Optimization of LDPC-Coded Turbo CDMA Systems

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***Abstract*—We consider the analysis and design of low-density parity-check (LDPC) codes for turbo multiuser detection in multipath code division multiple access (CDMA) channels. We develop techniques for computing the probability density function (pdf) of the extrinsic messages at the output of the soft-input soft-output (SISO) multiuser detectors as a function of the pdf of input extrinsic messages, user spreading codes, channel impulse responses, and signal-to-noise ratios. Of particular interest is the soft interference cancellation plus minimum mean square error (SIC-MMSE) multiuser detector, for which the pdf of the extrinsic messages can be characterized analytically. For the case of additive white Gaussian noise (AWGN) channels, the extrinsic messages can be well approximated as symmetric Gaussian distributed. For the case of asynchronous multipath fading channels, the extrinsic messages can be approximated by a mixture of symmetric Gaussian distributions. We show that the expectation–maximization (EM) algorithm can be used to compute the parameters of this mixture. Using these techniques, we are able to accurately compute the thresholds for LDPC codes and design good irregular LDPC codes. Simulation results are in good agreement with the computed thresholds, and the designed irregular LDPC codes outperform regular ones significantly.**

***Index Terms*—CDMA, code optimization, iterative (turbo) receiver, low-density parity check (LDPC) code, multipath fading, multiuser detection.**

I. INTRODUCTION

MOST works on turbo multiuser detection are confined to the use of convolutional codes or parallel concatenated convolutional codes (PCCCs) [9]. Recent results [11], [12] show that carefully designed irregular low-density parity-check (LDPC) codes can outperform PCCCs for long code lengths and provide near-capacity performance on memoryless channels. It is then natural to attempt to design good LDPC code ensembles for turbo multiuser detection.

The main idea used in the design of LDPC codes is to employ the technique of density evolution [1], [12], where the pdf of extrinsic messages is computed as a function of iteration and the given degree profiles for the LDPC code in order to compute the thresholds (in SNR or E_b/N_o). Then, an optimization procedure is used to find optimum degree profiles that result in the least thresholds (or near capacity performance). It has been shown that for a small sacrifice in the resulting thresholds, the design procedure can be simplified by making the assumption that the messages (extrinsic information) at the output of the check nodes and the bit nodes have a Gaussian distribution [2]. For turbo multiuser detection, the LDPC codes will be used in conjunction with a soft-input soft-output (SISO) multiuser detector. In order to extend the aforementioned technique to design good LDPC codes for the turbo multiuser receiver, we need a technique to characterize the pdf of the extrinsic messages at the output of the detector as a function of the input pdf and channel characteristics. In this paper, we will primarily focus on the SISO multiuser detector based on soft interference cancellation (SIC) and instantaneous linear minimum mean square ereror (MMSE) filtering: a technique first proposed in [18]. Other receivers, i.e., the optimal detector and the matched filter, are also discussed. We show how to characterize the input–output pdfs of the extrinsic information *analytically* for these multiuser detectors and use this to design good LDPC codes.

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II. TURBO MULTIUSER RECEIVER FOR LDPC-CODED CDMA

We consider an LDPC-coded CDMA system with K users, employing normalized modulation waveforms $s_1, s_2, \ldots, s_K$, and signaling through their respective multipath channels with additive white Gaussian noise. The block diagram of the transmitter end of such a system is shown in the upper half of Fig. 1. The binary information data $\{d_k(m)\}$ for user k are LDPC encoded. The interleaved code bits of the kth user are binary phase shift keying symbol mapped. Each data symbol $b_k(i)$ is then modulated by a spreading waveform $s_{k,i}(t)$ and transmitted through its multipath channel. As shown in the lower part of Fig. 1, the overall receiver is an iterative receiver that performs turbo multiuser detection by passing extrinsic messages on the code bits between a SISO multiuser detector and an LDPC decoder. In each turbo iteration, several inner iterations are performed within the LDPC decoder during which extrinsic messages are passed along the edges in the biparitite graph of the code.

Notation: The variable L is used to refer to extrinsic messages (in log-likelihood form). The variable f is used to denote the pdf of the extrinsic information L, and m is used to denote the mean of L. Superscript (p, q) is used to denote quantities during the pth round of inner decoding within the LDPC decoder and qth stage of outer iteration between the LDPC decoder and the multiuser detector. For the quantities passed between the multiuser detector and the decoder, only one superscript q, namely, the turbo multiuser detection iteration number, is used. A subscript $m \rightarrow L$ denotes quantities passed from the multiuser detector to the LDPC decoder, and vice versa. Similarly, quantities passed between the bit nodes and the check nodes of the LDPC code are denoted by $b \rightarrow c$ and $b \leftarrow c$, respectively. The degree of the ith bit node is denoted by ν_i, and the degree of the ith check node is denoted by Δ_i. Denote

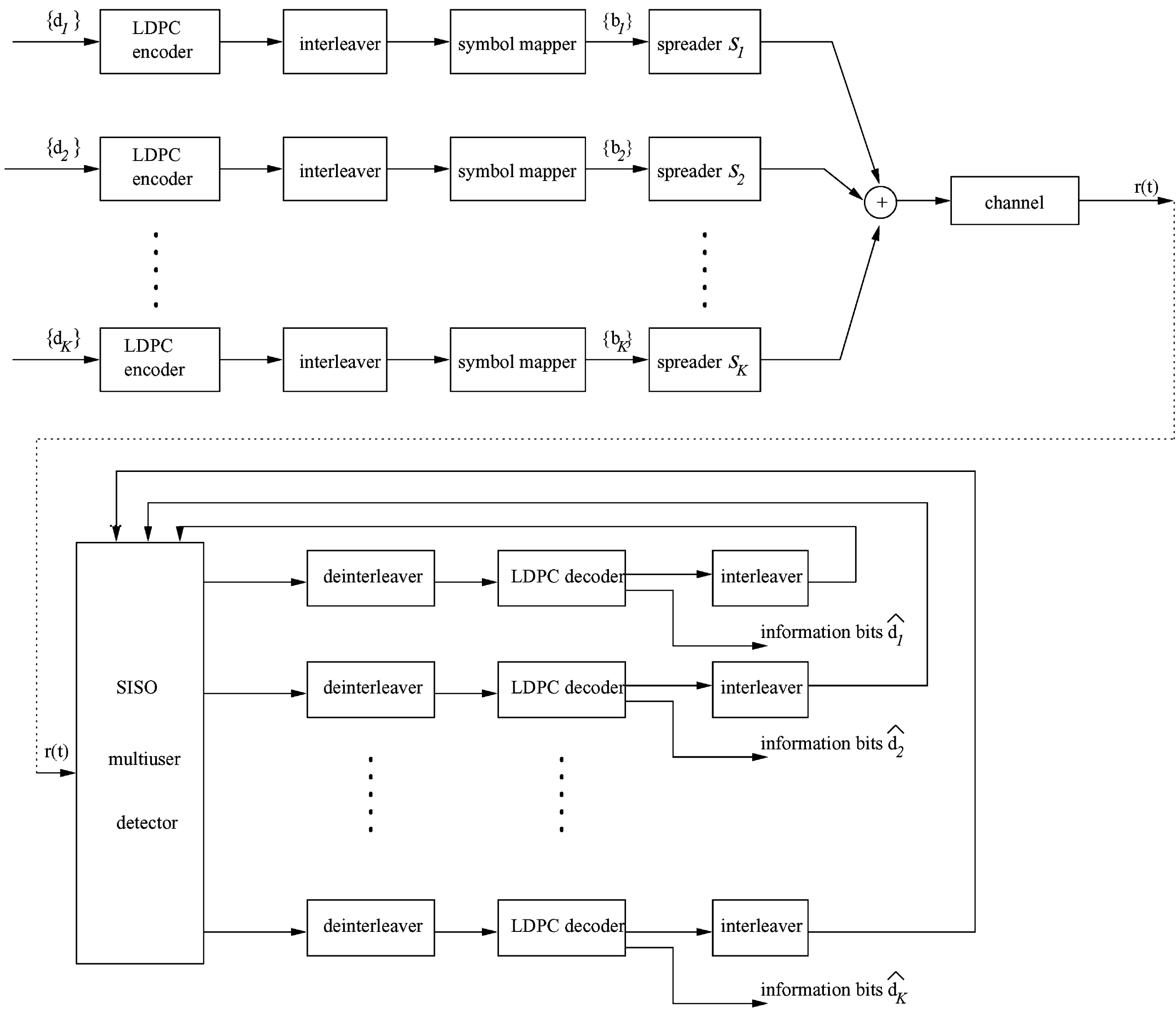

Fig. 1. LDPC-coded CDMA system with iterative receiver.

by $\{e^b_{i,1}, e^b_{i,2}, \ldots, e^b_{i,\nu_i}\}$ the set of edges connected to the ith bit node and by $\{e^c_{i,1}, e^c_{i,2}, \ldots, e^c_{i,\Delta_i}\}$ the set of edges connected to the ith check node. The particular edge or bit associated with an extrinsic information is denoted as the argument of L.

The turbo multiuser detection algorithm for LDPC-coded CDMA systems is as follows:

0) *Initialization:* $L^{0,0}_{b\leftarrow c}(e^b_{i,n}) = 0, \forall(i,n)$, and $L^0_{m\leftarrow L}[b_k(i)] = 0, \forall(i,k)$.

1) *Turbo multiuser detection iterations:* For $q = 1, 2, \ldots, Q$

1a) *SISO multiuser detection:* The SISO multiuser detector computes

$$L^q_{m\rightarrow L}[b_k(i)] = g\left(\{r(t)\}, \left\{L^{q-1}_{m\leftarrow L}[b_{k'}(i)]\right\}_{k'\neq k}\right) \quad (1)$$

where $g(\cdot)$ denotes the SISO multiuser detector.

1b) *LDPC decoding:* For $k = 1, 2, \ldots, K$

Iterate between bit node update and check node update: For $p = 1, 2, \ldots, P$

Bit node update: For each of the bit nodes i, and for all edges connected to it, compute

$$L^{p,q}_{b\rightarrow c}\left(e^b_{i,j}\right) = L^q_{m\rightarrow L}[b_k(i)] + \sum_{n=1,n\neq j}^{\nu_i} L^{p-1,q}_{b\leftarrow c}\left(e^b_{i,n}\right). \quad (2)$$

Check node update: For each of the check nodes i, and for all edges connected to it, compute [3]

$$L^{p,q}_{b\leftarrow c}\left(e^c_{i,j}\right) = 2\tanh^{-1}\left[\prod_{n=1,n\neq j}^{\Delta_i} \tanh\left(\frac{L^{p,q}_{b\rightarrow c}\left(e^c_{i,n}\right)}{2}\right)\right]. \quad (3)$$

1c) *Compute extrinsic messages passed back to the multiuser detector:*

$$L^q_{m\leftarrow L}[b_k(i)] = \sum_{n=1}^{\nu_i} L^{P,q}_{b\leftarrow c}\left(e^b_{i,n}\right). \quad (4)$$

1d) *Store check to bit messages:* For all edges, set

$$L^{0,q+1}_{b\leftarrow c}\left(e^b_{i,n}\right) = L^{P,q}_{b\leftarrow c}\left(e^b_{i,n}\right). \quad (5)$$

2) *Final hard decisions on information and parity bits:*

$$\hat{b}_k(i) = \text{sign}\left\{L^Q_{m\rightarrow L}[b_k(i)] + L^Q_{m\leftarrow L}[b_k(i)]\right\}. \quad (6)$$

III. SISO Multiuser Detectors

In this section, we outline three SISO multiuser detectors. For clarity, we first discuss these detectors in the context of a synchronous CDMA systems, in which the received (real-valued) signal is given by

$$r(t) = \sum_{k=1}^{K} A_k \sum_{i=0}^{M-1} b_k(i) s_{k,i}(t) + \sigma n(t) \tag{7}$$

where $s_{k,i}(t)$ is the spreading waveform of the kth user and ith symbol, and M is the number of the data symbols per user. A sufficient statistic for demodulating $\{b_k(i), k = 1, \ldots, K\}$ is given by

$$y_k(i) \triangleq \int_{iT}^{(i+1)T} s_{k,i}(t) r(t)\,\mathrm{d}t, \quad k = 1, \ldots, K. \tag{8}$$

Denote $\underline{y}(i) = [y_1(i), \ldots, y_K(i)]^T$. Then

$$\underline{y}(i) = \underline{R}(i)\underline{A}\underline{b}(i) + \sigma \underline{n}(i) \tag{9}$$

where $[\underline{R}(i)]_{k,l} \triangleq \int_{iT}^{(i+1)T} s_{k,i}(t) s_{l,i}(t)\mathrm{d}t$; $\underline{A} \triangleq \mathrm{diag}(A_1, \ldots, A_k)$; $\underline{b}(i) = [b_1(i), \ldots, b_K(i)]^T$; and $\underline{n}(i) \sim \mathcal{N}(\underline{0}, \underline{R}(i))$ is independent of $\underline{b}(i)$.

1) Exact SISO Multiuser Detector: [18]: Denote $\mathcal{B}_k^+ \triangleq \{(b_1, \ldots, b_{k-1}, +1, b_{k+1}, \ldots, b_K) : b_j \in \{+1, -1\}\}$. Similarly, define $\mathcal{B}_k^-$. We have the exact expression for the extrinsic messages from the multiuser detector in (10), shown at the bottom of the page.

2) SIC-MMSE SISO Multiuser Detector: [18]: A low-complexity approximate SISO multiuser detector was developed in [18], which is based on soft interference cancellation and instantaneous linear MMSE filtering and is summarized as follows. Denote $\underline{e}_k$ as the kth unit vector in $\mathbb{R}^K$. Define

$$\tilde{b}_j(i) \triangleq \tanh\left(\frac{1}{2} L_{m\leftarrow L}^{q-1}[b_j(i)]\right), \quad j = 1, \ldots, K \tag{11}$$

and

$$\underline{V}_k(i) \triangleq \sum_{j\neq k} A_j^2 [1 - \tilde{b}_j(i)^2] \underline{e}_j \underline{e}_j^T + A_k^2 \underline{e}_k \underline{e}_k^T. \tag{12}$$

Denote $\underline{\tilde{b}}(i) \triangleq [\tilde{b}_1(i) \cdots \tilde{b}_K(i)]^T$ and $\underline{\tilde{b}}_k(i) \triangleq \underline{\tilde{b}}(i) - \tilde{b}_k(i)\underline{e}_k$. Then, we have

$$L_{m\to L}^q[b_k(i)] = \frac{2 z_k(i)}{1 - \mu_k(i)}, \tag{13}$$

$$\text{where } z_k(i) = A_k \underline{e}_k^T [\underline{V}_k(i) + \sigma^2 \underline{R}(i)^{-1}]^{-1} [\underline{R}(i)^{-1} \underline{y}(i) - \underline{A}\underline{\tilde{b}}_k(i)]. \tag{14}$$

$$\mu_k(i) = A_k^2 \underline{e}_k^T \left[\underline{V}_k(i) + \sigma^2 \underline{R}(i)^{-1}\right]^{-1} \underline{e}_k. \tag{15}$$

3) SIC-MF SISO Multiuser Detector: A further simplification on the above SIC-MMSE detector is to skip the linear MMSE filtering step. In this case, the output is a scaled version of the matched filter output after ideal interference cancellation, which is given by

$$L_{m\to L}^q[b_k(i)] = \frac{2}{\gamma_k(i)} \left(y_k(i) - \sum_{j\neq k} A_j [\underline{R}(i)]_{k,j} \tilde{b}_j(i) \right) \tag{16}$$

$$\text{where } \gamma_k(i) = [\underline{R}(i)\underline{V}_k(i)\underline{R}(i) + \sigma^2 \underline{R}(i)]_{k,k} - 1. \tag{17}$$

4) Extension to Asynchronous CDMA With Multipath Fading: The received signal in an asynchronous CDMA system with multipath fading channels can be written as

$$r(t) = \sum_{k=1}^{K} A_k \sum_{i=0}^{M-1} b_k(i) \sum_{l=1}^{\ell_P} g_{kl}(i) s_{k,i}(t - \tau_{kl}) + \sigma n(t) \tag{18}$$

where ℓ_P is the number of resolvable paths in each user's channel; $g_{kl}(i)$ and τ_{kl} are, respectively, the complex gain corresponding to the ith symbol and the delay of the lth path of the kth user's channel. Assume that the multipath spread of any user signal is limited to at most Δ symbol intervals, where Δ is a positive integer. Define

$$\rho_{(k,l)(k',l')}^{[j]}(i) \triangleq \int_{-\infty}^{\infty} s_{k,i}(t - \tau_{kl}) s_{k',i-j}(t - \tau_{k'l'})\, dt \qquad -\Delta \leq j \leq \Delta. \tag{19}$$

The received signal $r(t)$ in (18) is first passed through a matched filter to obtain

$$\begin{aligned} z_{kl}(i) &\triangleq \int_{-\infty}^{\infty} r(t) s_{k,i}(t - \tau_{kl}) \\ &= \sum_{j=-\Delta}^{\Delta} A_{k'} b_{k'}(i+j) \sum_{l'=1}^{\ell_P} g_{k'l'}(i) \rho_{(k,l)(k',l')}^{[-j]}(i) \\ &\quad + \sigma u_{kl}(i) \end{aligned} \tag{20}$$

where $\{u_{kl}(i)\}$ are zero-mean complex Gaussian random sequences with covariance

$$\begin{aligned} E\{u_{kl}(i) u_{k'l'}(i')^*\} &= \int_{-\infty}^{\infty} s_{k,i}(t - \tau_{kl}) s_{k',i'}(t - \tau_{k'l'})\, dt \\ &= \rho_{(k,l)(k',l')}^{[i-i']}(i). \end{aligned} \tag{21}$$

Define the quantities shown at the bottom of the next page. We can then write (20) in the following vector form:

$$\underline{\zeta}(i) = \sum_{j=-\Delta}^{\Delta} \underline{R}^{[-j]}(i) \underline{G}(i+j) \underline{A}\underline{b}(i+j) + \underline{u}(i) \tag{22}$$

and from (21), the covariance matrix of the complex Gaussian vector sequence $\{\underline{u}(i)\}$ is $E\{\underline{u}(i)\underline{u}(i+j)^H\} = \underline{R}^{[-j]}(i)$. Define

$$\begin{aligned} L_{m\to L}^q[b_k(i)] &= \frac{2 A_k y_k(i)}{\sigma^2} \\ &+ \log \frac{\sum_{\underline{b}\in\mathcal{B}_k^+} \left\{ \exp[-\underline{b}^T \underline{A}\underline{R}(i)\underline{A}\underline{b}/(2\sigma^2)] \prod_{j\neq k} [1 + b_j \tanh(A_j y_j(i)/\sigma^2)] \left[1 + b_j \tanh\left(\frac{1}{2} L_{m\leftarrow L}^{q-1}[b_j(i)]\right)\right] \right\}}{\sum_{\underline{b}\in\mathcal{B}_k^-} \left\{ \exp[-\underline{b}^T \underline{A}\underline{R}(i)\underline{A}\underline{b}/(2\sigma^2)] \prod_{j\neq k} [1 + b_j \tanh(A_j y_j(i)/\sigma^2)] \left[1 + b_j \tanh\left(\frac{1}{2} L_{m\leftarrow L}^{q-1}[b_j(i)]\right)\right] \right\}}. \end{aligned} \tag{10}$$

$y_k(i) \triangleq \sum_{l=1}^{\ell} g_{kl}(i)^* z_{kl}(i)$. Then, $\underline{y}(i) \triangleq [y_1(i), \ldots, y_K(i)]^T$ is given by

$$\begin{aligned}\underline{y}(i) &\triangleq \underline{G}(i)^H \underline{\zeta}(i)\\ &= \sum_{j=-\Delta}^{\Delta} \underbrace{\underline{G}(i)^H \underline{R}^{[-j]}(i) \underline{G}(i+j)}_{\underline{H}^{[-j]}(i)} \underline{A}\underline{b}(i+j)\\ &\quad + \underbrace{\underline{G}(i)^H \underline{u}(i)}_{\underline{v}(i)}\end{aligned} \tag{23}$$

where $\underline{v}(i)$ is a sequence of zero-mean complex Gaussian vectors with covariance matrix

$$\begin{aligned}E\{\underline{v}(i)\underline{v}(i+j)^H\} &= \underline{G}(i)^H \underline{R}^{[-j]}(i) \underline{G}(i+j)\\ &\triangleq \underline{H}^{[-j]}(i).\end{aligned} \tag{24}$$

Now, define $\boldsymbol{H}(i) \triangleq [\underline{H}^{[1]}(i)\ \underline{H}^{[0]}(i)\ \underline{H}^{[-1]}(i)]$ ($K \times 3K$ matrix), $\boldsymbol{A} \triangleq \mathrm{diag}(\underline{A}, \underline{A}, \underline{A})$ ($3K \times 3K$ diagonal matrix), and $\boldsymbol{b}(i) \triangleq [\underline{b}(i-1)^T\ \underline{b}(i)^T\ \underline{b}(i+1)^T]^T$ ($3K$-vector). We can then write $\underline{y}(i)$ in (23) in a matrix form as

$$\underline{y}(i) = \boldsymbol{H}(i)\boldsymbol{A}\boldsymbol{b}(i) + \sigma \boldsymbol{v}(i) \tag{25}$$

where $\underline{v}(i) \sim \mathcal{N}_c(\underline{0}, \underline{H}^{[0]}(i))$. Based on (25), both the SIC-MMSE and the SIC-MF SISO multiuser detectors can be similarly applied as in the synchronous and additive white Gaussian noise (AWGN) cases. Specifically, the extrinsic information $L^q_{m\to L}[b_k(i)]$ is given by

$$L^q_{m\to L}[b_k(i)] = \frac{4\Re(\mu_k(i) z_k(i))}{\nu_k(i)^2} \tag{26}$$

where, as before, $z_k(i), \mu_k(i)$, and $\nu_k(i)$ are, respectively, the output, mean, and variance of the MMSE or matched filter (after soft interference cancellation).

IV. Distribution of Multiuser Extrinsic Messages

In this section, we describe how to compute the pdf of the extrinsic LLRs at the output of the SISO multiuser detector as a function of the pdf of the input *a priori* LLRs.

A. AWGN Channels

1) SIC-MMSE SISO Multiuser Detector: We first consider the SIC-MMSE SISO detector in a synchronous CDMA system. The extrinsic message in this case is given by (13). As discussed in [18], the output $z_k(i)$ of the instantaneous linear MMSE filter is well approximated by a Gaussian distribution. Hence, $L^q_{m\to L}[b_k(i)]$ has a Gaussian distribution with mean and variance given, respectively, by

$$\begin{aligned}E\{L^q_{m\to L}[b_k(i)]\} &= \left(\frac{2}{1-\mu_k(i)}\right) E\{z_k(i)\}\\ &= \frac{2\mu_k(i) b_k(i)}{1-\mu_k(i)}\end{aligned} \tag{27}$$

$$\begin{aligned}\mathrm{Var}\{L^q_{m\to L}[b_k(i)]\} &= \left(\frac{2}{1-\mu_k(i)}\right)^2 \mathrm{Var}\{z_k(i)\}\\ &= \frac{4\mu_k(i)}{1-\mu_k(i)}.\end{aligned} \tag{28}$$

Thus, the extrinsic message has a Gaussian distribution of the form $L^q_{m\to L}[b_k(i)] \sim \mathcal{N}(m_k(i)b_k(i), 2m_k(i))$, with $m_k(i) \triangleq 2\mu_k(i)/(1-\mu_k(i))$. Given $\underline{R}(i), \underline{A}$, and σ^2, and the *a priori* code bit LLR distribution $f^{q-1}_{m\leftarrow L}$. We can compute $\{m_k(i), k = 1, \ldots, K\}$ as follows:

For $j = 1, 2, \ldots, N$ (number of samples) and for $k = 1, 2, \ldots, K$, do the following.

- Draw i.i.d. $\Omega_k^{(j)} \sim f^{q-1}_{m\leftarrow L}$. Let $\underline{V}_k^{(j)} \triangleq \mathrm{diag}\{1 - \tanh(\Omega_1^{(j)}/2)^2, \ldots, 1 - \tanh(\Omega_{k-1}^{(j)}/2)^2, 1, 1 - \tanh(\Omega_{k+1}^{(j)}/2)^2, \ldots, 1 - \tanh(\Omega_K^{(j)}/2)^2\}$.
- Compute $\mu_k^{(j)} \triangleq A^2 \underline{e}_k^T [\underline{V}_k^{(j)} + \sigma^2 \underline{R}^{-1}]^{-1} \underline{e}_k$, and $m_k^{(j)} \triangleq 2\mu_k^{(j)} / \left(1 - \mu_k^{(j)}\right)$.

Finally, $m_k(i)$ is calculated as $m_k(i) \cong (1/N) \sum_{j=1}^N m_k^{(j)}$. Note that the *a priori* code bit LLR from the LDPC decoder is typically modeled as mixture symmetric Gaussian, i.e.,

$$f^{q-1}_{m\leftarrow L} = \sum_{\ell=1}^{L} \lambda_\ell \mathcal{N}(m_\ell, 2m_\ell) \tag{29}$$

where m_ℓ and $2m_\ell$ are, respectively, the mean and the variance of the ℓth component. Here, λ_ℓ is the fraction of the bit nodes of degree ℓ, and we assume that the output extrinsic LLR at a node of degree ℓ is symmetric Gaussian with mean m_ℓ [2], [15].

2) Exact and SIC-MF SISO Multiuser Detector: For these two detectors, simulations show that the extrinsic messages are also well approximated by symmetric Gaussian distributions. The means can be calculated via Monte Carlo as follows.

For $j = 1, 2, \ldots, N$ (number of samples)

- For $k = 1, \ldots, K$: Draw i.i.d. $b_k(i)^{(j)} \in \{+1, -1\}, n_k(i)^{(j)} \sim \mathcal{N}(0,1), \Omega_k^{(i)} \sim f^{q-1}_{m\leftarrow L}$; Set $L^{q-1}_{m\leftarrow L}[b_k(i)] = \Omega_k^{(i)} b_k(i)$; Compute $y_k(i)^{(j)}$ according to (9).

$$\underline{R}^{[j]}(i) \triangleq \begin{bmatrix} \rho^{[j]}_{(1,1)(1,1)}(i) & \cdots & \rho^{[j]}_{(1,1)(1,\ell_P)}(i) & \cdots & \rho^{[j]}_{(1,1)(K,1)}(i) & \cdots & \rho^{[j]}_{(1,1)(K,\ell_P)}(i)\\ \rho^{[j]}_{(2,1)(1,1)}(i) & \cdots & \rho^{[j]}_{(2,1)(1,\ell_P)}(i) & \cdots & \rho^{[j]}_{(2,1)(K,1)}(i) & \cdots & \rho^{[j]}_{(2,1)(K,\ell_P)}(i)\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \rho^{[j]}_{(K,\ell_P)(1,1)}(i) & \cdots & \rho^{[j]}_{(K,\ell_P)(1,\ell_P)}(i) & \cdots & \rho^{[j]}_{(K,\ell_P)(K,1)}(i) & \cdots & \rho^{[j]}_{(K,\ell_P)(K,\ell_P)}(i) \end{bmatrix}_{(K\ell_P \times K\ell_P)}$$

$$\begin{aligned}\underline{\zeta}(i) &\triangleq [z_{11}(i), \ldots, z_{1\ell_P}(i), \ldots, z_{K1}(i), \ldots, z_{K\ell_P}(i)]^T \quad [(K\ell_P \times 1)]\\ \underline{u}(i) &\triangleq [u_{11}(i), \ldots, u_{1\ell_P}(i), \ldots, u_{K1}(i), \ldots, u_{K\ell_P}(i)]^T \quad [(K\ell_P \times 1)]\\ \underline{g}_k(i) &\triangleq [g_{k1}(i), \ldots, g_{k\ell_P}(i)]^T \quad [(\ell_P \times 1)]\\ \underline{G}(i) &\triangleq \mathrm{diag}(\underline{g}_1(i), \ldots, \underline{g}_K(i)) \quad [(K\ell_P \times K)]\end{aligned}$$

- For $k=1,\ldots,K$: Compute the extrinsic information $L^q_{m\to L}[b_k(i)]^{(j)}$ according to (10) for the exact SISO detector or according to (14) for the SIC-MF SISO detector. Set $m_k^{(j)} = L^q_{m\to L}[b_k(i)]^{(j)} b_k(i)^{(j)}$.

We now demonstrate the validity of Gaussian assumption through following example. Consider estimating the pdf of extrinsic information at the output of the multiuser detector for a two-user synchronous system with ρ_{jk} fixed at 0.5 for $j \neq k$ when $E_s/N_o = -1$ dB. The pdf of the input *a priori* information to the multiuser detector is $f^{q-1}_{m\leftarrow L} = 0.05\mathcal{N}(0.1, 0.2) + 0.25\mathcal{N}(1.0, 2.0) + 0.7\mathcal{N}(5.0, 10.0)$. Fig. 2 shows the histograms of the extrinsic information at the optimal, SIC-MMSE, MF multiuser detectors by simulating the channel and the detector. The symmetric Gaussian pdf's with same means are also shown. It can be seen that the match is quite close for each detector, indicating that the underlying pdf is well approximated by the symmetric Gaussian.

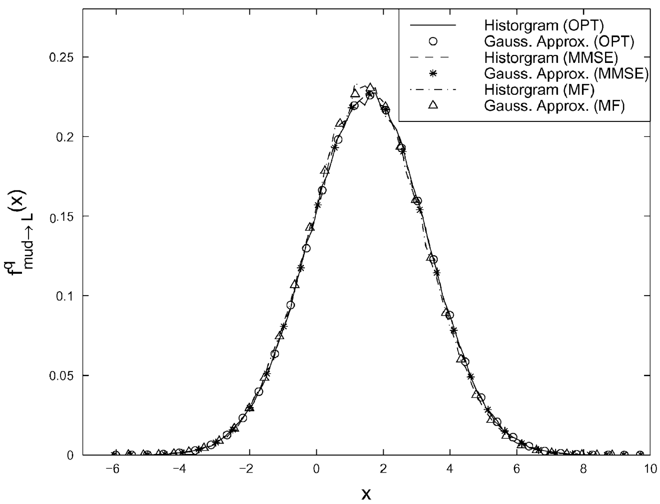

Fig. 2. Histograms for the multiuser detectors extrinsic information in a two-user synchronous CDMA system and the symmetric Gaussian approximations by Monte Carlo simulation.

B. Fading Channels

Consider the SIC-MMSE detector in a synchronous CDMA system with fading channels. Conditioned on the channels $\boldsymbol{\alpha}(i) = [\alpha_1(i),\ldots,\alpha_K(i)]$, the extrinsic message from the multiuser detector has a Gaussian distribution, i.e., $f^q_{m\to L}(\boldsymbol{\alpha}(i)) \sim \mathcal{N}(m(\boldsymbol{\alpha}(i)), 2m(\boldsymbol{\alpha}(i)))$, with $m(\boldsymbol{\alpha}(i)) \triangleq 4\mu_k(i)/(1-\mu_k(i))$. Hence, the pdf of the output extrinsic message is given by

$$f^q_{m\to L} \triangleq \int f^q_{m\to L}(\boldsymbol{\alpha}(i))p(\boldsymbol{\alpha}(i))\,\mathrm{d}\boldsymbol{\alpha}(i). \tag{30}$$

In general, the pdf in (30) cannot be well approximated as Gaussian. However, we can approximate $f^q_{m\to L}$ as a mixture of symmetric Gaussian pdf's, i.e., $f^q_{m\to L} \cong \sum_{j=1}^J \pi_j \mathcal{N}(m_j, 2m_j)$. Note that in the limit as $J\to\infty$, this can approximate (30) arbitrarily closely. For a fixed number of mixtures J, based on the observations $\Xi \triangleq \{\xi_t, t=1,\ldots,N\}$, the parameters $\boldsymbol{\theta} \triangleq \{\pi_j, m_j, j=1,\ldots,J\}$ can be estimated using the expectation–maximization (EM) algorithm as follows.

Denote $\phi(x;\mu,\sigma^2)$ as the pdf of an $\mathcal{N}(\mu,\sigma^2)$ random variable. Then, the maximum likelihood (ML) estimate of the parameters $\boldsymbol{\theta}$ is given by

$$\begin{aligned}\hat{\boldsymbol{\theta}} &= \arg\max_{\boldsymbol{\theta}:\sum_{j=1}^J \pi_j=1} \log p_{\boldsymbol{\theta}}(\Xi)\\ &= \arg\max_{\boldsymbol{\theta}:\sum_{j=1}^J \pi_j=1} \sum_{t=1}^N \log \sum_{j=1}^J \pi_j \phi(\xi_t; m_j, 2m_j).\end{aligned} \tag{31}$$

Direct solution to the above maximization problem is very difficult. The EM algorithm [4], [7] is an iterative procedure for solving this ML estimation problem. In the EM algorithm, the observation Ξ is termed *incomplete* data. The algorithm postulates that one has access to *complete* data $\boldsymbol{X}$, which is such that Ξ can be obtained through a many-to-one mapping. Typically, the complete data is chosen such that the conditional density $p_{\boldsymbol{\theta}}(\boldsymbol{X})$ is easy to obtain and optimize. Starting from some initial estimate $\boldsymbol{\theta}^{(0)}$, the EM algorithm solves the ML estimation problem (31) by the following iterative procedure.

- E-step: Compute $Q(\boldsymbol{\theta}\,|\,\boldsymbol{\theta}^{(i)}) = E_{\boldsymbol{\theta}^{(i)}}\{\log p_{\boldsymbol{\theta}}(\boldsymbol{X})\,|\,\Xi\}$.
- M-step: Solve $\boldsymbol{\theta}^{(i+1)} = \arg\max_{\boldsymbol{\theta}} Q(\boldsymbol{\theta}\,|\,\boldsymbol{\theta}^{(i)})$.

Define the following hidden data $\boldsymbol{Z} = \{\boldsymbol{z}_t, t=1,\ldots,N\}$, where $\boldsymbol{z}_t$ is a J-dimensional indicator vector such that $z_{t,j}=1$, if $\xi_t \sim \mathcal{N}(m_j, 2m_j)$, and $z_{t,j}=0$, otherwise. The complete data is then $\boldsymbol{X} \triangleq (\Xi, \boldsymbol{Z})$. We have $p_{\boldsymbol{\theta}}(\Xi,\boldsymbol{Z}) = \prod_{t=1}^N \prod_{j=1}^J [\pi_j \quad \phi(\xi_t; m_j, 2m_j)]^{z_{t,j}}$, where $\phi(x;\mu,\sigma^2) = (1/\sqrt{2\pi\sigma^2})e^{-((x-\mu)^2/2\sigma^2)}$; hence

$$\begin{aligned}\log p_{\boldsymbol{\theta}}(\Xi,\boldsymbol{Z}) &= \sum_{t=1}^N\sum_{j=1}^J z_{t,j}\log\pi_j\\ &\quad+\sum_{t=1}^N\sum_{j=1}^J z_{t,j}\left[-\frac{1}{2}\log 2m_j - \frac{(\xi_t-m_j)^2}{4m_j}\right]+C\end{aligned} \tag{32}$$

where C is some constant. The E-step can then be calculated as follows:

$$\begin{aligned}Q(\boldsymbol{\theta},\boldsymbol{\theta}') &\triangleq E_{\boldsymbol{\theta}'}\{\log p_{\boldsymbol{\theta}}(\Xi,\boldsymbol{Z})\,|\,\Xi\}\\ &= \sum_{t=1}^N\sum_{j=1}^J \hat{z}_{t,j}\left[\log\pi_j - \frac{1}{2}\log 2m_j\right.\\ &\quad\left.- \frac{(\xi_t-m_j)^2}{4m_j}\right]+C\end{aligned} \tag{33}$$

$$\begin{aligned}\text{where } \hat{z}_{t,j} &\triangleq E_{\boldsymbol{\theta}'}\{z_{t,j}\,|\,\Xi,\boldsymbol{\theta}'\}\\ &= \frac{\phi(\xi_t; m'_j, 2m'_j)\pi'_j}{\sum_{l=1}^J \phi(\xi_t; m'_l, 2m'_l)\pi'_l}.\end{aligned} \tag{34}$$

In addition, the M-step is calculated as follows:

$$\frac{\partial Q(\boldsymbol{\theta},\boldsymbol{\theta}')}{\partial \pi_j} = 0 \Rightarrow \pi_j = \frac{1}{N}\sum_{t=1}^N \hat{z}_{t,j}, \quad j=1,\ldots,J. \tag{35}$$

$$\frac{\partial Q(\boldsymbol{\theta},\boldsymbol{\theta}')}{\partial m_j} = 0 \Rightarrow m_j = -1 + \sqrt{1+\frac{\sum_{t=1}^N \hat{z}_{t,j}\xi_t^2}{\sum_{t=1}^N \hat{z}_{t,j}}}\qquad j=1,\ldots,J. \tag{36}$$

Finally, the EM algorithm for calculating the Gaussian mixture parameters for the extrinsic messages in fading channels is summarized as follows: Given the detector extrinsic messages $\{\xi_t\}$, the number of mixture components J, and the total number of EM iterations I, starting from the initial parameters $\boldsymbol{\theta}^{(0)}$, for $i = 1, \ldots, I$:

- Let $\boldsymbol{\theta}' = \boldsymbol{\theta}^{(i-1)}$, and calculate $\{\hat{z}_{t,j}, t = 1, \ldots, N; j = 1, \ldots, J\}$ according to (34).
- Calculate $\{\pi_j, j = 1, \ldots, J\}$ according to (35), and calculate $\{m_j, j = 1, \ldots, J\}$ according to (36). Set $\boldsymbol{\theta}^{(i)} = \boldsymbol{\theta}$.

The algorithm can be applied to the SISO multiuser detector in fading channels by letting $\xi_k(i) \triangleq b_k(i) L^q_{m\to L}[b_k(i)]$, where $L^q_{m\to L}[b_k(i)]$ is given by (26).

In the above EM algorithm, the number of mixture components J is fixed. Note that when J increases, $\log p_{\boldsymbol{\theta}}(\Xi)$ increases, or $-\log p_{\boldsymbol{\theta}}(\Xi)$ decreases. The minimum description length (MDL) principle can be used to select the optimal number of the components in a Gaussian mixture [6], [13]. In the MDL criterion, a penalty term $(J/2)\log N$ is introduced. In addition, the optimal number of components is given by

$$\hat{J}_{\text{MDL}} = \arg\min_J \left\{ -\log p_{\boldsymbol{\theta},J}(\Xi) + \frac{J}{2}\log N \right\}. \tag{37}$$

Hence, we can first set an upper bound of the number of mixture components, $J_{\max}$, and for each $J \le J_{\max}$, we run the above EM algorithm and calculate the corresponding MDL value. Finally, we choose the optimal J with the minimum MDL.

We now demonstrate the efficiency of the mixture Gaussian modeling of the multiuser detector extrinsic information developed in this section through the following example. Consider a five-user asynchronous CDMA system in an independently Rayleigh fading channel and an $E_s/N_o = 0$ dB employing MMSE multiuser detector when the input LLR distribution is $f^{q-1}_{m\leftarrow L} = 0.05\mathcal{N}(0.1, 0.2) + 0.25\mathcal{N}(1.0, 2.0) + 0.7\mathcal{N}(5.0, 10.0)$. The histogram of the multiuser detector output extrinsic information obtained using Monte Carlo simulations is plotted in Fig. 3. The approximation of the pdf using a mixture of symmetric Gaussian distributions computed via the EM algorithm is also shown in the figure. Note that the two curves are almost indistinguishable, indicating that the approximation is very accurate. On the other hand, a symmetric Gaussian pdf that has the same mean as that of the histogram is also shown. It is seen that such a single symmetric Gaussian approximation of the extrinsic information distribution is quite inaccurate. This confirms that the extrinsic information delivered by the SIC-MMSE multiuser detector in fading channels cannot be assumed to be Gaussian, whereas a mixture of symmetric Gaussian pdf offers a good approximation. In this example, the codeword length is $N = 20\,000$ for each user. The average number of mixture components given by the MDL criterion is $\hat{J}_{\text{MDL}} = 12$.

V. Design of LDPC Codes

A. Computing Threshold

In this section, we first describe how to compute the thresholds for LDPC codes with the afore mentioned receiver employing turbo multiuser detection. The main idea is to treat the

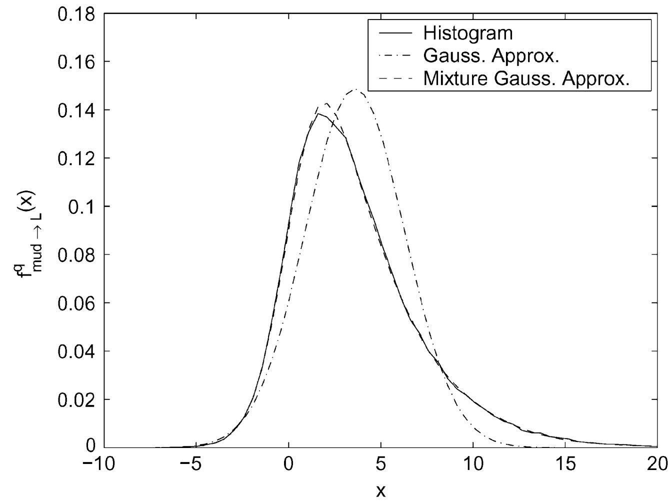

Fig. 3. Histogram for the SIC-MMSE multiuser detector extrinsic information in a five-user asynchronous CDMA system with fading and the approximations by a single symmetric Gaussian pdf and by a mixture of symmetric Gaussian pdf's obtained using the EM algorithm.

extrinsic LLRs as i.i.d random variables and to compute their pdf at each iteration [2], [5], [12]. In [2], the pdf of the extrinsic LLRs at each bit or check node was assumed to be Gaussian and symmetric (variance is twice the mean), and hence, it is sufficient to track the mean of the extrinsic LLRs. While this is a good approximation for the singler-user AWGN channel, this is not a good approximation for fading channels. Therefore, we will assume that the output of the multiuser detector and, hence, the input at every bit node is a mixture of symmetric Gaussian densities. We will show that this assumption allows us to track the pdfs of the extrinsic LLRs accurately without having to numerically convolve or evaluate pdfs. In computing the pdfs of the extrinsic LLRs, we will assume that the all-zeros codeword is transmitted but the coded bits are modulated in to ± 1 in a random order which is known to the receiver. Therefore, density evolution can still be performed, assuming the all-zeros codeword as reference, even though the overall system is not geometrically uniform. We next specify the procedure for computing the pdf's of the extrinsic messages passed around in the turbo multiuser detection algorithm described in Section II. Denote $\psi(x) \triangleq E\{\tanh[(1/2)\mathcal{N}(x, 2x)]\}$.

0) *Initialization:* Set $f^{0,0}_{b\leftarrow c}(x) = \delta(x)$, and $f^0_{m\leftarrow L}(x) = \delta(x)$.

1) *Turbo multiuser detection iterations:* For $q = 1, 2, \ldots, Q$

1a) *Compute the pdf of the multiuser detector extrinsic messages:* $f^q_{m\to L}$ is computed as a function of E_b/N_o and $f^{q-1}_{m\leftarrow L}$ using the appropriate procedure from Section IV (for static channels) or Section V (for fading channels) to obtain

$$f^q_{m\to L} = \sum_{j=1}^{J} \pi_j \mathcal{N}(\mu_j, 2\mu_j). \tag{38}$$

1b) *Compute the pdf of the LDPC extrinsic messages:*

1bi) *Iterate between bit node update and check node update:* For $p = 1, 2, \ldots, P$

◇ *At a bit node of degree i*: The pdf of the extrinsic LLR passed along an edge connected to a bit node of degree

i is denoted by $f^{p,q}_{b\to c,i}$. From (2), we can see that $f^{p,q}_{b\to c,i}$ is given by

$$f^{p,q}_{b\to c,i} = f^{q}_{m\to L} \otimes f^{p-1,q}_{b\leftarrow c}{}^{\otimes(i-1)} \tag{39}$$

where $\otimes$ denotes convolution, and $(\,\cdot\,)^{\otimes i}$ denotes i-fold convolution. We can simplify this by making the assumption that the output extrinsic from the bit node of degree i, excluding the contribution from the channel, is Gaussian. The same assumption has been made in [2]. That is

$$\begin{aligned} f^{p,q}_{b\to c,i} &= f^{q}_{m\to L} \otimes \mathcal{N}\left((i-1)m^{p-1,q}_{b\leftarrow c}, 2(i-1)m^{p-1,q}_{b\leftarrow c}\right) \\ &= \sum_{j=1}^{J} \pi_j \mathcal{N}\left(\mu_j + (i-1)m^{p-1,q}_{b\leftarrow c}\right. \\ &\qquad \left. 2\left[\mu_j + (i-1)m^{p-1,q}_{b\leftarrow c}\right]\right). \end{aligned} \tag{40}$$

The pdf of the extrinsic message passed from the bit to check nodes along an edge is then

$$\begin{aligned} f^{p,q}_{b\to c} &= \sum_{j=2}^{d_{l,\max}} \lambda_i f^{p,q}_{b\to c,i} \\ &= \sum_{j=1}^{J}\sum_{i=2}^{d_{l,\max}} \pi_j \lambda_i \mathcal{N}\left(\mu_j + (i-1)m^{p-1,q}_{b\leftarrow c}\right. \\ &\qquad \left. 2\left[\mu_j + (i-1)m^{p-1,q}_{b\leftarrow c}\right]\right). \end{aligned} \tag{41}$$

◇ *At check node of degree* j: Assume that the ith check node is of degree j and that the extrinsic LLR at the output of this check node is Gaussian with mean $m^{p,q}_{b\leftarrow c,j}$. To compute $m^{p,q}_{b\leftarrow c,j}$, we take the expectation on both sides of (3) and get

$$\begin{aligned} &E\left\{\tanh\left(L^{p,q}_{b\leftarrow c}(e^c_{i,r})/2\right)\right\} \\ &\quad = E\left\{\left[\prod_{k=1,k\neq r}^{j} \tanh\left(L^{p,q}_{b\to c}\left(e^c_{i,k}\right)/2\right)\right]\right\} \\ &\quad = \left[E\left\{\tanh\left(L^{p,q}_{b\to c}\left(e^c_{i,k}\right)/2\right)\right\}\right]^{j-1} \end{aligned} \tag{42}$$

where (42) follows from the fact that $L^{p,q}_{b\to c}(e^c_{i,k})$ and $L^{p,q}_{b\to c}(e^c_{i,s})$ are identically distributed and are independent for $k \neq s$. Since the distribution of $L^{p,q}_{b\leftarrow c}(e^c_{i,r})$ will be same for all r, distribution), we can drop r. Therefore

$$m^{p,q}_{b\leftarrow c,j} = \sum_j \rho_j\, \psi^{-1}\left[\left(\sum_{l=1}^{J}\sum_{i=2}^{d_{l,\max}} \pi_l\, \lambda_i\, \psi\left(m^{p,q}_{b\to c,i}\right)\right)^{j-1}\right]. \tag{43}$$

1bii) *Message passed back to the multiuser detector:* At bit node of degree i, by taking expectation on both sides of (4), we get $m^{q}_{m\leftarrow L}(i) = im^{p-1,q}_{b\leftarrow c}$. Since $\tilde{\lambda}_i$ of the *nodes* have degree i

$$f^{q}_{m\leftarrow L} = \sum_{i=2}^{d_{l,\max}} \tilde{\lambda}_i \mathcal{N}\left(m^{q}_{m\leftarrow L}(i), 2m^{q}_{m\leftarrow L}(i)\right). \tag{44}$$

The threshold is defned as the minimum E_b/N_o for which the mean $m^{Q}_{m\leftarrow L}$ or $m^{P,Q}_{b\leftarrow c}$ tends to ∞. That is

$$(E_b/N_o)_{\mathrm{th}} = \min E_b/N_o : \lim_{N\to\infty}\lim_{Q\to\infty} m^{P,Q}_{b\leftarrow c} \to \infty. \tag{45}$$

B. Design of LDPC Codes

The procedure for computing the threshold for a given degree profile $(\lambda(x), \rho(x))$ can be used in conjunction with an optimization procedure to design optimal LDPC codes for the multiuser detection. The idea is to find optimal $\lambda(x)$ and $\rho(x)$ such that the threshold is minimized. Note that the rate of the LDPC code is $R = 1 - (\int_0^1 \rho(x)\,dx / \int_0^1 \lambda(x)\,dx)$. If a rate of R_o is required, the optimization problem can be stated as follows: Find $\lambda(x)$ and $\rho(x)$ such that we minimize E_b/N_o subject to the following constraints: 1) $1 - (\int_0^1 \rho(x)\,dx / \int_0^1 \lambda(x)\,dx) = R_o$, and 2) $m^{P,Q}_{b\leftarrow c} \to \infty$ [computed using (38)–(44)].

A nonlinear optimization procedure called differential evolution [10], [11] has been used to perform this optimization. This technique involves choosing several candidates for $\lambda(x)$ and $\rho(x)$ and computing thresholds for each pair during the optimization. Without the Gaussian mixture assumption for the extrinsic LLR pdfs, the pdf's have to be evaluated numerically within the LDPC decoder and by using Monte Carlo in the multiuser detector. However, with this assumption, only the means of the components in the mixture need to be evaluated, which is a very significant reduction in complexity. This is a key advantage of the SIC-MMSE multiuser detection since the output pdf from the multiuser detector can be computed relatively easily.

VI. Results

A. Two-User Synchronous CDMA System With Periodic Spreading Sequences

We first present results for a two-user synchronous CDMA system in AWGN channel. With periodic spreading sequences, the cross correlation matrix is fixed. Set $\rho_{jk} = 0.5$ for $j \neq k$. Three different receivers were simulated (i.e., optimal, SIC-MMSE, and matched filter). The theoretical thresholds for a (3, 6) rate (1/2) regular LDPC code, and the simulation results for a (3, 6) rate (1/2) regular LDPC code of length $N = 100\,000$ bits are shown in Figs. 4–6. It is seen that the actual simulation results are within 0.2 dB of the theoretical thresholds for three different detectors, indicating that the Gaussian assumption and the characterization of the input–output pdf of the multiuser detector extrinsic information is quite accurate. Optimum degree profiles were computed for the same channel using algorithms and the technique discussed in Section V. The optimum degree profile for optimal multiuser detector was $\lambda(x) = 0.255\,869x + 0.144\,624x^2 + 0.069\,450x^3 + 0.076\,102x^4 + 0.028\,071x^5 + 0.048\,760x^6 + 0.013\,873x^8 + 0.021\,780x^9 + 0.026\,336x^{12} + 0.015\,840x^{13} + 0.011\,756x^{17} + 0.287\,539x^{19}$, and $\rho(x) = 0.721\,952x^7 + 0.278\,048x^8$. The resulting threshold is shown in Fig. 4. The performance of a randomly constructed LDPC code with the optimum degree profile of length $N = 100\,000$ is also shown in Fig. 4. It is seen that the performance is about 0.15 dB from the threshold at BER of 10^{-6}. The optimum degree profile for MMSE multiuser detector was $\lambda(x) = 0.244\,164x^1 +$

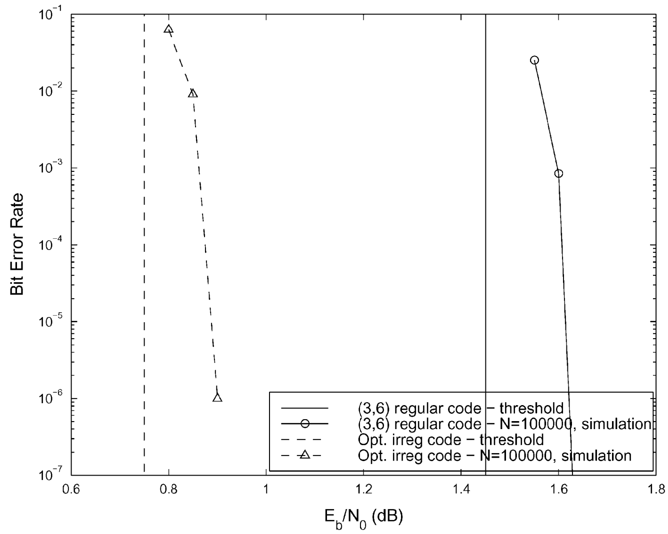

Fig. 4. Thresholds and simulation results for the (3, 6) regular LDPC codes and for the optimum irregular LDPC codes in a two-user synchronous system with optimal receiver.

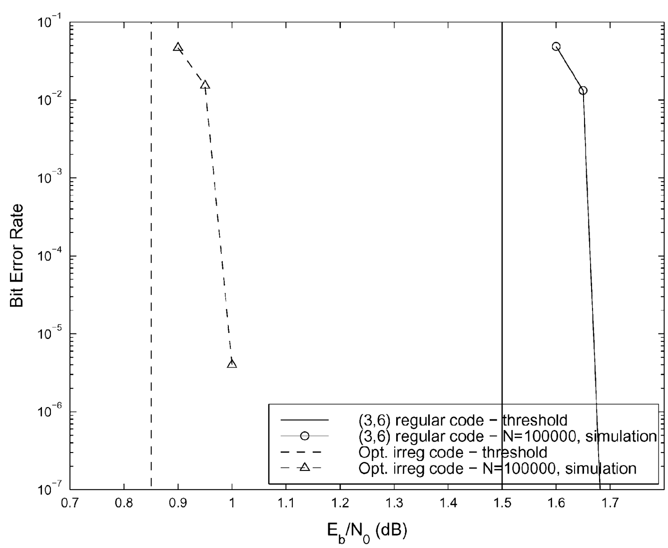

Fig. 5. Thresholds and simulation results for the (3, 6) regular LDPC codes and for the optimum irregular LDPC codes in a two-user synchronous system with MMSE receiver.

$0.168\,476x^2 + 0.093\,799x^3 + 0.047\,910x^4 + 0.022\,387x^5 + 0.025\,506x^6 + 0.015\,051x^8 + 0.038\,143x^9 + 0.035\,043x^{10} + 0.014\,559x^{11} + 0.024\,455x^{14} + 0.011\,258x^{17} + 0.259\,249x^{19}$ and $\rho(x) = 0.775\,041x^7 + 0.224\,959x^8$. The performance of the constructed irregular LDPC code, which is shown in Fig. 5, is around 0.15 dB from the threshold. The performance of the MMSE receiver is only 0.1 dB worse than optimal receiver. The optimum degree profile for the MF detector was $\lambda(x) = 0.259\,484x + 0.146\,658x^2 + 0.114\,980x^3 + 0.056\,695x^4 + 0.023\,731x^5 + 0.019\,086x^6 + 0.015\,051x^8 + 0.042\,996x^9 + 0.034\,469x^{10} + 0.016\,299x^{11} + 0.033\,337x^{14} + 0.010\,720x^{17} + 0.226\,495x^{19}$ and $\rho(x) = x^7$. As shown in Fig. 6, the simulation result of the randomly constructed LDPC code is about 0.15 dB from the threshold 1.25 dB. The results presented here show that the irregular codes provide about 0.5 dB better performance than the regular codes.

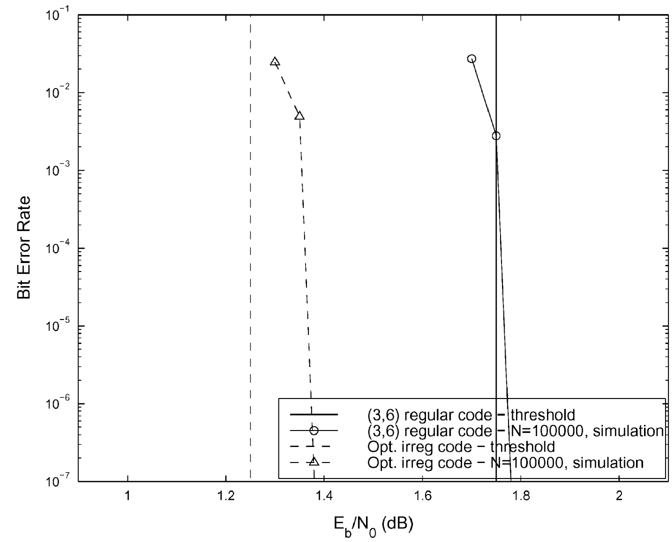

Fig. 6. Thresholds and simulation results for the (3, 6) regular LDPC codes and for the optimum irregular LDPC codes in a two-user synchronous system with MF receiver.

B. Achievable Information Rate

The achievable information rate for a two-user synchronous CDMA system with binary modulation can be computed for a given E_s/N_o and $\rho_{12} = \rho$ as follows. The equivalent signal space diagram for the two-user system can be obtained by projecting the received signal on to two basis functions $\phi_1(t) = -(\rho/\sqrt{1-\rho^2})s_1(t) + (1/\sqrt{1-\rho^2})s_2(t)$ and $\phi_2(t) = s_1(t)$ [16]. The four points in the two-dimensional signal space corresponding to the transmitted bits $(-1,-1), (-1,1), (1,-1)$, and $(1,1)$ can then be shown to be $\boldsymbol{x}_0 = [-\sqrt{1-\rho^2}, -1-\rho], \boldsymbol{x}_1 = [\sqrt{1-\rho^2}, -1+\rho], \boldsymbol{x}_2 = [-\sqrt{1-\rho^2}, 1-\rho], \boldsymbol{x}_3 = [\sqrt{1-\rho^2}, 1+\rho]$. The sufficient statistic $\underline{y}$ can be expressed as

$$\underline{y} = \underline{x} + \underline{n}, \quad \underline{x} \in \{\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3\} \tag{46}$$

with the choice of basis functions given above $\underline{n} \sim \mathcal{N}(\underline{0}, \sigma^2 \boldsymbol{I}_2)$, where $\sigma^2 = N_o/(2E_s)$ and $\boldsymbol{I}_2$ is the 2×2 identity matrix. For noncooperative coding between the two users, the information rate is maximized by the equiprobable distribution $p(\underline{x} = \boldsymbol{x}_i) = 1/4, \forall i$. The achievable information rate can be computed using

$$\begin{aligned} I_{\text{two-user}} &= h(\underline{y}) - h(\underline{y}\,|\,\underline{x}) && (47)\\ &= \sum_i -\frac{1}{4}\int_{\mathcal{R}^2} p(\underline{y}\,|\,\boldsymbol{x}_i)\ \log_2(p(\underline{y}\,|\,\boldsymbol{x}_i))\,d\underline{y} \\ &\quad - \log_2(2\pi e \sigma^2) \text{ bits.} && (48) \end{aligned}$$

The integral in (48) can be computed numerically after noting that $p(\underline{y}\,|\,\boldsymbol{x}_i)$ is $\mathcal{N}(\boldsymbol{x}_i, \sigma^2 \boldsymbol{I}_2)$.

For $\rho_{12} = 0.5$, the required E_b/N_o to achieve 0.5 bits/user/channel use is 0.46 dB. The threshold for the optimized irregular LDPC code with the optimal receiver (in Fig. 4) is less than 0.3 dB away corroborating the effectiveness of the proposed design methodology.

1) Five-User Synchronous System With Aperiodic Spreading Sequence: Next, we present some simulation results for five-user synchronous system using aperiodic spreading in the AWGN channel. For each user, the spreading code is a random code with processing gain $N_c = 10$, which varies with symbol i. The

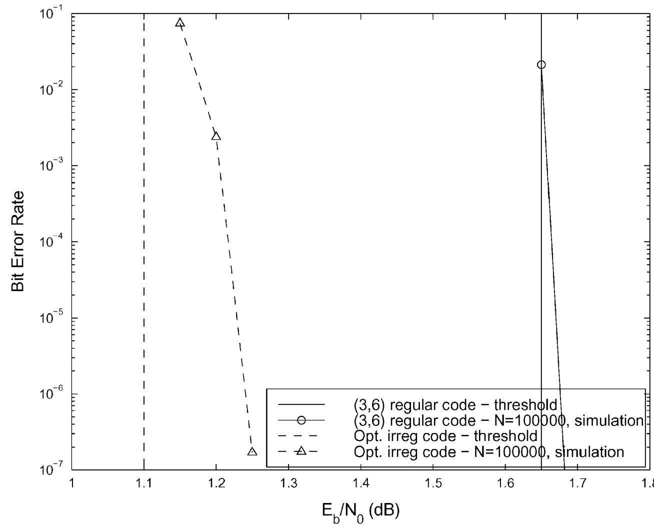

Fig. 7. Thresholds and simulation results for the (3, 6) regular LDPC codes and for the optimum irregular LDPC codes in a five-user synchronous system with MMSE receiver.

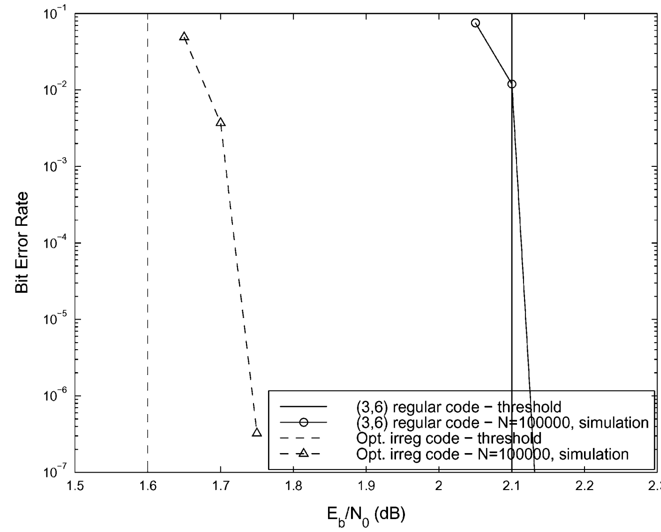

Fig. 8. Thresholds and simulation results for the (3, 6) regular LDPC codes and for the optimum irregular LDPC codes in a five-user synchronous system with MF receiver.

randomly chosen spreading sequence is an accurate model when a pseudonoise sequence spans many symbol periods [17]. With aperiodic random spreading, the cross correlation matrix after the matched filter is dynamically change symbol by symbol. The theoretical thresholds for the $(3,6)$ rate-$(1/2)$ regular LDPC code with maximum number of iterations between the multiuser detector and decoder $P = 30$ are shown in Figs. 7 and 8 with MMSE and MF receiver, respectively. Both receivers have the performance for the regular LDPC code within 0.05 dB from the thresholds. The irregular LDPC code was designed, and the resulting optimum degree profiles of MMSE receiver with $d_{l_{\max}} = 20$ was $\lambda(x) = 0.257\,632x^1 + 0.154\,468x^2 + 0.070\,704x^3 + 0.076\,268x^4 + 0.028\,071x^5 + 0.048\,760x^6 + 0.015\,496x^8 + 0.021\,936x^9 + 0.028\,388x^{12} + 0.015\,840x^{13} + 0.016\,623x^{17} + 0.265\,814x^{19}$ and $\rho(x) = 0.867\,663x^7 + 0.132\,337x^8$. The threshold for the above degree profile and simulation results for a randomly constructed LDPC code of length $N = 100\,000$ are shown in Fig. 7. It is seen that the simulation results agree well with the theoretical thresholds and that the irregular LDPC code is about 0.65 dB better than the (3, 6) regular LDPC code, indicating the usefulness of the proposed techniques for designing good LDPC codes. The optimum degree profile for the MF detector was $\lambda(x) = 0.291\,442x^1 + 0.146\,740x^2 + 0.017\,766x^3 + 0.050\,683x^4 + 0.066\,068x^5 + 0.044\,891x^7 + 0.015\,843x^8 + 0.052\,557x^9 + 0.019\,073x^{11} + 0.022\,182x^{12} + 0.016\,018x^{16} + 0.013\,343x^{17} + 0.243\,395x^{19}$ and $\rho(x) = x^7$. As shown in Fig. 8, the performance is only 0.1 dB from the threshold, and the irregular LDPC codes are 0.5 dB better than the (3, 6) regular LDPC code.

C. Five-User Asynchronous System in Fading

Finally, we consider a five-user asynchronous CDMA system in random fading channel with aperiodic random spreading. Each user's channel contains four paths, i.e., $\ell_P = 4$. The relative path power gains are $0, -3, -6$, and -9 dB, and the relative delay is $0, T_c, 2T_c, 3T_c$. The theoretical thresholds for a (3, 6) rate-$(1/2)$ regular LDPC code and simulation results for a randomly constructed regular LDPC code of

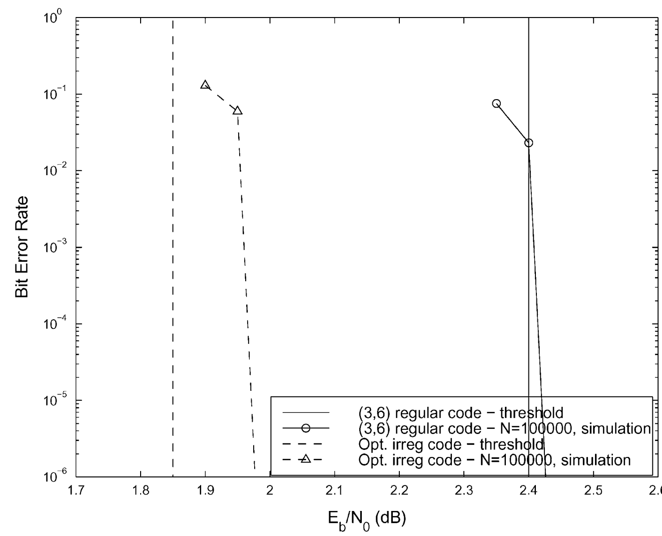

Fig. 9. Thresholds and simulation results for the (3, 6) regular LDPC codes and for the optimum irregular LDPC codes in a five-user asynchronous system with fading using the MMSE receiver.

length $N = 100\ 000$ are shown in Fig. 9 for the MMSE receiver and in Fig. 10 for the MF receiver. It is seen that the simulated BER performance matched quite well with the thresholds, indicating that the threshold computation is fairly accurate. Then, we designed optimal degree profiles with $d_{l_{\max}} = 20$ and rate-$(1/2)$ for both receivers. The resulting optimal degree profiles for the MMSE receiver were $\lambda(x) = 0.246\,553x^1 + 0.146\,658x^2 + 0.093\,799x^3 + 0.022\,387x^5 + 0.019\,086x^6 + 0.015\,051x^8 + 0.038\,143x^9 + 0.032\,978x^{10} + 0.014\,559x^{11} + 0.017\,916x^{14} + 0.010\,720x^{17} + 0.294\,241x^{19}$ and $\rho(x) = 0.562\,658x^7 + 0.437\,342x^8$. The resulting optimal degree profiles for the MF receiver were $\lambda(x) = 0.281\,703x^1 + 0.146\,740x^2 + 0.041\,345x^3 + 0.050\,683x^4 + 0.066\,068x^5 + 0.045\,004x^7 + 0.020\,060x^8 + 0.047\,036x^9 + 0.014\,923x^{11} + 0.014\,942x^{12} + 0.017\,153x^{16} + 0.013\,343x^{17} + 0.241\,000x^{19}$ and $\rho(x) = x^7$ The simulation results for a randomly constructed LDPC code with these degree profiles for a length of $N = 100\ 000$ are shown in Figs. 9 and 10. At a BER of 10^{-6},

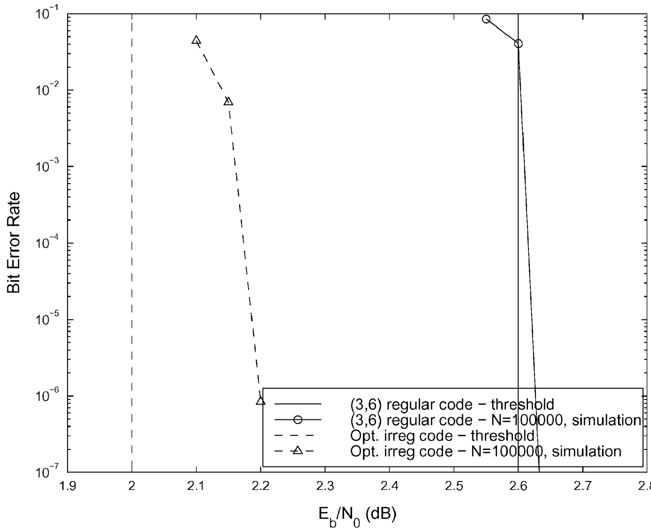

Fig. 10. Thresholds and simulation results for the (3, 6) regular LDPC codes and for the optimum irregular LDPC codes in a five-user asynchronous system with fading using the MF receiver.

the performance is about 0.2 dB away from the thresholds. The irregular codes outperform the regular ones by about 0.6 dB for both receivers. These results show that by using the EM algorithm, we can accurately model the extrinsic information as a mixture of Gaussian densities and use this to design good irregular LDPC codes.

VII. Conclusion

In this paper, we have shown how to characterize the pdf of the extrinsic information at the output of the multiuser detector as a function of the pdf of the input extrinsic information, the E_b/N_o, and the cross correlation matrix of spreading codes for CDMA systems through AWGN channels or multipath fading channels. For the synchronous system in AWGN, we have shown that the pdf can be assumed to be symmetric Gaussian, whereas for asynchronous system with multipath fading, the pdf can be approximated as a mixture of symmetric Gaussian densities. Then, we have shown how to compute the thresholds for a given irregular LDPC code degree profile and to design good irregular LDPC codes. In all cases, the computed thresholds match very well with simulations, and the designed irregular codes significantly outperform regular LDPC codes. The differences between computed thresholds and the simulations are within 0.2 dB. From the simulation, the performance of the designed irregular codes are about 0.6 dB closer to the capacity than regular LDPC codes for the synchronous sytem through AWGN and 0.45 dB for the asynchronous CDMA system with multipath fading. Finally, we note that the proposed framework can also be applied to optimize the turbo equalization systems [8] and turbo BLAST systems [14].

References

[1] J. Chen and M. P. C. Fossorier, "Density evolution for two improved bp-based decoding algorithms of LDPC codes," *IEEE Commun. Lett.*, vol. 6, no. 5, pp. 208–210, May 2002.

[2] S. Y. Chung, T. Richardson, and R. Urbanke, "Analysis of sum-product decoding of low-density parity check codes using a Gaussian approximation," *IEEE Trans. Inf. Theory*, vol. 47, no. 2, pp. 657–670, Feb. 2001.

[3] J. Hagenauer, E. Offer, C. Meason, and M. Mörz, "Decoding and equalization with analog nonlinear networks," *Eur. Trans. Telecom.*, vol. 10, no. 6, pp. 659–680, Nov.–Dec. 1999.

[4] E. L. Lehmann and G. Casella, *Theory of Point Estimation*. New York: Springer-Verlag, 1998.

[5] M. Luby, M. Mitzenmacher, A. Shokrollahi, and D. Spielman, "Analysis of low-density codes and improved designs using irregular graphs," in *Proc. ACM Symp. Theory Comput.*, Dallas, TX, 1998, pp. 249–258.

[6] P. McKenzie and M. Alder, "Selecting the optimal number of components for a Gaussian mixture mode," in *Proc. IEEE Int. Symp. Inf. Theory*, 1994, p. 393.

[7] G. J. McLachlan and T. Krishnan, *The EM Algorithm and Extensions*. New York: Wiley, 1997.

[8] K. R. Narayanan, X. Wang, and G. Yue, "LDPC code design for MMSE turbo equalization," in *Proc. IEEE Int. Symp. Inform. Theory*, Lausanne, Switzerland, Jun. 2002.

[9] H. V. Poor, "Turbo multiuser detection: A primer," *J. Commun. Networks*, vol. 3, no. 3, pp. 196–201, Sep. 2001.

[10] K. Price and R. Storn, "Differential evolution—A simple and efficient heuristic for global optimization over continuous spaces," *J. Global Optim.*, vol. 11, pp. 341–359, 1997.

[11] T. Richardson, A. Shokrohalli, and R. Urbanke, "Design of capacity approaching irregular low density parity check codes," *IEEE Trans. Inf. Theory*, vol. 47, no. 2, pp. 619–637, Feb. 2001.

[12] T. Richardson and R. Urbanke, "Capacity of low density parity check codes under message passing decoding," *IEEE Trans. Inf. Theory*, vol. 47, no. 2, pp. 599–618, Feb. 2001.

[13] J. Rissanen, *Stochastic Complexity in Statistical Inquiry*, Singapore: World Scientific, 1989.

[14] M. Sellathurai and S. Haykin, "Turbo-BLAST for wireless communications: Theory and experiments," *IEEE Trans. Signal Process.*, vol. 50, no. 10, pp. 2538–2546, Oct. 2002.

[15] S. ten Brink, "Convergence of iterative decoding," *Elec. Lett.*, vol. 35, no. 13, pp. 1117–1118, Jun. 1999.

[16] *Multiuser Detection*, Cambridge, U.K.: Cambridge Univ. Press, 1998, p. 111.

[17] S. Verdu and S. Shamai, "Spectral efficiency of CDMA with random spreading," *Electron. Lett.*, vol. 45, no. 2, pp. 622–640, Mar. 1999.

[18] X. Wang and H. V. Poor, "Iterative (turbo) soft interference cancellation and decoding for coded CDMA," *IEEE Trans. Commun.*, vol. 47, no. 7, pp. 1046–1061, Jul. 1999.

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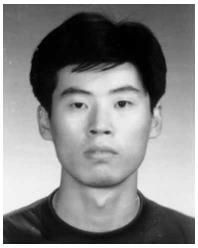

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