Generating Non-Redundant Association Rules

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ABSTRACT

The traditional association rule mining framework produces many redundant rules. The extent of redundancy is a lot larger than previously suspected. We present a new framework for associations based on the concept of *closed* frequent itemsets. The number of non-redundant rules produced by the new approach is exponentially (in the length of the longest frequent itemset) smaller than the rule set from the traditional approach. Experiments using several "hard" as well as "easy" real and synthetic databases confirm the utility of our framework in terms of reduction in the number of rules presented to the user, and in terms of time.

Categories and Subject Descriptors

H.2.8 [**Database Management**]: Applications—*Data Mining*; I.2.6 [**Artificial Intelligence**]: Learning

1. INTRODUCTION

Association rule discovery, a successful and important mining task, aims at uncovering all frequent patterns among transactions composed of data attributes or items. Results are presented in the form of rules between different sets of items, along with metrics like the joint and conditional probabilities of the antecedent and consequent, to judge a rule's importance.

It is widely recognized that the set of association rules can rapidly grow to be unwieldy, especially as we lower the frequency requirements. The larger the set of frequent itemsets the more the number of rules presented to the user, many of which are redundant. This is true even for sparse datasets, but for dense datasets it is simply not feasible to mine all possible frequent itemsets, let alone to generate rules, since they typically produce an exponential number of frequent itemsets; finding long itemsets of length 20 or 30 is not uncommon [2].

Prior research has mentioned that the traditional association rule mining framework produces too many rules, but the extent of redundancy is a lot larger than previously suspected. More concretely, the number of redundant rules are exponential in the length of the longest frequent itemset. We present a new framework for association rule mining based on the concept of *closed* frequent itemsets. The set of all closed frequent itemsets can be orders of magnitude smaller than the set of all frequent itemsets, especially for real (dense) datasets. At the same time, we don't loose any information; the closed itemsets uniquely determine the set of all frequent itemsets and their *exact* frequency. Note that using the maximal frequent itemsets results in loss of information, since subset frequency is not available. We show that the new framework produces exponentially (in the length of the longest frequent itemset) fewer rules than the traditional approach, again without loss of information. Our framework allows us to mine even dense datasets, where it is not feasible to find all frequent itemsets. Finally, the rule set we produce is a *generating* set, i.e., all possible association rules can be inferred from them using operations like transitivity and augmentation.

Experiments using several "hard" or dense, as well as sparse databases confirm the utility of our framework in terms of reduction in the number of rules presented to the user, and in terms of time. We show that closed itemsets can be found in a fraction of the time it takes to mine all frequent itemsets (with improvements of more than 100 times), and the number of rules returned to the user can be smaller by a factor of 3000 or more! (the gap widens for lower frequency values).

1.1 Related Work

There has been a lot of research in developing efficient algorithms for mining frequent itemsets [1, 2, 4, 9, 15, 21]. Most of these algorithms enumerate all frequent itemsets. Using these for rule generation produces many redundant rules, as we will show later. Some methods only generate maximal frequent itemsets [2, 9]. Maximal itemsets cannot be used for rule generation, since support of subsets is required for confidence computation. While it is easy to make one more data scan to gather the supports of all subsets, we still have the problem of many redundant rules. Further, for all these methods it is simply not possible to find rules in dense datasets which may easily have frequent itemsets of length 20 and more [2]. In contrast the set of *closed* frequent itemsets can be orders of magnitude smaller than the set of all frequent itemsets, and it can be used to generate rules even in dense domains.

In general, most of the association mining work has concentrated on the task of mining frequent itemsets. Rule generation has received very little attention. There has been some work in pruning discovered association rules by forming rule covers [17]. Other work addresses the problem of mining interesting association rules [8, 3, 10, 12]. The approach taken is to incorporate user-specified constraints on the kinds of rules generated or to define objective metrics of interestingness. As such these works are complimentary to our approach here. Furthermore, they do not address the issue of rule redundancy or of constructing a generating set.

A preliminary study of the idea of using closed frequent itemsets to generate rules was presented by us in [20]. This paper substantially improves on those ideas, and also presents experimental results to support our claims. Independently, Pasquier et al. have also used closed itemsets for association mining [13, 14]. However, they mainly concentrate on the discovery of frequent closed itemsets, and do not report any experiments on rule mining. We on the other hand are specifically interested in generating a smaller rule set, after mining the frequent closed itemsets. Furthermore, we recently proposed the CHARM algorithm [19] for mining all closed frequent itemsets. This algorithm outperforms, by orders of magnitude, the AClose method proposed by Pasquier et al [14], as well as the Apriori [1] method for mining all frequent itemsets. In this paper we do not present the CHARM algorithm, since our main goal is rule generation; we simply use the output it produces.

The notion of closed frequent sets has its origins in the elegant mathematical framework of formal concept analysis (FCA). A number of algorithms have been proposed within FCA for generating all the closed sets of a binary relation [5]. However, these methods have only been tested on very small datasets. Further, these algorithms generate all the closed sets, and thus have to be adapted to enumerate only the frequent concepts. The foundations of rule generation (in FCA) were studied in [11], but no experimentation on large sets was done. Our characterization of the generating set of association rules is different, and we also present an experimental verification. Other work has extended the FCA approach to incorporate incremental rule mining [6], and recent work has addressed the issue of extracting association rule bases [16].

The rest of the paper is organized as follows. Section 2 describes the association mining task. Section 3 presents the notion of closed itemsets. Section 4 looks at the problem of eliminating redundant rules. We experimentally validate our approach in Section 5. The proofs for all theorems have been omitted due to lack of space; these are available in [18].

2. ASSOCIATION RULES

The association mining task can be stated as follows: Let $\mathcal{I} = \{1, 2, \cdots, m\}$ be a set of items, and let $\mathcal{T} = \{1, 2, \cdots, n\}$ be a set of transaction identifiers or *tids*. The input database is a binary relation $\delta \subseteq \mathcal{I} \times \mathcal{T}$. If an item i occurs in a transaction t, we write it as $(i, t) \in \delta$, or alternately as $i\delta t$. Typically the database is arranged as a set of transactions, where each transaction contains a set of items. For example, consider the database shown in Figure 1, used as a running example in this paper. Here $\mathcal{I} = \{A, C, D, T, W\}$, and

$\mathcal{T} = \{1, 2, 3, 4, 5, 6\}$. The second transaction can be represented as $\{C\delta 2, D\delta 2, W\delta 2\}$; all such pairs from all transactions, taken together form the binary relation δ. A set $X \subseteq \mathcal{I}$ is called an *itemset*, and a set $Y \subseteq \mathcal{T}$ is called a *tidset*. For convenience we write an itemset $\{A, C, W\}$ as ACW, and a tidset $\{2, 4, 5\}$ as 245. The *support* of an itemset X, denoted $\sigma(X)$, is the number of transactions in which it occurs as a subset. An itemset is *frequent* if its support $\sigma(X) \geq minsup$, where *minsup* is a user-specified minimum support threshold.

An *association rule* is an expression $A \xrightarrow{p} B$, where A and B are itemsets. The *support* of the rule is $\sigma(A \cup B)$ (i.e., the joint probability of a transaction containing both A and B), and the *confidence* $p = \sigma(A \cup B)/\sigma(A)$ (i.e., the conditional probability that a transaction contains B, given that it contains A). A rule is frequent if the itemset $A \cup B$ is frequent. A rule is *confident* if $p \geq minconf$, where where *minconf* is a user-specified minimum threshold.

DISTINCT DATABASE ITEMS				
Jane Austen	Agatha Christie	Sir Arthur Conan Doyle	Mark Twain	P. G. Wodehouse
A	C	D	T	W

DATABASE		ALL FREQUENT ITEMSETS	
Transcation	Items	MINIMUM SUPPORT = 50%	
1	A C T W		
2	C D W	**Support**	**Itemsets**
3	A C T W	100% (6)	C
4	A C D W	83% (5)	W, CW
5	A C D T W	67% (4)	A, D, T, AC, AW CD, CT, ACW
6	C D T	50% (3)	AT, DW, TW, ACT, ATW CDW, CTW, ACTW

Figure 1: Generating Frequent Itemsets

Association rule mining consists of two steps [1]: 1) Find all frequent itemsets, and 2) Generate high confidence rules.

Finding frequent itemsets This step is computationally and I/O intensive. Consider Figure 1, which shows a bookstore database with six customers who buy books by different authors. It shows all the frequent itemsets with $minsup = 50\%$ (i.e., 3 transactions). $ACTW$ and CDW are the maximal frequent itemsets (i.e., not a subset of any other frequent itemset).

Let $|\mathcal{I}| = m$ be the number of items. The search space for enumeration of all frequent itemsets is 2^m, which is exponential in m. One can prove that the problem of finding a frequent set of a certain size is NP-Complete, by reducing it to the balanced bipartite clique problem, which is known to be NP-Complete [20]. However, if we assume that there is a bound on the transaction length, the task of finding all frequent itemsets is essentially linear in the database size, since the overall complexity in this case is given as $O(r \cdot n \cdot 2^l)$, where $|\mathcal{T}| = n$ is the number of transactions, l is the length of the longest frequent itemset, and r is the number of maximal frequent itemsets.

Generating confident rules This step is relatively straightforward; rules of the form $Y \xrightarrow{p} X - Y$, are generated for all frequent itemsets X, for all $Y \subset X$, $Y \neq \emptyset$, and provided $p \geq minconf$. For example, from the frequent itemset
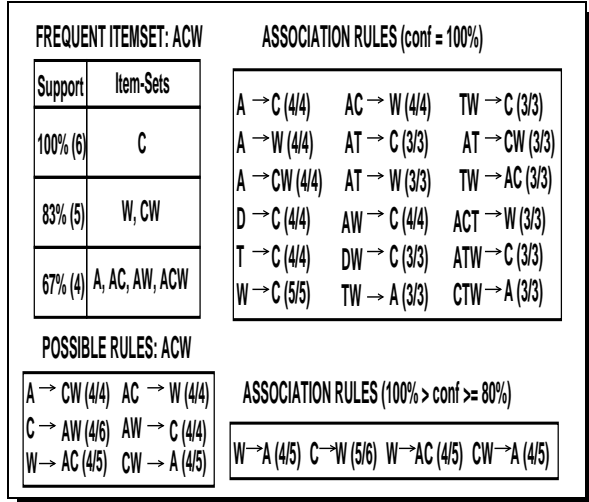
Figure 2: Generating Confident Rules

Figure 3: Frequent Itemsets

ACW we can generate 6 possible rules (all of them have support of 4): $A \xrightarrow{1.0} CW, C \xrightarrow{0.67} AW, W \xrightarrow{0.8} AC, AC \xrightarrow{1.0} W, AW \xrightarrow{1.0} C$, and $CW \xrightarrow{0.8} A$. This process is also shown pictorially in Figure 2. Notice that we need access to the support of all subsets of ACW to generate rules from it. To obtain all possible rules we need to examine each frequent itemset and repeat the rule generation process shown above for ACW. Figure 2 shows the set of all other association rules with confidence above or equal to *minconf* = 80%.

For an itemset of size k there are $2^k - 2$ potentially confident rules that can be generated. This follows from the fact that we must consider each subset of the itemset as an antecedent, except for the empty and the full itemset. The complexity of the rule generation step is thus $O(f \cdot 2^l)$, where f is the number of frequent itemsets, and l is the longest frequent itemset.

3. CLOSED FREQUENT ITEMSETS

In this section we describe the concept of closed frequent itemsets, and show that this set is necessary and sufficient to capture all the information about frequent itemsets, and has smaller cardinality than the set of all frequent itemsets.

Let $(P, \leq)$ be an ordered set with the binary relation $\leq$, and let S be a subset of P. An element $u \in P$ ($l \in P$) is an *upper bound* (*lower bound*) of S if $s \leq u$ ($s \geq l$) for all $s \in S$. The least upper bound is called the **join** of S, and is denoted as $\bigvee S$, and the greatest lower bound is called the **meet** of S, and is denoted as $\bigwedge S$. If $S = \{x, y\}$, we also write $x \vee y$ for the join, and $x \wedge y$ for the meet. An ordered set $(L, \leq)$ is a **lattice**, if for any two elements x and y in L, the join $x \vee y$ and meet $x \wedge y$ always exist. L is a *complete lattice* if $\bigvee S$ and $\bigwedge S$ exist for all $S \subseteq L$. Any finite lattice is complete.

Let $\mathcal{P}$ denote the power set of S (i.e., the set of all subsets of S). The ordered set $(\mathcal{P}(S), \subseteq)$ is a complete lattice, where the meet is given by set intersection, and the join is given by set union. For example the partial orders $(\mathcal{P}(\mathcal{I}), \subseteq)$, the set of all possible itemsets, and $(\mathcal{P}(\mathcal{T}), \subseteq)$, the set of all possible
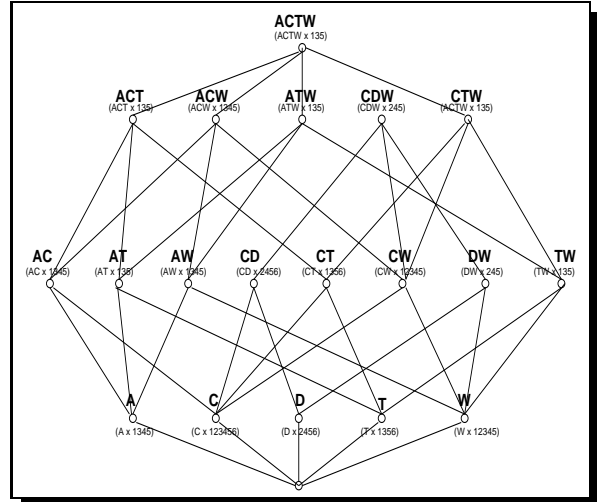
tidsets are both complete lattices. Figure 3 shows the lattice [1] of all frequent itemsets we found in our example database.

Let the binary relation $\delta \subseteq \mathcal{I} \times \mathcal{T}$ be the input database for association mining. Let $X \subseteq \mathcal{I}$, and $Y \subseteq \mathcal{T}$. The mappings
$$t : \mathcal{I} \mapsto \mathcal{T}, \quad t(X) = \{y \in \mathcal{T} \mid \forall x \in X, \ x\delta y\}$$
$$i : \mathcal{T} \mapsto \mathcal{I}, \quad i(Y) = \{x \in \mathcal{I} \mid \forall y \in Y, \ x\delta y\}$$
define a *Galois connection* between $\mathcal{P}(\mathcal{I})$ and $\mathcal{P}(\mathcal{T})$. We denote a $X, t(X)$ pair as $X \times t(X)$, and a $i(Y), Y$ pair as $i(Y) \times Y$. Figure 4 illustrates the two mappings. The mapping $t(X)$ is the set of all transactions (tidset) which contain the itemset X, similarly $i(Y)$ is the itemset that is contained in all the transactions in Y. For example, $t(ACW) = 1345$, and $i(245) = CDW$. In terms of individual elements $t(X) = \bigcap_{x \in X} t(x)$, and $i(Y) = \bigcap_{y \in Y} i(y)$. For example $t(ACW) = t(A) \cap t(C) \cap t(W) = 1345 \cap 123456 \cap 12345 = 1345$. Also $i(245) = i(2) \cap i(4) \cap i(5) = CDW \cap ACDW \cap ACDTW = CDW$. The Galois connection satisfies the following properties (where $X, X_1, X_2 \in \mathcal{P}(\mathcal{I})$ and $Y, Y_1, Y_2 \in \mathcal{P}(\mathcal{T})$): 1) $X_1 \subseteq X_2 \Rightarrow t(X_1) \supseteq t(X_2)$, 2) $Y_1 \subseteq Y_2 \Rightarrow i(Y_1) \supseteq i(Y_2)$, 3) $X \subseteq i(t(X))$ and $Y \subseteq t(i(Y))$. For example, for $ACW \subseteq ACTW, t(ACW) = 1345 \supseteq 135 = t(ACTW)$. For $245 \subseteq 2456, i(245) = CDW \supseteq CD = i(2456)$. Also, $AC \subseteq i(t(AC)) = i(1345) = ACW$.

Let S be a set. A function $c : \mathcal{P}(S) \mapsto \mathcal{P}(S)$ is a *closure operator* on S if, for all $X, Y \subseteq S$, c satisfies the following properties: 1) Extension: $X \subseteq c(X)$. 2) Monotonicity: if $X \subseteq Y$, then $c(X) \subseteq c(Y)$. 3) Idempotency: $c(c(X)) = c(X)$. A subset X of S is called *closed* if $c(X) = X$. Let $X \subseteq \mathcal{I}$ and $Y \subseteq \mathcal{T}$. Let $c_{it}(X)$ denote the composition of the two mappings $i \circ t(X) = i(t(X))$. Dually, let $c_{ti}(Y) = t \circ i(Y) = t(i(Y))$. Then $c_{it} : \mathcal{P}(\mathcal{I}) \mapsto \mathcal{P}(\mathcal{I})$ and $c_{ti} : \mathcal{P}(\mathcal{T}) \mapsto \mathcal{P}(\mathcal{T})$ are both closure operators.

We define a **closed itemset** as an itemset X that is that same as its closure, i.e., $X = c_{it}(X)$. For example the itemset ACW is closed. A **closed tidset** is a tidset $Y = c_{ti}(Y)$. For example,

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[1] Only meet is defined on frequent sets, while the join may not exist. For example, $AC \wedge AT = AC \cap AT = A$ is frequent. But, while $AC \vee AT = AC \cup AT = ACT$ is frequent, $AC \cup DW = ACDW$ is not frequent.
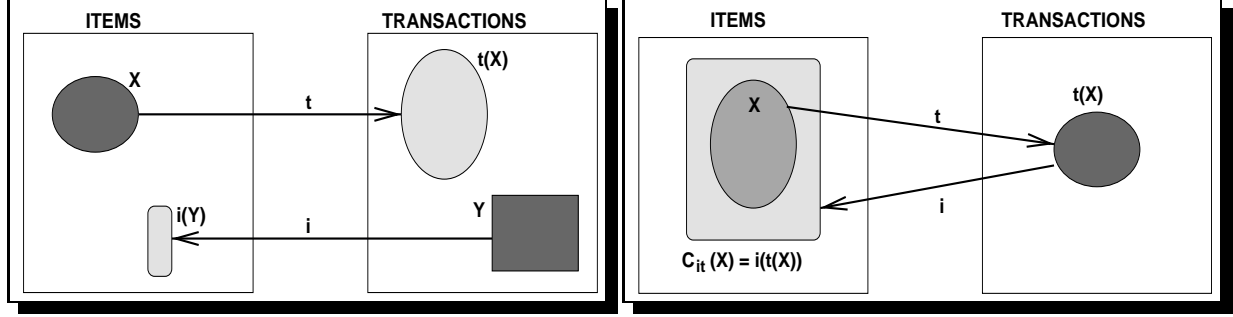
Figure 4: A) Galois Connection, B) Closed Itemset: Round-Trip

the tidset 1345 is closed.

The mappings c_{it} and c_{ti}, being closure operators, satisfy the three properties of extension, monotonicity, and idempotency. We also call the application of $i \circ t$ or $t \circ i$ a *round-trip*. Figure 4 illustrates this round-trip starting with an itemset X. For example, let $X = AC$, then the extension property say that X is a subset of its closure, since $c_{it}(AC) = i(t(AC)) = i(1345) = ACW$. Since $AC \neq c_{it}(AC) = ACW$, we conclude that AC is not closed. On the other hand, the idempotency property say that once we map an itemset to the tidset that contains it, and then map that tidset back to the set of items common to all tids in the tidset, we obtain a closed itemset. After this no matter how many such round-trips we make we cannot extend a closed itemset. For example, after one round-trip for AC we obtain the closed itemset ACW. If we perform another round-trip on ACW, we get $c_{it}(ACW) = i(t(ACW)) = i(1345) = ACW$.

For any closed itemset X, there exists a closed tidset given by Y, with the property that $Y = t(X)$ and $X = i(Y)$ (conversely, for any closed tidset there exists a closed itemset). We can see that X is closed by the fact that $X = i(Y)$, then plugging $Y = t(X)$, we get $X = i(Y) = i(t(X)) = c_{it}(X)$, thus X is closed. Dually, Y is closed. For example, we have seen above that for the closed itemset ACW the associated closed tidset is 1345. Such a closed itemset and closed tidset pair $X \times Y$ is called a **concept**.

A concept $X_1 \times Y_1$ is a *subconcept* of $X_2 \times Y_2$, denoted as $X_1 \times Y_1 \leq X_2 \times Y_2$, iff $X_1 \subseteq X_2$ (iff $Y_2 \subseteq Y_1$). Let $\mathcal{B}(\delta)$ denote the set of all possible concepts in the database. Then the ordered set $(\mathcal{B}(\delta), \leq)$ is a complete lattice, called the *Galois* lattice. For example, Figure 5 shows the Galois lattice for our example database, which has a total of 10 concepts. The least element is $C \times 123456$ and the greatest element is $ACDTW \times 5$. The mappings between the closed pairs of itemsets and tidsets are anti-isomorphic, i.e., concepts with large cardinality itemsets have small tidsets, and vice versa.

The concept generated by a single item $x \in \mathcal{I}$ is called an *item concept*, and is given as $\mathcal{C}_i(x) = c_{it}(x) \times t(x)$. Similarly, the concept generated by a single transaction $y \in \mathcal{T}$ is called a *tid concept*, and is given as $\mathcal{C}_t(y) = i(y) \times c_{ti}(y)$. For example, the item concept $\mathcal{C}_i(A) = i(t(A)) \times t(A) = i(1345) \times 1345 = ACW \times 1345$. Further, the tid concept $\mathcal{C}_t(2) = i(2) \times t(i(2)) = CDW \times t(CDW) = CDW \times 245$.

In Figure 5 if we relabel each node with the item concept or tid concept that it is equivalent to, then we obtain a lattice with *minimal labelling*, with item or tid labels, as shown in the figure in bold letters. Such a relabelling reduces clutter in the lattice diagram, which provides an excellent way of visualizing the structure of the patterns and relationships that exist between items. We shall see its benefit in the next section when we talk about high confidence rules extraction.

It is easy to reconstruct the concepts from the minimal labelling. Consider the tid concept $\mathcal{C}_t(2) = X \times Y$. To obtain the closed itemset X, we append all item labels reachable below it. Conversely, to obtain the closed tidset Y we append all labels reachable above $\mathcal{C}_t(2)$. Since W, D and C are all the labels reachable by a path below it, $X = CDW$ forms the closed itemset. Since 4 and 5 are the only labels reachable above $\mathcal{C}_t(2)$, $Y = 245$; this gives us the concept $CDW \times 245$, which matches the concept shown in the figure.

3.1 Frequent Closed Itemsets vs. Frequent Itemsets

We begin this section by defining the join and meet operation on the concept lattice (see [5] for the formal proof): The set of all concepts in the database relation δ, given by $(\mathcal{B}(\delta), \leq)$ is a (complete) lattice with join and meet given by

join: $(X_1 \times Y_1) \vee (X_2 \times Y_2) = c_{it}(X_1 \cup X_2) \times (Y_1 \cap Y_2)$

meet: $(X_1 \times Y_1) \wedge (X_2 \times Y_2) = (X_1 \cap X_2) \times c_{ti}(Y_1 \cup Y_2)$

For the join and meet of multiple concepts, we simply take the unions and joins over all of them. For example, consider the join of two concepts, $(ACDW \times 45) \vee (CDT \times 56) = c_{it}(ACDW \cup CDT) \times (45 \cap 56) = ACDTW \times 5$. On the other hand their meet is given as, $(ACDW \times 45) \wedge (CDT \times 56) = (ACDW \cap CDT) \times c_{ti}(45 \cup 56) = CD \times c_{ti}(456) = CD \times 2456$. Similarly, we can perform multiple concept joins or meets; for example, $(CT \times 1356) \vee (CD \times 2456) \vee (CDW \times 245) = c_{it}(CT \cup CD \cup CDW) \times (1356 \cap 2456 \cap 245) = c_{it}(CDTW) \times 5 = ACDTW \times 5$.

We define the support of a closed itemset X or a concept $X \times Y$ as the cardinality of the closed tidset $Y = t(X)$, i.e, $\sigma(X) = |Y| = |t(X)|$. A closed itemset or a concept is *frequent* if its support is at least *minsup*. Figure 6 shows all the frequent concepts with *minsup* = 50% (i.e., with tidset cardinality at least 3). All frequent itemsets can be determined by the join operation on the frequent item concepts. For example, since join of item concepts D and T, $\mathcal{C}_i(D) \vee \mathcal{C}_i(T)$, doesn't
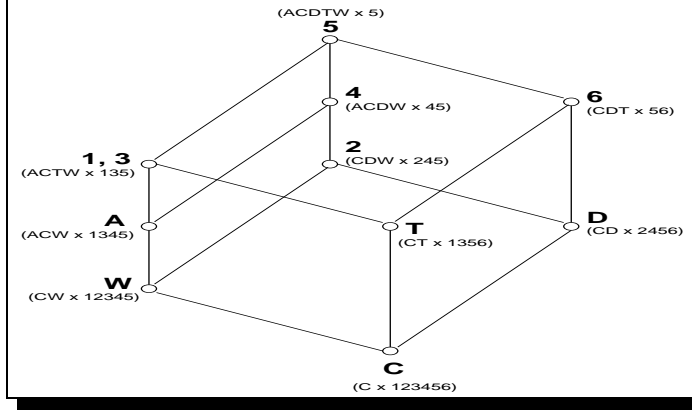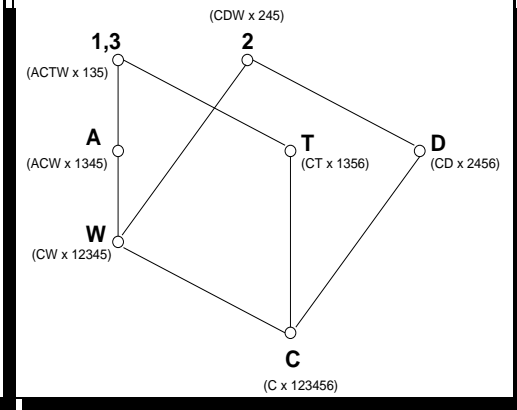
Figure 5: Galois Lattice of Concepts

Figure 6: Frequent Concepts

exist, DT is not frequent. On the other hand, $\mathcal{C}_i(A) \vee \mathcal{C}_i(T) = ACTW \times 135$, thus AT is frequent. Furthermore, the support of AT is given by the cardinality of the resulting concept's tid-set, i.e., $\sigma(AT) = |t(AT)| = |135| = 3$.

LEMMA 1. *An itemset's (X) support is equal to the support of its closure, i.e., $\sigma(X) = \sigma(c_{it}(X))$.*

This theorem (independently reported in [13]) states that all frequent itemsets are uniquely determined by the frequent closed itemsets (or frequent concepts). Furthermore, the set of frequent closed itemsets is bounded above by the set of frequent itemsets, and is typically much smaller, especially for dense datasets. For very sparse datasets, in the worst case, the two sets may be equal. To illustrate the benefits of closed itemset mining, contrast Figure 3, showing the set of all frequent itemsets, with Figure 6, showing the set of all closed frequent itemsets (or concepts). We see that while there are only 7 closed frequent itemsets, in contrast there are 19 frequent itemsets. This example clearly illustrates the benefits of mining the closed frequent itemsets. [2]

4. RULE GENERATION

Recall that an association rule is of the form $X_1 \xrightarrow{p} X_2$, where $X_1, X_2 \subseteq \mathcal{I}$. Its support equals $|t(X_1 \cup X_2)|$, and its confidence is given as $p = P(X_2|X_1) = |t(X_1 \cup X_2)|/|t(X_1)|$. We are interested in finding all high support and high confidence rules. It is widely recognized that the set of such association rules can rapidly grow to be unwieldy. In this section we will show how the closed frequent itemsets help us form a generating set of rules, from which all other association rules can be inferred. Thus, only a small and easily understandable set of rules can be presented to the user, who can later selectively derive other rules of interest.

Before we proceed, we need to formally define what we mean by a redundant rule. Let R_i denote the rule $X_1^i \xrightarrow{p_i} X_2^i$. We say that a rule R_1 is more general than a rule R_2, denoted

$R_1 \preceq R_2$ provided that R_2 can be generated by adding additional items to either the antecedent or consequent of R_1, i.e., if $X_1^1 \subseteq X_1^2$ and $X_2^1 \subseteq X_2^2$. Now let $\mathcal{R} = \{R_1, \cdots, R_n\}$ be a set of rules, such that all their confidences are equal, i.e., $p_i = p$, $\forall i$. Then we say that a rule R_j is **redundant** if there exists some rule R_i, such that $R_i \preceq R_j$. The non-redundant rules in the collection $\mathcal{R}$ are those that are most general.

We now show how to eliminate the redundant association rules, i.e., rules having the same support and confidence as some more general rule. In the last section, we showed that the support of an itemset X equals the support of its closure $c_{it}(X)$. Thus it suffices to consider rules *only* among the frequent concepts. In other words the rule $X_1 \xrightarrow{p} X_2$ is exactly the same as the rule $c_{it}(X_1) \xrightarrow{p} c_{it}(X_2)$.

Another observation that follows from the concept lattice is that it is sufficient to consider rules among adjacent concepts, since other rules can be inferred by transitivity, that is:

LEMMA 2. **Transitivity:** *Let X_1, X_2, X_3 be frequent closed itemsets, with $X_1 \subseteq X_2 \subseteq X_3$. If $X_1 \xrightarrow{p} X_2$ and $X_2 \xrightarrow{q} X_3$, then $X_1 \xrightarrow{pq} X_3$.*

In the discussion below, we consider two cases of association rules, those with 100% confidence, i.e., with $p = 1.0$, and those with $p < 1.0$.

4.1 Rules with Confidence = 100%

LEMMA 3. *An association rule $X_1 \xrightarrow{1.0} X_2$ has confidence $p = 1.0$ if and only if $t(X_1) \subseteq t(X_2)$.*

This theorem says that all 100% confidence rules are those that are directed from a super-concept ($X_1 \times t(X_1)$) to a sub-concept ($X_2 \times t(X_2)$),i.e., down-arcs, since it is in precisely these cases that $t(X_1) \subseteq t(X_2)$ (or $X_1 \subseteq X_2$). Consider the item concepts $\mathcal{C}_i(W) = CW \times 12345$ and $\mathcal{C}_i(C) = C \times 123456$. The rule $W \xrightarrow{1.0} C$ is a 100% confidence rule. Note that if we take the itemset closure on both sides of the rule, we obtain $CW \xrightarrow{1.0} C$, i.e., a rule between closed itemsets, but since the antecedent and consequent are not disjoint in this case, we prefer to write the rule as $W \xrightarrow{1.0} C$, although both rules are exactly the same. Figure 7 shows some of the other rules among adjacent concepts with 100% confidence.
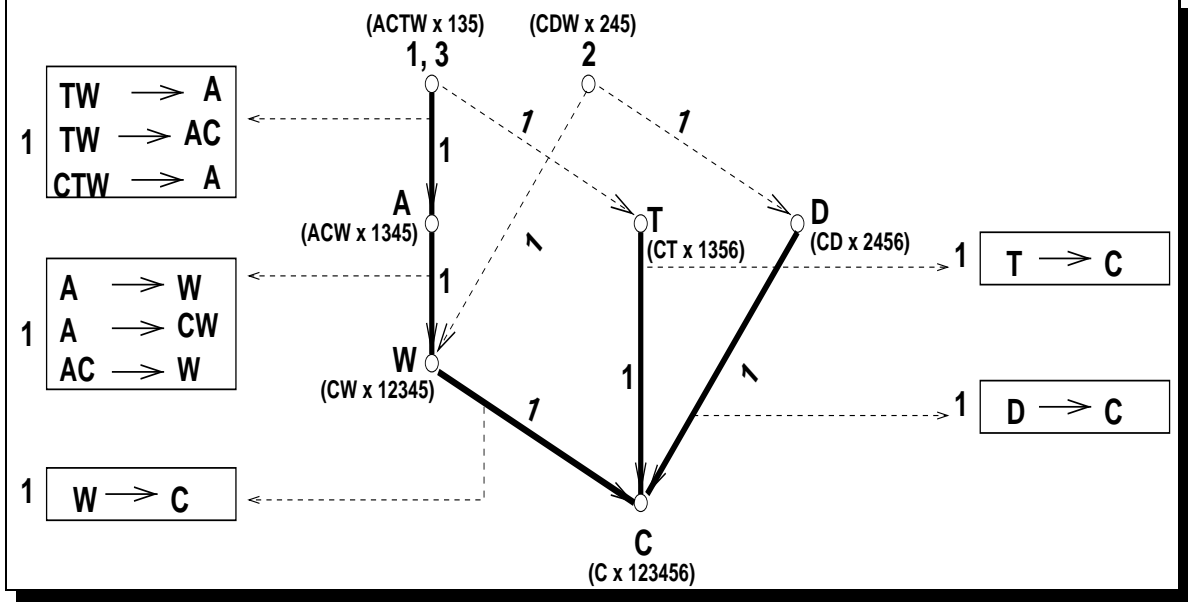
Figure 7: Rules with 100% Confidence

We notice that some down-arcs are labeled with more than one rule. In such cases, all rules within a box are equivalent, and we prefer the rule that is most general. For example, consider the rules $TW \xrightarrow{1.0} A$, $TW \xrightarrow{1.0} AC$, and $CTW \xrightarrow{1.0} A$. $TW \xrightarrow{1.0} A$ is more general than the latter two rules. since the latter two are obtained by adding one (or more) items to either the antecedent or consequent of $TW \xrightarrow{1.0} A$. In fact, we can say that the addition of C to either the antecedent or the consequent has no effect on the support or confidence of the rule. Thus, according to our definition, we say that the other two rules redundant.

THEOREM 1. *Let $\mathcal{R} = \{R_1, \cdots, R_n\}$ be a set of rules with 100% confidence ($p_i = 1.0$, $\forall i$), such that $I_1 = c_{it}(X_1^i \cup X_2^i)$, and $I_2 = c_{it}(X_2^i)$ for all rules R_i. Let R_I denote the 100% confidence rule $I_1 \xrightarrow{1.0} I_2$. Then all the rules $R_i \neq R_I$ are more specific than R_I, and thus are* redundant.

Let's apply this theorem to the three rules we considered above. For the first rule $c_{it}(TW \cup A) = c_{it}(ATW) = ACTW$. Similarly for the other two rules we see that $c_{it}(TW \cup AC) = c_{it}(ACTW) = ACTW$, and $c_{it}(CTW \cup A) = c_{it}(ACTW) = ACTW$. Thus for these three rules we get the closed itemset $I_1 = ACTW$. By the same process we obtain $I_2 = ACW$. All three rules correspond to the arc between the tid concept $\mathcal{C}_t(1, 3)$ and the item concept $\mathcal{C}_i(A)$. Finally $TW \xrightarrow{1.0} A$ is the most general rule, and so the other two are redundant.

A set of such general rules constitutes a *generating set*, i.e., a rule set, from which all other 100% confidence rules can inferred. Note that in this paper we do not address the question of eliminating self-redundancy within this generating set, i.e., there may still exist rules in the generating set that can be derived from other rules in the set. In other words we do not claim anything about the minimality of the generating set; that is the topic of a forthcoming paper. See [7, 11, 16] for more information on generating a base set (or minimal generating

set) of rules.

Figure 7 shows the generating set in bold arcs, which includes the 5 most general rules $\{TW \xrightarrow{1.0} A, A \xrightarrow{1.0} W, W \xrightarrow{1.0} C, T \xrightarrow{1.0} C, D \xrightarrow{1.0} C\}$ (the down-arcs that have been left out produce rules that cannot be written with disjoint antecedent and consequent. For example, between $\mathcal{C}_t(2)$ and $\mathcal{C}_i(D)$, the most general rule is $DW \xrightarrow{1.0} D$. Since the antecedent and consequent are not disjoint, as required by definition, we discard such rules). All other 100% confidence rules can be derived from this generating set by application of simple inference rules. For example, we can obtain the rule $A \xrightarrow{1.0} C$ by transitivity from the two rules $A \xrightarrow{1.0} W$ and $W \xrightarrow{1.0} C$. The rule $DW \xrightarrow{1.0} C$ can be obtained by augmentation of the two rules $W \xrightarrow{1.0} C$ and $D \xrightarrow{1.0} C$, etc. One can easily verify that all the 18 100% confidence rules produced by using frequent itemsets, as shown in Figure 2, can be generated from this set of 5 rules, produced using the closed frequent itemsets!

4.2 Rules with Confidence < 100%

We now turn to the problem of finding a generating set for association rules with confidence less than 100%. As before, we need to consider only the rules between adjacent concepts. But this time the rules correspond to the up-arcs, instead of the down-arcs for the 100% confidence rules, i.e., the rules go from sub-concepts to super-concepts.

Consider Figure 8. The edge between item concepts $\mathcal{C}_i(C)$ and $\mathcal{C}_i(W)$ corresponds to $C \xrightarrow{0.83} W$. Rules between non-adjacent concepts can be derived by transitivity. For example, for $C \xrightarrow{p} A$ we can obtain the value of p using the rules $C \xrightarrow{q=5/6} W$ and $W \xrightarrow{r=4/5} A$. We have $p = qr = 5/6 \cdot 4/5 = 2/3 = 0.67$.

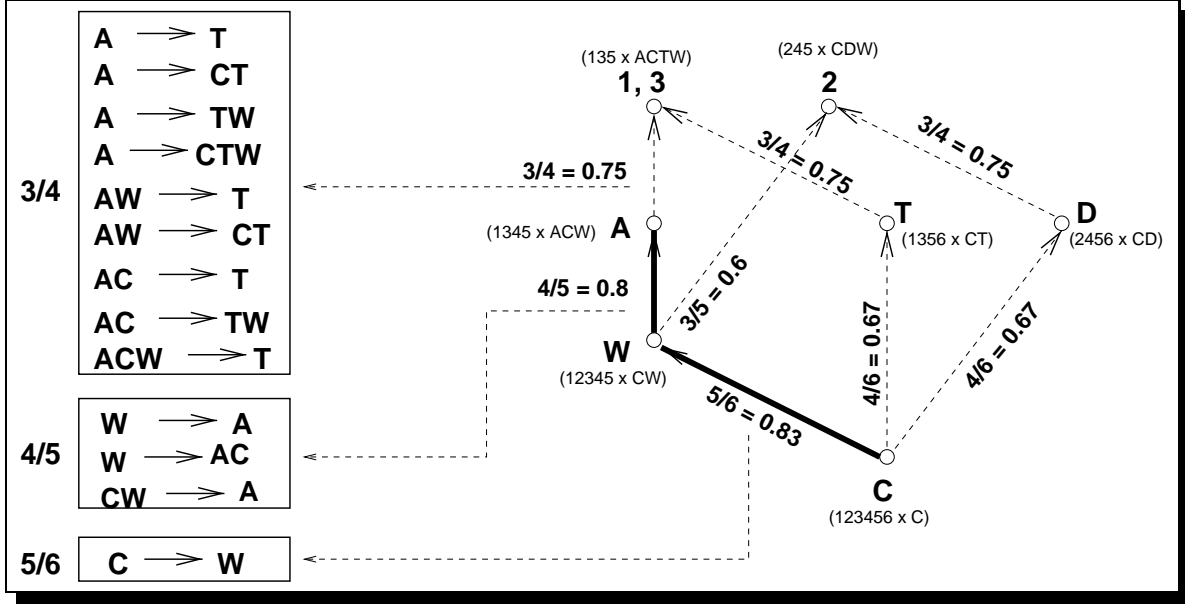THEOREM 2. *Let $\mathcal{R} = \{R_1, \cdots, R_n\}$ be a set of rules*

Figure 8: Rules with Confidence $< 100\%$

with confidence $p < 1.0$, such that $I_1 = c_{it}(X_1^i)$, and $I_2 = c_{it}(X_1^i \cup X_2^i)$ for all rules R_i. Let R_I denote the rule $I_1 \xrightarrow{p} I_2$. Then all the rules $R_i \neq R_I$ are more specific than R_I, and thus are redundant.

This theorem differs from that of the 100% confidence rules to account for the up-arcs. Consider the rules produced by the up-arc between item concepts $\mathcal{C}_i(W)$ and $\mathcal{C}_i(A)$. We find that for all three rules, $I_1 = c_{it}(W) = c_{it}(CW) = CW$, and $I_2 = c_{it}(W \cup A) = c_{it}(W \cup AC) = c_{it}(CW \cup A) = ACW$. The support of the rule is given by $|t(I_1 \cup I_2)| = |t(ACW)| = 4$, and the confidence given as $|t(I_1 \cup I_2)|/|t(I_1)| = 4/5 = 0.8$. Finally, since $W \xrightarrow{0.8} A$ is the most general rule, the other two are redundant. Similarly for the up-arc between $\mathcal{C}_i(A)$ and $\mathcal{C}_t(1, 3)$, we get the general rule $A \xrightarrow{0.75} T$. The other 8 rules in the box are redundant!

The set of all such general rules forms a generating set of rules from which other rules can be inferred. The two bold arrows in Figure 8 constitute a generating set for all rules with $0.8 \leq p < 1.0$. Due to the transitivity property, we only have to consider arcs with confidence at least *minconf* = 0.8. No other rules can be confident at this level.

By combining the generating set for rules with $p = 1.0$, shown in Figure 7 and the generating set for rules with $1.0 > p \geq 0.8$, shown in Figure 8, we obtain a generating set for all association rules with *minsup* = 50%, and *minconf* = 80%: $\{TW \xrightarrow{1.0} A, A \xrightarrow{1.0} W, W \xrightarrow{1.0} C, T \xrightarrow{1.0} C, D \xrightarrow{1.0} C, W \xrightarrow{0.8} A, C \xrightarrow{0.83} W\}$.

It can be easily verified that all the association rules shown in Figure 2, for our example database from Figure 1, can be derived from this set. Using the closed itemset approach we produce 7 rules versus the 22 rules produced in traditional association mining. To see the contrast further, consider the set of all possible association rules we can mine. With *minsup* =

50%, the least value of confidence can be 50% (since the maximum support of an itemset can be 100%, but any frequent subset must have at least 50% support; the least confidence value is thus 50/100 = 0.5). There are 60 possible association rules versus only 13 in the generating set (5 rules with $p = 1.0$ in Figure 7, and 8 rules with $p < 1.0$ in Figure 8)

4.3 Complexity of Rule Generation: Traditional vs. New Framework

The complexity of rule generation in the traditional framework is $O(f \cdot 2^l)$, exponential in the length l of the longest frequent itemset (f is the total number of frequent itemsets). On the other hand using the closed itemset framework, the number of non-redundant rules is linear in the number of closed itemsets. To see how much savings are possible using closed frequent itemsets, lets consider the case where the longest frequent itemset has length l; with all 2^l subsets also being frequent.

In the traditional association rule framework, we would have to consider for each frequent itemset all its subsets as rule antecedents. The total number of rules generated in this approach is given as $\sum_{i=0}^{l} \binom{l}{i} \cdot 2^{l-i} \leq \sum_{i=0}^{l} \binom{l}{i} \cdot 2^l = 2^l \sum_{i=0}^{l} \binom{l}{i} = 2^l \cdot 2^l = 0(2^{2l})$.

On the other hand the number of non-redundant rules produced using closed itemsets is given as follows. Let's consider two extreme cases: In the best case, there is only one closed itemset, i.e., all 2^l subsets have the same support as the longest frequent itemset. Thus all rules between itemsets must have 100% confidence. The closed itemset approach doesn't produce any rule; it just lists the closed itemset with its frequency, with the implicit assumption that all possible rules from this itemset have 100% confidence. This corresponds to a reduction in the number of rules by a factor of $O(2^{2l})$.

In the worst case, all 2^l frequent itemsets are also closed. In this case there can be no 100% confidence rules and all ($<$

100% confidence) rules point upwards, i.e., from subsets to their immediate supersets. For each subset of length k we have k rules from each of its $k-1$ length subsets to that set. The total number of rules generated is thus $\sum_{i=0}^{l} \binom{l}{i} \cdot (l-i) \leq \sum_{i=0}^{l} \binom{l}{i} \cdot l = O(l \cdot 2^l)$. Thus we get a reduction in the number of rules by of a factor of $O(2^l/l)$, i.e., asymptotically exponential in the length of the longest frequent itemset.

5. EXPERIMENTAL EVALUATION

All experiments described below were performed on a 400MHz Pentium PC with 256MB of memory, running RedHat Linux 6.0. Algorithms were coded in C++. Table 1 shows the characteristics of the real and synthetic datasets used in our evaluation. The real datasets were obtained from IBM Almaden (www.almaden.ibm.com/cs/quest/demos.html). All datasets except the PUMS (pumsb and pumsb*) sets, are taken from the UC Irvine Machine Learning Database Repository. The PUMS datasets contain census data. pumsb* is the same as pumsb without items with 80% or more support. The mushroom database contains characteristics of various species of mushrooms. Finally the connect and chess datasets are derived from their respective game steps. Typically, these real datasets are very dense, i.e., they produce many long frequent itemsets even for very high values of support.

Database	# Items	Record Length	# Records
chess	76	37	3,196
connect	130	43	67,557
mushroom	120	23	8,124
pumsb*	7117	50	49,046
pumsb	7117	74	49,046
T20I12D100K	1000	20	100,000
T40I8D100K	1000	40	100,000
T10I4D100K	1000	10	100,000
T20I4D100K	1000	20	100,000

Table 1: Database Characteristics

We also chose a few synthetic datasets (also available from IBM Almaden), which have been used as benchmarks for testing previous association mining algorithms. These datasets mimic the transactions in a retailing environment. Usually the synthetic datasets are sparse when compared to the real sets. We used two dense and two sparse (the last two rows in Table 1) synthetic datasets for our study.

5.1 Traditional vs. Closed Framework

Consider Table 2 and 3, which compare the traditional rule generation framework with the closed itemset approach proposed in this paper. The tables shows the experimental results along a number of dimensions: 1) total number of frequent itemsets vs. closed frequent itemsets, 2) total number of rules in the traditional vs. new approach, and 3) total time taken for mining all frequent itemsets (using Apriori) and the closed frequent itemsets (using CHARM).

Table 2 shows that the number of closed frequent itemsets can be much smaller than the set of all frequent itemsets. For the support values we look at here, we got reductions (shown in the Ratio column) in the cardinality upto a factor of 45. For lower support values the gap widens rapidly [19]. It is noteworthy, that CHARM finds these closed sets in a fraction of

the time it takes Apriori to mine all frequent itemsets as shown in Table 2. The reduction in running time ranges upto a factor of 145 (again the gap widens with lower support). For the sparse sets, and for high support values, the closed and all frequent set coincide, but CHARM still runs faster than Apriori.

Table 3 shows that the reduction in the number of rules (with all possible consequent lengths) generated is drastic, ranging from a factor of 2 to more than 3000 times! Incidentally, these ratios are roughly in agreement with the complexity formula we presented in Section 4.3. For example, consider the mushroom dataset. At 40% support, the longest frequent itemset has length 7. The complexity figure predicts a reduction in the number of rules by a factor of $2^7/7 = 128/7 = 18$, which is close to the ratio of 15 we got empirically. Similarly for 20% support, we expect a reduction of $2^{15}/15 = 2185$, and empirically it is 3343.

We also computed how many single consequent rules are generated by the traditional approach. We then compared these with the non-redundant rule set from our approach (with possibly multiple consequents). The table also shows that even if we restrict the traditional rule generation to a single item consequent, the reduction with the closed itemset approach is still substantial, with upto a factor of 66 reduction (once again, the reduction is more for lower supports). It is worth noting that, even though for sparse synthetic sets the closed frequent itemsets is not much smaller than the set of all frequent itemsets, we still get upto a factor of 5 reduction in the number of rules generated.

The results above present all possible rules that are obtained by setting *minconf* equal to the *minsup*. Figure 9 shows the effect of *minconf* on the number of rules generated. It shows that most of the rules have very high confidence; as the knee of the curves show, the vast majority of the rules have confidences between 95 and 100 percent! This is a particularly distressing result for the traditional rule generation framework. The new approach produces a rule set that can be orders of magnitude smaller. In general it is possible to mine closed sets using CHARM for low values of support, where it is infeasible to find all frequent itemsets. Thus, even for dense datasets we can generate rules, which may not be possible in the traditional approach.

6. CONCLUSIONS

This paper has demonstrated in a formal way, supported with experiments on several datasets, the well known fact that the traditional association rule framework produces too many rules, most of which are redundant. We proposed a new framework based on closed itemsets that can drastically reduce the rule set, and that can be presented to the user in a succinct manner.

This paper opens a lot of interesting directions for future work. For example we plan to use the concept lattice for interactive visualization and exploration of a large set of mined associations. Keep in mind that the frequent concept lattice is a very concise representation of all the frequent itemsets and the rules that can be generated from them. Instead of generating all possible rules, we plan to generate the rules on-demand, based on the user's interests. Finally, there is the issue of developing a

Database	Sup	Len	Number of Itemsets			Running Time		
			#Freq	#Closed	Ratio	Apriori	CHARM	Ratio
chess	80%	10	8227	5083	1.6	18.54	1.92	9.7
chess	70%	13	48969	23991	2.0	213.03	8.17	26.1
connect	97%	6	487	284	1.7	19.7	4.15	4.7
connect	90%	12	27127	3486	7.8	2084.3	43.8	47.6
mushroom	40%	7	565	140	4.0	1.56	0.28	5.6
mushroom	20%	15	53583	1197	44.7	167.5	1.2	144.4
pumsb*	60%	7	167	68	2.5	11.4	1.0	11.1
pumsb*	40%	13	27354	2610	10.5	847.9	17.1	49.6
pumsb	95%	5	172	110	1.6	19.7	1.7	11.7
pumsb	85%	10	20533	8513	2.4	1379.8	76.1	18.1
T20I12D100K	0.5%	9	2890	2067	1.4	6.3	5.1	1.2
T40I8D100K	1.5%	13	12088	4218	2.9	41.6	15.8	2.6
T10I4D100K	0.5%	5	1073	1073	1	2.0	1.1	1.8
T10I4D100K	0.1%	10	27532	26806	1.03	32.9	8.3	4.0
T20I4D100K	1.0%	6	1327	1327	1	6.7	4.8	2.6
T20I4D100K	0.25%	10	30635	30470	1.01	32.8	10.7	3.1

Table 2: Number of Itemsets and Running Time (Sup=*minsup*, Len=longest frequent itemset)

Database	Sup	Len	All Possible Rules			Rules with one Consequent	
			#Traditional	#Closed	Ratio	#Traditional	Ratio
chess	80%	10	552564	27711	20	44637	2
chess	70%	13	8171198	152074	54	318248	2
connect	97%	6	8092	1116	7	1846	1.7
connect	90%	12	3640704	18848	193	170067	9
mushroom	40%	7	7020	475	15	1906	4.0
mushroom	20%	15	19191656	5741	3343	380999	66
pumsb*	60%	7	2358	192	12	556	3
pumsb*	40%	13	5659536	13479	420	179638	13
pumsb	95%	5	1170	267	4	473	2
pumsb	85%	10	1408950	44483	32	113089	3
T20I12D100K	0.5%	9	40356	2642	15	6681	3
T40I8D100K	1.5%	13	1609678	11379	142	63622	6
T10I4D100K	0.5%	5	2216	1231	1.8	1231	1.0
T10I4D100K	0.1%	10	431838	86902	5.0	90350	1.04
T20I4D100K	1.0%	6	2736	1738	1.6	1738	1.0
T20I4D100K	0.25%	10	391512	89963	4.4	90911	1.01

Table 3: Number of Rules (all vs. consequent of length 1) (Sup=*minsup*, Len=longest itemset)

theory for extracting a base, or a minimal generating set, for all the rules.

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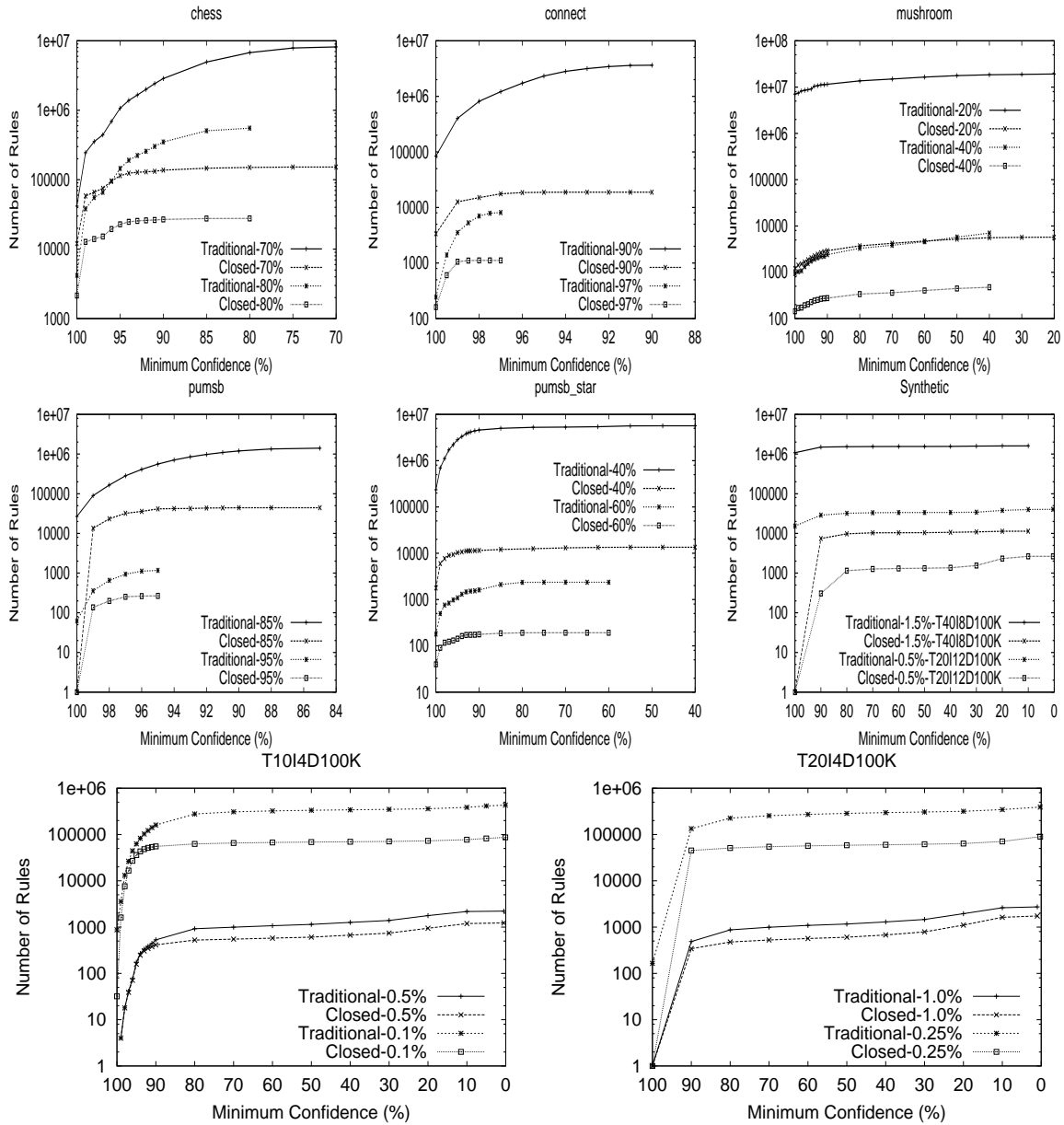
Figure 9: Number of Rules: Traditional vs. Closed Itemset Framework

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