Optimal Sequential Detection with Stochastic Energy Constraint

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***Abstract*—The sequential detection problem in wireless sensors powered by energy harvested from the enviroment is considered. Assuming that a unit of energy arrives with probability p at each time instant, two different problem setups, namely minimizing the weighted sum of detection delay and error probabilities and minimizing the average detection delay subject to error probabilities constraints, are studied. Optimal decision rules including energy allocation, stopping rule and terminal decision rules are derived for both setups. For both setups, we show that the sensor will take samples immediately after energy arrives. However, the numbers of samples that the sensor will take for these two setups behave differently as p changes. For the first setup, the number of samples increases as p increase. For the second setup, the number of samples the sensor will take is the same for different values of p.**

I. Introduction

The green concept is a rapidly growing idea in a wide spread of industrial and technological field. Energy harvesting wireless sensor network is a promising green solution in wireless communication field. Rather than using batteries, these sensors use energy harvester to convert the ambient energy, such as solar, mechanical or thermal energy to electrical power [1]. Apart from their energy being cleaner, the energy harvesters also reduce the cost associated with powering systems and extend the life expectancy of wireless sensors. However, the energy received in each time instant is a random variable, which brings new optimization problems and challenges. Signal detection is a basic application for wireless sensor networks. It is widely used in structural health monitoring (SHM) [2], environment monitoring [3], industrial monitoring [4], cognitive radio, etc. Hence, it is critical to design optimal detection schemes for sensors powered by random arriving energies.

In this paper, we revisit the classic sequential detection problem [5] in an energy harvester wireless sensor. The wireless sensor observes an independent and identically distributed (i.i.d.) sequence generated by one of two distributions Q_0 or Q_1 and wishes to test hypothesis H_1 that the sequence generated by Q_1 against hypothesis H_0 that the sequence generated by Q_0. Each observation consumes a unit of energy collected by the energy harvester. The whole system must be operated under the causal energy constraint, namely energy cannot be spent before it arrives. We assume the random energy arriving conforms to the Bernoulli distribution with probability p at each time instant. Our goal is to minimize a objective function related to detection delay and detection errors by optimizing: 1) energy allocation schemes that decide how to spend the energy; 2) stopping rules that determine when to stop the test; and 3) terminal decision rules that make final hypothesis decisions. We note that the classic sequential detection problem is a special case of the problem considered here with $p = 1$.

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In this paper, we consider two setups for the objective function. Each of them is motivated by many applications. The first setup is to minimize a weighted sum of the average detection delay and error probabilities. The second setup is to minimize the average detection delay subject to constraints on the error probabilities. Under both setups, we show that it is optimal to immediately take samples once energy arrives. Furthermore, we show that the celebrated SPRT algorithm is optimal for both scenarios. However, the number of samples the sensor will take respect to the energy arriving rate p behaves differently for these two setups. For the first setup, the optimal solution always makes a tradeoff between the cost of average time delay and the cost of error probabilities. When the p is small, the sensor tends to make a smaller number of samples. This is due to the fact that if p is small, the delay between two energy arriving (and hence two allowable observations) is large, and hence the sensor tends to make a smaller number observations to avoid excessive cost on the delay. As p increases, the delay between two observations decreases, and the sensor tends to take more samples to reduce the decision errors. On the other hand, for the second setup, the average number of samples that will take is determined only by the error probability requirements but not the energy arriving probability p. This is due to the fact that the number of samples the sensor has to take to satisfy the error probability constraints is independent of the energy arriving process.

The remainder of the paper is organized as follows. The mathematical model is given in Section II. Section III presents the solution for these two setups; and in Section IV, we discuss the performance of these optimal solutions. Numerical examples are given in Section V. Finally, Section VI offers concluding remarks.

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II. Model

We consider a statistical sequence $\{X_i : i = 1, 2, \ldots\}$ that consists of real and independent and identically distributed (i.i.d.) random variables. We assume that the sequence is generated from one of the two statistical hypotheses:

$$H_0 : X_i \sim Q_0, \quad i = 1, 2, \ldots$$

versus

$$H_1 : X_i \sim Q_1, \quad i = 1, 2, \ldots$$

where the index i is time instant, Q_0 and Q_1 are two distinct but equivalent distributions on $(\mathbb{R}, \mathcal{B})$. We use P_0 and P_1 to denote the measure probability for Q_0 and Q_1, respectively. We assume the null hypothesis H_0 occurs with prior probability $1-\pi_0$ and the alternative hypothesis H_1 with prior π_0.

We consider the sequential detection setup to determine which hypothesis to be accepted. After making each observation, the system can take one of the following three actions: 1) To accept the H_0. 2) To accept the H_1. 3) To continue the process by taking an additional observation. If decision 1) or 2) is made, the process is terminated. We use τ to denote the stopping rule used by the sensor to decide whether the test should be terminated or not, and δ_τ to denote the terminal decision rule that will decide which H_i is true. For the wireless sensor, each observation will consume a unit of energy. We assume that the wireless sensor does not have energy at the initial time $i = 0$, and all the energies needed are collected by the energy harvester. We further assume that the energy harvester works independently in each time instant, and the energy arrives randomly which obeys Bernoulli distribution, i.e., if we denote S_i as the performance of the energy harvester at time instant i, we have

$$S_i = \begin{cases} 0 & \text{with probability } 1-p \\ 1 & \text{with probability } p \end{cases},$$

where $S_i = 1$ represents the energy harvester collects a unit of energy from the ambient environment, while $S_i = 0$ means the energy harvester gets nothing.

The wireless sensor can decide how to allocate these collected energies. For example, the energy can be spent on taking observation as soon as it is harvested; or the energy can be stored in a rechargeable battery for future usage. However, the energy allocation must obey the causality constraint, that is *the energy cannot be used before it is harvested.* We denote the energy allocation scheme as μ_i, whose value is taken from $\{0, 1\}$. $\mu_i = 1$ means that the wireless sensor spends a unit of energy on taking observation at time i, while $\mu_i = 0$ means no energy is spent (or no observation is taken). The energy constraint can be represented as

$$\sum_{j=1}^{i} S_j \geq \sum_{j=1}^{i} \mu_j \quad i = 1, 2, \ldots. \tag{1}$$

The whole procedure is illustrated in Figure 1. The sequence $\{a_l\}$ denotes the time instants when energies arrive, i.e., $S_{a_l} = 1$. The sequence $\{b_l\}$ denotes the time instants when observations are taken, i.e., $\mu_{b_l} = 1$. The sequence observed by the wireless sensor is denoted as $\left\{X_l^{(a_l,b_l)}, l = 1, 2, \ldots, k\right\}$, which means the l^{th} observation $X_l^{(a_l,b_l)}$ is taken at time b_l while the consumed energy arrives at time a_l.

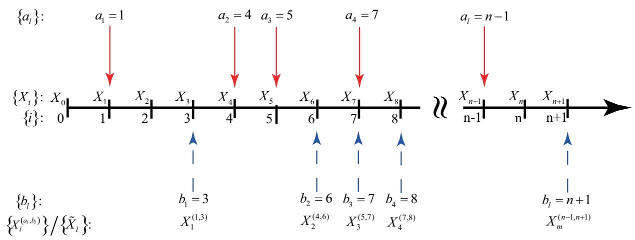

Fig. 1: A typical operation of the sensor powered by energy harvested

Two different setups, referred as Setup I and Setup II, are considered in this paper. In Setup I, we try to minimize a weighted sum of average detection delay and error probabilities. The cost caused by error decision making can be represented as

$$R(\delta_\tau, \tau) = (1-\pi_0)c_{10}P_0(\delta_\tau = 1) + \pi_0 c_{01} P_1(\delta_\tau = 0),$$

where c_{01} and c_{10} denotes the cost of Type I error and Type II error, respectively. There is a linear cost $c > 0$ for the time delay. The problem under Setup I can be described as

$$\begin{aligned} &\inf_{\{\mu_i\},\tau,\delta_\tau} \left\{ c\mathbb{E}\{\tau\} + R(\delta_\tau, \tau) \right\}, \\ &\text{s.t.} \quad \sum_{j=1}^{i} S_j \geq \sum_{j=1}^{i} \mu_j \quad i = 1, 2, \ldots. \end{aligned} \tag{2}$$

In Setup II, we try to minimize the average time delay under given error probabilities. The optimization problem can be stated as follows:

$$\begin{aligned} &\min_{\{\mu_i\},\tau,\delta_\tau} \mathbb{E}\{\tau\}, \\ &\text{s.t.} \quad P_0(\delta_\tau = 1) \leq \alpha, \\ &\qquad P_1(\delta_\tau = 0) \leq \beta, \\ &\qquad \sum_{j=1}^{i} S_j \geq \sum_{j=1}^{i} \mu_j \quad i = 1, 2, \ldots \end{aligned} \tag{3}$$

III. The optimal solutions

A. The optimal energy allocation scheme

We first show that for both setups, the optimal power allocation scheme is to use the energy as soon as they arrive.

Lemma 3.1: For the optimization problem (2) and (3), the optimal energy allocation strategy is to spend the energy on taking observation as soon as the energy is harvested, i.e. $\mu_i = S_i$ for any $i = 1, \ldots, \tau$ or $a_l = b_l, l = 1, \ldots, k$.

Proof: We give the proof for Setup I. The proof for Setup II follows a similar procedure, which we omit here. For any $\{a_l\}$ and $\{b_l\}$ we have the inequalities: $a_l < a_{l+1}$, $b_l < b_{l+1}$ and $b_l \geq a_l$.

For an arbitrary but given $\{a_l\}$, we consider two different sample sequences, say $\left\{X_l^{(a_l,b_l)}\right\}$ and $\left\{X_l^{(a_l,a_l)}\right\}$, produced by different power allocation schemes. These two samples have the same statistical property since they have the same joint cumulative distribution function (cdf), i.e.,

$$\begin{aligned} & F_b(X_1^{(a_1,b_1)} \le x_1, \dots, X_l^{(a_l,b_l)} \le x_l, \dots) \\ = & F_a(X_1^{(a_1,a_1)} \le x_1, \dots, X_l^{(a_l,a_l)} \le x_l, \dots), \quad l = 1, 2, \dots, \end{aligned}$$

Therefore, $\left\{X_l^{(a_l,b_l)}\right\}$ and $\left\{X_l^{(a_l,a_l)}\right\}$ can have the same optimal stopping time k, which we will use in the following. Further,

$$P_b(k = j) = P_a(k = j), \quad j = 1, 2, \dots,$$

where P_a and P_b are the probability measurement of k under the samples $\left\{X_l^{(a_l,b_l)}\right\}$ and $\left\{X_l^{(a_l,a_l)}\right\}$, respectively. For a given but arbitrary stopping time $k \in \mathcal{K}$, it has been proved the optimal terminal decision rule is

$$\delta_k^o = \begin{cases} 1 & \text{if } \pi_k \ge \frac{c_{01}}{c_{01}+c_{10}} \\ 0 & \text{if } \pi_k \le \frac{c_{01}}{c_{01}+c_{10}} \end{cases}$$

and the corresponding cost of error decision is

$$\inf_{\delta \in \mathcal{D}} R(k, \delta) = \mathbb{E}\left\{\min\left\{c_{10}\pi_k, c_{01}(1 - \pi_k)\right\}\right\}.$$

Therefore, $\left\{X_l^{(a_l,b_l)}\right\}$ and $\left\{X_l^{(a_l,a_l)}\right\}$ have the same $\inf_{\delta \in \mathcal{D}} R(k, \delta)$.

Now, we consider the total cost of these two different energy power allocation schemes. The average time delay caused by $\left\{X_l^{(a_l,b_l)}\right\}$ is

$$\begin{aligned} \mathbb{E}(\tau_b) &= \mathbb{E}(b_k) = \sum_{j=1}^{\infty} \mathbb{E}(b_j | k = j) P_b(k = j) \\ &\ge \sum_{j=1}^{\infty} \mathbb{E}(a_j | k = j) P_a(k = j) = \mathbb{E}(a_k) = \mathbb{E}(\tau_a), \end{aligned}$$

which is larger than the average time delay caused by $\left\{X_l^{(a_l,a_l)}\right\}$. But the cost of decision error

$$\begin{aligned} \inf R(\delta, \tau_b) &= \inf R(\delta, b_k) = \inf R(\delta, k) \\ &= \inf R(\delta, a_k) = \inf R(\delta, \tau_a) \end{aligned}$$

are the same. Therefore, the average cost of $\left\{X_l^{(a_l,b_l)}\right\}$ is no less than that of $\left\{X_l^{(a_l,a_l)}\right\}$. The equality holds only when $b_l = a_l$, or equivalently $\mu_i = S_i$ for any i. This completes the proof. ■

According to the above lemma, we can omit the upper index of $\left\{X_l^{(a_l,b_l)}\right\}$ and denote it as $\left\{\tilde{X}_l\right\}$.Obviously, if $\tilde{X}_k$ is the last observation to taken by wireless sensor, the time delay τ for decision making is $\tau = a_k$.

Denote $N_l = a_l - a_{l-1}$ as the time interval between two successive observations, the probability mass function of N_l

$$P(N_l = n) = (1 - p)^{n-1} p,$$

and the average time delay between two successive observations is

$$\mathbb{E}(N) = \sum_{n=1}^{\infty} n * P(N = n) = \frac{1}{p}.$$

Therefore, the average time delay can be represented as

$$\begin{aligned} \mathbb{E}(\tau) &= \mathbb{E}(a_k) = \mathbb{E}\left(\sum_{l=1}^{k}(a_l - a_{l-1})\right) \\ &= \mathbb{E}\left(\sum_{l=1}^{k} N_l\right) = \frac{1}{p}\mathbb{E}(k), \end{aligned} \tag{4}$$

The last equality is the well known Wald Equation. Therefore, to study the average time delay τ is equivalent to study the stopping time k.

B. The problem under Setup I: total cost minimization

Using the conclusion of Lemma 3.1 and (4), the objective function (2) can be rewritten as

$$\inf_{k \in \mathcal{K} \delta_k \in \mathcal{D}} \left\{\frac{c}{p}\mathbb{E}(k) + R(\delta_k, k)\right\}. \tag{5}$$

This problem is a canonical sequential problem, which has been solved in [6]. For the completeness of this paper, we cite the important expressions and conclusions. This problem can be converted to a Markov optimal stopping problem:

$$\begin{aligned} & J_l(\pi_l) = \\ & \min\left\{\min\left\{c_{01}\pi_l, c_{10}(1 - \pi_l)\right\}, \frac{c}{p} + \mathbb{E}\left\{J_{l+1}(\pi_{l+1}) | \pi_l\right\}\right\}, \end{aligned}$$

where $J_l(\pi_l)$ is the minimum expected cost at the l^{th} observation and π_l is the posterior probability that the sequence is generated by Q_1 after observing $\left\{\tilde{X}_1, \dots, \tilde{X}_l\right\}$. The optimal stopping rule is described in the following theorem:

Theorem 3.2: For the sequential detection problem described in (5), the optimal stopping time is

$$k_{opt} = \inf\left\{k > 0 : \pi_k \notin \left[\pi^L, \pi^U\right]\right\}, \tag{6}$$

where π^L and π^U can be decided by

$$\begin{aligned} \pi^L &= \sup\left\{0 \le \pi \le 1/2 : c_{01}\pi = J_0(\pi_0)\right\} \\ \pi^U &= \inf\left\{1/2 \le \pi \le 1 : c_{10}(1 - \pi) = J_0(\pi_0)\right\} \end{aligned}$$

C. The problem under Setup II: average time delay minimization

By Lemma 3.1, the problem (3) can be rewritten as

$$\begin{aligned} & \min \mathbb{E}\left\{k\right\}, \\ \text{s.t.} \quad & P_0(\delta_k = 1) \le \alpha \\ & P_1(\delta_k = 0) \le \beta, \end{aligned} \tag{7}$$

which is a canonical problem studied in [6]. The optimal solution is the well known sequential probability ratio test (SPRT). The optimal stopping rule is given as

$$k_{opt} = \inf\left\{k > 0 : \Lambda_k \notin [A, B]\right\}, \tag{8}$$

where, $\Lambda_k = \Pi_{l=1}^{k} \frac{q_1(\tilde{X}_l)}{q_0(\tilde{X}_l)}$ is the likelihood ratio of the first k observations received by the wireless sensor. The boundary A and B are approximately given as $A \approx \beta/(1-\alpha)$ and $B \approx (1-\beta)/(\alpha)$.

IV. Performance Analysis

In the previous section, we developed the optimal power allocation, the optimal stopping strategies and decision rules for both setups. It is easy to see that their performances depend on the energy harvesting probability p. In this section, we study their performances with respect to p.

We first consider the performance of the problem under Setup II. Its error probabilities are fixed, and the average time delay can be represented as

$$\mathbb{E}_i(\tau_{opt}) = \frac{1}{p}\mathbb{E}_i(T), \quad i = 1, 2, \tag{9}$$

where, $\mathbb{E}_i(\cdot)$ denotes the average under the hypothesis H_i, $i = 1, 2$, and T is the stopping time when $p = 1$. The expression of $\mathbb{E}_i(T)$ is studied in [6]. From (9), the average time delay is inversely proportional to p. Because the system needs sufficient observations to satisfy the error probability requirements, the smaller p is, the more difficult it is to take an observation, the longer, therefore, the average time delay is needed.

The performance of the problem under Setup I is difficult to evaluate by a close form formula. Here, we illustrate the relationship between p and its performance by a numerical calculation. In the numerical calculation, we adopt the equal priors and uniform costs, i.e., $\pi_0 = 0.5$ and $c_{01} = c_{10} = 1$. The statistical sequence is generated from Gaussian distribution $N(0, \sigma^2)$ and we want to test the hypothesis $H_0 : \sigma^2 = 1$ against the hypothesis $H_1 : \sigma^2 = 5$. We choose $c = 0.01$.

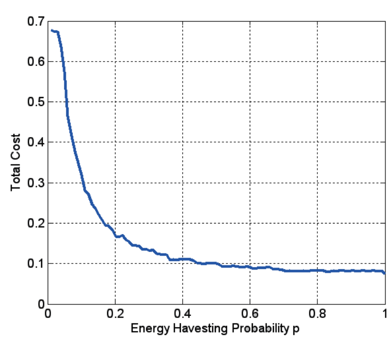

Fig. 2: The minimized total cost vs energy harvesting probability

Figure 2 illustrates the relationship between the minimized total cost and the energy harvesting probability p. As we expected, the total cost is a decreasing function of p. In order to gain a better understanding of the interaction between average time delay and error probabilities, we simulate their relationship with p separately. In Figure 3, the red dot line and the green dot-dash line represent $\mathbb{E}_0(\tau_{opt})$ and $\mathbb{E}_1(\tau_{opt})$ under Setup I. In Figure 4, the red dot line and the green dot-dash line

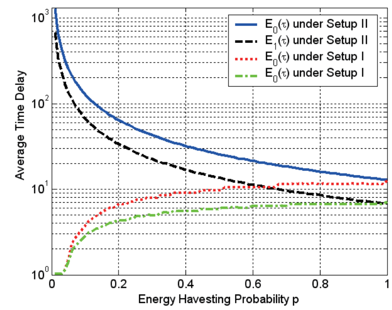

Fig. 3: The average time delay vs energy harvesting probability

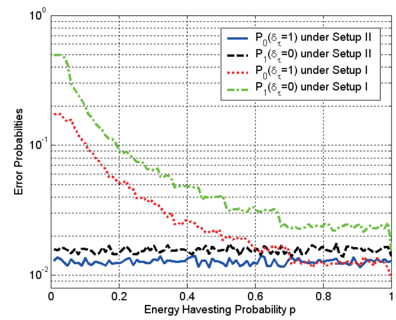

Fig. 4: The error probability vs energy harvesting probability

represent the Type I and Type II error probabilities under Setup I, respectively. Generally speaking, the average time delay is an increasing function of p, while the error probabilities are decreasing. From (5), the cost of each observation is c/p. When p is small, the cost of the average time delay dominates the total cost. In this case, the optimal solution tends to bear a larger error and to adopt a small sample size. Specially, when p is close to zero, the system makes a random guess at the first time instant $i = 1$ without taking any observations. In contrast, when p is large, the cost of each observation is relative small. To increase the sample size tends to reduce the cost of error decision. The overall effect is to achieve a decreasing function of the total cost. *To make a tradeoff between error probability and average time delay* is the most obvious feature of the optimal solution under Setup I. Figure 3 and Figure 4 also show the average time delay and error probabilities under Setup II. The blue solid and the black dash lines in these two figures are the simulation results. These two figures show that the error probabilities are maintained in constant levels for all $p \in (0, 1]$, but the average time delay increasing unboundedly when p goes to zero. This is a significant difference from the Setup I. *To maintain qualified error probabilities regardless of the cost of average time delay* is the feature of the solution under setup II. These two figures also show these two problem setups have the same performance when $p = 1$.

V. Numerical Simulation

In this section, we give a numerical example to illustrate the results in previous section. In this example, we assume that under H_0, the observations are i.i.d. Gaussian random variables with mean 0 and variance σ^2. Under H_1, the observations are i.i.d. Gaussian random variables with mean 0 and variance $\sigma^2 + P$. We fixed the $p = 0.7$, and we examine the performance of these two setups under different SNR.

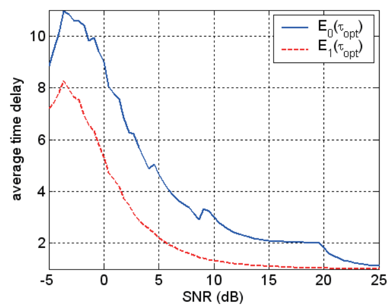

Fig. 5: $\mathbb{E}_i(\tau)$ vs SNR when $c = 0.01$ under Setup I

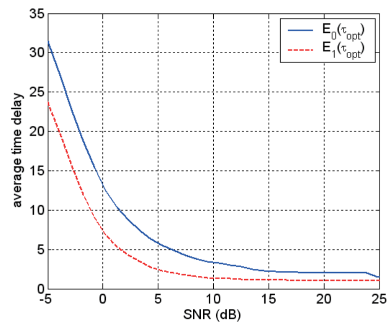

Fig. 6: $\mathbb{E}_i(\tau)$ vs SNR when $c = 0.003$ under Setup I

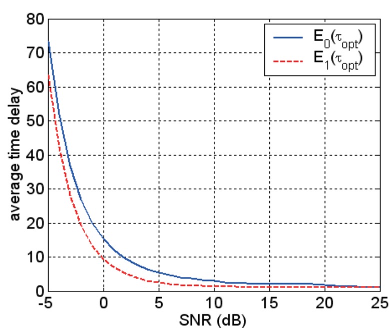

Fig. 7: $\mathbb{E}_i(\tau)$ vs SNR when $\alpha = 0.001, \beta = 0.005$ under Setup II

Figure 5 and 6 show the simulation results under the Setup I. In the simulation, we adopt the equal priors and uniform costs, and we choose $c = 0.01$. Figure 5 shows the relationship between the average time delay and SNR. The function generally decreases when the SNR goes higher. Hence the higher SNR is, the easier it is to distinguish between different distributions. However, it is not a monotonic function. This is a phenomenon of taking tradeoff between error probabilities and average time delay. Specifically, this can be explained as when the SNR is very low, the information provided by taking more observation does not justify the cost required to take these observation. Figure 6 shows the relationship between average time delay and SNR when $c = 0.003$, as we find that the average time delay is generally larger than that in Figure 5, and the function is monotonic decreasing in the present SNR scale. This is mainly due to the reason that the cost of taking more observations is smaller here.

Figure 7 is a simulation for the problem under Setup II, which shows the relationship between the average time delay and the SNR. In the simulation, we choose $\alpha = 0.005$ and $\beta = 0.01$. The function is a monotonic decreasing function of p, and we should notice when the SNR is low, average time delay is very large. This is because the wireless sensor need a considerable sample size to meet the error probability requirements when the two distributions are very close to each other.

VI. Conclusion

We have considered the sequential detection problem in energy harvested wireless sensor. The energy arriving is modeled as Bernoulli distribution with probability p. Motivated by various applications, we considered two setups in this paper. The first one is to minimize the total cost caused by the average time delay and the error decision. The second setup is to minimize the average time delay subjected to the constrains on error probabilities. We have shown that the optimal power allocation scheme for both setups is to spend the energy on taking observation as soon as it is harvested. The performance of these two setups have been shown to be quite different, especially when p is small or SNR is low. The optimal solution for the first setup always makes a tradeoff between average time delay and error probabilities. In the small p or low SNR case, it tends to bear larger error probabilities rather than to wait longer time. However, the solution for the second setup always keeps a proper sample size to make the error probabilities within the acceptable levels.

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