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Physical Premium Principle: A New Way for Insurance Pricing

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Abstract: In our previous work we suggested a way for computing the non-life insurance premium. The probable surplus of the insurer company assumed to be distributed according to the canonical ensemble theory. The Esscher premium principle appeared as its special case. The difference between our method and traditional principles for premium calculation was shown by simulation. Here we construct a theoretical foundation for the main assumption in our method, in this respect we present a new (physical) definition for the economic equilibrium. This approach let us to apply the maximum entropy principle in the economic systems. We also extend our method to deal with the problem of premium calculation for correlated risk categories. Like the Buhlman economic premium principle our method considers the effect of the market on the premium but in a different way.

Keywords: Economic Equilibrium; Statistical Equilibrium; Physical Entropy; Premium Calculation Principles; Economic premium principle.

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1 Introduction

Methods developed in physics are widely used for modeling and data analysis of the financial market. This approach to quantitative economy was adopted by economists in nineteen century and the beginning of the twentieth century, however it was only during the last decade that physicists turned their attention to problems of such nature [1]. Among all branches of physics, the statistical mechanics appears as the most suitable context for studying the dynamics of complex systems such as financial market [2, 3]. There are many cases which demonstrate the power of statistical mechanics in exploring dynamics of the financial market. Recently Sornette and Zhou [4] show the advantage of applying the principles of statistical mechanics to price prediction in the US market.

The study of insurance with the aid of ideas borrowed from statistical mechanics was begun by the work of author [5, 6, 7, 8]. In this paper we try to elucidate the theoretical foundation of our previous work and explain how this approach can be used for pricing the insurance. In what follows we first describe the equilibrium in the economic systems from physical viewpoint then we suggest a new way for calculation of the insurance premium. Like the economic model of Bühlmann [9, 10] the role of market is taken into the account when premium is assigned to a category of risks. The Esscher principle also appears as a special case of this method.

2 Economic Systems in Equilibrium

In the neo classic economic model, competition between demand and supply forces determines the price dynamics in a financial market. When there is no excess demand in the market the price becomes stationary in the time. This situation is nominated as equilibrium by the economists [11]. Analysis of time series of prices is a way to test equilibrium condition in the market empirically. Last two decade incredible amounts of data have been gathered from the financial markets and to date we have no evidence for stable equilibrium in any markets in the sense of the economists viewpoint (Walrasian equilibrium) [12, 13]. In the following we describe a new way to interpret the equilibrium in the economic systems then we use it to obtain the wealth distribution in the market. The similar way was previously adopted by D. Foley in his maximum entropy exchange equilibrium theory [14, 15, 16, 17, 18]. Although we have not found any convincing interpretation yet for the entropy, its relation to the price and its dependence on the quantity of assets in the real market, but we hope find them by mining the financial data.

The conservative exchange market can be considered as a large number of economic agents which are interacting with each other through buying and selling. In this market productive activities don't exist therefore in each trading only money is exchanged . We consider the behavior of one of the agents for example an insurance company; all other agents may be regarded as its environment. The exchange assumption for the market means, the environment absorbs the money that the agent loses and will supply the agent's incomes.

$$w_a + w_e = W_m = Const. \tag{1}$$

The quantities w_a,w_e and W_m are the wealth (income) of agent, its environment and total money in the market respectively.

In a given duration, the agent has many ways to acquire amounts of money as a result of random loses and incomes in its trading. The quantity $\Gamma_a(w_a)$ represents number of these ways. The environment has also $\Gamma_e(w_e)$ ways to possess amounts of money as its wealth. The number of trading ways for agent/environment is correlated positively with his wealth. The state of the market is given by two quantities w_a andw_e, the market has $\Gamma_m(w_a, w_e)$ ways of reaching this specified state. Clearly,

$$\Gamma_m(w_a, w_e) = \Gamma_a(w_a)\Gamma_e(w_e). \tag{2}$$

Our common sense tells us, at any time the market chooses any one of these ways with equal probability because no reason exists for preferring some of them. The physical entropy up to a constant factor may be defined as logarithm of available ways for the market in a given state;

$$S(w_a) = \ln \Gamma_m(w_a, W_m - w_a e). \tag{3}$$

By definition, in equilibrium state the market entropy reach to its maximum value [19]. This means, the agent and its environment have maximum options for buying or selling in equilibrium state.

$$\frac{\partial S(w_a)}{\partial w_a} = 0. \tag{4}$$

The agent's wealth in the equilibrium state, W_a, is obtained by solving the above equation. Combination of the eqs. 4 and 2 lead us to the following equality.

$$\left(\frac{\partial \ln \Gamma_a(w_a)}{\partial w_a}\right)_{w_a=W_a} = \left(\frac{\partial \ln \Gamma_e(w_e)}{\partial w_e}\right)_{w_e=W_e}. \tag{5}$$

We use $\partial w_e/\partial w_a = -1$ in the derivation of above equality.

The value of parameter $\partial \ln \Gamma(w)/\partial w$ is denoted by the symbol β, whence the condition for equilibrium becomes,

$$\beta_a = \beta_e. \tag{6}$$

The above condition is similar to the zero law of thermodynamics which states two systems in thermal contact will have the same temperature when they reach to the equilibrium. It is easily understood the parameter β is analogous to inverse temperature in thermal systems.

The equality in eq. 6 is useless unless the parameter $\beta_a(\beta_e)$ can be expressed in terms of measurable quantities of the agent (environment). For an insurance company we express this parameter in terms of the initial wealth of company, the mean claim size and the ultimate ruin probability.

The above approach is different from general economic equilibrium theory. We use the statistical (physical) interpretation for the equilibrium in the economic systems instead of its Walrasian (mechanical) picture. The latter description is inadequate to explain some real features of the market [20, 13] but the former one may elucidate some of them [21, 14, 15, 16].

3 The Canonical Ensemble Theory in Economics

What is the probability that an agent possesses the specified amount $W_a^{(r)}$ when it is in equilibrium with its environment? From basic probability theory we know it should be directly proportional to the number of possible ways corresponding with the market state.

$$Pr(W_a^{(r)}) = \frac{\Gamma_a(W_a^{(r)})\Gamma_e(W_e^{(r)})}{\int_0^{W_m} \Gamma_a(w)\Gamma_e(W_m - w)dw}. \quad (7)$$

The environment is supposed to have much money in comparison to the agent wealth,

$$\frac{W_a^{(r)}}{W_m} \ll 1. \quad (8)$$

It is clear that $\Gamma_e(W_e^{(r)})$ is also much larger than $\Gamma_a(W_a^{(r)})$ hence the eq. 7 up to a constant coefficient; can be approximated as,

$$Pr(W_a^{(r)}) \approx \Gamma_e(W_m - W_a^{(r)}). \quad (9)$$

We can expand the logarithm of above equation around the value .

$$\begin{aligned} \ln Pr(W_a^{(r)}) &= \ln \Gamma_e(W_m - W_a^{(r)}) && (10) \\ &= \ln \Gamma_e(W_m) + \left(\frac{\partial \ln \Gamma_e(w)}{\partial w}\right)_{w=W_m} \left(-W_a^{(r)}\right) + \cdots \\ &\approx \ln \Gamma_e(W_m) - \beta W_a^{(r)}. && (11) \end{aligned}$$

The first term in right hand side of the eq.10 is a constant number and the second term is product of the parameter β and the agent wealth. The eq. 8 insures that other terms in the above expansion are small with respect to these leading terms. By a simple algebraic manipulation we obtain the desired result.

$$Pr(W_a^{(r)}) = \frac{e^{-\beta W_a^{(r)}}}{\sum_r e^{-\beta W_a^{(r)}}}. \quad (12)$$

The accessible equilibrium states of the market make an ensemble of possible values for the agent wealth. The index r indicates members of this ensemble. This is what the physicists called the canonical ensemble.

Eq. 12 was proposed theoretically in ref. [22] and confirmed by simulation [22, 23] and by empirical data [24, 25, 26].

4 The Insurance Pricing

The insurance is a contract between insurer and policy holder. Any happening loss incurred on the insured party over a specified period of time, T, is covered by the insurer, in return for an amount of money received as premium. The wealth of insurer at the end of this period is,

$$W(t) = U + S(T). \quad (13)$$

Where U is the insurer initial wealth and $S(T)$ shows his total surplus when the policy duration is over.

The insurance policy may include different types of the loss events; like as fire, car, natural hazards and so on. Each loss category has its own premium. The holder should pay premiums for those risk categories that are covered by his insurance policy and the insurer company also compensates the loss for holder upon his claim. The difference between the received premium and the payment for the claims makes the category surplus. The total surplus is resulting from summation of all categories surplus.

$$\begin{aligned} S(T) &= \sum_{\alpha} S_{\alpha}(T) \\ &= \sum_{\alpha}(p_{\alpha} I_{\alpha}(T) - \sum_{j=1}^{N_{\alpha}(T)} X_j^{\alpha}). \end{aligned} \tag{14}$$

The summation goes over different categories. $I_{\alpha}(T)$ is number of the issued policies in the α category . It is also equal to the number of policy holders in this category. These people should pay premium, p_{α}, to the insurer. They have legal right to claim an appropriate relief, X_j^{α}, in accordance with their policies. The number of these claims is symbolized by $N_{\alpha}(T)$. The quantities, $I_{\alpha}(T)$,$N_{\alpha}(T)$ and X_j^{α} are random variables.

The probability for acquiring the surplus $S(T)$ by the insurer can be derived regarding to eq. 12.

$$Pr(S(T)) = \frac{e^{-\beta S(T)}}{\sum e^{-\beta S(T)}}. \tag{15}$$

The summation goes over all possible values for surplus. The parameter β is positive to ensure that extreme values for surplus have small probability. Unlike the traditional premium principles, the number of issued policies in eq. 13 is a random variable and indicates the competition in the market. It may be decreased due to an increment in the premium and will be increased when the insurer reduces his prices. In the case of constant number of policy holders we obtain the result similar to what the Bhlmann [9, 10] arrived at in his economic model for insurance pricing.

$$Pr(Z(T)) = \frac{e^{\beta Z(T)}}{\sum e^{\beta Z(T)}}. \tag{16}$$

The $Z(T)$ is called the aggregate loss and for a specified period is defined as,

$$Z(T) = \sum_{\alpha} \sum_{j=1}^{N_{\alpha}(T)} X_j^{\alpha}. \tag{17}$$

The insurer naturally aims at maximizing its profit, hence its surplus, when the insurance contract is over, should be positive or zero at least. This condition may be expressed mathematically only as an average form.

$$< S(T) >= \frac{\sum S(T) e^{-\beta S(T)}}{\sum e^{-\beta S(T)}} = 0. \tag{18}$$

For practical purpose it is better to consider the constrained form of the above equation, In this respect we assume the surplus vanishes for all categories.

$$< S_\alpha(T) >= \frac{\sum S_\alpha(T) e^{-\beta S(T)}}{\sum e^{-\beta S(T)}} = 0. \tag{19}$$

When there is no correlation between the number of issued policies, number and size of the claims in different categories, the eq. is reduced to [9],

$$< S_\alpha(T) >= \frac{\sum S_\alpha(T) e^{-\beta S_\alpha(T)}}{\sum e^{-\beta S_\alpha(T)}} = 0. \tag{20}$$

In the above equation sum over all possible values for surplus in the equilibrium state is understood.

The set of equations like as the eq. 4 may be solved numerically to compute the premiums for all categories. We need first to determine the value of the β parameter for such calculation. We will come back to this matter subsequently. In special case when the number of policy holders is constant, the above equation changes to the Esscher principle for premium calculation [27].

$$p_\alpha = \frac{1}{I_\alpha} \frac{\sum Z_\alpha(T) e^{\beta Z_\alpha(T)}}{\sum e^{\beta Z_\alpha(T)}}. \tag{21}$$

The Esscher principle(transform) has tremendous success not only in the insurance context but also in the asset pricing [28].

If the parameter β vanishes, premium will be equal to the mean value of the loss events per number of the policy holders, in such case it is nominated as the net premium [27].

$$p_{0\alpha} = \frac{1}{I_\alpha} E(Z_\alpha(T)). \tag{22}$$

In the following we restrict ourselves to a specified category of insurance policies for simplicity. We assume this category is uncorrelated to others. Henceforth the unnecessary subscripts will be dropped.

The net premium is a limiting value. If an insurer supplies his insurance with a price less than the net premium it will be certainly come to ruin after a finite time. The ruin is probable for other prices. The ultimate ruin probability, ε, shows the probability of the ruin for an insurer in an infinite time interval. It depends on the ratio of the premium to the net premium, the initial wealth of the insurer and also on statistics of the claims (their number and size) [27, 29].

If the number of claims during the insurance contract has Poisson distribution and the claims size distributed exponentially then the ultimate ruin probability is [29],

$$\varepsilon = \frac{p_0}{p} \exp\left[-\frac{U}{\mu}\left(\frac{p}{p_0} - 1\right)\right]. \tag{23}$$

The value of parameter β is necessary for premium calculation. By a simple dimensional analysis one can establish that it must be proportional to the inverse of the contracts duration T and the initial wealth of insurer. Through eqs. 4 (or 21) and 23 we can also relate it to the ruin probability. Regarding to this fact we can determined the β parameter in terms of the initial wealth, U, the mean claim size, μ and the ultimate ruin probability, ε.

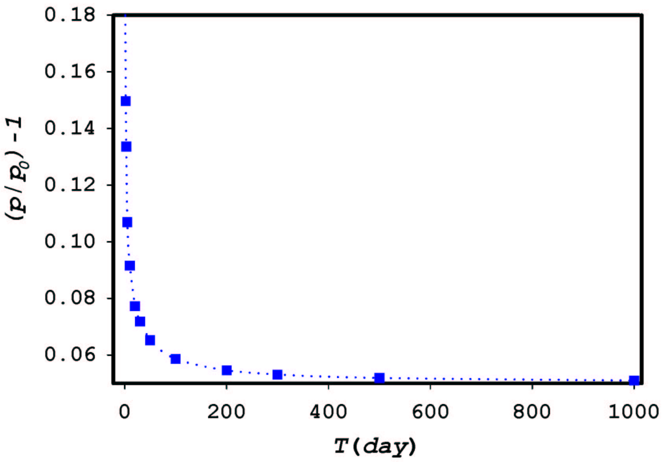

Figure 1: The loading parameter, $(p/p_0) - 1$, versus the contract duration for large β parameter. This figure shows the dependence of the premium on the period of insurance contract. It justifies our experience in trading. The premium of a risk category is not change linearly with the contract duration. The long term contract is more advantageous than some short term contraction. The loading parameter is used to get rid of the monetary unit.

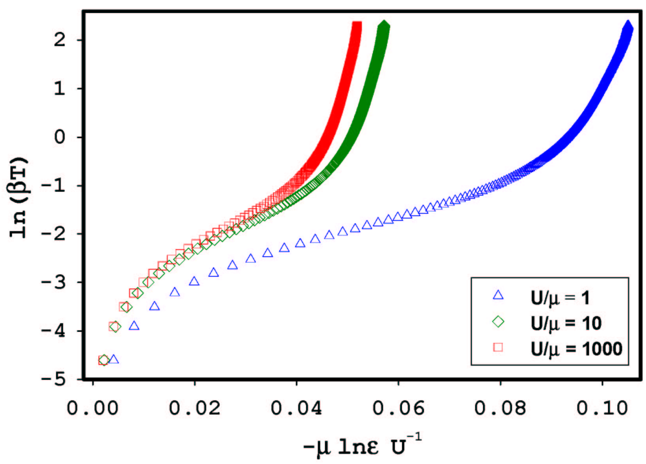

Figure 2: Dependence of the $\ln \beta T$ on the ruin probability, the initial wealth and the mean claim size. The β parameter for apparently is very small for the wealthier insurer. Such a company may offer the insurance with a low price.

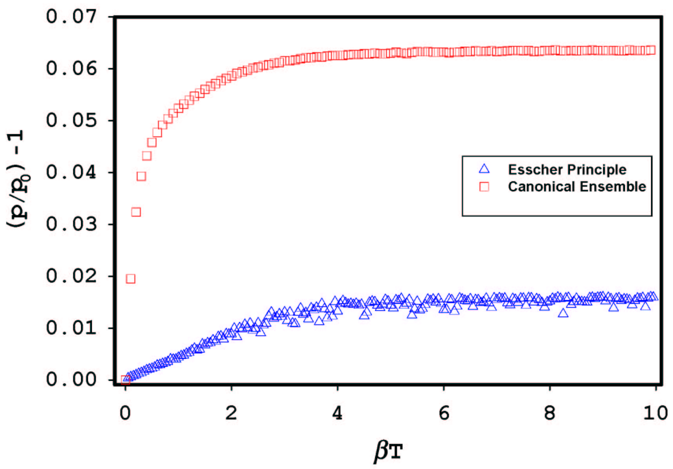

Figure 3: The loading parameter, $(p/p_0) - 1$, versus the, βT, parameter. The squares display the results of the canonical ensemble theory and the triangles correspond to the Esscher premium principle. The difference between two curves is the result of the market effects.

In the following we present the simulation results for a special case of car insurance. The distribution of the random variables , X_j, $N(T)$, $I(T)$ are extracted from the reports of Iran Central Insurance Company for the year 2000 [30]. The claims size, is exponentially distributed. The distribution of the time interval between the claims has also exponential form. This shows the number of claims during the insurance contract, $N(T)$, has Poisson distribution [29]. We have a poor statistics about the number of issued policies but analysis of data from recent years shows it is distributed uniformly.

The relation between the loading parameter, $(p/p_0) - 1$, and the contract duration, T, is plotted in Fig. 1. The loading parameter is used instead of premium to eliminate dependence of the premium on the monetary unit. It displays what we expected in real cases. Our usual trading experience tell us that a long term contract is more advantageous than some short term ones for a given period of time.

As we already mentioned, if we specify the ruin probability, the initial wealth and the mean claim size the β parameter will be determined. Fig. 2 displays the relation between these parameters. The initial wealth is measured in terms of the mean claim size to avoid working with the monetary unit. The variation in number of the policy holders influences the premium. Fig.3 is plotted to show this effect, the value of the premium which is obtained from Esscher formula is less than that we found in our method.

5 Conclusion

Apart from traditional picture of the economic equilibrium which deal with the quarrel between supply and demand, we introduce a new definition for the equilibrium in an economic system based on the statistical mechanics. The wealth distribution takes exponential form in a conservative exchange market. Empirical evidence regarding to this fact has been reported by Yakovenko and his colleagues. We use the same distribution for the surplus of an insurance company then suggested a method for premium calculation. Unlike to other premium principles we let the number of the policy holders to change randomly. The role of the surrounding market is involved in our method by this consumption. Bülmann previously had considered the market effect by a different method in his economic premium principle. Here we retrieve his result and also find the Esscher principle, the well known method in actuary, as a special case in our method.

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