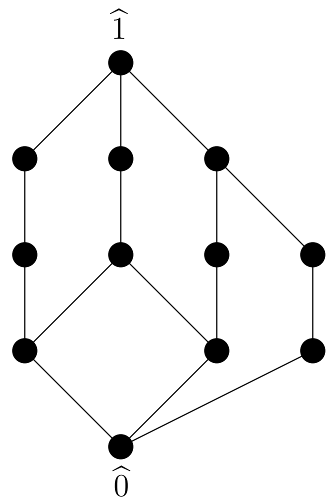

Figure 1: A barely branching graded planar poset

Theorem 4.5 *The s-polynomial of a barely branching graded poset P has non-negative coefficients.*

Proof: Assume P has rank $n+1$, and let us draw in the plane and enumerate its saturated chains just like in [4]. This means that for each fixed $p \in P$ we number the edges (p, q) in the Hasse diagram left to right, and then we enumerate the saturated chains in lexicographic order of the labels. It was shown in [4] that under these circumstances we may associate a set S to each saturated chain C in such a way that a subset of C appears first in C if an only if the set of ranks T of the subset has a nonempty intersection with every interval of $\mathcal{I}[S]$. Here $\mathcal{I}[S]$ is the unique family of intervals $\{[a_1, b_1], \ldots, [a_k, b_k]\}$ satisfying $S = \cup_{i=1}^k [a_i, b_i]$ and $a_{i+1} \geq b_i + 2$ for $i \leq k-1$. The set S associated to the saturated chain

$$C : p_0 := \widehat{0} \prec p_1 \prec p_2 \prec \cdots \prec p_n \prec p_{n+1} := \widehat{1}$$

is determined as follows. Let $\mathcal{I}_C$ consist of all intervals $[u,v]$ for which there is a saturated chain $p_{u-1} \prec p'_u \prec p'_{u+1} \prec \cdots \prec p'_v \prec p_{v+1}$ in $[p_{u-1}, p_{v+1}]$ such that the label of the cover relation (p_{u-1}, p'_u) is less than the label of (p_{u-1}, p_u). (Equivalently, the saturated chain

$$C' : \widehat{0} \prec p_1 \prec \cdots \prec p_{u-1} \prec p'_u \prec \cdots p'_v \prec p_v \prec \cdots p_n \prec \widehat{1}$$

precedes C in the lexicographic order.) By the results in [4] that a subset of C does not occur in any preceding saturated chain if and only of the set T of ranks of its elements has a nonempty intersection with every element of $\mathcal{I}_C$. Obviously we get the same condition if we require only $T \cap I \neq \emptyset$ for the minimal intervals I in $\mathcal{I}_C$. Finally, it was also shown in [4] that, for a graded planar poset, the family of minimal intervals of $\mathcal{I}_C$ is always of the form $\mathcal{I}[S]$ for some $S \subseteq \{1, \ldots, n\}$. (If the union of two minimal intervals of $\mathcal{I}_C$ is an interval then some edges are bound to cross in the Hasse diagram.)

We claim that the barely branching property implies that in the above enumeration every family of intervals $\mathcal{I}_C$ is either empty or has a unique minimum interval. In fact, if for some saturated chain

$$C : p_0 := \widehat{0} \prec p_1 \prec p_2 \prec \cdots \prec p_n \prec p_{n+1} := \widehat{1}$$

the family $\mathcal{I}_C$ has at least two minimal intervals $[a_1, b_1]$ and $[a_2, b_2]$ then by what was said above we may assume $a_2 \geq b_1 + 2$. Consider now the element $p := p_{b_1+1}$ of rank $b_1 + 1$ in C. Since $[a_1, b_1] \in \mathcal{I}_C$ implies the existence of at least two saturated chains in $[p_{a_1-1}, p_{b_1+1}] \subseteq [\widehat{0}, p]$, and $[a_2, b_2] \in \mathcal{I}_C$ implies the existence of at least two saturated chains in $[p_{a_2-1}, p_{b_2+1}] \subseteq [p, \widehat{1}]$, the definition of a barely branching planar poset is violated by p.

It is thus sufficient to show that every saturated chain C for which $\mathcal{I}_C$ is empty or has a unique minimal interval increases the s-polynomial. Since for any graded poset P we have

$$f_{j-1}\left(\triangle\left(P \setminus \{\widehat{0}, \widehat{1}\}\right)\right) = \sum_{\substack{T \subseteq \{1,\ldots,n\} \\ |T| = j}} f_T(P) \tag{4}$$

in the case $\mathcal{I}_C = \emptyset$ we need to show that the polynomial

$$\sum_{T \subseteq \{1,\ldots,n\}} \binom{x}{|T|} = \sum_{j=0}^{n} \binom{n}{j} \cdot \binom{x}{j} = \sum_{j=0}^{n} \binom{n}{n-j} \cdot \binom{x}{j} = \binom{x+n}{n}$$

has non-negative coefficients. This is obviously true. Similarly, in the case when the unique minimal interval of $\mathcal{I}_C$ is the interval I we need to show that the polynomial

$$\sum_{\substack{T \subseteq \{1,\ldots,n\} \\ T \cap I \neq \emptyset}} \binom{x}{|T|}$$

has non-negative coefficients. Introducing $k := |I|$ we may write

$$\sum_{\substack{T\subseteq\{1,\ldots,n\}\\ T\cap I\neq\emptyset}} \binom{x}{|T|} = \sum_{T\subseteq\{1,\ldots,n\}} \binom{x}{|T|} - \sum_{\substack{T\subseteq\{1,\ldots,n\}\\ T\cap I=\emptyset}} \binom{x}{|T|} = \binom{x+n}{n} - \sum_{j=0}^{n-k} \binom{n-k}{j}\binom{x-k}{j}$$

$$= \binom{x+n}{n} - \sum_{j=0}^{n-k} \binom{n-k}{n-k-j}\binom{x-k}{j} = \binom{x+n}{n} - \binom{x+n-k}{n-k}.$$

Now our Theorem follows from the fact that the polynomial

$$\binom{x+n}{n} - \binom{x+n-k}{n-k} = \frac{(x+n-k)\cdots(x+1)}{(n-k)!}\cdot\frac{(x+n)\cdots(x+n-k+1) - n\cdots(n-k+1)}{n\cdots(n-k+1)}$$

has non-negative coefficients. (The subtracted product in the numerator of the second factor is the constant term of the first polynomial in the same numerator.) $\square$

Remark 4.6 In [4] the number p_S is introduced for the number of saturated chains satisfying $\mathcal{I}_C = \mathcal{I}[S]$. The resulting vector $(p_S : S \subseteq \{1,\ldots,n\})$ is a planar analogue of the flag h-vector (whose definition may be also found in [4] but was widely used before). In terms of this flag p-vector the barely branching condition implies $p_S = 0$ whenever $|\mathcal{I}[S]| \geq 2$.

Remark 4.7 Consider an arbitrary graded planar poset of rank 4. The contribution of a saturated chain C satisfying $\mathcal{I}_C = \mathcal{I}[\{1,3\}]$ to the s-polynomial is $\binom{x}{2} + \binom{x}{3} = x^3/6 - x/6$ since the only subsets of $\{1,2,3\}$ containing $\{1,3\}$ are $\{1,3\}$ itself and $\{1,2,3\}$. The a sequence of planer posets $P_N := P(3,\mathcal{I}[\{1,3\}],N)$, as defined in [4, Definition 4], has the property

$$\lim_{N\to\infty}\frac{1}{N^2}p_S(P_N) = \delta_{\{1,3\},S}$$

where $\delta_{\{1,3\},S}$ is the Kronecker delta symbol. Thus, for a sufficiently large N, the coefficient of x in the s-polynomial of the graded planar poset P_N must be negative.

Our last example is a "smaller" class of graded posets, giving rise to a "false hope".

Proposition 4.8 *The face lattice of any* 3*-dimensional convex polytope has a non-negative* s*-vector.*

Proof: Using (4) we may rewrite the s-polynomial of an arbitrary graded poset P of rank 4 in terms of its flag f-vector as follows.

$$s_P(x) = 1 + \left(f_{\{1\}} + f_{\{2\}} + f_{\{3\}}\right)\cdot\binom{x}{1} + \left(f_{\{1,2\}} + f_{\{2,3\}} + f_{\{1,3\}}\right)\cdot\binom{x}{2} + f_{\{1,2,3\}}\cdot\binom{x}{3}.$$

The face lattice of a convex polytope is Eulerian, hence we may use the Bayer-Billera relations from Theorem 1.1, and replace each of $f_{\{1,2\}}$, $f_{\{2,3\}}$, and $f_{\{1,3\}}$ with $f_{\{1,2,3\}}/2 = 2f_{\{2\}}$, the number $f_{\{1,2,3\}}$ with $4f_{\{2\}}$, and the sum $f_{\{1\}} + f_{\{3\}}$ with $f_{\{2\}} + 2$. Thus we obtain

$$\begin{aligned} s_P(x) &= 1 + \left(2f_{\{2\}} + 2\right) \cdot \binom{x}{1} + 6f_{\{2\}} \cdot \binom{x}{2} + 4f_{\{2\}} \cdot \binom{x}{3} \\ &= 1 + \left(\frac{1}{3}f_{\{2\}} + 2\right) \cdot x + f_{\{2\}} \cdot x^2 + \frac{4}{6}f_{\{2\}} \cdot x^3, \end{aligned}$$

a polynomial with positive coefficients. □

The "false hope" arising from Proposition 4.8 is that perhaps the face lattice of any convex polytope has a non-negative s-vector. This turns out to be false even for 4-dimensional polytopes. In fact, any Eulerian poset P of rank 5, we may use (4) and the Bayer-Billera relations from Theorem 1.1 to express the f-vector $\triangle(P \setminus \{\widehat{0}, \widehat{1}\})$ in terms of the flag f-vector of P as follows. (To alleviate our notation, we write f_S instead of $f_S(P)$.)

$$\begin{aligned} f_0\left(\triangle\left(P \setminus \{\widehat{0}, \widehat{1}\}\right)\right) &= f_{\{1\}} + f_{\{2\}} + f_{\{3\}} + f_{\{4\}} \\ &= 2f_{\{1\}} + 2f_{\{3\}}, \\ f_1\left(\triangle\left(P \setminus \{\widehat{0}, \widehat{1}\}\right)\right) &= f_{\{1,2\}} + f_{\{1,3\}} + f_{\{1,4\}} + f_{\{2,3\}} + f_{\{2,4\}} + f_{\{3,4\}} \\ &= f_{\{1,2\}} + f_{\{1,3\}} + \left(f_{\{1,3\}} - f_{\{1,2\}} + 2f_{\{1\}}\right) + f_{\{1,3\}} + 2f_{\{3\}} \\ &= 2f_{\{1\}} + 2f_{\{3\}} + 3f_{\{1,3\}}, \\ f_2\left(\triangle\left(P \setminus \{\widehat{0}, \widehat{1}\}\right)\right) &= f_{\{1,2,3\}} + f_{\{1,2,4\}} + f_{\{1,3,4\}} + f_{\{2,3,4\}} = 8 \cdot f_{\{1,3\}}, \\ f_3\left(\triangle\left(P \setminus \{\widehat{0}, \widehat{1}\}\right)\right) &= f_{\{1,2,3,4\}} = 4 \cdot f_{\{1,3\}}. \end{aligned}$$

As a consequence, the polynomial $s_P(x)$ may be written as

$$s_P(x) = 1 + \left(2f_{\{1\}} + 2f_{\{3\}}\right) \cdot \binom{x}{1} + \left(2f_{\{1\}} + 2f_{\{3\}} + 3f_{\{1,3\}}\right)\binom{x}{2} + 8 \cdot f_{\{1,3\}}\binom{x}{3} + 4 \cdot f_{\{1,3\}}\binom{x}{4}.$$

Collecting terms yields

$$s_P(x) = 1 + \left(f_{\{1\}} + f_{\{3\}} + \frac{f_{\{1,3\}}}{6}\right) \cdot x + \left(f_{\{1\}} + f_{\{3\}} - \frac{2 \cdot f_{\{1,3\}}}{3}\right) \cdot x^2 + \frac{f_{\{1,3\}}}{3} \cdot x^3 + \frac{f_{\{1,3\}}}{6} \cdot x^4.$$

Corollary 4.9 *An Eulerian poset of rank* 5 *has a positive* s*-polynomial if and only if its flag* f*-vector satisfies* $2f_{\{1,3\}} \leq 3f_{\{1\}} + 3f_{\{3\}}$.

If P is the face lattice of a 4-polytope then $f_{\{1\}}$ stands here for the number of vertices (elements of rank 1), and $f_{\{3\}}$ for the number of 2-faces. In most of the the literature on 4-polytopes the dimensions (rather than the ranks) of the faces are used to index the flag f-vector, so the condition of Corollary 4.9 would be written as

$$2f_{02} \leq 3f_0 + 3f_2. \tag{5}$$

This inequality is not satisfied by all 4-polytopes, at least two known classes of counterexamples exists.

Example 4.10 In [19] G. M. Ziegler constructs polytopes $P(n, r)$ as projected products of polygons, whose flag f-vector (indexed by dimensions) satisfies

$$(f_0, f_1, f_2, f_3; f_{03}) = (4n, 4rn, 5rn - 6 + 4r, rn - 2 + 4r; 16rn - 16n) \cdot \frac{1}{4}n^{r-1}$$

For large values of n and r, the vector $(f_0, f_1, f_2, f_3; f_{03})$ is approximately equal to $(0, 4, 5, 1; 16) \cdot \frac{rn^r}{4}$, and so we have

$$f_{02} = f_{03} + 2f_1 - 2f_0 \sim (16 + 2 \cdot 4 - 2 \cdot 0) \cdot \frac{rn^r}{4} = 24 \cdot \frac{rn^r}{4}$$

while $3f_0+3f_2$ is only about $15\frac{rn^r}{4}$. (I thank Prof. Ziegler for communicating this example in response to my question.)

The other set of counterexamples consists of cubical polytopes, hence there is no cubical analogue of Theorem 4.2.

Definition 4.11 *A graded poset P is a* cubical poset *if for every $p \in P \setminus \{\widehat{1}\}$ the interval $[\widehat{0}, p]$ is isomorphic to the face lattice of a cube. A convex polytope is cubical if its face lattice is a cubical poset.*

Let us note that cubical polytopes satisfy $f_{02} = 4f_2$ and so (5) is equivalent to

$$f_2 \leq 3f_0. \tag{6}$$

Example 4.12 M. Joswig and G. M. Ziegler have shown [13] that neighborly cubical polytopes exists: for any $n \geq d \geq 2r + 2$ there is a cubical convex d-polytope, whose r-skeleton is combinatorially equivalent to that of the n-dimensional cube. For $d = 4$ the neighborly cubical polytopes exhibit the fact that the facet-vertex ratio f_3/f_0 is not bounded. Using the Bayer-Billera relations from Theorem 1.1 we may rewrite (6) as $f_1 + f_3 - f_0 \leq 3f_0$ or, equivalently,

$$f_1/f_0 + f_3/f_0 \leq 4.$$

This inequality is violated by any neighborly cubical polytope satisfying $f_3/f_0 > 0$.

It is worth noting that, although general s-positivity fails for 4-dimensional polytopes, there seems to be no easier way to find a counterexample than using some of the most recent and highly non-trivial constructions. The neighborly cubical polytopes mentioned in Example 4.12 actually disproved an existing conjecture on the face numbers of a cubical polytope, while the polytopes mentioned in Example 4.10 were constructed with the purpose to get close to the suspected boundary of the cone of flag f-vectors of 4-polytopes. The class of polytopes with a non-negative s-polynomials is reasonably large since, as a consequence of Theorem 4.2 and the invariance under taking the dual noted at the beginning of Section 3, we have the following.

Corollary 4.13 *The face lattice of a simplicial or simple polytope has a non-negative s-vector.*

As it is underscored by Ehrenborg and Fox [7], taking the Cartesian product of two polytopes induces taking the diamond product of their face lattices. (At this point, Ehrenborg and Fox cite Kalai [14] who uses the term *lower truncated product* instead of diamond product and *upper truncated product* instead of the dual diamond product.) Thus, Theorem 3.2 has the following corollary.

Corollary 4.14 *The class of polytopes having a non-negative s-vector is closed under taking the Cartesian product.*

5 Cubical posets and an identity for Bernoulli polynomials

As noted before Example 4.12, the s-polynomial of a *cubical poset* does not necessarily have non-negative coefficients. Nevertheless, computing the s-polynomial of a cubical poset may be achieved by using a more general method, which may serve in the future as a tool to prove the non-negativity of the s-vector of some other classes of graded posets. This method consists of calculating the s-vector of an arbitrary graded poset P from the s-vectors of some of its proper intervals.

Lemma 5.1 *The s-polynomial $s_P(x)$ of any graded poset P satisfies*

$$s_P(x+1) - s_P(x) = \sum_{p \in P \setminus \{\widehat{0},\widehat{1}\}} s_{[\widehat{0},p]}(x).$$

Proof: As noted in the proof of Theorem 4.2, the j-element chains in $P \setminus \{\widehat{0},\widehat{1}\}$ may be enumerated by first choosing the top element p of the chain, and then choosing a $(j-1)$-element chain in the open interval $(\widehat{0},p)$. Thus we have

$$s_P(x) = 1 + \sum_{j=1}^{n} \sum_{p \in P \setminus \{\widehat{0},\widehat{1}\}} f_{j-2}\left(\triangle\left(\widehat{0},p\right)\right) \cdot \binom{x}{j}.$$

Substituting $x+1$ for x in the above equation and taking the difference yields

$$s_P(x+1) - s_P(x) = \sum_{j=1}^{n} \sum_{p \in P \setminus \{\widehat{0},\widehat{1}\}} f_{j-2}\left(\triangle\left((\widehat{0},p)\right)\right) \cdot \left(\binom{x+1}{j} - \binom{x}{j}\right).$$

As a consequence of the well-known identity

$$\binom{x+1}{j} - \binom{x}{j} = \binom{x}{j-1}$$

the polynomial associated to each p is the s-polynomial of $[\widehat{0}, p]$. $\square$

Corollary 5.2 *For any positive integer x we have*

$$s_P(x) = 1 + \sum_{p \in P \setminus \{\widehat{0}, \widehat{1}\}} \sum_{y=0}^{x-1} s_{[\widehat{0},p]}(y).$$

In fact, we have $s_P(x) = s_P(0) + \sum_{y=0}^{x-1}(s_P(y+1) - s_P(y))$, so the Corollary follows from Lemma 5.1 and the obvious identity $s_P(0) = 1$.

Corollary 5.2 is especially useful when we consider a class of posets for which every interval of the form $[\widehat{0}, p]$ belongs to a smaller class, as it is the case for simplicial or cubical posets. If this smaller class is selected in such a way that even[1] the polynomial $\sum_{y=0}^{x-1} s_{[\widehat{0},p]}(y)$ has non-negative coefficients then the s-polynomial of the entire poset has also non-negative coefficients. This is stronger non-negativity is true for Boolean algebras, but false for the face lattices of cubes.

In order to obtain an explicit formula for the s-polynomial of a cubical poset, we now calculate the s-polynomial of a cube.

Proposition 5.3 *The s-polynomial of the face lattice of a d-dimensional cube is $(2x+1)^d$.*

Proof: As noted at the beginning of Section 3, the s-polynomial of a graded poset is equal to the s-polynomial of its dual. The dual of the face lattice of a d-cube is the face lattice of a d-dimensional cross-polytope. This latter poset is simplicial of rank $d+1$, the number of its faces of rank k is $\binom{d}{k} \cdot 2^k$. Using Theorem 4.2 we obtain that the s-polynomial is $1 + \sum_{k=1}^{d} \binom{d}{k} 2^k x^k = (2x+1)^d$. $\square$

As a consequence of Corollary 5.2 and Proposition 5.3 we obtain the following formula for the s-polynomial of a cubical poset P of rank $n+1$. For a positive integer x, we must have

$$s_P(x) = 1 + \sum_{k=1}^{n} f_{\{k\}} \cdot \sum_{y=0}^{x-1} (2y+1)^{k-1}. \tag{7}$$

This formula may be simplified using Bernoulli polynomials.

[1]The use of the word "even" is justified as follows. If $\psi(x)$ is a polynomial then the function $\phi(x) = \sum_{y=0}^{x-1} \psi(y)$ is a polynomial function, determined up to its constant term. The polynomial $\phi(x)$ may be equivalently defined by the equation $\phi(x+1) - \phi(x) = \psi(x)$. Clearly, if $\phi(x)$ has non-negative coefficients, the same holds for $\psi(x)$.

Definition 5.4 *The Bernoulli polynomials $B_n(x)$ $(n = 0, 1, \ldots)$ are given by the generating function*

$$\frac{t \cdot e^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \cdot \frac{t^n}{n!}.$$

It is well-known that the Bernoulli polynomials may be used to express the sum of fixed powers of consecutive integers. For any positive integer m and n we have

$$\sum_{k=1}^{m} k^n = \frac{B_{n+1}(m+1) - B_{n+1}(0)}{n+1}. \tag{8}$$

See Abramowitz and Stegun [1, 23.1.2 and 23.1.4]. Hence it is not surprising that Bernoulli polynomials may be also used to express the sum of fixed powers of consecutive odd integers.

Lemma 5.5 *For any positive integer x and k we have*

$$\sum_{y=0}^{x-1} (2y+1)^{k-1} = \frac{2^{k-1}}{k} \cdot \left(B_k \left(x + \frac{1}{2} \right) - B_k \left(\frac{1}{2} \right) \right).$$

This Lemma is easy to derive directly, and it is also a special case of the following more general formula of G. Dattoli, S. Lorenzutta, and C. Cesarano [6, Formula (2.3)]:

$$\sum_{n=0}^{N-1} (x + ny)^r = \frac{y^r}{r+1} \cdot \left(B_{r+1} \left(N + \frac{x}{y} \right) - B_{r+1} \left(\frac{x}{y} \right) \right) \quad \text{if } y \neq 0. \tag{9}$$

Using Lemma 5.5 we may rewrite equation (7) in a more compact form and summarize our findings in the following Theorem.

Theorem 5.6 *The s-polynomial of a cubical poset of rank $n+1$ is given by*

$$s_P(x) = 1 + \sum_{k=1}^{n} f_{\{k\}} \cdot \frac{2^{k-1}}{k} \cdot \left(B_k \left(x + \frac{1}{2} \right) - B_k \left(\frac{1}{2} \right) \right).$$

Note that equation (7) was only stated for positive integer values of x while in Theorem 5.6 we do not make such restriction. As usual, this may be justified by the fact that two polynomials from $\mathbb{R}[x]$ are equal if they yield the same value for infinitely many substitutions.

Theorem 5.6 provides a second alternative to compute the s-polynomial of the face lattice of a cube. Thus, together with Proposition 5.3, it yields the following identity for Bernoulli polynomials:

Corollary 5.7 *The Bernoulli polynomials satisfy*

$$1 + 2^n \sum_{k=1}^{n} \frac{\binom{n}{k-1}}{k} \cdot \left(B_k \left(x + \frac{1}{2} \right) - B_k \left(\frac{1}{2} \right) \right) = (2x+1)^n \quad \textit{for all } n \geq 0.$$

In fact, the number of $(k-1)$-dimensional (=rank k) faces in an n-cube is $f_{\{k\}} = \binom{n}{k-1} \cdot 2^{n+1-k}$, and multiplying this number with $2^{k-1}/k$ yields a factor of $\binom{n}{k-1} \cdot 2^n/k$.

6 Concluding remarks

Reasons to continue exploring the s-polynomials of face lattices of polytopes are presented at the end of Section 4. Corollary 5.2, together with (9) indicates that Bernoulli polynomials are likely to play an important role in s-polynomial calculations of graded posets, not only in the cubical case. Finally, let us remind the reader that another way to calculate the a sum of powers $1^r + \cdots + n^r$ is by means of the Eulerian numbers $A_{r,i}$ expressing the number of permutations of r numbers with i descents. The formula may be found by multiplying both sides of the well-known identity

$$\sum_{k=0}^{\infty} k^r \cdot t^k = \frac{\sum_{i=0}^{r-1} A_{r,i} t^{i+1}}{(1-t)^{r+1}}$$

(see, e.g. [5, Section 6.5]) by $(1-t)^{-1}$. This approach may also yield some interesting formulas for cubical posets, since the Eulerian numbers have an interpretation as volumes of certain slices of a hypercube (see Stanley [17] and the generalization by Ehrenborg, Readdy and Steingrímsson [11]). Since even among cubical polytopes some do not have a non-negative s-vector, one would need to find a special property that would guarantee s-positivity and see whether in that special setting the use of Eulerian numbers and their interpretations as volumes is helpful. We leave this question as a challenge to the reader.

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References

[1] M. Abramowitz and I. A. Stegun, "Handbook of Mathematical Functions," National Bureau of Standards, Washington, D.C., issued 1964, Tenth Printing, 1972, with corrections.

[2] M. M. Bayer and L. J. Billera, Generalized Dehn–Sommerville relations for polytopes, spheres and Eulerian partially ordered sets, *Invent. Math.* **79** (1985) 143–157.

[3] M. Bayer and A. Klapper, A new index for polytopes, *Discrete Comput. Geom.* **6** (1991) 33–47.

[4] L. J. Billera and G. Hetyei, Decompositions of partially ordered sets, *Order* **17** (2000) 141–166.

[5] L. Comtet, "Advanced Combinatorics", D. Reidel Publishing Co. Dodrecht, Holland, 1974.

[6] G. Dattoli, S. Lorenzutta, and C. Cesarano, Finite sum and generalized forms of Bernoulli polynomials, *Rend. Mat. Appl. (7)* **19** (1999) 385–391.

[7] R. Ehrenborg and H. Fox, Inequalities for cd-indices of joins and products of polytopes, *Combinatorica* **23** (2003), 427–452.

[8] R. Ehrenborg and M. Readdy, Coproducts and the cd-index, J. Algebraic Combin. **8** (1998), 273–299.

[9] R. Ehrenborg and M. Readdy, Homology of Newtonian Coalgebras, *European J. of Combin.* **23** (2002), 919–927.

[10] R. Ehrenborg and M. Readdy, The Tchebyshev Transforms of the First and Second Kind, manuscript in preparation.

[11] R. Ehrenborg, M. Readdy, and E. Steingrímsson, Mixed volumes and slices of the cube, *J. Combin. Theory Ser. A* **81** (1998), 121–126.

[12] G. Hetyei, Polyspherical complexes, preprint 2004, `http://www.math.uncc.edu/preprint/2004/2004_10.pdf`

[13] M. Joswig and G. M. Ziegler, Neighborly Cubical Polytopes, *Discrete Comput. Geom.* **24** (2000), 325–444.

[14] G. Kalai A new basis for polytopes, *J. Combin. Theory Ser. A* **49** (1988) 191–209.

[15] J. P. May, "A concise course in algebraic topology," The University of Chicago Press, Chicago and London, 1999.

[16] R. P. Stanley, "Combinatorics and Commutative Algebra," second ed., Birkhäuser Boston, 1996.

[17] R. P. Stanley, Eulerian partitions of a unit hypercube, in: "Higher Combinatorics, Proceedings of the NATO Advanced Study Institute, Berlin, West-Germany, September 1-10, 1976" (M. Aigner, Ed.), p. 49, Reidel, Dordrecht/Boston, 1977.

[18] R. P. Stanley, Flag f-vectors and the cd-index, *Math. Z.* **216** (1994) 483–499.

[19] G. M. Ziegler, Projected Products of Polygons, *Electronic Research Announcements AMS*, **10** (2004) 122–134.