Notes on Subsets of $\{1, 2, ..., n\}$ that Contain No Consecutive Integers

Dennis Walsh
Middle Tennessee State University

Let $[n]$ denote the set of the first n positive integers, that is, $[n] = \{1, ..., n\}$. There are 2^n subsets of $[n]$. How many of these subsets contain no consecutive integers? In particular, how many size-k subsets contain no consecutive integers? Let $c(n, k)$ denote the number of size-k subsets that contain no consecutive integers.

Example 1. (i) If $n = 8$ and $k = 2$, the subsets are

$$\begin{array}{rrrrrr} \{1,3\}, & \{1,4\}, & \{1,5\}, & \{1,6\}, & \{1,7\}, & \{1,8\} \\ & \{2,4\}, & \{2,5\}, & \{2,6\}, & \{2,7\}, & \{2,8\} \\ & & \{3,5\}, & \{3,6\}, & \{3,7\}, & \{3,8\} \\ & & & \{4,6\}, & \{4,7\}, & \{4,8\} \\ & & & & \{5,7\}, & \{5,8\} \\ & & & & & \{6,8\} \end{array}$$

giving us a count $c(8, 2) = 1 + 2 + ... + 6 = \binom{7}{2} = 21$.

(ii) If $n = 8$ and $k = 3$, the subsets are

$$\begin{array}{l} \{1,3,5\},\ \{1,3,6\},\ \{1,3,7\},\ \{1,3,8\} \\ \{1,4,6\},\ \{1,4,7\},\ \{1,4,8\} \\ \{1,5,7\}.\ \{1,5,8\} \\ \{1,6,8\} \\ \{2,4,6\},\ \{2,4,7\},\ \{2,4,8\} \\ \{2,5,7\},\ \{2,5,8\} \\ \{2,6,8\} \\ \{3,5,7\},\ \{3,5,8\} \\ \{3,6,8\} \\ \{4,6,8\}. \end{array}$$

There are $c(8, 3) = \binom{5}{2} + \binom{4}{2} + \binom{3}{2} + \binom{2}{2} = \binom{6}{3} = 20$ subsets.

(iii). If $n = 8$ and $k = 4$ the subsets are

$$\begin{array}{l} \{1,3,5,7\},\ \{1,3,5,8\},\ \{1,3,6,8\},\ \{1,4,6,8\} \\ \{2,4,6,8\} \end{array}$$

There are $c(8, 4) = \binom{5}{4} = 5$ subsets. $\diamond$

I. Size-2 Subsets that Contain No Consecutive Integers

We first look at the case of size-2 subsets. For example, there are 6 size-2 subsets of $\{1,2,3,4,5\}$, namely, $\{1,3\},\ \{1,4\},\ \{1,5\},\ \{2,4\},\ \{2,5\},$ and $\{3,5\}$. Hence $c(5,2)=6$. To find a general formula for $c(n,2)$, we use the principle of inclusion/exclusion to obtain

$$\begin{aligned} c(n,2) &= \text{the number of all size-2 subsets} \\ &\quad - \text{the number of subsets that contain } i \text{ and } i+1 \text{ where } i \in [n-1] \\ &= \binom{n}{2} - (n-1) \\ &= \binom{n-1}{2}. \qquad (1) \end{aligned}$$

An alternative approach to formula (1) uses the number of size-2 subsets that contain no consecutive integers and that has minimum element m which we denote by $c(n,2,m)$. Clearly $c(n,2,m)=n-m-1$ since the subsets being counted are $\{m,m+2\},\{m,m+3\},\ldots,\{m,m+n-m\}$. Therefore,

$$\begin{aligned} c(n,2) &= \sum_{m=1}^{n-2} c(n,2,m) \\ &= \sum_{m=1}^{n-2} (n-m-1) \\ &= \sum_{j=1}^{n-2} j \quad \text{(upon reversing the order of summation)} \\ &= \binom{n-1}{2}. \qquad (2) \end{aligned}$$

[See OEIS integer sequence A161680, at http://oeis.org/A161680, which sequence is given by $a(n)=\binom{n}{2}$, and $\binom{n}{2}$ = number of size-2 subsets of $\{0,1,\ldots,n\}$ that contain no consecutive integers.]

Initial Values for $c(n,2)$.

n	0	1	2	3	4	5	6	7	8	9	10	11
$c(n,2)$	0	0	0	1	3	6	10	15	21	28	36	45

II. Derivation of A General Formula for $c(n,k)$, the Number of Size-k Subsets of $[n]$ that Contain No Consecutive Integers

Consider a string of symbols consisting of k A's and $(n+1-2k)$ B's. There are $\binom{n+1-k}{k}$ ways to shuffle the symbols in the string. For each resulting string, replace each B symbol in the string with the number 0. If $k=2,$ replace the first A with the numbers 1 0 and the last A with the numbers 0 1. If $k \geq 3$, replace the first and the last A's in the string with the number 1, replace the second A with the numbers 0 1 0, and if there are any more A's, replace each of them with the numbers 1 0. The resulting string $<s_1 s_2 ... s_n>$ is a string of k ones and $(n-k)$ zeros with no consecutive ones. The corresponding subset S of $[n]$ is given by $S = \{i \in [n] : s_i = 1\}$. Since the string has no consecutive ones, S has no consecutive integers. Hence there are $\binom{n+1-k}{k}$ such subsets S. We have derived the following theorem.

Theorem 1. The number c of size-k subsets of $[n]$ that contain no consecutive integers is given by

$$c = c(n,k) = \binom{n+1-k}{k}. \quad (4)$$

Example. Suppose $n=7$ and $k=3$. Then $c(7,3) = \binom{8-3}{3} = \binom{5}{3} = 10.$
Look at all the shufflings of AAABB:

Shuffling	Resulting Binary Strings	Corresponding Subsets
AAABB	1010100	$\{1,3,5\}$
AABAB	1010010	$\{1,3,6\}$
AABBA	1010001	$\{1,3,7\}$
ABAAB	1001010	$\{1,4,6\}$
ABABA	1001001	$\{1,4,7\}$
ABBAA	1000101	$\{1,5,7\}$
BAAAB	0101010	$\{2,4,6\}$
BAABA	0101001	$\{2,4,7\}$
BABAA	0100101	$\{2,5,7\}$
BBAAA	0010101	$\{3,5,7\}$

$\diamond$

A Table of Initial Values for $c(n,k)$, $0 \le k \le \lfloor \frac{n+1}{2} \rfloor$.

$n\backslash k$	0	1	2	3	4	5	6	7
0	1							
1	1	1						
2	1	2						
3	1	3	1					
4	1	4	3					
5	1	5	6	1				
6	1	6	10	4				
7	1	7	15	10	1			
8	1	8	21	20	5			
9	1	9	28	35	15	1		
10	1	10	36	56	35	6		
11	1	11	45	84	70	21	1	
12	1	12	55	120	126	56	7	
13	1	13	66	165	210	126	28	1

Note. The diagonals form rows of Pascal's triangle and the recursive formula $c(n,k) = c(n-1,k) + c(n-2,k-1)$ holds. The *Maple* code to generate the values in the table above is given by

```
>seq(seq(binomial(n+1-k,k),k=0..floor(n/2+1/2)),n=0..13);
```

Generating functions for $c(n,k)$

The generating function g_k for $c(n,k)$ when k is fixed is given by

$$g_k(x) = \frac{x^{2k-1}}{(1-x)^{k+1}}. \tag{5}$$

Derivation.

$$\begin{aligned} g_k(x) &= \sum_{n=0}^{\infty} c(n,k)x^n = \sum_{n=2k-1}^{\infty} \binom{n+1-k}{k} x^n = \sum_{j=0}^{\infty} \binom{j+k}{k} x^{j+2k-1} \\ &= x^{2k-1} \sum_{j=0}^{\infty} \binom{j+k}{k} x^j = x^{2k-1} \left(\frac{1}{1-x}\right)^{k+1} \\ &= \frac{x^{2k-1}}{(1-x)^{k+1}} \end{aligned}$$

$\diamond$

These generating functions generate the column sequences for the $c(n,k)$ table of values. For example,

$$g_3(x) = \frac{x^5}{(1-x)^4} = x^5 + 4x^6 + 10x^7 + 20x^8 + 35x^9 + 56x^{10} + 84x^{11} + 120x^{12} + 165x^{13} + ...$$

III. The Number of Subsets of $\{1,...,n\}$ that Contain No Consecutive Integers

To find the number $c(n)$ of all subsets of $[n]$ that contain no consecutive integers, we sum $c(n,k)$ over all subset sizes k :

$$c(n) = \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} c(n,k) = \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \binom{n+1-k}{k} = F_{n+2}, \qquad (6)$$

the $(n+2)$nd Fibonacci number. We note that $F_n = \frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$, and therefore

$$c(n) = \frac{(1+\sqrt{5})^{n+2} - (1-\sqrt{5})^{n+2}}{2^{n+2}\sqrt{5}}. \qquad (7)$$

Table of Initial Values for $c(n)$

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
$c(n)$	1	2	3	5	8	13	21	34	55	89	144	233	377	610

Note that $c(n)$ satisfies the recursive formula $c(n+2) = c(n+1) + c(n)$ with $c(0) = 1$ and $c(1) = 2$.

Generating function for $c(n)$

$$g(x) = \sum_{n=0}^{\infty} c(n)x^n = x\sum_{n=0}^{\infty} c(n-1)x^{n-1} + x^2\sum_{n=0}^{\infty} c(n-2)x^{n-2}$$

$$= x\,(\sum_{n=1}^{\infty} c(n-1)x^{n-1} + c(-1)x^{-1})$$

$$+ x^2(\sum_{n=2}^{\infty} c(n-2)x^{n-2} + c(-2)x^{-2} + c(-1)x^{-1})$$

$$= x(g(x) + \tfrac{1}{x}) + x^2(g(x) + 0 + \tfrac{1}{x})$$

$$= xg(x) + 1 + x^2 g(x) + x,$$

which implies $g(x) = \frac{1+x}{1-x-x^2}$. (8)