ON THE DIVERSITY OF THE NAIVE LATTICE DECODER

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***Abstract*—"Naive Lattice Decoding" (NLD) and its low-complexity approximations such as lattice reduction-aided linear decoders represent an alternative to Maximum Likelihood lattice decoders for MIMO systems. Their diversity order has been investigated in recent works. These showed that the NLD achieves only the receive diversity and that MMSE-GDFE left preprocessing followed by NLD or its approximations achieves the maximum diversity. All the theoretical results have so far focused on the diversity order but this is not the only relevant parameter to achieve good performance and the coding gain also needs to be considered. In addition, up to now there has not been any numerical analysis of the actual performance of these techniques for the coded systems for moderate SNR.**

In this paper, we consider MIMO systems using high-dimensional perfect space-time codes. We show that by adding MMSE-GDFE preprocessing, the NLD has a loss of only 1.5 dB with respect to optimal decoding in the case of the Perfect Code 4×4. However, even with MMSE-GDFE preprocessing, the performance of lattice-reduction aided linear receivers is still very poor for high-dimensional lattices.

***Index Terms*—Diversity, Perfect Codes, Naive Lattice Decoder, MMSE-GDFE, LLL-reduction.**

I. Introduction

In the last decade, a great interest has been accorded to wireless transmission systems using multiple antennas at the transmitter and the receiver. These systems offer higher data rates as well as performance gain thanks to transmit and receive diversity techniques. In order to exploit the benefits of MIMO systems, space-time (ST) codes based on algebraic structures have been developed. For example, the perfect codes are a class of ST codes which are full-rate, full-rank and satisfy the property of the non-vanishing determinant. The linear structure of these codes allows their decoding based on the lattice point representation. Optimal performances can be obtained when the maximum likelihood (ML) criterion is considered. Many algorithms have been implemented to perform the ML detection, for example, the Sphere Decoder (SD) and the Schnorr-Euchner are employed. However, the higher the lattice dimension or the constellation size is, the more these decoders become complex, which limits their use.

An alternative to these techniques consists in using suboptimal decoders such as the ZF (Zero Forcing), the ZF-DFE (Zero Forcing-Decision Feedback Equalizer) or the MMSE (Minimum Mean Square Error). These decoders have low complexity, but they don't preserve the diversity order and therefore they have suboptimal performances.

So, is it possible to achieve maximal receive and transmit diversity using suboptimal decoders?

This question has been addressed by the recent studies [10] and [5]. The authors of [10] introduced the Naive Lattice Decoder (NLD), also called Lattice Decoder in [4], as a relaxed version of the ML detection in which the decoded points don't necessarily take into account the finite constellation. The theoretical survey achieved in [10] shows that the Naive Lattice Decoder represents a suboptimal solution to the diversity problem and reaches only the maximal receive diversity. According to [5], the maximal transmit diversity can not be achieved because the Naive Lattice Decoder and its approximations (such as lattice-reduction aided linear decoders) do not take the constellation constraint into account and a shaping problem occurs. In order to solve the shaping problem, the authors of [5] propose the MMSE-GDFE left preprocessing. According to the survey of [5], the MMSE-GDFE followed by a lattice decoder such as the Naive Lattice Decoder, achieves asymptotically optimal performance and maximal diversity. Up to now, all the analytical results focused on the diversity order of such decoders, however achieving diversity does not guarantee good performances for moderate SNR and the coding gain of these techniques should also be considered.

Numerical simulations were carried out in [9] in order to analyze this performance in the case of uncoded systems which only have the receive diversity; to the best of our knowledge, there has not been any numerical result confirming what has been proposed in theory and analyzing the performance for moderate SNR when space-time coding is employed.

In this paper, we consider the cases of the Golden Code and the Perfect Code 4×4, and present numerical results that confirm the theoretical findings of [10] and [5] as far as the diversity order is considered. On the other side, we show that while the MMSE-GDFE followed by Naive Lattice Decoding also achieves excellent coding gain, the coding gain of its suboptimal approximations is very poor for high lattice dimensions.

II. System Model

We consider within this work an $n_t \times n_r$ MIMO system where n_t and n_r denote respectively the number of transmit and receive antennas, and $n_r \geq n_t$. The transmitted bits are mapped onto symbols belonging to a finite constellation $\mathcal{A}$ (in the simulation results, we consider the case of a QAM constellation). We consider the case of space-time coding using the Golden Code [2] and the Perfect Code 4×4 [7].

The constructed codeword $\mathbf{X} \in \mathbb{C}^{n_t \times T}$ is then transmitted by the n_t antennas during T time slots. The received codeword has the following expression:

$$\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{W} \tag{1}$$

where $\mathbf{H} \in \mathbb{C}^{n_r \times n_t}$ represents the complex channel matrix. The channel matrix elements are modeled as independent identically distributed (i.i.d.) complex Gaussian random variables with zero mean and unit variance. $\mathbf{H}$ is supposed to be known at the receiver. Besides, $\mathbf{W}$ accounts for the additive Gaussian noise whose entries are i.i.d. Gaussian with zero mean and variance σ^2 per complex dimension.
We can convert the system model given in (1) into a real-valued system as follows. Consider the vectors $\mathbf{x_1}$, $\mathbf{y_1}$ and $\mathbf{w_1}$ that are obtained by column-wise vectorization of the matrices $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{W}$ respectively. The vector $\mathbf{x_1}$ can be written as

$$\mathbf{x_1} = \mathbf{\Phi}\mathbf{s}', \tag{2}$$

where $\mathbf{\Phi} \in \mathbb{C}^{n_t T \times n_t T}$ is the generator matrix of the space-time code, and $\mathbf{s}'$ is the vector of information symbols. Let $\mathbf{H}' = \mathbf{I}_T \otimes \mathbf{H}$ be an $n_r T \times n_t T$-block diagonal matrix whose blocks are equal to $\mathbf{H}$. Then we can write:

$$\mathbf{y_1} = \mathbf{H}'\mathbf{\Phi}\mathbf{s}' + \mathbf{w_1} = \mathbf{H}_{eq}\mathbf{s}' + \mathbf{w_1}. \tag{3}$$

Finally, by applying a complex-to-real transformation respectively to $\mathbf{y_1}$, $\mathbf{H}_{eq}$, $\mathbf{s}'$ and $\mathbf{w_1}$, we get the real equivalent system

$$\mathbf{y} = \mathbf{M}\mathbf{s} + \mathbf{w}, \tag{4}$$

where $\mathbf{M}$ is the $2Tn_t \times 2Tn_r$ real-valued channel matrix and $\mathbf{y}$ and $\mathbf{w}$ denote respectively the $2Tn_r$ real-valued received signal and the $2Tn_r$ noise vector.
In the rest of this paper we consider a symmetric MIMO system where $n_t = n_r = T$ and we define n by $n = 2n_t^2$.

III. Lattice Decoding and ML Detection in MIMO Systems

From the expression of the received signal given in (4), we conclude that the latter can be viewed as a point of the lattice generated by $\mathbf{M}$ perturbed by the noise vector $\mathbf{w}$. Consequently, the MIMO detection problem can be reduced to a lattice decoding problem. In order to achieve optimal performances, the ML criterion should be used. Following this criterion, we search the estimate $\hat{\mathbf{s}}_{ML}$ which satisfies [3]:

$$\hat{\mathbf{s}}_{ML} = \underset{\mathbf{s} \in \mathcal{A}^n}{\operatorname{argmin}} \|\mathbf{y} - \mathbf{M}\mathbf{s}\|^2 \tag{5}$$

Equation (5) is equivalent to solving a Closest Vector Problem (CVP) in the lattice Λ generated by $\mathbf{M}$.

The most well-known ML decoding algorithms are the Sphere Decoder (SD) [11] and Schnorr-Euchner algorithm [1]. Nonetheless, the optimality of the SD results in high computational complexity which increases as the number of antennas or the constellation size grows [8]-[6], which limits the use of this detection approach. One can also solve a relaxed version of (5) by searching the estimate in the lattice $\mathbb{Z}^n$ and solving:

$$\hat{\mathbf{s}}_{NLD} = \underset{\mathbf{s} \in \mathbb{Z}^n}{\operatorname{argmin}} \|\mathbf{y} - \mathbf{M}\mathbf{s}\|^2 \tag{6}$$

This decoding approach is known as the Naive Lattice Decoding, called also Lattice Decoding in [4], and does not guarantee that $\hat{\mathbf{s}}_{NLD}$ belongs to the constellation $\mathcal{A}^n$ which results in decoding errors. In the following section, we investigate the performances of the Naive Lattice Decoder by numerical simulations.

IV. Performances of the Naive Lattice Decoder

In this section, we start by analyzing the performances of the Naive Lattice Decoder in a first subsection. The second subsection is dedicated to studying the approximations of the NLD, and the last one deals with the MMSE-GDFE preprocessing.

A. *Naive Lattice Decoding*

The authors of [10] led a theoretical survey showing that the Naive Lattice Decoder does not attain the optimal diversity order and computed an upper bound of its reached diversity which is given by:

$$d_{NLD} \leq n_r \left(n_r - n_t + 1\right) \tag{7}$$

We recall that the maximal diversity of coded MIMO systems is

$$d_{max} = n_t n_r \tag{8}$$

From (7), when $n_t = n_r$, the Naive Lattice Decoder achieves only the receive diversity order n_r. In the light of these theoretical results, we examine the performance of this decoder in case of the Golden Code presented in Fig. 1. As we can see, the diversity order can't be observed for moderate SNR range. Nevertheless, we notice that the gap between the NLD and the ML detection exceeds $4.5\,\mathrm{dB}$ at a Bit Error Rate BER$= 10^{-5}$ when $16-$QAM constellations are used. In the following table we present numerical results in order to quantify the loss of NLD compared to ML for the Golden Code at a BER$= 10^{-3}$ for different constellation sizes.

constellation	4-QAM	16-QAM	64-QAM	256-QAM
gap (dB)	5.8	4.58	2.9	0.45

As shown in the table, this loss decreases as a function of the constellation size. Intuitively, large signal constellations are almost indistinguishable from lattices for moderate SNR.
Now, considering the case of the Perfect Code 4×4, simulations of the NLD presented in Fig. 2 confirm the loss of transmit diversity. In fact, this decoding approach not only doesn't achieve full diversity, but also has a huge gap compared to ML detection, which exceeds $13.5\,\mathrm{dB}$ for BER$= 10^{-3}$.

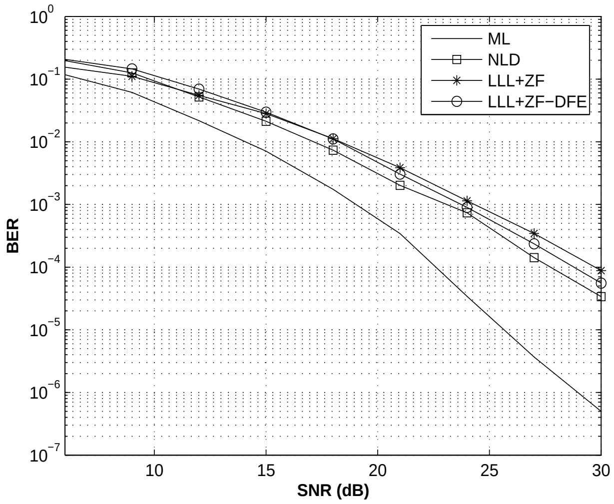

Fig. 1. Performance comparison of ML decoding, NLD decoding and LLL-aided decoding for the Golden Code using $16-$QAM constellations.

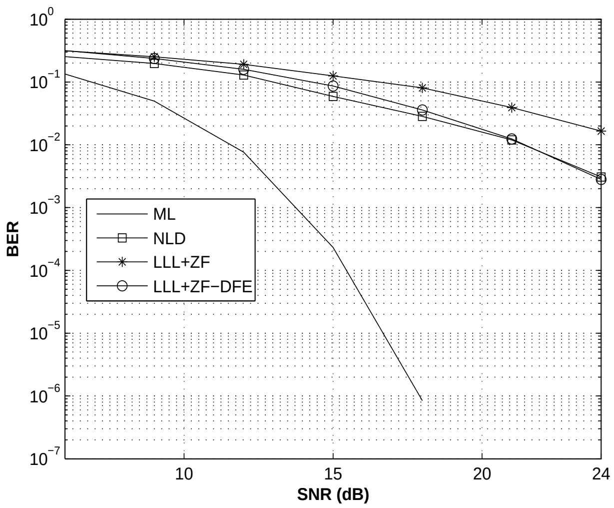

Fig. 2. Performance comparison of ML decoding, NLD decoding and LLL-aided decoding for for the Perfect Code 4×4 using $16-$QAM constellations.

B. Approximations of the Naive Lattice Decoder

A theoretical survey done in [5] proved that lattice-reduction-aided linear decoders represent an approximation of the Closest Vector Problem. Particularly, the LLL reduction followed by the ZF or ZF-DFE detectors represents an approximation of the lattice decoding or the Naive Lattice Decoder. In the case of the Golden Code and the Perfect Code 4×4, we see in Fig. 1 and Fig. 2, that the LLL reduction followed by the ZF or the ZF-DFE decoders are extremely close to the performance of the NLD.

The performances of the NLD and its approximations given by the LLL+ZF and the LLL+ZF-DFE decoders confirm that the loss of the maximal diversity order is due to the shaping problem and to the out-of constellation errors caused by the fact that the NLD and its approximations don't take into consideration the constellation bounds.
In the next subsection we study by simulations the effects of the MMSE-GDFE preprocessing on the performances of the NLD and its approximations, since this left preprocessing is known to solve the shaping problem [5].

C. MMSE-GDFE preprocessed Naive Lattice Decoding

First of all, let us outline the MMSE-GDFE left preprocessing principle. We define the augmented channel matrix $\tilde{\mathbf{M}}$ with respect to the Signal to Noise Ratio $\rho = \frac{E_s}{\sigma^2}$ as:

$$\tilde{\mathbf{M}} = \begin{bmatrix} \mathbf{M} \\ \frac{1}{\sqrt{\rho}}\mathbf{I} \end{bmatrix}$$

Consider the QR decomposition $\tilde{\mathbf{M}} = \tilde{\mathbf{Q}}\mathbf{R} = \begin{bmatrix} \mathbf{Q}_1 \\ \mathbf{Q}_2 \end{bmatrix}\mathbf{R}$, where $\tilde{\mathbf{Q}} \in \mathbb{R}^{2n \times n}$ is an orthogonal matrix, $\mathbf{R} \in \mathbb{R}^{n \times n}$ is upper triangular and $\mathbf{Q}_1 \in \mathbb{R}^{n \times n}$ is not necessarily an orthogonal matrix. Then we can write: $\mathbf{M} = \mathbf{Q}_1\mathbf{R}$.
The MMSE-GDFE preprocessing transforms the decoding problem in (5) into the non-equivalent problem of finding :

$$\hat{\mathbf{s}}_{MMSE-GDFE} = \underset{\mathbf{s} \in \mathcal{A}^n}{\operatorname{argmin}} \left\|\mathbf{Q}_1^t\mathbf{y} - \mathbf{R}\mathbf{s}\right\|^2 \qquad (9)$$

Besides, we define the $\alpha-$regularized decoders as the decoders that solve the CVP in the lattice $\mathbb{Z}^n$ with respect to the modified metric:

$$\hat{\mathbf{s}}_{Reg} = \underset{\mathbf{s} \in \mathbb{Z}^n}{\operatorname{argmin}} \left(\|\mathbf{y} - \mathbf{M}\ \mathbf{s}\|^2 + \alpha \|\mathbf{s}\|^2\right) \qquad (10)$$

One can easily prove that $\hat{\mathbf{s}}_{MMSE-GDFE} = \hat{\mathbf{s}}_{Reg}$, that is the MMSE-GDFE preprocessing is equivalent to α-regularized decoding for $\alpha = \frac{1}{\rho}$ [5].
The theoretical result of [5] shows that with MMSE-GDFE preprocessing, the approximations of the $\alpha-$regularized decoders achieve the maximum diversity. This leads us to analyse the impact of the MMSE-GDFE on the performance of the Naive Lattice Decoder and of the LLL-aided suboptimal decoders ZF and ZF-DFE for the coded systems. The main contribution of this paper consists in validating the results of [5] and estimating the coding gain by simulations.
Starting with the case of the Golden Code, where the lattice dimension is 8, we present in Fig. 3 the performances of both MMSE-GDFE preprocessed NLD and the MMSE-GDFE followed by the LLL+ZF-DFE decoder. We see that this preprocessing corrects the errors caused by the out-of-constellation events and achieves the optimal transmit diversity. Besides, as far as the coding gain is concerned, we notice that the preprocessing decreases the gap between the NLD and the ML which reaches $2\,\mathrm{dB}$ at BER$=10^{-4}$ when 16-QAM constellations are used, and in the case of the LLL+ZF-DFE, the MMSE-GDFE brings a gain of $2\,\mathrm{dB}$ at BER$=10^{-4}$.

In the case of the Perfect Code 4×4 where the lattice dimension is 32, the MMSE-GDFE leads to a great improvement in terms of diversity order and performance of the NLD. In fact, as shown in Fig. 4, where we consider the NLD and the MMSE-GDFE+NLD, with left preprocessing, the NLD

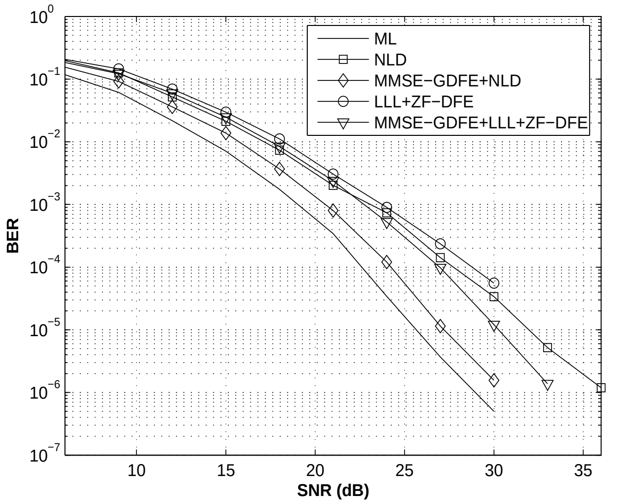

Fig. 3. Bit Error Rate vs. SNR for the Golden Code using $16-$QAM constellation.

recovers the full diversity order and its gap compared to the ML detection is noticeably reduced. For a BER$=10^{-3}$ the loss was about $13.6\,\mathrm{dB}$ and with preprocessing, it is reduced to only $1.5\,\mathrm{dB}$ for the $16-$QAM constellations, i.e the MMSE-GDFE brings a performance gain of more than $12\,\mathrm{dB}$ for the NLD.

However, as we can see from the same figure, applying the MMSE-GDFE preprocessing to the LLL-aided ZF-DFE decoder does not entirely solve the shaping problem for moderate SNR. In fact, even though the theoretical results of [5] predict that the maximal diversity order can be achieved with this method, for practical ranges of the SNR and for high dimensional coded systems, which is the case of the Perfect Code 4×4, the performance is still extremely poor. We explain this with the fact that LLL reduction is not efficient for high lattice dimensions because it performs size reduction only locally on consecutive pairs of columns.

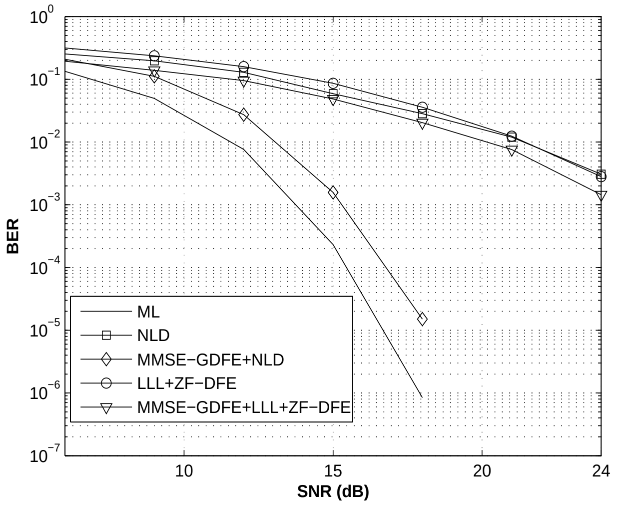

Fig. 4. Bit Error Rate vs. SNR for the Perfect Code 4×4 using $16-$QAM constellation.

This is bad news for the purpose of designing practical decoders, since Naive Lattice Decoding is mostly a theoretical tool and doesn't provide an advantage in terms of complexity compared to ML decoding, as was shown in [9].

V. Conclusion and perspectives

In this paper we led a numerical analysis of the performances of Lattice Decoders in terms of the achieved diversity and the coding gain. The proposed results concerning the diversity confirmed that the NLD and its suboptimal approximations achieve only the receive diversity and that the loss of the transmit diversity is due to the shaping problem caused by neglecting the constellation constraint. Besides, simulation results validated that by adding the left preprocessing MMSE-GDFE, the Naive Lattice Decoder recovers the maximal diversity and offers very good performance. However, even with MMSE-GDFE preprocessing, the gap of LLL-reduced receivers compared to ML is still very large when the lattice dimension increases.

Why is the combination of LLL reduction and MMSE-GDFE preprocessing suboptimal? Can this be explained by the fact that the LLL reduction is not efficient and does not provide good approximations for lattice decoding for high dimensional lattices? The answer to these questions will be the subject of future works in which we will investigate more efficient lattice reduction methods.

References

[1] E. Agrell, T. Eriksson, A. Vardy, and K. Zeger, "Closest point search in lattices", *IEEE Transactions On Information Theory*, Vol. 48, pp. 2201-2214, August 2002.

[2] J-C. Belfiore, G. Rekaya and E. Viterbo, "The Golden Code: A 2×2 Full-Rate Space-Time Code With Nonvanishing Determinants", *IEEE Transactions On Information Theory*, Vol. 51, No. 4, April 2005.

[3] M. O. Damen, H. El Gamal and G. Caire, "On Maximum likelihood detection and the search for the closest lattice point", *IEEE Transactions On Information Theory*, pp. 2389-2402, October 2003.

[4] H. El Gamal, G. Caire, and M.O. Damen, "Lattice coding and decoding achieves the optimal diversity-multiplexing tradeoff of MIMO channels", *IEEE Transactions on Information Theory*, vol 50 n.6, June 2004

[5] J. Jalden, P. Elia, "DMT optimality of LR-aided linear decoders for a general class of channels, lattice designs, and system models", *IEEE Transactions On Information Theory*, Vol. 56, No. 10, 2010.

[6] J. Jalden, B. Ottersten, "On the Complexity of Sphere Decoding In Digital Communications", *IEEE Transactions on Signal Processing*, Vol. 53, pp. 1474-1484, 2005.

[7] F. Oggier, G. Rekaya, J-C. Belfiore and E. Viterbo, "Perfect Space Time Block Codes", *IEEE Transactions On Information Theory*, Vol. 52, No. 9, September 2009.

[8] G. Rekaya and J.-C. Belfiore, "On the Complexity of ML lattice decoders for decoding linear full rate spacetime codes", *IEEE International Symposium On Information Theory (ISIT)*, Yokohama, Japan, July 2003.

[9] C. Studer, D. Seethaler and H. B'olcskei, "Finite Lattice-Size Effects in MIMO Detection", *42nd Asilomar Conference on Signals, Systems and Computers*, pp. 2032-2037, October 2008.

[10] M. Taherzadeh and A. Khandani, "On the Limitations of The Naive Lattice Decoding", *IEEE Transactions on Information Theory*, vol 56 n. 10, October 2010

[11] E. Viterbo and J. Boutros, "A Universal Lattice Code Decoder for Fading Channels", *IEEE Transaction On Information Theory*, Vol. 45, No. 5, pp. 1639-1642, July 1999.