Non-deterministic algebraic structures for soft computing

I. P. Cabrera, P. Cordero, and M. Ojeda-Aciego

Dept. Matemática Aplicada, Universidad de Málaga. Spain*
{ipcabrera,pcordero,aciego}@uma.es

Abstract. The need of considering non-determinism in theoretical computer science has been claimed by several authors in the literature. The notion of non-deterministic automata as a formal model of computation is widely used, but the specific study of non-determinism is useful, for instance, for natural language processing, in describing interactive systems, for characterizing the flexibility allowed in the design of a circuit or a network, etc. The most suitable structures for constituting the foundation of this theoretical model of computation are non-deterministic algebras. The interest on these generalized algebras has been growing in recent years, both from a crisp and a fuzzy standpoint. This paper presents a survey of these structures in order to foster its applicability for the development of new soft computing techniques.

Keywords: Non-determinism, multialgebras, hyperalgebras, non-deterministic algebras

1 Hyperstructures, multistructures and nd-structures

The difficulty of handling non-determinism has been sometimes avoided by simulating it using specific algorithms on deterministic automata. Nonetheless, the need of developing a formal theory which considers non-determinism as an inherent aspect of computation, instead of merely simulating it, is widely accepted.

A usual direct approach to non-determinism is the use of multialgebras [25, 41,48], also called multivalued algebras or hyperalgebras, in which the arguments of the operations are individuals and the result is a set of possible outcomes.

Hyperalgebra, or hyperstructure theory, was introduced in [36] when Marty defined hypergroups, began to analyze their properties, and applied them to groups, rational fractions and algebraic functions. Nowadays, a number of different hyperstructures have been widely studied from both the theoretical and applicative point of view, and for their use in applied mathematics and artificial intelligence; note, however, that an important obstacle for the study of these structures is the lack of consensus in the terminology. These hyperstructures can be roughly classified as follows:

* Partially supported by Spanish Ministry of Science project TIN09-14562-C05-01 and Junta de Andalucía project P09-FQM-5233.

- *Generalizations of group theory.* The most general hyperstructure, the hypergroupoid, is just a nonempty set H endowed with a binary operation $H \times H \to \mathcal{P}(H) \smallsetminus \{\varnothing\}$. *Semihypergroups*, *quasihypergroups*, *hypergroups*, *join spaces*, etc, are different classes of hypergroupoids with different sets of additional requirements. A theoretical study of these and other structures and a wide survey of its applications can be found in [14], which describes applications to geometry, graph theory, fuzzy set theory, rough set theories, cryptography, codes, etc. Recently, several results relating hypergroups and fuzzy set theory have been obtained, see [16, 18, 19, 49].
- *Extensions of ring theory.* In this topic, the most referenced structures are hyperrings and hyperfields, which were defined by Krasner in [30, 31] and have been applied to geometry and number theory [11]. A weakening of these structures (multiring and multifield) was introduced in [33].
- *Lattice-related structures.* A number of structures are inspired in lattice theory although not all of them are proper extensions of the structure of lattice, for instance, *nearlattices* [10], *near lattices* [43], *hyperlattices* [29], or *superlattices* [38].

  Specially interesting, in this context, is the structure of multilattice (see Section 2), which provides a convenient generalization of lattices both from the algebraic and the coalgebraic points of view.

It is remarkable that most of the structures above consider that the codomain of the operations is always a nonempty set. This restriction does not suit certain applications, and that is why we introduced the non-deterministic algebras (briefly, *nd-algebras*) [34] by considering operations of type $A_1 \times \cdots \times A_n \to \mathcal{P}(A)$. Thus, a non-deterministic groupoid (or *nd-groupoid*) is just a hypergroupoid in which the restriction of the images being nonempty is dropped.

Among the applications which demand nd-operations we can find a number of them requiring partial ordered sets (*posets*) which are not lattices, but have similar properties. The notion of partially ordered set has proven to be very relevant in modern mathematics, perhaps being lattice theory one of the best examples. Note, however, that it is not difficult to find situations in which posets arise that are not lattices as, for example, in *divisibility* theory, *special relativity* theory, ... These posets, although lacking a proper lattice structure, share some of their properties.

2 Multilattices: algebraically and coalgebraically

It was Benado [3] who firstly proposed an approach to generalizing the notion of lattice in which the supremum and the infimum are replaced by the set of minimal upper bounds, named *multisupremum*, and the set of maximal lower bounds, named *multiinfimum*, respectively. This structure is called *multilattice*. Notice that the operators which compute the multi-suprema and multi-infima in a poset provide precisely nd-groupoids or, if we have for granted that at least a multi-supremum always exists, a hypergroupoid. Although other generalizations of the notion of lattice have been developed so far, see above, we are focusing our attention on multilattices because of their computational properties.

The idea underlying the algebraic study of multilattices is the development of a new theory involving *non-deterministic operators* as a framework for formalizing key notions in computer science and artificial intelligence. For instance, non-determinism has been considered under the combination of modal and temporal logics to be used in communication systems; new results have been recently obtained in database theory as well. A lot of effort is being put in this area, as one can still see recent works dealing with non-determinism both from the theoretical and from the practical point of view [27, 47].

Although Benado's original motivation was purely theoretical (he used multilattices to work with *Dedekind connections*, *Schreier's refinement* theorem and *evaluation* theory) multilattices (and relatives such as multisemilattices) have been identified in several disparate research areas: (1) in the field of automated deduction, specifically when devising a theory about implicates and implicants for certain temporal logics during the development of automated theorem provers for those logics [13]; (2) unification for logical systems, whose starting point was the existence of a most general unifier for any unifiable formula in Boolean logic: Ghilardi [22] proved that there are no most general unifiers in intuitionistic propositional calculus but a finite set of maximal general unifiers instead; and (3) multilattices play important roles in computation, for instance the set of words builded from an alphabet by considering the *"be a subword"* ordering.

As stated above, the notions of ordered and algebraic multilattice were introduced by Benado in [3]. An alternative algebraic characterization was introduced by Hansen in [24] and, later, Johnston studies ideals and distributivity on this algebras [26]. However, the first applicable algebraic characterization is relatively recent, Martínez *et al.* [34], and it reflects much better the corresponding classical theory about lattices than those given previously. Moreover, this algebraic characterization allows natural definitions of related structures such as *multisemilattices* and, in addition, is better suited for applications. For instance, [46] shows several examples in process semantics where the carrier set has the structure of multilattice, and Medina *et al.* [37] developed a general approach to fuzzy logic programming based on a multilattice as underlying set of truth-values for the logic.

Certain abstract structures can be thought of both algebraically and coalgebraically. The context and the aims of the work usually indicates which framework one should consider; for instance, when non-deterministic behavior is assumed, the coalgebraic framework is generally preferred because it appears to fit more naturally, since coalgebras are becoming an ideal framework for formalization in diverse branches of computer science (Kripke structures, labeled transition systems, various types of non-deterministic automata, etc).

Following this trend, we started a research line consisting in the development of a coalgebraic view of several mathematical structures of interest for the handling of non-determinism, in particular, for multilattices. In [8], we have defined a suitable class of coalgebras, the ND-coalgebras, and developed a thorough analysis of the required properties in order to achieve a convenient coalgebraic characterization of multilattices which complements the algebraic one given in [35]. The class of ND-coalgebras can be regarded as a collection of coalgebras underlying non-deterministic situations, and creates a setting in which many other structures could be suitably described.

3 Congruences, homomorphisms and ideals on non-deterministic structures

In traditional mathematics, congruences, homomorphisms and ideals are usually considered as different views of the same phenomenon, as stated by the so-called isomorphism theorems. Note, however, that in the realm of nd-structures there are several plausible generalizations of these notions which do not necessarily preserve the existing relationships in the classical case.

The study of congruences is important both from a theoretical standpoint and for its applications in the field of logic-based approaches to uncertainty. Regarding applications, the notion of congruence is intimately related to the foundations of fuzzy reasoning and its relationships with other logics of uncertainty [21]. More focused on the theoretical aspects of computer science, some authors [2, 40] have pointed out the relation between congruences, fuzzy automata and determinism. There have also been studies on qualitative reasoning about the morphological relation of congruence. A spatial congruence relation is introduced in [15] which, moreover, provides an algebraic structure to host relations based on it.

3.1 Crisp approaches

To begin with, a discussion on the most suitable extension of the notions of congruence and homomorphism on a given nd-structure is needed. In [6], we consider the notion of homomorphism on nd-groupoids and how it preserves the different subhyperstructures. Likewise, in this general framework, the relation between nd-homomorphisms and crisp congruences on a hyper-structure is investigated. In [4], we dealt with congruences on a hypergroupoid or nd-groupoid. Specifically, the set of congruences on an nd-groupoid need not be a lattice unless we assume some extra properties. This problem led us to review some related literature and, as a result, we found one counter-example even in the context of crisp congruences on a hypergroupoid. The previous example motivated the search for a sufficient condition which granted the structure of complete lattice for the set of congruences on a hypergroupoid and, by extension, on an nd-groupoid; this property turned out to be that the underlying nd-structure should be a certain sort of multisemilattice.

The next step in this context is to study congruence relations in the more general structure of multilattices, together with a suitable definition of homomorphism. In [12] the classical relationship between homomorphisms and congruences was suitably adapted, as well as a proof that the set of congruences of a certain class of multilattices is a complete lattice.

In a subsequent work, the focus was put on the notion of ideal. This is not a trivial matter since several definitions have been proposed for the notion of ideal of a multilattice: for instance, one can find the notion of s-ideals introduced by Rachůnek, or the l-ideals of Burgess, or the m-ideals given by Johnston [26, 42]. In [7], we introduced an alternative definition more suitable for extending the classical results about congruences and homomorphisms. This approach led to generalize the result about the lattice structure of the set of congruences to be applied to *any* multilattice.

3.2 Fuzzy approaches

The systematic generalization of crisp concepts to the fuzzy case has proven to be an important theoretical tool for the development of new methods of reasoning under uncertainty, imprecision and lack of information. One can find disparate extensions of classical algebraic structures to a fuzzy framework in the literature; moreover, recently, hyperstructures and fuzzy theory are being studied jointly, giving rise to the so-called fuzzy hyperalgebra and, consequently, several areas within artificial intelligence and soft computing have been benefitted from the new results obtained [1, 32, 44, 50, 51].

Regarding the generalization level, since the inception of fuzzy sets and fuzzy logic, there have been approaches to consider underlying sets of truth-values more general than the unit interval; for instance, consider the L-fuzzy sets introduced in [23], where L is a complete lattice. Furthermore, one can even consider the study of M-fuzzy sets where M has the structure of a multilattice.

Several papers have been published about the lattice of fuzzy congruences on different algebraic structures [17, 20, 39, 45]. A previous step before studying the fuzzy congruences on multilattices and the suitable generalizations of the concept of L-fuzzy and M-fuzzy congruence, is to define fuzzy congruence relations on nd-groupoids.

Our generalization to the context of nd-groupoids is introduced in [5], following the trend initiated in [4]. Concerning the study of the lattice structure of fuzzy congruence relations, the main result obtained is a set of conditions guaranteeing that the set of fuzzy congruences on an nd-groupoid is a complete lattice, since in general this is not always the case.

Unlike the development of the fuzzy versions of other crisp concepts in mathematics like congruence relation, the fuzzy extension of the notion of function has been studied from several standpoints, and this fact complicates the choice of the most suitable definition of fuzzy homomorphism: the most convenient definition seems to depend on particular details of the underlying algebraic structure under consideration. The definition of fuzzy function introduced in [28] is used in [6] in order to establish the relation between fuzzy congruences and perfect fuzzy homomorphisms, leading to a fuzzy version of the canonical decomposition theorem for certain class of fuzzy homomorphisms. Specifically, a given $\varphi\colon A \to B$ in this class can be decomposed[1] as $\varphi = \iota \circ \bar{\varphi} \circ \pi$ where $\pi\colon A \to A/\rho_\varphi$ is the

[1] Note that all the notions involved in the decomposition are fuzzy.

fuzzy projection from A to its quotient set over the kernel congruence relation ρ_φ induced by φ, $\bar{\varphi}\colon A/\rho_\varphi \to \operatorname{Im}\varphi$ is the induced isomorphism, and $\iota\colon \operatorname{Im}\varphi \to B$ is the inclusion.

The previous approaches are extended to the general theory of hyperrings in [9], where the theory of hyperrings and fuzzy homomorphisms between them is studied. Specifically, isomorphism theorems are established which relate fuzzy homomorphisms between hyperrings, fuzzy congruences and fuzzy hyperideals.

4 Conclusions

(Multi, hyper, nd)-algebras provide a suitable theory for the foundation of non-determinism. Although this theory originated in 1934, currently a lot of effort has been put on them, mostly due to its applicability, especially in computer science: the current trend being the fuzzy extension of hyperalgebra and its relation to soft computing.

In this work, we have reviewed a class of these structures in order to foster its applicability for the development of new soft computing techniques. Specifically, a brief survey of the most cited hyperalgebras in the literature has been presented. Then, the notion of non-deterministic algebra (nd-algebra) is introduced; this is a general notion which includes, in a common framework, algebras, partial algebras and hyperalgebras. Later, the focus is put on two important classes of nd-algebras: multisemilattices and multilattices. The importance of these structures is due to the fact that they extend the classical results about lattice theory to a wide range of partially ordered sets and they appear in several areas in theoretical computer science. The final section is devoted to the recent advances related to congruences (and its relatives, homomorphisms and ideals) on non-deterministic structures, due to its intrinsic interest both from a theoretical standpoint and for its applications in the field of logic-based approaches to uncertainty.

References

1. R. Ameri and T. Nozari. Fuzzy hyperalgebras. *Computers and Mathematics with Applications*, 61(2):149–154, 2011.
2. R. Bělohlávek. Determinism and fuzzy automata. *Information Sciences*, 143:205–209, 2002.
3. M. Benado. Les ensembles partiellement ordonnés et le théorème de raffinement de Schreier. I. *Čehoslovack. Mat. Ž.*, 4(79):105–129, 1954.
4. I. P. Cabrera, P. Cordero, G. Gutiérrez, J. Martínez, and M. Ojeda-Aciego. Congruence relations on some hyperstructures. *Annals of Mathematics and Artificial Intelligence*, 56(3–4):361–370, 2009.
5. I. P. Cabrera, P. Cordero, G. Gutiérrez, J. Martínez, and M. Ojeda-Aciego. Fuzzy congruence relations on nd-groupoids. *International Journal on Computer Mathematics*, 86:1684–1695, 2009.
6. I. P. Cabrera, P. Cordero, G. Gutiérrez, J. Martínez, and M. Ojeda-Aciego. On congruences and homomorphisms on some non-deterministic algebras. In *Proc of Intl Conf on Fuzzy Computation*, pages 59–67, 2009.

7. I. P. Cabrera, P. Cordero, G. Gutiérrez, J. Martínez, and M. Ojeda-Aciego. On congruences, ideals and homomorphisms over multilattices. pages 299–304. In EUROFUSE Workshop Preference Modelling and Decision Analysis, 2009.
8. I. P. Cabrera, P. Cordero, G. Gutiérrez, J. Martínez, and M. Ojeda-Aciego. A coalgebraic approach to non-determinism: applications to multilattices. *Information Sciences*, 180:4323–4335, 2010.
9. I. P. Cabrera, P. Cordero, G. Gutiérrez, J. Martínez, and M. Ojeda-Aciego. On fuzzy homomorphisms between hyperrings. In *XV Congreso Español Sobre Tecnologías Y Lógica Fuzzy – ESTYLF 2010*, 2010.
10. I. Chajda and M. Kolařík. Nearlattices. *Discrete Math.*, 308(21):4906–4913, 2008.
11. A. Connes and C. Consani. The hyperring of adèle classes. *J. Number Theory*, 131(2):159–194, 2011.
12. P. Cordero, G. Gutiérrez, J. Martínez, M. Ojeda-Aciego, and I. P. Cabrera. Congruence relations on multilattices. In *Intl FLINS Conference on Computational Intelligence in Decision and Control, FLINS'08*, pages 139–144, 2008.
13. P. Cordero, G. Gutiérrez, J. Martínez, and I. P. de Guzmán. A new algebraic tool for automatic theorem provers. *Annals of Mathematics and Artificial Intelligence*, 42(4):369–398, 2004.
14. P. Corsini and V. Leoreanu. *Applications of hyperstructure theory*. Kluwer, 2003.
15. M. Cristani. The complexity of reasoning about spatial congruence. *Journal of Artificial Intelligence Research*, 11:361–390, 1999.
16. I. Cristea and B. Davvaz. Atanassov's intuitionistic fuzzy grade of hypergroups. *Information Sciences*, 180(8):1506 – 1517, 2010.
17. P. Das. Lattice of fuzzy congruences in inverse semigroups. *Fuzzy Sets and Systems*, 91(3):399–408, 1997.
18. B. Davvaz, P. Corsini, and V. Leoreanu-Fotea. Fuzzy n-ary subpolygroups. *Computers & Mathematics with Applications*, 57(1):141 – 152, 2009.
19. B. Davvaz and V. Leoreanu-Fotea. Applications of interval valued fuzzy n-ary polygroups with respect to t-norms (t-conorms). *Computers & Mathematics with Applications*, 57(8):1413 – 1424, 2009.
20. T. Dutta and B. Biswas. On fuzzy congruence of a near-ring module. *Fuzzy Sets and Systems*, 112(2):399–408, 2000.
21. B. Gaines. Fuzzy reasoning and the logics of uncertainty. In *Proc. of ISMVL'76*, pages 179–188, 1976.
22. S. Ghilardi. Unification in intuitionistic logic. *The Journal of Symbolic Logic*, 64(2):859–880, 1999.
23. J. Goguen. L-fuzzy sets. *J. Math. Anal. Appl.*, 18:145–174, 1967.
24. D. J. Hansen. An axiomatic characterization of multilattices. *Discrete Math.*, 33(1):99–101, 1981.
25. W. H. Hesselink. A mathematical approach to nondeterminism in data types. *ACM Trans. Program. Lang. Syst.*, 10:87–117, January 1988.
26. I. J. Johnston. Some results involving multilattice ideals and distributivity. *Discrete Math.*, 83(1):27–35, 1990.
27. J. Khan and A. Haque. Computing with data non-determinism: Wait time management for peer-to-peer systems. *Computer Communications*, 31(3):629–642, 2008.
28. F. Klawonn. Fuzzy points, fuzzy relations and fuzzy function. In V. Novák and I. Perfilieva, editors, *Discovering World with Fuzzy Logic*, pages 431–453. Physica-Verlag, 2000.
29. M. Konstantinidou and J. Mittas. An introduction to the theory of hyperlattices. *Math. Balkanica*, 7:187–193, 1977.

30. M. Krasner. Approximation des corps values complets de caracteristique p $\neq 0$ par ceux de caracteristique 0. In *Colloque d'algebre superieure*, pages 129–206. Centre Belge de Recherches Mathematiques Etablissements, 1957.
31. M. Krasner. A class of hyperrings and hyperfields. *Internat. J. Math. & Math. Sci.*, 6(2):307–312, 1983.
32. X. Ma, J. Zhan, and V. Leoreanu-Fotea. On (fuzzy) isomorphism theorems of Γ-hyperrings. *Computers and Mathematics with Applications*, 60(9):2594–2600, 2010.
33. M. Marshall. Real reduced multirings and multifields. *Journal of Pure and Applied Algebra*, 205(2):452 – 468, 2006.
34. J. Martínez, G. Gutiérrez, I. P. de Guzmán, and P. Cordero. Generalizations of lattices via non-deterministic operators. *Discrete Math.*, 295(1-3):107–141, 2005.
35. J. Martínez, G. Gutiérrez, I. Pérez de Guzmán, and P. Cordero. Multilattices via multisemilattices. In *Topics in applied and theoretical mathematics and computer science*, pages 238–248. WSEAS, 2001.
36. F. Marty. Sur une généralisation de la notion de groupe. In *Proceedings of* 8th Congress Math. Scandinaves, pages 45–49, 1934.
37. J. Medina, M. Ojeda-Aciego, and J. Ruiz-Calviño. Fuzzy logic programming via multilattices. *Fuzzy Sets and Systems*, 158(6):674–688, 2007.
38. J. Mittas and M. Konstantinidou. Sur une nouvelle généralisation de la notion de treillis: les supertreillis et certaines de leurs propriétés générales. *Ann. Sci. Univ. Clermont-Ferrand II Math*, 25:61–83, 1989.
39. V. Murali. Fuzzy congruence relations. *Fuzzy Sets and Systems*, 41(3):359–369, 1991.
40. T. Petković. Congruences and homomorphisms of fuzzy automata. *Fuzzy Sets and Systems*, 157:444–458, 2006.
41. H. E. Pickett. Homomorphisms and subalgebras of multialgebras. *Pacific Journal of Mathematics*, 21(2):327–342, 1967.
42. J. Rachůnek. 0-idéaux des ensembles ordonnés. *Acta Univ. Palack. Fac. Rer. Natur.*, 45:77–81, 1974.
43. D. Schweigert. Near lattices. *Math. Slovaca*, 32(3):313–317, 1982.
44. K. Sun, X. Yuan, and H. Li. Fuzzy hypergroups based on fuzzy relations. *Computers and Mathematics with Applications*, 60(3):610–622, 2010.
45. Y. Tan. Fuzzy congruences on a regular semigroup. *Fuzzy Sets and Systems*, 117(3):399–408, 2001.
46. D. Vaida. Note on some order properties related to processes semantics. I. *Fund. Inform.*, 73(1-2):307–319, 2006.
47. D. Varacca and G. Winskel. Distributing probability over non-determinism. *Mathematical Structures in Computer Science*, 16(1):87–113, 2006.
48. M. Walicki and S. Meldal. A complete calculus for the multialgebraic and functional semantics of nondeterminism. *ACM Trans. Program. Lang. Syst.*, 17:366–393, 1995.
49. S. Yamak, O. Kazancı, and B. Davvaz. Applications of interval valued t-norms (t-conorms) to fuzzy n-ary sub-hypergroups. *Information Sciences*, 178(20):3957 – 3972, 2008.
50. S. Yamak, O. Kazancı, and B. Davvaz. Normal fuzzy hyperideals in hypernearrings. *Neural Computing and Applications*, 20(1):25–30, 2011.
51. Y. Yin, J. Zhan, D. Xu, and J. Wang. The L-fuzzy hypermodules. *Computers and Mathematics with Applications*, 59(2):953–963, 2010.