THE *spt*-FUNCTION OF ANDREWS

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Abstract. Recently, Andrews introduced the function $s(n) = spt(n)$ which counts the number of smallest parts among the integer partitions of n. We show that its generating function satisfies an identity analogous to Ramanujan's mock theta identities. As a consequence, we are able to completely determine the parity of $s(n)$. Using another type of identity, one based on Hecke operators, we obtain a complete multiplicative theory for $s(n)$ modulo 3. These congruences confirm unpublished conjectures of Garvan and Sellers. Our methods generalize to all integral moduli.

1. Introduction and Statement of Results

Recently, Andrews [3] introduced the function $s(n) = spt(n)$ which counts the number of smallest parts among the integer partitions of n. As an illustration, one sees that $s(4) = 10$ by examining the partitions of 4 below:

$$\underline{4},\quad 3+\underline{1},\quad \underline{2}+\underline{2},\quad 2+\underline{1}+\underline{1},\quad \underline{1}+\underline{1}+\underline{1}+\underline{1}.$$

The generating function for $s(n)$ is

$$S(z) := \sum_{n=0}^{\infty} s(n)q^n = \frac{1}{(q)_\infty} \cdot \sum_{n=1}^{\infty} \frac{q^n \cdot \prod_{m=1}^{n-1}(1-q^m)}{1-q^n} = q + 3q^2 + 5q^3 + 10q^4 + \cdots, \tag{1.1}$$

where $q := e^{2\pi i z}$, and $(q)_\infty := \prod_{n=1}^{\infty}(1-q^n)$. Andrews obtained an elegant description of $s(n)$ in terms of the partition function $p(n)$, and the 2nd rank moment function[1]

$$N_2(n) := \sum_{m\in\mathbb{Z}} m^2 N(m,n).$$

The *rank* of a partition is its largest part minus its number of parts, and $N(m,n)$ denotes the number of partitions of n with rank m. Andrews proved that

$$s(n) = np(n) - \frac{1}{2}N_2(n). \tag{1.2}$$

In view of the recent works on ranks (for example, see [8, 9] among other subsequent works), one suspects that (1.2) can be placed within the framework of the theory of Maass forms. In a recent preprint, Bringmann [5] indeed produces a harmonic weak Maass form related to $q^{-1}S(24z)$. As in [8, 9], she makes great use of this result to

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[1]Moments were introduced by Atkin and Garvan [4] in their work on partition ranks and cranks.

obtain asymptotics, and congruences modulo primes $\ell > 3$. Recent works by Bringmann, Garvan, and Mahlburg [7, 13] give more results concerning such congruences.

Here we investigate $s(n)$ modulo 2 and 3. To place this in proper context, we note that very little is known about $p(n)$ modulo 2 and 3. This difficulty stems from the fact that there are no known methods of mapping the generating function for $p(n)$ to spaces of holomorphic modular forms mod 2 or 3, where the powerful theory of Galois representations gives results for primes $\ell \geq 5$ (see [1, 2, 15]). Bringmann's work shows that these technical difficulties persist for $s(n)$ modulo 2 and 3.

We circumvent these problems by invoking the "principle" of Ramanujan's mock theta function identities. Loosely speaking, this is the fact that the difference of two harmonic weak Maass forms, whose non-holomorphic parts agree, is a weakly holomorphic modular form (for example, see [10, 12, 14]). Our task is to find such pairs of harmonic weak Maass forms whose difference contains the generating function for $s(n)$ as a component. These pairs provide further relations which are presently unavailable in the case of $p(n)$, thereby allowing us to leverage information such as congruences.

Remark. The methods here apply to all integral moduli ℓ, not just $\ell = 2$ and 3. For brevity, we restrict attention to these troublesome primes.

We require two further mock theta functions. The first arises from the nearly modular Eisenstein series $E_2(z)$. Define $D(z)$ by

$$D(z) := \frac{q^{-\frac{1}{24}}}{(q)_\infty}\left(1 - 24\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}\right) = \frac{q^{-\frac{1}{24}}}{(q)_\infty}E_2(z) = q^{-\frac{1}{24}} - 23q^{\frac{23}{24}} - \cdots . \tag{1.3}$$

The second mock theta function $L(z)$ is defined by

$$\begin{aligned} L(z) &:= \frac{(q^6)_\infty^2(q^{24})_\infty^2}{(q^{12})_\infty^5}\cdot\left(\sum_{n\in\mathbb{Z}}\frac{(12n-1)q^{6n^2-\frac{1}{24}}}{1-q^{12n-1}} - \sum_{n\in\mathbb{Z}}\frac{(12n-5)q^{6n^2-\frac{25}{24}}}{1-q^{12n-5}}\right) \\ &= q^{\frac{23}{24}} + q^{\frac{47}{24}} + q^{\frac{71}{24}} - 4q^{\frac{95}{24}} - 6q^{\frac{119}{24}} + 12q^{\frac{143}{24}} - \cdots . \end{aligned} \tag{1.4}$$

Our first result provides a modular linear combination of these mock theta functions.

Theorem 1.1. *In the notation above, the function*

$$D(24z) - 12L(24z) - 12q^{-1}S(24z) = q^{-1} - 47q^{23} - 142q^{47} - 285q^{71} - 547q^{95} - \cdots$$

is a weight 3/2 weakly holomorphic modular form on $\Gamma_0(576)$ *with Nebentypus* $\left(\frac{12}{\bullet}\right)$.

Remark. Using the Dedekind eta-function $\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n)$, in Section 3 we shall construct a modular function $\widetilde{F}(z)$ with the property that

$$D(24z) - 12L(24z) - 12q^{-1}S(24z) = \widetilde{F}(24z)\cdot q^{23}(-q^{24})_\infty(q^{96})_\infty(q^{144})_\infty(q^{288})_\infty. \tag{1.5}$$

Now we turn to questions of congruence. In [3], Andrews proved that

$$\begin{aligned} s(5n+4) &\equiv 0 \pmod 5, \\ s(7n+5) &\equiv 0 \pmod 7, \\ s(13n+6) &\equiv 0 \pmod{13}. \end{aligned}$$

Modulo primes $\ell > 3$, there are infinitely many congruences of the form

$$s(an+b) \equiv 0 \pmod{\ell}.$$

Bringmann essentially proves this in [5]. To obtain this result, one simply combines her work for $N_2(n)$ with results for $p(n)$ [1, 2, 15] using Lemmas 2.63 and 3.30 of [16]. Garvan [13] has obtained explicit examples modulo 11, and a forthcoming paper by Bringmann, Garvan, and Mahlburg [7] promises further results.

For the moduli $\ell = 2$ and 3, different techniques are required. Theorem 1.1 allows us to determine the parity of $s(n)$, a result that proves phenomena first observed numerically by both Garvan and Sellers.

Theorem 1.2. *We have that $s(n)$ is odd if and only if $24n - 1 = pm^2$, where m is an integer and $p \equiv 23 \pmod{24}$ is prime.*

Remark. Note that $s(n)$ is even for almost all n (in terms of arithmetic density).

Ramanujan-type congruences follow immediately from Theorem 1.2. For example, if $3 < \ell \not\equiv 23 \pmod{24}$ is prime, and if $0 < k < \ell$, then for all n, $s(\ell^2 n + \ell k - (\ell^2-1)/24) \equiv 0 \pmod{2}$. For example, when $\ell = 7$, one has

$$\begin{aligned} s(49n+5) &\equiv s(49n+12) \equiv s(49n+19) \equiv s(49n+26) \\ &\equiv s(49n+33) \equiv s(49n+40) \equiv 0 \pmod{2}. \end{aligned} \tag{1.6}$$

Remark. By (1.2), Theorem 1.2 gives congruences for $N_2(n)$ for even integers n.

Garvan conjectured a surprising multiplicative congruence modulo 3. Here we prove this phenomenon by leveraging a Maass form using the Hecke algebra, a method which plays a central role in recent work of the second author and Bruinier on Heegner divisors and derivatives of modular L-functions (see the proof of Theorem 7.6 of [11]). For notational convenience, if $p \geq 5$ is prime, let

$$\delta(p) := (p^2-1)/24. \tag{1.7}$$

In what follows, we let $\left(\frac{\bullet}{\circ}\right)$ denote the Legendre symbol.

Theorem 1.3. *If $p \geq 5$ is prime, then for every non-negative integer n we have*

$$s(p^2n - \delta(p)) + \left(\frac{3-72n}{p}\right)s(n) + ps\left(\frac{n+\delta(p)}{p^2}\right) \equiv \left(\frac{3}{p}\right)(1+p)s(n) \pmod{3}.$$

Remark. Theorem 1.3 asserts that $q^{-1}S(24z) \pmod{3}$ is a weight 3/2 Hecke eigenform with Nebentypus. We shall see that it is related to $\eta(6z)^4 = \sum_{n=1}^{\infty} a(n)q^n$, and the elliptic curve

$$E: \quad y^2 = x^3 + 1.$$

Remark. A straightforward calculation involving Theorem 1.3, combined with Dirichlet's Theorem on primes in arithmetic progressions, easily implies

$$\#\{0 \leq n \leq X \ : \ s(n) \equiv r \pmod{3}\} \gg \begin{cases} X & \text{if } r \equiv 0 \pmod{3}, \\ \frac{\sqrt{X}}{\log X} & \text{if } r \equiv 1, 2 \pmod{3}. \end{cases}$$

Remark. Theorem 1.3 also yields Ramanujan congruences thanks to the fact that the Legendre symbol satisfies $\left(\frac{n}{p}\right) = 0$ when $p \mid n$. Specifically, let $p > 3$ be a prime such that $p \equiv 2 \pmod 3$. If $0 < k < p-1$, then for all n, $s(p^4 n + p^3 k - (p^4-1)/24) \equiv 0 \pmod 3$. For example, when $p = 5$ we have

$$\begin{aligned} s(625n+99) &\equiv s(625n+224) \\ &\equiv s(625n+349) \equiv s(625n+474) \equiv 0 \pmod 3. \end{aligned}$$

When $3 \mid n$, Theorem 1.3 and (1.2) implies congruences for $N_2(n) \pmod 3$.

In Section 2 we prove Theorem 1.1, and in Section 3 we produce (1.5). In Section 4 we provide a proof of Theorem 1.2, and in Section 5 we conclude with a proof of Theorem 1.3.

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2. The new mock theta identity

Theorem 1.1 follows easily from the lemma in this section which expresses "completed" combinations of the series $L(24z)$, $D(24z)$, and $q^{-1}S(24z)$ as harmonic weak Maass forms. We then use these forms to produce a *weakly holomorphic modular form*, a meromorphic form whose poles (if any) are supported at cusps.

2.1. Preliminaries.

We first recall the notion of a harmonic weak Maass form (see also [6]-[10]). We suppose $k \in \frac{1}{2} + \mathbb{Z}$, and define ϵ_v, for v odd, by

$$\epsilon_v := \begin{cases} 1 & \text{if } v \equiv 1 \pmod 4, \\ i & \text{if } v \equiv 3 \pmod 4. \end{cases} \tag{2.1}$$

As usual, we let

$$\Delta_k := -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right) \tag{2.2}$$

be the weight k hyperbolic Laplacian, where $z = x + iy$ with $x, y \in \mathbb{R}$. If N is a positive integer and $\psi \pmod{4N}$ is a Dirichlet character, then a *harmonic weak Maass form* of weight k on $\Gamma_0(4N)$ with Nebentypus ψ is any smooth function $f : \mathbb{H} \to \mathbb{C}$ satisfying:

(1) For all $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4N)$ and all $z \in \mathbb{H}$, we have

$$f(Az) = \psi(d) \left(\frac{c}{d}\right)^{2k} \epsilon_d^{-2k} \, (cz+d)^k \, f(z).$$

(2) We have that $\Delta_k f = 0$.

(3) The function $f(z)$ has at most linear exponential growth at all cusps.

Ramanujan's "Lost" Notebook contains a number of identities involving Ramanujan's original mock theta functions. By the "mock theta conjectures", we refer to the surprising identities involving the 5th order mock theta functions. These identities remained unproven until the important work of Hickerson [14] in 1988.

Recently, a complete understanding of the modular properties of Ramanujan's mock theta functions (see [8], [9], [18] [19]) has been obtained. These functions are holomorphic projections of harmonic weak Maass forms. Thanks to this theory, we have a conceptual understanding of the phenomenon of the mock theta conjectures. Such identities arise when two harmonic weak Maass forms share a non-holomorphic part. Folsom [12] has obtained a short proof of the original identities arguing in this way, and Bringmann, Ono, and Rhoades [10] have developed this further to obtain many infinite families of modular forms as Eulerian series. Theorem 1.1 is a new mock theta identity.

2.2. **Proof of Theorem 1.1.** Theorem 1.1 follows immediately from the next lemma.

Lemma 2.1. *The two real analytic functions*

$$\mathcal{M}(z) := q^{-1}S(24z) - \frac{D(24z)}{12} - \frac{i}{4\pi\sqrt{2}} \int_{-\overline{z}}^{i\infty} \frac{\eta(24\tau)}{(-i(\tau+z))^{\frac{3}{2}}} d\tau,$$
$$\mathcal{N}(z) := L(24z) + \frac{i}{4\pi\sqrt{2}} \int_{-\overline{z}}^{i\infty} \frac{\eta(24\tau)}{(-i(\tau+z))^{\frac{3}{2}}} d\tau$$

are weight 3/2 harmonic weak Maass forms on $\Gamma_0(576)$ *with Nebentypus* $\left(\frac{12}{\bullet}\right)$.

Remark. We refer to the period integrals

$$\pm \frac{i}{4\pi\sqrt{2}} \int_{-\overline{z}}^{i\infty} \frac{\eta(24\tau)}{(-i(\tau+z))^{\frac{3}{2}}} d\tau$$

as the *non-holomorphic parts* of these Maass forms.

Remark. We note that Lemma 2.1 corrects a sign error in front of the non-holomorphic part of $\mathcal{M}(z)$ in [5].

Proof. The claim about $\mathcal{M}(z)$ follows from [5, Thm. 1.1], (1.2), and the identity

$$\sum_{n=0}^{\infty} np(n)q^n = \frac{1}{(q)_\infty} \sum_{n=1}^{\infty} \frac{nq^n}{1-q^n} = \frac{1-E_2(z)}{24(q)_\infty}.$$

The claim about $\mathcal{N}(z)$ is easily derived from [6, Thm. 2.1, Prop. 3.1], the fact that $\psi_{a,b}(\tau) = \psi_{b-a,b}(\tau)$ for $0 < a < b$ as defined in [6, Eqn. (1.20)], and the two identities

$$\eta(24z) = \sum_{k\in\mathbb{Z}} (-1)^k q^{(6k+1)^2} \quad \text{and} \quad \Theta_0(z) = \sum_{n\in\mathbb{Z}} q^{n^2} = \frac{\eta(2z)^5}{\eta(z)^2\eta(4z)^2}.$$

□

3. Explicit Description of the modular form

By Theorem 1.1, we have that

$$\eta(24z)\cdot(D(24z) - 12L(24z) - 12q^{-1}S(24z)) = 1 - 48q^{24} - 96q^{48} - 96q^{72} - \cdots$$

is a weight 2 weakly holomorphic modular form on $\Gamma_0(576)$ whose non-zero Fourier coefficients are supported on multiples of 24. By standard facts (for example, see [16]), it equals $\Omega(24z)$ for some modular form $\Omega(z)$ on $\Gamma_0(24)$. We obtain an explicit description of $\Omega(z)$. To this end we define modular functions $F_1(z)$ and $F_2(z)$. First we let

$$\begin{aligned}
f_1(z) &:= \frac{\eta(2z)^2\eta(4z)^2}{\eta(z)\eta(3z)\eta(8z)\eta(24z)} = q^{-1} + 1 + 2q^2 - q^3 - 2q^4 + 4q^5 - \cdots,\\
f_2(z) &:= \frac{\eta(4z)^4\eta(6z)^4}{\eta(z)\eta(2z)^2\eta(3z)\eta(8z)\eta(12z)^2\eta(24z)} = q^{-1} + 1 + 4q + 6q^2 + 11q^3 + \cdots,\\
f_3(z) &:= \frac{\eta(2z)^3\eta(3z)^9}{\eta(z)^3\eta(6z)^9} = q^{-1} + 3 + 6q + 4q^2 - 3q^3 - 12q^4 - 8q^5 + \cdots,\\
f_4(z) &:= \frac{\eta(z)^2\eta(3z)^2}{\eta(4z)^2\eta(12z)^2} = q^{-1} - 2 - q + 7q^3 - 9q^5 + \cdots,\\
f_5(z) &:= \frac{\eta(2z)^2\eta(4z)^2\eta(6z)^2\eta(12z)^2}{\eta(z)^2\eta(3z)^2\eta(8z)^2\eta(24z)^2} = q^{-1} + 2 + 3q + 8q^2 + 11q^3 + 16q^4 + 31q^5 + \cdots.
\end{aligned}$$

As usual, if m is a positive integer, let $U(m)$ denote the Atkin-U-operator defined by

$$\left(\sum b(n)q^n\right) \mid U(m) := \sum b(mn)q^n. \tag{3.1}$$

The modular function $F_1(z)$ is given by

(3.2)
$$\begin{aligned}
F_1(z) = \sum_{n=-1}^{\infty} A_1(n)q^n &:= -6f_1 - 28f_1|U(2) + 24f_1|U(3) - 18f_2 + 14f_3 + 15f_4 - 4f_5\\
&= q^{-1} - 24 - 95q - 88q^2 - 213q^3 - 856q^4 - 1003q^5 - 904q^6 - 1694q^7 - \cdots.
\end{aligned}$$

To define $F_2(z)$, we require

$$f_6(z) := \frac{\eta(4z)^4\eta(6z)^2}{\eta(2z)^2\eta(12z)^4} = q^{-1} + 2q + q^3 - 2q^7 - \cdots,$$

$$f_7(z) := \frac{\eta(z)^4\eta(4z)^4\eta(6z)^4}{\eta(2z)^4\eta(3z)^4\eta(12z)^4} = q^{-1} - 4 + 6q - 4q^2 - 3q^3 + 12q^4 - 8q^5 - \cdots,$$

$$f_8(z) := \frac{\eta(3z)^4\eta(4z)^4}{\eta(z)^4\eta(12z)^4} = q^{-1} + 4 + 14q + 36q^2 + 85q^3 + 180q^4 + 360q^5 + \cdots,$$

$$f_9(z) := \frac{\eta(z)^4\eta(2z)^4}{\eta(3z)^4\eta(6z)^4} = q^{-1} - 4 - 2q + 28q^2 - 27q^3 - 52q^4 + 136q^5 - \cdots.$$

Using these functions and $F_1(z)$, we define $F_2(z)$ by

(3.3)

$$\begin{aligned} F_2(z) = \sum_{n=-1}^{\infty} A_2(n)q^n &:= 76 + F_1|U(2) - 2f_6 + 2f_7 + 6f_8 - 20f_3 + 14f_9 \\ &= -48 - 144q - 336q^2 - 720q^3 - 1584q^4 - 3168q^5 - 6192q^6 - 11520q^7 - \cdots. \end{aligned}$$

Using the coefficients of $F_1(z)$ and $F_2(z)$, define $\widetilde{F}(z)$ by

(3.4)

$$\begin{aligned} \widetilde{F}(z) :&= \sum_{n=0}^{\infty} A_1(2n-1)q^{2n-1} + F_2(2z) \\ &= q^{-1} - 48 - 95q - 144q^2 - 213q^3 - 336q^4 - 1003q^5 - 720q^6 - 1694q^7 - \cdots. \end{aligned}$$

Although these harmonic weak Maass forms are proven to be of level 576, all of their images under $\mathrm{SL}_2(\mathbb{Z})$ are known. For brevity they are not given here. Nevertheless, the holomorphic parts of these images are always reciprocals of explicit weight 1/2 theta functions (which are eta-products) multiplied with explicit Lambert-type series (see [5, 6]). By considering poles at cusps for these series (using the valence formula), it turns out that

$$\Omega(z) = \widetilde{F}(z) \cdot \eta(2z)\eta(4z)\eta(6z)\eta(12z),$$

which in turns implies that

$$D(24z) - 12L(24z) - 12q^{-1}S(24z) = \frac{\widetilde{F}(24z) \cdot \eta(48z)\eta(96z)\eta(144z)\eta(288z)}{\eta(24z)}. \tag{3.5}$$

Remark. It would be very interesting to know whether there is a simpler description of $\widetilde{F}(z)$ involving just one or two "natural" modular functions.

4. Parity of $s(n)$

We first reduce the proof of Theorem 1.2 to an elementary congruence.

Lemma 4.1. *Theorem 1.2 is equivalent to the congruence*

$$q^{-1}S(24z) \equiv L(24z) \pmod 2.$$

Proof. It suffices to prove that

$$L(24z) \equiv \sum_{p\equiv 23 \pmod{24} \text{ prime}} \sum_{\substack{m\geq 1\\ \gcd(m,6)=1}} q^{pm^2} \pmod 2.$$

Since

$$\frac{(q^6)^2_\infty (q^{24})^2_\infty}{(q^{12})^5_\infty} \equiv 1 \pmod 2,$$

it follows that

$$\begin{aligned} L(24z) &\equiv \sum_{n\in\mathbb{Z}} \frac{q^{144n^2-1}}{1-q^{288n-24}} + \sum_{n\in\mathbb{Z}} \frac{q^{144n^2-25}}{1-q^{288n-120}} \pmod 2 \\ &\equiv \sum_{\substack{m\geq 1\\ -12m<d<0\\ \gcd(d,6)=1}} q^{144m^2-d^2} \equiv \sum_{\substack{m\geq 1\\ -12m<d<0\\ \gcd(d,6)=1}} q^{(12m+d)(12m-d)} \pmod 2. \end{aligned}$$

The proof now follows easily by rewriting the expression $(12m+d)(12m-d)$ as a sum over divisors of integers. □

Proof of Theorem 1.2. By Lemma 4.1, it suffices to prove that

$$q^{-1}S(24z) \equiv L(24z) \pmod 2.$$

By Theorem 1.1 and (1.5) (or (3.5)), it suffices to show that

$$D(24z) \equiv \frac{\widetilde{F}(24z)\cdot \eta(48z)\eta(96z)\eta(144z)\eta(288z)}{\eta(24z)} \pmod{24}.$$

Since $\eta(24z) = \sum_{n=1}^{\infty} \left(\frac{12}{n}\right) q^{n^2} = q + \cdots$, and since $\eta(24z)D(24z) \equiv 1 \pmod{24}$, we must show that

$$\widetilde{F}(24z)\cdot \eta(48z)\eta(96z)\eta(144z)\eta(288z) = 1 - 48q^{24} - 96q^{48} - \cdots \equiv 1 \pmod{24}.$$

Since the non-zero coefficients of this form are supported on exponents which are multiples of 24, it follows that $\Psi(z) := \widetilde{F}(z)\cdot \eta(2z)\eta(4z)\eta(6z)\eta(12z)$ is a weakly holomorphic modular form of weight 2 on $\Gamma_0(24)$. From the definitions in the introduction, or a simple analysis of the orders at cusps of the form $\widetilde{F}(z)$, it follows that $\Psi(z)\eta(z)^{24}E_4(z)E_6(z)$ is a weight 24 cusp form on $\Gamma_0(24)$, where $E_4(z)$ and $E_6(z)$ are the classical Eisenstein series

$$E_4(z) = 1 + 240\sum_{n=1}^{\infty}\sum_{d\mid n} d^3 q^n \qquad \text{and} \qquad E_6(z) = 1 - 504\sum_{n=1}^{\infty}\sum_{d\mid n} d^5 q^n.$$

Since $E_4(z) \equiv E_6(z) \equiv 1 \pmod{24}$, the claim is implied by the following congruence between two weight 24 modular forms:

$$\Psi(z)\eta(z)^{24}E_4(z)E_6(z) \equiv \eta(z)^{24}E_6(z)^2 \pmod{24}.$$

This is easily verified by a result of Sturm (for example, see Th. 2.58 of [16]). □

5. Sketch of the proof of Theorem 1.3

Since many of the methods of the proof already appear previously in this paper, and since this theorem is merely about congruences rather than explicit formulas, for brevity we provide just a brief sketch of the proof.

The classical weight 3/2 Hecke operators $T(m^2)$ act (for example, see Section 7 of [11]) on the Maass form $\mathcal{M}(z)$ from Lemma 2.1. Since $\eta(24z)$ is a Hecke eigenform, it follows that the non-holomorphic part of this Maass form is also an eigenform with explicit eigenvalues. Combined with elementary facts about p-adic modular forms (e.g. that E_2 is a weight two 3-adic modular form, and the algebra structure of p-adic modular forms [17]), we may, for every prime $p \geq 5$, combine $\mathcal{M}(z)$ with $\mathcal{M}(z)|T(p^2)$ to obtain a weakly holomorphic modular form. Numerical calculations, combined with Hecke theory and standard facts about p-adic modular forms reveal that

$$q^{-1}S(24z)|T(25) \equiv 0 \pmod 3,$$
$$q^{-1}S(24z)|T(49) \equiv q^{-1}S(24z) \pmod 3.$$

Such formulas, combined with the corresponding Hecke action on the other summands of $\mathcal{M}(z)$, allow us to conclude that $q^{-1}S(24z)$ is the reduction modulo 3 of a weakly holomorphic modular form. It turns out to be the reduction of a cusp form whose image modulo 3 under the Shimura correspondence is the 12-quadratic twist of the newform $\eta(6z)^4 = \sum_{n=1}^{\infty} a(n)q^n \in S_2(\Gamma_0(36))$. The Shimura correspondence is Hecke equivariant, and so it suffices to show, for primes $p \geq 5$, that $a(p) \equiv 1 + p \pmod 3$.

The newform $\eta(6z)^4$ corresponds to the $\mathbb{Q}(\sqrt{-3})$ CM elliptic curve

$$E: \quad y^2 = x^3 + 1.$$

The desired congruence follows by the theory of Hecke Grössencharakters, which in this case is equivalent to multiplying out q-series identities of Euler and Jacobi. One uses the fact that the ring of integers of $\mathbb{Q}(\sqrt{-3})$ is a UFD. Alternatively, one may employ the fact that the reduction map modulo p for elliptic curves is injective on torsion. In this case, the proof follows since the torsion subgroup of $E/\mathbb{Q}$ has non-trivial 3-rank. In particular, the torsion subgroup is $\mathbb{Z}/6\mathbb{Z}$, which implies that $a(p) \equiv 1 + p \pmod 6$.

References

[1] S. Ahlgren, *Distribution of the partition function modulo composite integers M*, Math. Annalen, **318** (2000), pages 795-803.

[2] S. Ahlgren and K. Ono, *Congruence properties for the partition function*, Proc. Natl. Acad. Sci., USA **98**, No. 23 (2001), pages 12882-12884.

[3] G. E. Andrews, *The number of smallest parts in the partitions of n*, J. reine Angew. Math., accepted for publication.

[4] A. O. L. Atkin and F. Garvan, *Relations between the ranks and cranks of partitions*, Ramanujan J. **7** (2003), pages 343-366.

[5] K. Bringmann, *On the explicit construction of higher deformations of partition statistics*, Duke Math. J., **144** (2008), pages 195-233.

[6] K. Bringmann, A. Folsom, and K. Ono, q-series and weight $3/2$ Maass forms, preprint.
[7] K. Bringmann, F. Garvan, and K. Mahlburg, *Partition statistics and quasiweak Maass forms*, Int. Math. Res. Notices, accepted for publication.
[8] K. Bringmann and K. Ono, *Dyson's ranks and Maass forms*, Ann. of Math., in press.
[9] K. Bringmann and K. Ono, *The $f(q)$ mock theta function conjecture and partition ranks*, Invent. Math. **165** (2006), pages 243-266.
[10] K. Bringmann, K. Ono, and R. Rhoades, *Eulerian series as modular forms*, J. Amer. Math. Soc., **21** (2008), pages 1085-1104.
[11] J. Bruinier and K. Ono, *Heegner divisors, L-functions, and harmonic weak Maass forms*, preprint.
[12] A. Folsom, *A short proof of the mock theta conjectures using Maass forms*, Proc. Amer. Math. Soc., **136** (2008), pages 4143-4149.
[13] F. Garvan, *Congruences for Andrews' smallest parts partition function and new congruences for Dyson's rank*, Int. J. Number Th., accepted for publication.
[14] D. Hickerson, *A proof of the mock theta conjectures*, Invent. Math. **94** (1988), pages 639-660.
[15] K. Ono, *Distribution of the partition function modulo m*, Ann. of Math. **151** (2000), pages 293-307.
[16] K. Ono, *The web of modularity: arithmetic of the coefficients of modular forms and q-series*, Conference Board of the Mathematical Sciences **102**, Amer. Math. Soc. (2004).
[17] J.-P. Serre, *Formes modulaires et fonctions zêta p-adiques*, Springer Lect. Notes in Math. **350** (1973), pages 191-268.
[18] S. P. Zwegers, *Mock theta functions*, Ph.D. Thesis, Universiteit Utrecht, 2002.
[19] S. P. Zwegers, *Mock ϑ-functions and real analytic modular forms*, q-series with applications to combinatorics, number theory, and physics (Ed. B. C. Berndt and K. Ono), Contemp. Math. **291**, Amer. Math. Soc., (2001), pages 269-277.

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