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Dissipative differential inclusions, set-valued energy storage and supply rate maps, and stability of discontinuous feedback systems

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ABSTRACT

In this paper, we develop dissipativity notions for dynamical systems with discontinuous vector fields. Specifically, we consider dynamical systems with Lebesgue measurable and locally essentially bounded vector fields characterized by differential inclusions involving Filippov set-valued maps specifying a set of directions for the system velocity and admitting Filippov solutions with absolutely continuous curves. In particular, we introduce a generalized definition of dissipativity for discontinuous dynamical systems in terms of set-valued supply rate maps and set-valued storage maps consisting of locally Lebesgue integrable supply rates and Lipschitz continuous storage functions, respectively. In addition, we introduce the notion of a set-valued available storage map and a set-valued required supply map, and show that if these maps have closed convex images they specialize to single-valued maps corresponding to the smallest available storage and the largest required supply of the differential inclusion, respectively. Furthermore, we show that all system storage functions are bounded from above by the largest required supply and bounded from below by the smallest available storage, and hence, a dissipative differential inclusion can deliver to its surroundings only a fraction of its generalized stored energy and can store only a fraction of the generalized work done to it. Moreover, extended Kalman–Yakubovich–Popov conditions, in terms of the discontinuous system dynamics, characterizing dissipativity via generalized Clarke gradients and locally Lipschitz continuous storage functions are derived. Finally, these results are then used to develop feedback interconnection stability results for discontinuous systems thereby providing a generalization of the small gain and positivity theorems to systems with discontinuous vector fields.

1. Introduction

Dissipativity theory is a system-theoretic concept that provides a powerful framework for the analysis and control design of dynamical systems based on generalized system energy considerations. In particular, dissipativity theory exploits the notion that numerous physical dynamical systems have certain input–output and state properties related to conservation, dissipation, and transport of mass and energy. Such conservation laws are prevalent in dynamical systems, in general, and feedback control systems, in particular. The dissipation hypothesis on dynamical systems results in a fundamental constraint on the system dynamical behavior, wherein the stored energy of a dissipative dynamical system is at most equal to sum of the initial energy stored in the system and the total externally supplied energy to the system. Thus, the energy that can be

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extracted from the system through its input–output ports is less than or equal to the initial energy stored in the system, and hence, there can be no internal creation of energy; only conservation or dissipation of energy is possible.

The key foundation in developing dissipativity theory for nonlinear dynamical systems with continuously differentiable flows was presented by Willems [1,2] in his seminal two-part paper on dissipative dynamical systems. In particular, Willems [1] introduced the definition of dissipativity for general nonlinear dynamical systems in terms of a *dissipation inequality* involving a generalized system power input, or *supply rate*, and a generalized energy function, or *storage function*. The dissipation inequality implies that the increase in generalized system energy over a given time interval cannot exceed the generalized energy supply delivered to the system during this time interval. The set of all possible system storage functions is convex and every system storage function is bounded from below by the *available system storage* and bounded from above by the *required energy supply*.

Dissipativity theory along with Lyapunov stability theory for feedback interconnections of dissipative systems has been extensively developed for continuous dynamical systems possessing continuously differentiable flows [3]. However, numerous engineering applications give rise to discontinuous dynamical systems. Specifically, in impact mechanics the motion of a dynamical system is subject to velocity jumps and force discontinuities leading to nonsmooth dynamical systems [4,5]. In mechanical systems subject to unilateral constraints on system positions [6], discontinuities occur naturally through system-environment interactions. Alternatively, control of networks and control over networks with dynamic topologies also give rise to discontinuous systems [7]. Specifically, link failures or creations in network systems result in switchings of the communication topology leading to dynamical systems with discontinuous right-hand sides. In addition, open-loop and feedback controllers also give rise to discontinuous dynamical systems. In particular, bang–bang controllers discontinuously switch between maximum and minimum control input values to generate minimum-time system trajectories [8], whereas sliding mode controllers [9,10] use discontinuous feedback control for system stabilization. In switched systems [11,12], switching algorithms are used to select an appropriate plant (or controller) from a given finite parameterized family of plants (or controllers) giving rise to discontinuous systems. As for dynamical systems with continuously differentiable flows [3], dissipativity theory can play a fundamental role in addressing robustness, disturbance rejection, stability of feedback interconnections, and optimality for discontinuous dynamical systems.

In light of the fact that energy notions involving conservation, dissipation, and transport also arise naturally for discontinuous systems, it seems natural that dissipativity theory can play a key role in the analysis and control design of discontinuous dynamical systems. Specifically, as in the analysis of continuous dynamical systems with continuously differentiable flows, dissipativity theory for discontinuous dynamical systems can involve conditions on system parameters that render an input, state, and output system dissipative. In addition, robust stability for discontinuous dynamical systems can be analyzed by viewing a discontinuous dynamical system as an interconnection of discontinuous dissipative dynamical subsystems. Alternatively, discontinuous dissipativity theory can be used to design discontinuous feedback controllers that add dissipation and guarantee stability robustness allowing discontinuous stabilization to be understood in physical terms.

In [13], the authors extend the notion of dissipativity theory to impulsive and hybrid dynamical systems possessing left-continuous flows using generalized storage functions and hybrid supply rates. The overall approach provides an interpretation of a generalized energy balance for impulsive and hybrid dynamical systems in terms of the stored or accumulated system generalized energy, the dissipated energy over the continuous-time dynamics, and the dissipated energy at the resetting instants. Extensions of dissipativity theory to vector dissipativity notions using vector storage functions and vector supply rates for analyzing large-scale interconnected systems are considered in [14]. More recently passivity theory for switched dynamical systems described by a family of subsystems parameterized by a finite index set are discussed in [15–18].

In this paper, we extend the results of [18] to develop dissipativity notions for dynamical systems with discontinuous vector fields. Specifically, we consider dynamical systems with Lebesgue measurable and locally essentially bounded vector fields characterized by differential inclusions involving Filippov set-valued maps specifying a set of directions for the system velocity and admitting Filippov solutions with absolutely continuous curves. In particular, we introduce a generalized definition of dissipativity for discontinuous dynamical systems in terms of set-valued supply rate maps and set-valued storage maps consisting of locally Lebesgue integrable supply rates and locally Lipschitz continuous storage functions, respectively. The collection of storage functions and supply rates satisfy a set of dissipation inequalities reflecting the fact that the dissipated generalized energies of a discontinuous dissipative system are nonnegative and are given by the difference of what is supplied and what is stored.

In addition, we introduce the notion of a set-valued available storage map and a set-valued required supply map, and show that if these set-valued maps have closed convex images they specialize to single-valued maps corresponding to the *smallest available storage* and the *largest required supply* of the dissipative differential inclusion, respectively. Furthermore, we show that all system storage functions are bounded from above by the largest required supply and bounded from below by the smallest available storage, and hence, a dissipative differential inclusion can deliver to its surroundings only a fraction of its generalized stored energy and can store only a fraction of the generalized work done to it. Moreover, we develop analogous results for lossless differential inclusions as well as specialize our results to switched dynamical systems.

Finally, we develop extended Kalman–Yakubovich–Popov conditions in terms of the discontinuous system dynamics for characterizing dissipativity via generalized Clarke gradients of locally Lipschitz continuous storage functions for discontinuous systems. In addition, using the concepts of dissipativity for discontinuous dynamical systems with appropriate storage maps and supply rate maps, we construct nonsmooth Lyapunov functions for discontinuous feedback systems by appropriately combining the storage maps for the forward and feedback subsystems. General stability criteria are given for Lyapunov, asymptotic, and exponential stability for feedback interconnections of discontinuous dynamical systems. In the case where the supply rate map consists of supply rates involving net system power or weighted input–output energy, these results provide extensions of the positivity and small gain theorems to discontinuous dynamical systems. The consideration of nonsmooth Lyapunov functions for proving stability of feedback interconnections of discontinuous systems is an important extension to classical stability theory of dissipative feedback systems since, as shown in [19], there exist nonsmooth dynamical systems whose equilibria cannot be proved to be stable using standard continuously differentiable Lyapunov function theory.

2. Notation and mathematical preliminaries

The notation used in this paper is fairly standard. Specifically, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}^n$ denotes the set of $n \times 1$ real column vectors, $\overline{\mathbb{Z}}_+$ denotes the set of nonnegative integers, and $(\cdot)^{\mathrm{T}}$ denotes transpose. We write $\partial \mathcal{S}$, $\overline{\mathcal{S}}$, and $|\mathcal{S}|$ to denote the boundary, the closure, and the cardinality of the subset $\mathcal{S} \subset \mathbb{R}^n$, respectively. Furthermore, we write $\|\cdot\|$ for the Euclidean vector norm on $\mathbb{R}^n$, $\mathcal{B}_\varepsilon(\alpha), \alpha \in \mathbb{R}^n, \varepsilon > 0$, for the *open ball centered* at α with *radius* ε, $\operatorname{dist}(p, \mathcal{M})$ for the distance from a point p to the set $\mathcal{M}$, that is, $\operatorname{dist}(p, \mathcal{M}) \triangleq \inf_{x \in \mathcal{M}} \|p - x\|$, and $x(t) \to \mathcal{M}$ as $t \to \infty$ to denote that the trajectory $x(t)$ approaches the set $\mathcal{M}$, that is, for every $\varepsilon > 0$ there exists $T > t_0$ such that $\operatorname{dist}(x(t), \mathcal{M}) < \varepsilon$ for all $t > T$. Finally, the notions of openness, convergence, continuity, and compactness that we use throughout the paper refer to the topology generated on $\mathbb{R}^n$ by the norm $\|\cdot\|$.

In this paper, we consider differential inclusions $\mathcal{G}$ of the form[1]

$$\dot{x}(t) \in \mathcal{F}(x(t), u(t)), \quad x(t_0) = x_0, \ \text{a.e. } t \geq t_0, \tag{1}$$

$$u(t) \in \mathcal{U}, \tag{2}$$

$$y(t) \in \mathcal{H}(x(t), u(t)), \tag{3}$$

where $\mathcal{U}$ is a set of admissible (control) inputs consisting of Lebesgue measurable U-valued functions on the semi-infinite interval $[t_0, \infty)$, or, in the case where $U = U(x)$, differential inclusions $\mathcal{G}$ of the form

$$\dot{x}(t) \in \tilde{\mathcal{F}}(x(t)), \quad x(t_0) = x_0, \ \text{a.e. } t \geq t_0, \tag{4}$$

$$y(t) \in \tilde{\mathcal{H}}(x(t)), \tag{5}$$

where $\tilde{\mathcal{F}}(x) \triangleq \{f(x, u) : f(x, u) \in \mathcal{F}(x, u), u \in U(x)\}$ and $\tilde{\mathcal{H}}(x) \triangleq \{h(x, u) : h(x, u) \in \mathcal{H}(x, u), u \in U(x)\}$, and, for every $t \geq t_0, x(t) \in \mathcal{D} \subseteq \mathbb{R}^n, u(t) \in U \subseteq \mathbb{R}^m$, and $y(t) \in Y \subseteq \mathbb{R}^l$. Here $\mathcal{F} : \mathcal{D} \times U \to \mathcal{B}(\mathbb{R}^n)$ and $\mathcal{H} : \mathcal{D} \times U \to \mathcal{B}(\mathbb{R}^l)$ are *Filippov set-valued maps* that assign sets to points, where $\mathcal{B}(\mathbb{R}^q)$ denotes the collection of all subsets of $\mathbb{R}^q$. Hence, $\mathcal{F}$ and $\mathcal{H}$ are mappings from $\mathcal{D} \times U$ to subsets of $\mathbb{R}^n$ and $\mathbb{R}^l$, respectively. The set $\tilde{\mathcal{F}}(x)$ captures all of the directions in $\mathbb{R}^n$ that can be generated at x with inputs $u(\cdot) \in \mathcal{U}$. The inputs $u(\cdot)$ can be selected as either $u : [t_0, \infty) \to U, u : \mathcal{D} \to U$, or $u : [t_0, \infty) \times \mathcal{D} \to U$. We assume that $\tilde{\mathcal{F}}(x)$ is an upper semicontinuous, nonempty, convex, and compact set for all $x \in \mathbb{R}^n$. That is, for every $x \in \mathcal{D}$ and every $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $z \in \mathbb{R}^n$ satisfying $\|z - x\| \leq \delta$, $\tilde{\mathcal{F}}(z) \subseteq \tilde{\mathcal{F}}(x) + \varepsilon \overline{\mathcal{B}}_1(0)$. This assumption is mainly used to guarantee the existence of Filippov solutions to (4) [20]. An absolutely continuous function $x : [t_0, \tau] \to \mathbb{R}^n$ is said to be a *Filippov solution* [20] of (4) on the interval $[t_0, \tau]$ with initial condition $x(t_0) = x_0$, if $x(t)$ satisfies (4) for almost all $t \in [t_0, \tau]$.[2,3] Hence, the tangent vector to a Filippov solution, when it exists, lies in the convex closure of the limiting values of the system vector field $f(\cdot, \cdot)$ in progressively smaller neighborhoods around the solution point. Dynamical systems of the form given by (4) are called *differential inclusions* in the literature [22] and, for every state $x \in \mathbb{R}^n$, they specify a *set* of possible evolutions of $\mathcal{G}$ rather than a single one.

Since the Filippov set-valued map given by (4) is upper semicontinuous with nonempty, convex, and compact values, and $\tilde{\mathcal{F}}$ is also locally bounded [20, p. 85], it follows that Filippov solutions to (4) exist [20, Theorem 1, p. 77]. Recall that the Filippov solution $t \mapsto x(t)$ to (4) is a *right maximal solution* if it cannot be extended (either uniquely or nonuniquely) forward in time. We assume that all right maximal Filippov solutions to (4) exist on $[t_0, \infty)$, and hence, we assume that

[1] Note that the differential inclusion (1) and (2) with measurable $u(\cdot) \in \mathcal{U}$ subsumes the standard open system $\dot{x}(t) = f(x(t), u(t))$, where $f : \mathcal{D} \times U \to \mathbb{R}^n$ and $u(t) \in \mathcal{U}$, by simply considering $\tilde{\mathcal{F}}(x) \triangleq \{f(x, u) : u \in \mathcal{U}\}$. In particular, the Filippov–Wazewski relaxation theorem [20] implies that, under mild hypothesis on $f(\cdot, \cdot)$, an arc $x(\cdot)$ satisfies (1) if and only if there exists a measurable function $u(\cdot) \in \mathcal{U}$ such that $\dot{x}(t) = f(x(t), u(t))$ holds for almost all $t \geq t_0$.

[2] For the closed system $\mathcal{G}_c$ given by $\dot{x}(t) = f(x(t))$ for almost all $t \geq t_0$, where $f : \mathcal{D} \to \mathbb{R}^n$ is Lebesgue measurable and locally essentially bounded, $x : [t_0, \tau) \to \mathbb{R}^n$ is a Filippov solution of $\mathcal{G}_c$ if $x(t)$ satisfies the differential inclusion $\dot{x}(t) \in \mathcal{F}(x(t))$, a.e. $t \in [t_0, \tau]$, where $\mathcal{F}(x) = \mathcal{K}[f](x) \triangleq \bigcap_{\delta > 0} \bigcap_{\mu(\mathcal{S}) = 0} \overline{\mathrm{co}}\{f(\mathcal{B}_\delta(x)) \setminus \mathcal{S}\}$, $x \in \mathbb{R}^n$, $\mu(\cdot)$ denotes the Lebesgue measure in $\mathbb{R}^n$, "$\overline{\mathrm{co}}$" denotes convex closure, and $\bigcap_{\mu(\mathcal{S})} = 0$ denotes the intersection over all sets $\mathcal{S}$ of Lebesgue measure zero. Equivalently, it follows from Theorem 1 of [21] that there exists a set $\mathcal{N}_f \subset \mathbb{R}^n$ of measure zero such that, for every $\mathcal{W} \subset \mathbb{R}^n$ of measure zero, $\mathcal{K}[f](x) = \overline{\mathrm{co}}\{\lim_{i \to \infty} f(x_i) : x_i \to x, x_i \notin \mathcal{N}_f \cup \mathcal{W}\}$. (Since f is locally essentially bounded, $\mathcal{K}[f](\cdot)$ is upper semicontinuous and has nonempty, compact, and convex values.) Thus, Filippov solutions are limits of solutions to $\mathcal{G}_c$ with f averaged over progressively smaller neighborhoods around the solution point, and hence, allow solutions to be defined at points where f itself is not defined.

[3] Alternatively, we can consider Krasovskii solutions of (4) wherein the possible misbehavior of the derivative of the state on null measure sets is not ignored; that is, $\mathcal{K}[f](x)$ in footnote 2 is replaced with $\mathcal{K}[f](x) = \bigcap_{\delta > 0} \overline{\mathrm{co}}\{f(\mathcal{B}_\delta(x))\}$ and where f is assumed to be locally bounded.

(4) is *forward complete*. Recall that (4) is forward complete if and only if the Filippov solutions to (4) are uniformly globally sliding time stable [23, Lemma 1, p. 182]. An *equilibrium point* of (4) is a point $x_e \in \mathbb{R}^n$ such that $0 \in \tilde{\mathcal{F}}(x_e)$. It is easy to see that x_e is an equilibrium point of (4) if and only if the constant function $x(\cdot) = x_e$ is a Filippov solution of (4). We denote the set of equilibrium points of (4) by $\mathcal{E}$. Since the set-valued map $\tilde{\mathcal{F}}(\cdot)$ is upper semicontinuous, it follows that $\mathcal{E}$ is closed.

To develop stability and dissipativity theory for discontinuous dynamical systems of the form given by (1)–(3), we need to introduce the notion of generalized derivatives and gradients. Here we focus on Clarke generalized derivatives and gradients [24].

Definition 2.1 (*[24,25]*)**.** Let $V : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous function. The *Clarke upper generalized derivative* of $V(\cdot)$ at x in the direction of $v \in \mathbb{R}^n$ is defined by

$$V^o(x, v) \triangleq \limsup_{y \to x, h \to 0^+} \frac{V(y + hv) - V(y)}{h}. \tag{6}$$

The *Clarke generalized gradient* $\partial V : \mathbb{R}^n \to \mathcal{B}(\mathbb{R}^{1\times n})$ of $V(\cdot)$ at x is the set

$$\partial V(x) \triangleq \overline{\text{co}} \left\{ \lim_{i \to \infty} \nabla V(x_i) : x_i \to x,\ x_i \notin \mathcal{N} \cup \mathcal{S} \right\}, \tag{7}$$

where co denotes the convex hull, ∇ denotes the nabla operator, $\mathcal{N}$ is the set of measure zero of points where ∇V does not exist, $\mathcal{S}$ is any subset of $\mathbb{R}^n$ of measure zero, and the increasing unbounded sequence $\{x_i\}_{i \in \overline{\mathbb{Z}}_+} \subset \mathbb{R}^n$ converges to $x \in \mathbb{R}^n$.

Note that (6) always exists. Furthermore, note that it follows from Theorem 2.5.1 of [24] that (7) is well defined and consists of all convex combinations of all the possible limits of the gradient at neighboring points where V is differentiable. In addition, note that since $V(\cdot)$ is Lipschitz continuous, it follows from Rademacher's theorem [26, Theorem 6, p. 281] that the gradient $\nabla V(\cdot)$ of $V(\cdot)$ exists almost everywhere, and hence, $\nabla V(\cdot)$ is bounded. Thus, since for each $x \in \mathbb{R}^n$, $\partial V(x)$ is convex, closed, and bounded, it follows that $\partial V(x)$ is compact.

In order to state the main results of this paper, we need some additional notation and definitions. Given a locally Lipschitz continuous function $V : \mathbb{R}^n \to \mathbb{R}$, the *set-valued Lie derivative* $\mathcal{L}_{\tilde{\mathcal{F}}} V : \mathbb{R}^n \to \mathcal{B}(\mathbb{R})$ of V with respect to $\tilde{\mathcal{F}}$ at x [25,27] is defined as

$$\begin{aligned} \mathcal{L}_{\tilde{\mathcal{F}}} V(x) &\triangleq \left\{ a \in \mathbb{R} : \text{ there exists } v \in \tilde{\mathcal{F}}(x) \text{ such that } p^{\mathrm{T}} v = a \text{ for all } p^{\mathrm{T}} \in \partial V(x) \right\} \\ &\subseteq \bigcap_{p^{\mathrm{T}} \in \partial V(x)} p^{\mathrm{T}} \tilde{\mathcal{F}}(x). \end{aligned} \tag{8}$$

If $\tilde{\mathcal{F}}$ is convex with compact values, then $\mathcal{L}_{\tilde{\mathcal{F}}} V(x),\ x \in \mathbb{R}^n$, is a closed and bounded, possibly empty, interval in $\mathbb{R}$. If $V(\cdot)$ is continuously differentiable at x, then $\mathcal{L}_{\tilde{\mathcal{F}}} V(x) = \{\nabla V(x) \cdot v : v \in \tilde{\mathcal{F}}(x)\}$. In the case where $\mathcal{L}_{\tilde{\mathcal{F}}} V(x)$ is nonempty, we use the notation $\max \mathcal{L}_{\tilde{\mathcal{F}}} V(x)$ (resp., $\min \mathcal{L}_{\tilde{\mathcal{F}}} V(x)$) to denote the largest (resp., smallest) element of $\mathcal{L}_{\tilde{\mathcal{F}}} V(x)$. Furthermore, we adopt the convention $\max \varnothing = -\infty$. Finally, recall that a function $V : \mathbb{R}^n \to \mathbb{R}$ is *regular* at $x \in \mathbb{R}^n$ [24, Definition 2.3.4] if, for all $v \in \mathbb{R}^n$, the right directional derivative $V'_+(x, v) \triangleq \lim_{h \to 0^+} \frac{1}{h}[V(x + hv) - V(x)]$ exists and $V'_+(x, v) = V^o(x, v)$. V is called *regular* on $\mathbb{R}^n$ if it is regular at every $x \in \mathbb{R}^n$.

3. Dissipative differential inclusions: input–output and state properties

In this section, we extend the notion of dissipativity of dynamical systems with continuously differentiable flows to develop the concept of dissipativity and losslessness for differential inclusions. Specifically, we extend the classical definitions of dissipativity and losslessness [1,2] for dynamical systems with continuously differentiable flows to discontinuous systems and develop dissipativity theory for differential inclusions in terms of integral and generalized derivative inequalities with respect to set-valued supply rate maps. Since a special case of (1)–(3) involves switched dynamical systems [15,16], in our definition of dissipativity and losslessness we consider multiple supply rates and multiple storage functions. Specifically, for switched systems each subsystem can have its individual supply rate, and hence, its own storage function. Thus, for Filippov dynamical systems, in general, and switched dynamical systems, in particular, using the same storage function and the same supply rate for all subsystems can be very restrictive as they may be very difficult to find or may not exist. For further details of this fact, see [15,16].

Definition 3.1. (i) The differential inclusion $\mathcal{G}$ given by (1)–(3) is *weakly exponentially dissipative* (resp., *weakly dissipative*) *with respect to the set-valued supply rate map* $\mathcal{S}_{\mathrm{r}} : U \times Y \to \mathcal{B}(\mathbb{R})$, consisting of r locally Lebesgue integrable functions[4]

[4] More generally, a countably infinite number of supply rates can also be considered. A similar remark also holds for the available storage, storage, and required supply functions introduced later in the paper. This, for example, would correspond to the case where we are considering a switched dynamical system with a countably infinite number of subsystems.

$s_k : U \times Y \to \mathbb{R}$, $k \in \{1, \ldots, r\}$, called *supply rates*, if there exist constants $\varepsilon_k > 0$ (resp., $\varepsilon_k = 0$) such that the *dissipation inequality*

$$0 \le \int_{t_0}^{t} e^{\varepsilon_k \sigma} s_k(u(\sigma), y(\sigma))\mathrm{d}\sigma, \quad t_0 \le \sigma \le t, \tag{9}$$

is satisfied for at least one $s_k(\cdot,\cdot) \in \mathcal{S}_\mathrm{r}$ with $s_k(0, 0) = 0$ and $u(\cdot) \in \mathcal{U}$, and at least one Filippov solution $x(t)$, $t \ge t_0$, of $\mathcal{G}$ with $x(t_0) = 0$.

(ii) The differential inclusion $\mathcal{G}$ of the form (1)–(3) is *strongly exponentially dissipative* (resp., *strongly dissipative*) *with respect to the set-valued supply rate map* $\mathcal{S}_\mathrm{r} : U \times Y \to \mathcal{B}(\mathbb{R})$ if the dissipation inequality (9) is satisfied with $\varepsilon_k > 0$ (resp., $\varepsilon_k = 0$) for every Filippov solution $x(t)$, $t \ge t_0$, of $\mathcal{G}$ with $x(t_0) = 0$ by at least one $s_k(\cdot,\cdot) \in \mathcal{S}_\mathrm{r}$ with $s_k(0, 0) = 0$, $k \in \{1, \ldots, r\}$, and $u(\cdot) \in \mathcal{U}$.

(iii) The differential inclusion $\mathcal{G}$ of the form (1)–(3) is *weakly lossless with respect to the set-valued supply rate map* $\mathcal{S}_\mathrm{r} : U \times Y \to \mathcal{B}(\mathbb{R})$ if $\mathcal{G}$ is weakly dissipative with respect to the set-valued supply rate map $\mathcal{S}_\mathrm{r}$ and the dissipation inequality (9) is satisfied as an equality for $\varepsilon_k = 0$, at least one $s_k(\cdot,\cdot) \in \mathcal{S}_\mathrm{r}$ with $s_k(0, 0) = 0$, $k \in \{1, \ldots, r\}$, and $u(\cdot) \in \mathcal{U}$, and with $x(t_0) = x(t) = 0$ along at least one Filippov solution of $\mathcal{G}$.

(iv) The differential inclusion $\mathcal{G}$ of the form (1)–(3) is *strongly lossless with respect to the set-valued supply rate map* $\mathcal{S}_\mathrm{r} : U \times Y \to \mathcal{B}(\mathbb{R})$ if $\mathcal{G}$ is strongly dissipative with respect to the set-valued supply rate map $\mathcal{S}_\mathrm{r}$ and the dissipation inequality (9) is satisfied as an equality for $\varepsilon_k = 0$, along every Filippov solution of $\mathcal{G}$ by at least one $s_k(\cdot,\cdot) \in \mathcal{S}_\mathrm{r}$ with $s_k(0, 0) = 0$, $k \in \{1, \ldots, r\}$, and $u(\cdot) \in \mathcal{U}$, and with $x(t_0) = x(t) = 0$.

Note that every strongly dissipative (resp., strongly lossless) differential inclusion is weakly dissipative (resp., weakly lossless); however, the converse is not true. In most cases it is sufficient to consider the notion of weak dissipativity for a given differential inclusion. In this case, a single-valued supply rate map suffices in Definition 3.1. However, when considering a switched dynamical system as a family of parameterized systems and the dissipativity of every system is needed, then it is necessary to use the stronger notion of strong dissipativity, which requires that the set of all Filippov solutions of $\mathcal{G}$ to be *epimorphic* (i.e., a surjective homomorphism) to the set-valued supply rate map $\mathcal{S}_\mathrm{r}$. That is, the mapping from the set of all Filippov solutions of $\mathcal{G}$ to the set of supply rates $\mathcal{S}_\mathrm{r}$ that takes each Filippov solution into its own supply rate is a surjection.

Next, define the *available storage functions* $V_{\mathrm{a}_k} : \mathcal{D} \to \mathbb{R}$ of the differential inclusion $\mathcal{G}$ by

$$V_{\mathrm{a}_k}(x_0) \triangleq -\inf_{u(\cdot), T \ge t_0} \int_{t_0}^{T} e^{\varepsilon_k t} s_k(u(t), y(t))\mathrm{d}t = \sup_{u(\cdot),\, T \ge t_0} \left[-\int_{t_0}^{T} e^{\varepsilon_k t} s_k(u(t), y(t))\mathrm{d}t \right], \tag{10}$$

where $s_k(\cdot,\cdot) \in \mathcal{S}_\mathrm{r}$, $k \in \{1, \ldots, r\}$, and $x(t)$, $t \ge t_0$, is a Filippov solution of $\mathcal{G}$ with $x(t_0) = x_0$ and admissible input $u(\cdot) \in \mathcal{U}$. Furthermore, define the *set-valued available storage map* $\mathcal{V}_\mathrm{a} : \mathcal{D} \to \mathcal{B}(\mathbb{R})$ of the differential inclusion $\mathcal{G}$ by $\mathcal{V}_\mathrm{a}(x) \triangleq \{V_{\mathrm{a}_1}(x), \ldots, V_{\mathrm{a}_r}(x)\}$, $x \in \mathcal{D}$. The supremum in (10) is taken over all admissible inputs $u(\cdot)$, all time $t \ge t_0$, and all Filippov system trajectories with initial value $x(t_0) = x_0$ and terminal value left free. Note that for every $k \in \{1, \ldots, r\}$, $V_{\mathrm{a}_k}(x) \ge 0$ for all $x \in \mathcal{D}$ since $V_{\mathrm{a}_k}(x)$ is the supremum over a set of numbers containing the zero element $(T = t_0)$.

Next, define the *least available storage map* of the differential inclusion $\mathcal{G}$ by

$$V_{\mathrm{as}}(x_0) \triangleq \left\{ v \in \mathcal{V}_\mathrm{a}(x_0) : v = \inf_{w \in \mathcal{V}_\mathrm{a}(x_0)} w \right\} = \left\{ v \in \mathcal{V}_\mathrm{a}(x_0) : v = -\sup_{w \in (-\mathcal{V}_\mathrm{a}(x_0))} w \right\}. \tag{11}$$

Note that if $\mathcal{B}(\mathbb{R})$ is a reflexive strictly convex space[5] and $\mathcal{V}_\mathrm{a}(x)$ has closed convex images, then it follows that the least available storage map is single-valued [28, p. 360]. We call this map the *smallest available storage* of the differential inclusion $\mathcal{G}$. For the remainder of the paper we assume that $\mathcal{B}(\mathbb{R})$ is a reflexive strictly convex space. It follows from Corollary 9.3.3 of [28] that if $\mathcal{V}_\mathrm{a} : \mathcal{D} \to \mathcal{B}(\mathbb{R})$ is a continuous set-valued map with nonempty closed convex images, then the smallest available storage is continuous. In other words, if $\mathcal{V}_\mathrm{a}(\cdot)$ is continuous and takes its values in a compact subset of a strictly convex reflexive Banach space, then the smallest available storage $V_{\mathrm{as}}(x_0)$ is continuous. In addition, it follows from (11) that the smallest available storage of a differential inclusion $\mathcal{G}$ is the maximum amount of storage, or generalized stored energy, which can be extracted from the differential inclusion $\mathcal{G}$ at any time T. In the definition of $V_{\mathrm{as}}(\cdot)$, the infimum is taken elementwise over all $V_{\mathrm{a}_k}(\cdot) \in \mathcal{V}_\mathrm{a}$, $k \in \{1, \ldots, r\}$, since $s_k(\cdot,\cdot) \in \mathcal{S}_\mathrm{r}$, $k \in \{1, \ldots, r\}$, which implies that for different elements of $\mathcal{V}_\mathrm{a}(\cdot)$ the infimum is calculated separately. Moreover, note that $V_{\mathrm{as}}(x) \ge 0$ for all $x \in \mathcal{D}$ since $V_{\mathrm{as}}(x)$, $x \in \mathcal{D}$, is the infimum over a set of nonnegative numbers.

[5] Recall that if $\mathcal{B}(\mathbb{R}^q)$ is a Hilbert space, then $\mathcal{B}(\mathbb{R}^q)$ is a reflexive strictly convex space; that is, for all $x, y \in \mathcal{B}(\mathbb{R}^q)$ that are not colinear, $\|x + y\| < \|x\| + \|y\|$.

The following definitions are needed later in the paper.

Definition 3.2. A differential inclusion $\mathcal{G}$ is *weakly* (resp., *strongly*) *completely reachable* if for every $x_0 \in \mathcal{D} \subseteq \mathbb{R}^n$ there exists a finite time $t_i < t_0$ and an admissible input $u(t)$ defined on $[t_i, t_0]$ such that at least one (resp., every) Filippov solution $x(t),\ t \geq t_i$, of $\mathcal{G}$ can be driven from $x(t_i) = 0$ to $x(t_0) = x_0$. $\mathcal{G}$ is *weakly* (resp., *strongly*) *completely null controllable* if for every $x_0 \in \mathcal{D} \subseteq \mathbb{R}^n$ there exists a finite time $t_f > t_0$ and an admissible input $u(t)$ defined on $[t_0, t_f]$ such that at least one (resp., every) Filippov solution $x(t),\ t \geq t_0$, of $\mathcal{G}$ can be driven from $x(t_0) = x_0$ to $x(t_f) = 0$.

Definition 3.3. Consider the differential inclusion $\mathcal{G}$ given by (1)–(3).

(i) A set-valued map $\mathcal{V}_s : \mathcal{D} \to \mathcal{B}(\mathbb{R})$ consisting of q nonnegative definite, locally Lipschitz continuous, and regular functions $V_i : \mathcal{D} \to \mathbb{R},\ i \in \{1, \ldots, q\}$, is called a *weak set-valued storage map*, if the dissipation inequality

$$e^{\varepsilon_k t} V_i(x(t)) \leq e^{\varepsilon_k t_0} V_i(x(t_0)) + \int_{t_0}^{t} e^{\varepsilon_k \sigma} s_k(u(\sigma), y(\sigma))\mathrm{d}\sigma, \quad t \geq t_0, \tag{12}$$

is satisfied for at least one $V_i(\cdot) \in \mathcal{V}_s(x)$ with $V_i(0) = 0$, where $i \in \{1, \ldots, q\}$, at least one $s_k(\cdot,\cdot) \in \mathcal{S}_r$ with $s_k(0, 0) = 0$, where $k \in \{1, \ldots, r\}$, and at least one Filippov solution $x(t),\ t \geq t_0$, of $\mathcal{G}$ with $u(\cdot) \in \mathcal{U}$. The functions $V_i(\cdot) \in \mathcal{V}_s(x),\ i \in \{1, \ldots, q\}$, satisfying (12) are called the *storage functions* of $\mathcal{G}$ and are denoted as $V_{s_i}(x),\ x \in \mathcal{D}$.

(ii) A set-valued map $\mathcal{V}_s : \mathcal{D} \to \mathcal{B}(\mathbb{R})$ is called a *strong set-valued storage map*, if the dissipation inequality (12) is satisfied for every Filippov solution $x(t),\ t \geq t_0$, of $\mathcal{G}$ with $u(\cdot) \in \mathcal{U}$, by at least one $V_{s_i}(\cdot) \in \mathcal{V}_s(x)$ with $V_{s_i}(0) = 0$, where $i = 1, \ldots, q$, and at least one $s_k(\cdot,\cdot) \in \mathcal{S}_r$ with $s_k(0, 0) = 0$, where $k = 1, \ldots, r$.

Note that, since for $i \in \{1, \ldots, q\}$ and $k \in \{1, \ldots, r\}$, $V_{s_i}(0) = 0$, it follows that

$$\int_{t_0}^{t} e^{\varepsilon_k \sigma} s_k(u(\sigma), y(\sigma))\mathrm{d}\sigma \geq e^{\varepsilon_k t} V_{s_i}(x(t)) \geq 0, \quad x \in \mathcal{D},\ V_{s_i}(\cdot) \in \mathcal{V}_s(x),$$

and hence, the existence of a weak (resp., strong) set-valued storage map implies weak (resp., strong) dissipativity. Inequality (12) is known as the *dissipation inequality* and reflects the fact that some of the supplied generalized energies to the system $\mathcal{G}$ are stored, and some are dissipated. The dissipated generalized energies are nonnegative and are given by the difference of what is supplied and what is stored. In addition, the amount of generalized stored energies are a function of the state of the dynamical system.

In the remainder of the paper, we drop the adjectives "weak" and "strong" as well as the index k whenever a statement holds for both the weak and strong cases. If $V_{s_i}(\cdot)$ is locally Lipschitz continuous and regular, then an equivalent statement for the dissipativeness of $\mathcal{G}$ involving supply rates $s_k(u, y)$ is

$$\dot{V}_{s_i}(x(t)) \leq s_k(u(t), y(t)), \quad \text{a.e. } t \geq 0, \tag{13}$$

or, equivalently, $\dot{V}_{s_i}(x) \leq s_k(u, y)$, where

$$\dot{V}_{s_i}(x) = \left.\frac{\mathrm{d}}{\mathrm{d}t} V_{s_i}(\psi(t, x, u))\right|_{t=0} \triangleq \limsup_{h \to 0^+} \frac{V_{s_i}(\psi(h, x, u)) - V_{s_i}(x)}{h} \tag{14}$$

for every $x \in \mathbb{R}^n$ such that the limit in (14) exists, denotes the upper right directional Dini derivative of $V_{s_i}(x)$ along the Filippov state trajectories $\psi(t, x, u)$ of (1) through $x \in \mathcal{D}$ with $u(\cdot) \in \mathcal{U}$ at $t = 0$. Alternatively, an equivalent statement for exponential dissipativeness of $\mathcal{G}$ involving supply rates $s_k(u, y)$ is

$$\dot{V}_{s_i}(x(t)) + \varepsilon_k V_{s_i}(x(t)) \leq s_k(u(t), y(t)), \quad \text{a.e. } t \geq 0. \tag{15}$$

The following lemma is needed for the next and subsequent results in the paper.

Lemma 3.1 ([25])**.** *Let $x : [t_0, t] \to \mathbb{R}^n$ be a Filippov solution of the differential inclusion (1) and let $V : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz continuous and regular. Then $\frac{\mathrm{d}}{\mathrm{d}\sigma} V(x(\sigma))$ exists for almost all $\sigma \in [t_0, t]$ and $\frac{\mathrm{d}}{\mathrm{d}\sigma} V(x(\sigma)) \in \mathcal{L}_{\mathcal{F}} V(x(\sigma))$ for almost all $\sigma \in [t_0, t]$.*

Proposition 3.1. *Consider the differential inclusion $\mathcal{G}$ given by (1)–(3) and let $V : \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz continuous and regular function such that $V(x) \geq 0$ for all $x \in \mathbb{R}^n$ and $V(0) = 0$. Assume there exist a Lebesgue measurable function $s : \mathbb{R}^m \times \mathbb{R}^l \to \mathbb{R}$ and a scalar $\varepsilon > 0$ (resp., $\varepsilon = 0$) such that*

$$\max \mathcal{L}_{\mathcal{F}} V(x) \leq -\varepsilon V(x) + s(u, y), \quad \text{a.a. } u \in \mathcal{U}. \tag{16}$$

Then $\mathcal{G}$ is strongly exponentially dissipative (resp., strongly dissipative) with respect to the set-valued supply rate map $\mathcal{S}_r = \{s(u, y)\}$.

Proof. It suffices to show that if (16) holds, then (12) holds on the interval $[t_0, t]$. To see this, let $x : [t_0, t] \to \mathbb{R}^n$ be a Filippov solution of (4) with initial condition $x(0) = x_0$. Now, since by Lemma 3.1 $\dot{V}(x(\sigma)) \le \max \mathcal{L}_{\mathcal{F}} V(x(\sigma))$ for almost all $\sigma \in [t_0, t]$, it follows from (16) that $\dot{V}(x(\sigma)) \le -\varepsilon V(x(\sigma)) + s(u(\sigma), y(\sigma))$ for almost all $\sigma \in [t_0, t]$, and hence,

$$e^{\varepsilon\sigma}(\dot{V}(x(\sigma)) + \varepsilon V(x(\sigma))) \le e^{\varepsilon\sigma} s(u(\sigma), y(\sigma)), \quad \text{a.e. } \sigma \in [t_0, t]. \tag{17}$$

Now, integrating (17), where the integral is a Lebesgue integral, it follows that (12) holds with $\varepsilon > 0$ (resp., $\varepsilon = 0$). □

Example 3.1. Consider the controlled discontinuous dynamical system $\mathcal{G}$ representing a mass sliding on a horizontal surface subject to a Coulomb frictional force. During sliding, the Coulomb frictional model states that the magnitude of the friction force is independent of the magnitude of the system velocity and is equal to the normal contact force times the coefficient of kinetic friction. The application of this model to a sliding mass on a horizontal frictional surface gives

$$\dot{x}(t) = -\text{sign}(x(t)) + u(t), \qquad x(0) = x_0, \quad \text{a.e. } t \ge 0, \tag{18}$$

$$y(t) = x(t). \tag{19}$$

Eq. (18) can be rewritten in the form of (1) with $\mathcal{F}(x, u) = \mathcal{K}[f](x) + u$ so that

$$\dot{x}(t) \in \mathcal{K}[f](x(t)) + u(t), \qquad x(0) = x_0, \quad \text{a.e. } t \ge 0, \tag{20}$$

where the Filippov set-valued map $\mathcal{K}[f] : \mathbb{R} \to \mathcal{B}(\mathbb{R})$ is given by

$$\mathcal{K}[f](x) \triangleq \begin{cases} -1, & x > 0, \\ [-1, 1], & x = 0, \\ 1, & x < 0. \end{cases} \tag{21}$$

Let $V_{s_1}(x) = x^2$ and $\mathcal{V}_s = \{V_{s_1}(x)\}$. Since

$$\begin{aligned} \dot{V}_{s_1}(x) \in \mathcal{L}_{\mathcal{F}} V_{s1}(x) &= \partial V_{s1}(x)(\mathcal{K}[f](x) + u) \\ &= 2x\mathcal{K}[f](x) + 2xu \\ &= -|x| + 2uy \\ &\le 2uy, \end{aligned} \tag{22}$$

it follows that $\max \mathcal{L}_{\mathcal{F}} V_{s_1}(x) \le 2uy$ for all Filippov solutions, which, by Proposition 3.1, implies that $\mathcal{G}$ is strongly dissipative with respect to the single-valued supply rate map $\mathcal{S}_r = \{2uy\}$.

Next, let $V_{s_2}(x) = |x|$ and $\mathcal{V}_s = \{V_{s_2}(x)\}$. Since

$$\begin{aligned} \dot{V}_{s_2}(x) \in \mathcal{L}_{\mathcal{F}} V_{s_2}(x) &= \begin{cases} -1 + \text{sign}(x)u, & x \ne 0, \\ 0, & x = 0, \end{cases} \\ &= -1 + u\,\text{sign}(y), \quad x \ne 0, \end{aligned} \tag{23}$$

it follows that $\max \mathcal{L}_{\mathcal{F}} V_{s_2}(x) \le u\,\text{sign}(y)$ for almost all $x \in \mathbb{R}$ and all Filippov solutions, which, by Proposition 3.1, implies that $\mathcal{G}$ is strongly dissipative with respect to the single-valued supply rate map $\mathcal{S}_r = \{u\,\text{sign}(y)\}$.

Alternatively, if we choose $\mathcal{V}_s = \{V_{s_1}(x), V_{s_2}(x)\}$ and $\mathcal{S}_r = \{2uy, u\,\text{sign}(y)\}$, then it follows that $\mathcal{G}$ is strongly dissipative with respect to the set-valued supply rate map $\mathcal{S}_r$. □

Next, we show that all of the available storage functions of $\mathcal{G}$ are finite and zero at the origin if and only if $\mathcal{G}$ is weakly (resp., strongly) dissipative.

Theorem 3.1. *Consider the differential inclusion $\mathcal{G}$ given by (1)–(3) and assume that $\mathcal{G}$ is weakly (resp., strongly) completely reachable. Then $\mathcal{G}$ is weakly (resp., strongly) dissipative with respect to the set-valued supply rate map $\mathcal{S}_r : U \times Y \to \mathcal{B}(\mathbb{R})$ if and only if the available storage functions $V_{a_k}(x_0)$ given by (10) are finite and $V_{a_k}(0) = 0$ for all $x_0 \in \mathcal{D}$, some $k \in \{1, \ldots, r\}$, and at least one (resp., every) Filippov solution of $\mathcal{G}$. Moreover, if $V_{a_k}(0) = 0$ and $V_{a_k}(x_0)$ are finite for all $x_0 \in \mathcal{D}$, some $k \in \{1, \ldots, r\}$, and at least one (resp., every) Filippov solution of $\mathcal{G}$, then $\mathcal{V}_a : \mathcal{D} \to \mathcal{B}(\mathbb{R})$ is a weak (resp., strong) set-valued storage map for $\mathcal{G}$. Finally, all storage functions $V_{s_i}(\cdot) \in \mathcal{V}_s(x)$, $i \in \{1, \ldots, q\}$, for $\mathcal{G}$ satisfy*

$$0 \le V_{as}(x) \le V_{s_i}(x), \quad x \in \mathcal{D}. \tag{24}$$

Proof. Suppose $V_{a_k}(0) = 0$ and $V_{a_k}(x_0)$, $x_0 \in \mathcal{D}$, are finite for some $k \in \{1, \ldots, r\}$. Now, it follows from (10) (with $T = t_0$) that $V_{a_k}(x_0) \ge 0$ for all $x_0 \in \mathcal{D}$ and some $k \in \{1, \ldots, r\}$. Next, let $x(t)$, $t \ge t_0$, be a Filippov solution to (1) with admissible input $u(t)$, $t \in [t_0, T]$. Since $-V_{a_k}(x_0)$, $x_0 \in \mathcal{D}$ and $k \in \{1, \ldots, r\}$, is given by the infimum over all admissible inputs $u(\cdot)$ in (10), it follows that for all admissible inputs $u(\cdot) \in \mathcal{U}$, $T > t_0$, and $k \in \{1, \ldots, r\}$,

$$-V_{a_k}(x(t_0)) \le \int_{t_0}^{T} s_k(u(t), y(t))\mathrm{d}t = \int_{t_0}^{t_f} s_k(u(t), y(t))\mathrm{d}t + \int_{t_f}^{T} s_k(u(t), y(t))\mathrm{d}t,$$

which implies

$$-V_{a_k}(x(t_0)) - \int_{t_0}^{t_f} s_k(u(t), y(t))dt \le \int_{t_f}^{T} s_k(u(t), y(t))dt.$$

Hence,

$$V_{a_k}(x(t_0)) + \int_{t_0}^{t_f} s_k(u(t), y(t))dt \ge -\inf_{u(\cdot),\, T\ge t_f} \int_{t_f}^{T} s_k(u(t), y(t))dt = V_{a_k}(x(t_f)) \ge 0, \quad k \in \{1, \ldots, r\}, \tag{25}$$

which implies that

$$\int_{t_0}^{t_f} s_k(u(t), y(t))dt \ge -V_{a_k}(x(t_0)), \quad k \in \{1, \ldots, r\}. \tag{26}$$

Thus, since by assumption $V_{a_k}(0) = 0$ for some $k \in \{1, \ldots, r\}$, the dissipation inequality (9) holds. Hence, it follows from Definition 3.1 that $\mathcal{G}$ is weakly dissipative with respect to the set-valued supply rate map $\mathcal{S}_r : U \times Y \to \mathcal{B}(\mathbb{R})$. Furthermore, $V_{a_k}(x),\ x \in \mathcal{D},\ k \in \{1, \ldots, r\}$, is a storage function for $\mathcal{G}$. Hence, by Definition 3.3, $\mathcal{V}_a : \mathcal{D} \to \mathcal{B}(\mathbb{R})$ is a weak set-valued storage map for $\mathcal{G}$.

Alternatively, if $V_{a_k}(0) = 0$ and $V_{a_k}(x_0),\ x_0 \in \mathcal{D},\ k \in \{1, \ldots, r\}$, is finite for all Filippov solutions to $\mathcal{G}$, then it follows from (26) that the dissipation inequality (9) is satisfied for all Filippov solutions to $\mathcal{G}$, and hence, by Definitions 3.1 and 3.3, $\mathcal{G}$ is strongly dissipative with respect to the set-valued supply rate map $\mathcal{S}_r : U \times Y \to \mathcal{B}(\mathbb{R})$ and $\mathcal{V}_a : \mathcal{D} \to \mathcal{B}(\mathbb{R})$ is a strong set-valued storage map for $\mathcal{G}$.

Conversely, suppose $\mathcal{G}$ is weakly dissipative with respect to the set-valued supply rate map $\mathcal{S}_r : U \times Y \to \mathcal{B}(\mathbb{R})$. Since $\mathcal{G}$ is weakly completely reachable it follows that for every $x_0 \in \mathcal{D}$ such that $x(t_0) = x_0$, there exist $\hat{t} \le t < t_0$ and an admissible input $u(\cdot) \in \mathcal{U}$ defined on $[\hat{t}, t_0]$ such that $x(\hat{t}) = 0$ and $x(t_0) = x_0$. Now, since $\mathcal{G}$ is weakly dissipative with respect to the set-valued supply rate map $\mathcal{S}_r : U \times Y \to \mathcal{B}(\mathbb{R})$ and $x(\hat{t}) = 0$, it follows that for some $s_k(u, y) \in \mathcal{S}_r,\ k \in \{1, \ldots, r\}$, and at least one Filippov solution of $\mathcal{G}$,

$$\int_{\hat{t}}^{T} s_k(u(t), y(t))dt \ge 0, \quad T > \hat{t},$$

or, equivalently,

$$\int_{t_0}^{T} s_k(u(t), y(t))dt \ge -\int_{\hat{t}}^{t_0} s_k(u(t), y(t))dt, \quad T > t_0,$$

which implies that there exist functions $W_k : \mathcal{D} \to \mathbb{R}$ such that

$$\int_{t_0}^{T} s_k(u(t), y(t))dt \ge W_k(x_0) > -\infty, \quad T > t_0,\ k \in \{1, \ldots, r\}. \tag{27}$$

Now, it follows from (27) that for all $x \in \mathcal{D}$ and some $k \in \{1, \ldots, r\}$,

$$V_{a_k}(x) = -\inf_{u(\cdot),\, T\ge t_0} \int_{t_0}^{T} s_k(u(t), y(t))dt \le -W_k(x),$$

and hence, the available storage function $V_{a_k}(x) < \infty$ for all $x \in \mathcal{D}$ and some $k \in \{1, \ldots, r\}$. Furthermore, with $x(t_0) = 0$, it follows that for all admissible inputs $u(t),\ t \ge t_0$, and some $k \in \{1, \ldots, r\}$,

$$\int_{t_0}^{T} s_k(u(t), y(t))dt \ge 0, \quad T \ge t_0,$$

which implies that

$$\sup_{u(\cdot),\, T\ge t_0} \left[-\int_{t_0}^{T} s_k(u(t), y(t))dt \right] \le 0,$$

or, equivalently, $V_{a_k}(x(t_0)) = V_{a_k}(0) \le 0$ for some $k \in \{1, \ldots, r\}$. However, since $V_{a_k}(x) \ge 0,\ x \in \mathcal{D}$ and $k \in \{1, \ldots, r\}$, it follows that $V_{a_k}(0) = 0,\ k \in \{1, \ldots, r\}$. Moreover, it follows from (25) that $V_{a_k}(x),\ x \in \mathcal{D}$ and $k \in \{1, \ldots, r\}$, is a storage function for $\mathcal{G}$. Thus, $\mathcal{V}_a : \mathcal{D} \to \mathcal{B}(\mathbb{R})$ is a weak set-valued storage map for $\mathcal{G}$.

Alternatively, if $\mathcal{G}$ is strongly completely reachable and strongly dissipative with respect to the set-valued supply rate map $\mathcal{S}_r : U \times Y \to \mathcal{B}(\mathbb{R})$, then the above analysis holds for all Filippov solutions to $\mathcal{G}$. Hence, $\mathcal{V}_a : \mathcal{D} \to \mathcal{B}(\mathbb{R})$ is a strong set-valued storage map for $\mathcal{G}$.

Finally, let $V_{s_i}(x)$ for all $x \in \mathcal{D}$ and some $i \in \{1, \ldots, q\}$ be a storage function for $\mathcal{G}$. Then it follows from (12) that, for all $T > t_0$ and $x_0 \in \mathcal{D}$, some $s_k(\cdot, \cdot) \in \mathcal{S}_r$, $k \in \{1, \ldots, r\}$, $i \in \{1, \ldots, q\}$, and at least one Filippov solution of $\mathcal{G}$,

$$V_{s_i}(x_0) \geq V_{s_i}(x(T)) - \int_{t_0}^{T} s_k(u(t), y(t))\mathrm{d}t \geq -\int_{t_0}^{T} s_k(u(t), y(t))\mathrm{d}t,$$

which implies

$$V_{s_i}(x_0) \geq \sup_{u(\cdot),\, T \geq t_0} \left[-\int_{t_0}^{T} s_k(u(t), y(t))\mathrm{d}t \right] = V_{a_k}(x_0) \geq V_{as}(x_0), \tag{28}$$

yielding (24). □

The following theorem provides sufficient conditions for guaranteeing that all of the storage functions of a given dissipative differential inclusion are positive definite. For this result we require the following definition.

Definition 3.4. A differential inclusion $\mathcal{G}$ is *weakly* (resp., *strongly*) *zero-state observable* if $u(t) \equiv 0$ and $y(t) \equiv 0$ implies $x(t) \equiv 0$ for at least one (resp., every) Filippov solution of $\mathcal{G}$.

Theorem 3.2. *Consider the differential inclusion $\mathcal{G}$ given by (1)–(3), and assume that $\mathcal{G}$ is weakly (resp., strongly) completely reachable and weakly (resp., strongly) zero-state observable. Furthermore, assume that $\mathcal{G}$ is weakly (resp., strongly) dissipative with respect to the set-valued supply rate map $\mathcal{S}_r : U \times Y \to \mathcal{B}(\mathbb{R})$ and there exist functions $\kappa_k : Y \to U$, $k \in \{1, \ldots, r\}$, such that $\kappa_k(0) = 0$ and $s_k(\kappa_k(y), y) < 0$, $y \neq 0$, $k \in \{1, \ldots, r\}$. Then the storage functions $V_{s_i}(x)$, $x \in \mathcal{D}$, $i \in \{1, \ldots, q\}$, are positive definite, that is, $V_{s_i}(0) = 0$ and $V_{s_i}(x) > 0$, $x \in \mathcal{D}$, $x \neq 0$, $i \in \{1, \ldots, q\}$.*

Proof. Suppose $\mathcal{G}$ is weakly dissipative with respect to the set-valued supply rate map $\mathcal{S}_r : U \times Y \to \mathcal{B}(\mathbb{R})$. Then it follows from Theorem 3.1 that the available storage functions $V_{a_k}(x)$ for all $x \in \mathcal{D}$ and some $k \in \{1, \ldots, r\}$ are storage functions of $\mathcal{G}$. Next, suppose there exists $\hat{x} \in \mathcal{D}$, $\hat{x} \neq 0$, such that $V_{a_k}(\hat{x}) = 0$ for some $k \in \{1, \ldots, r\}$. Since, by assumption, there exist functions $\kappa_k : Y \to U$ such that $\kappa_k(0) = 0$ and $s_k(\kappa_k(y), y) < 0$, $y \neq 0$, it follows that for some $k \in \{1, \ldots, r\}$,

$$0 = \sup_{u(\cdot), T \geq t_0} \left[-\int_{t_0}^{T} s_k(u(t), y(t))\mathrm{d}t \right] \geq \sup_{T \geq t_0} \left[-\int_{t_0}^{T} s_k(\kappa_k(y(t)), y(t))\mathrm{d}t \right] \geq 0, \quad \text{a.e. } t \geq t_0,$$

and hence,

$$s_k(\kappa_k(y(t)), y(t)) = 0, \quad \text{a.e. } t \geq t_0,\ k \in \{1, \ldots, r\}.$$

Since $\kappa_k(0) = 0$ and $s_k(\kappa_k(y), y) < 0$, $y \neq 0$, $k \in \{1, \ldots, r\}$, it follows that $y(t) = 0$ almost everywhere $t \geq t_0$. Now, since $\mathcal{G}$ is weakly zero-state observable it follows that $\hat{x} = 0$, and hence, $V_{a_k}(x) = 0$, $k \in \{1, \ldots, r\}$, if and only if $x = 0$. The result now follows from (28).

Alternatively, if $\mathcal{G}$ is strongly zero-state observable and strongly dissipative with respect to the set-valued supply rate map $\mathcal{S}_r : U \times Y \to \mathcal{B}(\mathbb{R})$, then the above analysis holds for all Filippov solutions to $\mathcal{G}$. Hence, it follows from (24) that all $V_{s_i}(\cdot) \in \mathcal{V}_s(x)$, $i \in \{1, \ldots, q\}$, are positive definite. □

Next, we consider a special case of the differential inclusions (1)–(3) involving switched dynamical systems. Specifically, consider the nonlinear dynamical systems given by

$$\dot{x}(t) = f_p(x(t)) + G_p(x(t))u_p(t), \qquad x(t_0) = x_0, \quad t \geq t_0, \tag{29}$$

$$y_p(t) = h_p(x(t)), \tag{30}$$

where, for every $t \geq t_0$, $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $f_p : \mathbb{R}^n \to \mathbb{R}^n$, $G_p : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, and $h_p : \mathbb{R}^n \to \mathbb{R}^l$ are locally Lipschitz continuous functions, and $p \in \mathcal{P} = \{1, \ldots, d\}$ is a finite index set. The family of nonlinear dynamical systems (29) and (30) can be rewritten as the *switched dynamical system* $\mathcal{G}_\sigma$ [15,17,18] given by

$$\dot{x}(t) = f_{\sigma(t)}(x(t)) + G_{\sigma(t)}(x(t))u_{\sigma(t)}(t), \quad \sigma(\cdot) \in \Sigma, \qquad x(t_0) = x_0, \quad t \geq t_0, \tag{31}$$

$$y_{\sigma(t)}(t) = h_{\sigma(t)}(x(t)), \tag{32}$$

where $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $t \geq t_0$, $f_\sigma : \mathbb{R}^n \to \mathbb{R}^n$, $G_\sigma : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $h_\sigma : \mathbb{R}^n \to \mathbb{R}^l$, $\sigma : [t_0, \infty) \to \mathcal{P}$ is a piecewise constant switching signal, and Σ denotes the set of switching signals. The switching signal σ effectively switches the right-hand side of (31) and (32) by selecting different subsystems from the parameterized family $\{f_p(x) + G_p(x)u_p$ and $h_p(x) : p \in \mathcal{P}\}$. We denote by t_i, $i = 1, 2, \ldots$, the consecutive discontinuities of σ which we call the *switching times* of (31) and (32). Our convention here is that $\sigma(\cdot)$ is left-continuous, that is, $\sigma(t^-) = \sigma(t)$, where $\sigma(t^-) \triangleq \lim_{h\to 0^-}(t + h)$.

The pair $(x, \sigma) : [t_0, \infty) \times \Sigma \to \mathbb{R}^n$ is a *solution* to the switched dynamical system (31) if $x(\cdot)$ is absolutely continuous and satisfies (31) for almost all $t \geq t_0$. Here, we assume that if there are infinitely many switching times, then there exists $\tau > 0$ such that for every $T \geq t_0$ there exists a positive integer l such that $t_{l+1} - \tau \geq t_l \geq T$. When $t \in [t_l, t_{l+1})$, $\sigma(t) = i_l$, that is, the i_lth subsystem is active. Hence, the trajectory $x(t)$ of the switched dynamical system (31) is defined as the trajectory $x_{i_l}(t)$ of the i_lth subsystem when $t \in [t_l, t_{l+1})$.

Note that the notion of strong dissipativity for differential inclusions subsumes the notion of dissipativity for switched dynamical systems with cross-supply rates introduced in [15]. This is due to the fact that if the switched dynamical system is strongly dissipative, then all of its subsystems are dissipative for all time $t \geq t_0$ whether or not they are active.

Proposition 3.2. *Consider the switched dynamical system $\mathcal{G}_\sigma$ given by (31) and (32) with switching times $\{t_1, \ldots, t_{c-1}\}$, where $t_c = T$ and $c < \infty$. Then all storage functions $V_{s_i}(\cdot) \in \mathcal{V}_s(x)$, $i \in \{1, \ldots, q\}$, for $\mathcal{G}_\sigma$ satisfy*

$$\sum_{l=0}^{c-1} V_{as}(x(t_l))_{[t_l,t_{l+1})} \leq \inf_{V_{s_i}(\cdot)\in\mathcal{V}_s} \left[\sum_{l=0}^{c-1} V_{s_i}(x(t_l))\right], \tag{33}$$

where

$$V_{as}(x(t_l))_{[t_l,t_{l+1})} \triangleq \inf_{k\in\{1,\ldots,r\}} \left\{- \inf_{u(\cdot),t_{l+1}\geq t_l} \int_{t_l}^{t_{l+1}} s_k(u(t), y(t))\mathrm{d}t\right\}.$$

Proof. Since (28) holds for all storage functions $V_{s_i}(\cdot) \in \mathcal{V}_s(x)$, $i = 1, \ldots, q$, it follows that over every time interval $[t_l, t_{l+1})$ with $l \in \{0, \ldots, c-1\}$,

$$V_{s_i}(x(t_l)) \geq \sup_{u(\cdot),t_{l+1}\geq t_l} \left[- \int_{t_l}^{t_{l+1}} s_k(u(t), y(t))\mathrm{d}t\right] = V_{a_k}(x(t_l))_{[t_l,t_{l+1})} \geq V_{as}(x(t_l))_{[t_l,t_{l+1})}. \tag{34}$$

Now, summing both sides of inequality (34) over the switching times and the initial time t_0, it follows that for every element $V_{s_i}(\cdot) \in \mathcal{V}_s(x)$ and $i \in \{1, \ldots, q\}$,

$$\sum_{l=0}^{c-1} V_{as}(x(t_l))_{[t_l,t_{l+1})} \leq \sum_{l=0}^{c-1} V_{s_i}(x(t_l)),$$

which implies that

$$\sum_{l=0}^{c-1} V_{as}(x(t_l))_{[t_l,t_{l+1})} \leq \inf_{V_{s_i}(\cdot)\in\mathcal{V}_s} \left[\sum_{l=0}^{c-1} V_{s_i}(x(t_l))\right],$$

yielding (33). □

Next, we introduce the concepts of a set-valued required supply map and required supply functions for the differential inclusion $\mathcal{G}$. Specifically, define the *required supply functions* $V_{r_k} : \mathcal{D} \to \mathbb{R}$ of the differential inclusion $\mathcal{G}$ by

$$V_{r_k}(x_0) \triangleq \inf_{u(\cdot),\, T\geq t_0} \int_{-T}^{t_0} e^{\varepsilon_k t} s_k(u(t), y(t))\mathrm{d}t, \tag{35}$$

where $s_k(\cdot,\cdot) \in \mathcal{S}_r$, $k \in \{1, \ldots, r\}$, and $x(t)$, $t \geq -T$, is a Filippov solution of $\mathcal{G}$ with $x(-T) = 0$ and $x(t_0) = x_0$. Furthermore, define the *set-valued required supply map* $\mathcal{V}_r : \mathcal{D} \to \mathcal{B}(\mathbb{R})$ of the differential inclusion $\mathcal{G}$ by $\mathcal{V}_r(x) = \{V_{r_1}(x), \ldots, V_{r_r}(x)\}$, $x \in \mathcal{D}$. The infimum in (35) is taken over all Filippov system trajectories starting from $x(-T) = 0$ at time $t = -T$ and ending at $x(t_0) = x_0$ at time $t = t_0$, and all times $t \geq t_0$ or, equivalently, over all admissible inputs $u(\cdot) \in \mathcal{U}$ which drive the differential inclusion $\mathcal{G}$ from the origin to x_0 over the time interval $[-T, t_0]$. If the system is not reachable from the origin, then we define $V_{r_k}(x_0) = \infty$, $k = 1, \ldots, r$.

Next, define the *greatest required supply map* of the differential inclusion $\mathcal{G}$ by

$$V_{rs}(x_0) \triangleq \left\{v \in \mathcal{V}_r(x_0) : v = \sup_{V_{r_k}(x_0)\in\mathcal{V}_r(x_0)} V_{r_k}(x_0)\right\}. \tag{36}$$

Recall that if $\mathcal{B}(\mathbb{R})$ is a reflexive strictly convex space and $\mathcal{V}_r(x)$ has closed convex images, then it follows that the greatest required supply map is single-valued. We call this map the *largest required supply* of the differential inclusion $\mathcal{G}$. It follows from (36) that the largest required supply of a differential inclusion is the minimum amount of generalized energy that has to be delivered to the system in order to transfer it from an initial state $x(-T) = 0$ to a given state $x(t_0) = x_0$. In the definition of $V_{rs}(\cdot)$, the supremum is taken elementwise over all $V_{r_k}(\cdot) \in \mathcal{V}_r(x)$, $k \in \{1, \ldots, r\}$, since $s_k(\cdot,\cdot) \in \mathcal{S}_r$, $k \in \{1, \ldots, r\}$, which implies that for different elements of $\mathcal{V}_r(x)$ the infimum is calculated separately. Note that since, with $x(t_0) = 0$, the infimum in (35) is zero, it follows that $V_{rs}(0) = 0$.

Next, we show that all of the required supply functions of $\mathcal{G}$ are finite and nonnegative if and only if the differential inclusion $\mathcal{G}$ is weakly (resp., strongly) dissipative. Moreover, we prove that all storage functions are bounded from above by the largest required supply and bounded from below by the smallest available storage, and hence, a dissipative differential inclusion can deliver to its surroundings only a fraction of its generalized stored energy and can store only a fraction of the generalized work done to it.

Theorem 3.3. *Consider the differential inclusion $\mathcal{G}$ given by (1)–(3) and assume that $\mathcal{G}$ is weakly (resp., strongly) completely reachable. Then $\mathcal{G}$ is weakly (resp., strongly) dissipative with respect to the set-valued supply rate map $\mathcal{S}_{\rm r} : U \times Y \to \mathcal{B}(\mathbb{R})$ if and only if $0 \le V_{{\rm r}_k}(x) < \infty$ for all $x_0 \in \mathcal{D}$, some $k \in \{1, \ldots, r\}$, and at least one (resp., every) Filippov solution of $\mathcal{G}$. Moreover, if the required supply functions $V_{{\rm r}_k}(x)$ are finite and nonnegative for all $x \in \mathcal{D}$, some $k \in \{1, \ldots, r\}$, and at least one (resp., every) Filippov solution of $\mathcal{G}$, then $\mathcal{V}_{\rm r} : \mathcal{D} \to \mathcal{B}(\mathbb{R})$ is a weak (resp., strong) set-valued storage map for $\mathcal{G}$. Finally, all storage functions $V_{{\rm s}_i}(\cdot) \in \mathcal{V}_{\rm s}(x)$, $i \in \{1, \ldots, q\}$, for $\mathcal{G}$ satisfy*

$$0 \le V_{\rm as}(x) \le V_{{\rm s}_i}(x) \le V_{\rm rs}(x) < \infty, \quad x \in \mathcal{D}. \tag{37}$$

Proof. Suppose $0 \le V_{{\rm r}_k}(x) < \infty$ for all $x \in \mathcal{D}$ and some $k \in \{1, \ldots, r\}$. Next, let $x(t)$, $t \in \mathbb{R}$, be a Filippov solution to (1)–(3) with admissible inputs $u(t)$, $t \in \mathbb{R}$, and $x(t_0) = x_0$. Since $V_{{\rm r}_k}(x)$, $x \in \mathcal{D}$ and $k \in \{1, \ldots, r\}$, is given by the infimum over all admissible inputs $u(\cdot) \in \mathcal{U}$ and $T > t_0$ in (35), it follows that for all admissible inputs $u(\cdot)$, $-T \le t \le t_0$, and $k \in \{1, \ldots, r\}$,

$$V_{{\rm r}_k}(x_0) \le \int_{-T}^{t_0} s_k(u(t), y(t)){\rm d}t = \int_{-T}^{t} s_k(u(\sigma), y(\sigma)){\rm d}\sigma + \int_{t}^{t_0} s_k(u(\sigma), y(\sigma)){\rm d}\sigma,$$

and hence, for $k \in \{1, \ldots, r\}$,

$$\begin{aligned} V_{{\rm r}_k}(x_0) &\le \inf_{u(\cdot),\, T \ge t} \int_{-T}^{t} s_k(u(\sigma), y(\sigma)){\rm d}\sigma + \int_{t}^{t_0} s_k(u(\sigma), y(\sigma)){\rm d}\sigma \\ &= V_{{\rm r}_k}(x(t)) + \int_{t}^{t_0} s_k(u(\sigma), y(\sigma)){\rm d}\sigma. \end{aligned} \tag{38}$$

This inequality together with the fact that $V_{{\rm r}_k}(0) = 0$, $k \in \{1, \ldots, r\}$, shows that $V_{{\rm r}_k}(x)$, $x \in \mathcal{D}$ and $k \in \{1, \ldots, r\}$, is a storage function for $\mathcal{G}$. Hence, $\mathcal{G}$ is weakly dissipative with respect to the set-valued supply rate map $\mathcal{S}_{\rm r} : U \times Y \to \mathcal{B}(\mathbb{R})$. Furthermore, since $V_{{\rm r}_k}(x)$, $x \in \mathcal{D}$ and $k \in \{1, \ldots, r\}$, is a storage function for $\mathcal{G}$, $\mathcal{V}_{\rm r} : \mathcal{D} \to \mathcal{B}(\mathbb{R})$ is a weak set-valued storage map for $\mathcal{G}$.

Alternatively, if $0 \le V_{{\rm r}_k}(x) < \infty$, $x \in \mathcal{D}$, for all Filippov solutions to $\mathcal{G}$, then $V_{{\rm r}_k}(x)$, $x \in \mathcal{D}$, $k \in \{1, \ldots, r\}$, satisfy the dissipation inequality (12). Hence, $\mathcal{G}$ is strongly dissipative with respect to the set-valued supply rate map $\mathcal{S}_{\rm r} : U \times Y \to \mathcal{B}(\mathbb{R})$ and $\mathcal{V}_{\rm r} : \mathcal{D} \to \mathcal{B}(\mathbb{R})$ is a strong set-valued storage map for $\mathcal{G}$.

Conversely, suppose $\mathcal{G}$ is weakly dissipative with respect to the set-valued supply rate map $\mathcal{S}_{\rm r} : U \times Y \to \mathcal{B}(\mathbb{R})$ and let $x_0 \in \mathcal{D}$. Since $\mathcal{G}$ is weakly completely reachable it follows that there exist $T > t_0$ and $u(t)$, $t \in [-T, t_0]$, such that $x(-T) = 0$ and $x(t_0) = x_0$. Hence, since $\mathcal{G}$ is weakly dissipative with respect to the set-valued supply rate map $\mathcal{S}_{\rm r} : U \times Y \to \mathcal{B}(\mathbb{R})$, it follows that for at least one $s_k(u, y) \in \mathcal{S}_{\rm r}$, $k \in \{1, \ldots, r\}$, at least one Filippov solution $x(t)$, and for all $T \ge t_0$,

$$0 \le \int_{-T}^{t_0} s_k(u(t), y(t)){\rm d}t,$$

and hence,

$$0 \le \inf_{u(\cdot),\, T \ge t_0} \left[\int_{-T}^{t_0} s_k(u(t), y(t)){\rm d}t \right],$$

which implies that

$$0 \le V_{{\rm r}_k}(x_0) < \infty, \quad x_0 \in \mathcal{D},\ k \in \{1, \ldots, r\}.$$

Furthermore, it follows from (38) that $V_{{\rm r}_k}(x)$, $x \in \mathcal{D}$, $k \in \{1, \ldots, r\}$, is a storage function for $\mathcal{G}$. Thus, $\mathcal{V}_{\rm a} : \mathcal{D} \to \mathcal{B}(\mathbb{R})$ is a weak set-valued storage map for $\mathcal{G}$.

Alternatively, if $\mathcal{G}$ is strongly completely reachable and strongly dissipative with respect to the set-valued supply rate map $\mathcal{S}_{\rm r} : U \times Y \to \mathcal{B}(\mathbb{R})$, then the above analysis holds for all Filippov solutions to $\mathcal{G}$. Hence, $0 \le V_{{\rm r}_k}(x) < \infty$ for all $x \in \mathcal{D}$, $k \in \{1, \ldots, r\}$. Moreover, $\mathcal{V}_{\rm r} : \mathcal{D} \to \mathcal{B}(\mathbb{R})$ is a strong set-valued storage map for $\mathcal{G}$.

Finally, let $V_{{\rm s}_i}(x)$, $x \in \mathcal{D}$, and some $i \in \{1, \ldots, q\}$ be a storage function for $\mathcal{G}$. Then it follows from Theorem 3.2 that

$$0 \le V_{\rm as}(x) \le V_{{\rm s}_i}(x), \quad x \in \mathcal{D},\ i \in \{1, \ldots, q\}.$$

Furthermore, for all $T \ge t_0$ such that $x(-T) = 0$ and $i \in \{1, \ldots, q\}$, it follows that

$$V_{{\rm s}_i}(x_0) \le V_{{\rm s}_i}(0) + \int_{-T}^{t_0} s_k(u(t), y(t)){\rm d}t,$$

for some $s_k(\cdot,\cdot) \in \mathcal{S}_{\rm r},\ k \in \{1,\ldots,r\}$, and hence,

$$V_{{\rm s}_i}(x_0) \le \inf_{u(\cdot),\, T\ge t_0} \int_{-T}^{t_0} s_k(u(t), y(t))\mathrm{d}t = V_{{\rm r}_k}(x_0) \le V_{\rm rs}(x_0) < \infty, \tag{39}$$

which implies (37). □

Next, we consider the switched dynamical system given by (31) and (32).

Proposition 3.3. *Consider the switched dynamical system $\mathcal{G}_\sigma$ given by (31) and (32) with switching times $\{t_1,\ldots,t_{c-1}\}$, where $t_c = t_0$ and $c < \infty$. Then all storage functions $V_{{\rm s}_i}(\cdot) \in \mathcal{V}_{\rm s}(x),\ i \in \{1,\ldots,q\}$, for $\mathcal{G}_\sigma$ satisfy*

$$\sup_{V_{{\rm s}_i}(\cdot)\in\mathcal{V}_{\rm s}} \left[\sum_{l=1}^{c} V_{{\rm s}_i}(x(t_l))\right] \le \sum_{l=1}^{c} V_{\rm rs}(x(t_l))_{[t_{l-1},t_l)}, \tag{40}$$

where

$$V_{\rm rs}(x(t_l))_{[t_{l-1},t_l)} \triangleq \sup_{k\in\{1,\ldots,r\}} \left\{\inf_{u(\cdot),t_{l-1}\le t_l} \int_{t_{l-1}}^{t_l} s_k(u(t), y(t))\mathrm{d}t\right\}.$$

Proof. Since (39) holds for all storage functions $V_{{\rm s}_i}(\cdot) \in \mathcal{V}_{\rm s}(x),\ i \in \{1,\ldots,q\}$, it follows that over every time interval $[t_{l-1}, t_l)$ with $l \in \{1,\ldots,c-1\}$,

$$V_{{\rm s}_i}(x(t_l)) \le \inf_{u(\cdot),t_{l-1}\le t_l} \int_{t_{l-1}}^{t_l} s_k(u(t), y(t))\mathrm{d}t = V_{{\rm r}_k}(x(t_l)) \le V_{\rm rs}(x(t_l)) < \infty. \tag{41}$$

Now, summing both sides of inequality (41) over the switching times and noting $t_c = t_0$, it follows that for every element $V_{{\rm s}_i}(\cdot) \in \mathcal{V}_{\rm s}(x)$ and $i \in \{1,\ldots,q\}$,

$$\sum_{l=1}^{c} V_{\rm rs}(x(t_l))_{[t_{l-1},t_l)} \ge \sum_{l=1}^{c} V_{{\rm s}_i}(x(t_l)),$$

which implies

$$\sum_{l=1}^{c} V_{\rm rs}(x(t_l))_{[t_{l-1},t_l)} \ge \sup_{V_{{\rm s}_i}(\cdot)\in\mathcal{V}_{\rm s}} \left[\sum_{l=1}^{c} V_{{\rm s}_i}(x(t_l))\right],$$

yielding (40). □

In light of Theorems 3.1 and 3.3 we have the following result on lossless differential inclusions.

Theorem 3.4. *Consider the differential inclusion $\mathcal{G}$ given by (1)–(3) and assume that $\mathcal{G}$ is weakly (resp., strongly) completely reachable to and from the origin. Then $\mathcal{G}$ is weakly (resp., strongly) lossless with respect to the set-valued supply rate map $\mathcal{S}_{\rm r} : U\times Y \to \mathcal{B}(\mathbb{R})$ if and only if the dissipation inequality (12) is satisfied as an equality for some $V_{{\rm s}_i}(\cdot) \in \mathcal{V}_{\rm s}(x),\ i \in \{1,\ldots,q\}$, and at least one (resp., every) Filippov solution of $\mathcal{G}$. Furthermore, if $\mathcal{G}$ is weakly (resp., strongly) lossless with respect to the set-valued supply rate map $\mathcal{S}_{\rm r} : U \times Y \to \mathcal{B}(\mathbb{R})$, then $V_{{\rm a}_k}(x) = V_{{\rm r}_k}(x)$ for all $x \in \mathcal{D}$, some $k \in \mathcal{K} \subseteq \{1,\ldots,r\}$, and at least one (resp., every) Filippov solution of $\mathcal{G}$, and hence, the storage functions $V_{{\rm s}_i}(x),\ x \in \mathcal{D}$ and $i \in \mathcal{I} \subseteq \{1,\ldots,q\}$, are given by*

$$V_{{\rm s}_i}(x_0) = -\int_{t_0}^{T_+} s_k(u(t), y(t))\mathrm{d}t = \int_{-T_-}^{t_0} s_k(u(t), y(t))\mathrm{d}t, \tag{42}$$

where $i \in \mathcal{I},\ k \in \mathcal{K},\ |\mathcal{I}| = |\mathcal{K}|$, and $x(t),\ t \ge t_0$, is a Filippov solution to (1)–(3) with admissible $u(\cdot) \in \mathcal{U}$ and $x(t_0) = x_0,\ x_0 \in \mathcal{D}$, for every $T_-, T_+ > |t_0|$ such that $x(-T_-) = 0$ and $x(T_+) = 0$.

Proof. Suppose $\mathcal{G}$ is weakly lossless with respect to the set-valued supply rate map $\mathcal{S}_{\rm r} : U \times Y \to \mathcal{B}(\mathbb{R})$. Since $\mathcal{G}$ is weakly completely reachable to and from the origin it follows that, for every $x_0 \in \mathcal{D}$, there exist $T_-, T_+ > |t_0|$, and $u(t) \in U,\ t \in [-T_-, T_+]$, such that $x(-T_-) = 0, x(T_+) = 0$, and $x(t_0) = x_0$. Now, it follows that for at least one $s_k(u, y) \in \mathcal{S}_{\rm r},\ k \in \{1,\ldots,r\}$, and at least one Filippov solution of $\mathcal{G}$,

$$\begin{aligned} 0 &= \int_{-T_-}^{T_+} s_k(u(t), y(t))\mathrm{d}t \\ &= \int_{-T_-}^{t_0} s_k(u(t), y(t))\mathrm{d}t + \int_{t_0}^{T_+} s_k(u(t), y(t))\mathrm{d}t \end{aligned}$$

$$\geq \inf_{u(\cdot),\, T\geq t_0} \int_{-T}^{t_0} s_k(u(t), y(t))\mathrm{d}t + \inf_{u(\cdot),\, T\geq t_0} \int_{t_0}^{T} s_k(u(t), y(t))\mathrm{d}t$$
$$= V_{\mathrm{r}_k}(x_0) - V_{\mathrm{a}_k}(x_0), \tag{43}$$

which implies that $V_{\mathrm{r}_k}(x_0) \leq V_{\mathrm{a}_k}(x_0)$ for all $x_0 \in \mathcal{D}$ and some $k \in \{1, \ldots, r\}$. However, since, by definition, $\mathcal{G}$ is weakly dissipative with respect to the set-valued supply rate map $\mathcal{S}_{\mathrm{r}} : U \times Y \to \mathcal{B}(\mathbb{R})$, it follows from (28) and (39) that $V_{\mathrm{a}_k}(x_0) \leq V_{\mathrm{s}_i}(x_0) \leq V_{\mathrm{r}_k}(x_0)$ for all $x_0 \in \mathcal{D}$ and some $k \in \{1, \ldots, r\}$ and $i \in \{1, \ldots, q\}$. Hence, the storage functions $V_{\mathrm{s}_i}(x_0),\ x_0 \in \mathcal{D}$ and $i \in \{1, \ldots, q\}$, satisfy $V_{\mathrm{a}_k}(x_0) = V_{\mathrm{s}_i}(x_0) = V_{\mathrm{r}_k}(x_0),\ x_0 \in \mathcal{D},\ k \in \mathcal{K} \subseteq \{1, \ldots, r\}$ and $i \in \mathcal{I} \subseteq \{1, \ldots, q\}$. Furthermore, it follows that the inequality in (43) is indeed an equality, which implies (42) and $|\mathcal{I}| = |\mathcal{K}|$.

Next, let $\hat{t},\ t, T \geq t_0$ be such that $\hat{t} < t < T$, $x(T) = 0$. Hence, for at least one $s_k(u, y) \in \mathcal{S}_{\mathrm{r}},\ k \in \{1, \ldots, r\}$, corresponding $V_{\mathrm{s}_i}(\cdot) \in \mathcal{V}_{\mathrm{s}},\ i \in \{1, \ldots, q\}$, and at least one Filippov solution of $\mathcal{G}$, it follows from (42) that

$$\begin{aligned} 0 &= V_{\mathrm{s}_i}(x(\hat{t})) + \int_{\hat{t}}^{T} s_k(u(\sigma), y(\sigma))\mathrm{d}\sigma \\ &= V_{\mathrm{s}_i}(x(\hat{t})) + \int_{\hat{t}}^{t} s_k(u(\sigma), y(\sigma))\mathrm{d}\sigma + \int_{t}^{T} s_k(u(\sigma), y(\sigma))\mathrm{d}\sigma \\ &= V_{\mathrm{s}_i}(x(\hat{t})) + \int_{\hat{t}}^{t} s_k(u(\sigma), y(\sigma))\mathrm{d}\sigma - V_{\mathrm{s}_i}(x(t)), \end{aligned}$$

which implies that (12) is satisfied as an equality.

Alternatively, if $\mathcal{G}$ is strongly completely reachable to and from the origin and strongly lossless with respect to the set-valued supply rate map $\mathcal{S}_{\mathrm{r}} : U \times Y \to \mathcal{B}(\mathbb{R})$, then the above analysis holds for all Filippov solutions of $\mathcal{G}$ with some $V_{\mathrm{a}_k}(\cdot) \in \mathcal{V}_{\mathrm{a}}(x), V_{\mathrm{r}_k}(\cdot) \in \mathcal{V}_{\mathrm{r}}(x)$, and $V_{\mathrm{s}_i}(\cdot) \in \mathcal{V}_{\mathrm{s}}(x),\ x \in \mathcal{D},\ k \in \{1, \ldots, r\}$, and $i \in \{1, \ldots, q\}$.

Conversely, if (12) is satisfied as an equality for some storage function $V_{\mathrm{s}_i}(\cdot) \in \mathcal{V}_{\mathrm{s}}(x),\ i \in \{1, \ldots, q\}$, and at least one Filippov solution of $\mathcal{G}$, then it follows from Definition 3.3 that $\mathcal{G}$ is weakly dissipative with respect to the set-valued supply rate map $\mathcal{S}_{\mathrm{r}} : U \times Y \to \mathcal{B}(\mathbb{R})$. Furthermore, for every $u(\cdot) \in \mathcal{U}$, $t \geq t_0$, and $x(t_0) = x(t) = 0$, it follows from (12) (with an equality) that for at least one $s_k(u, y) \in \mathcal{S}_{\mathrm{r}},\ k \in \{1, \ldots, r\}$, and at least one Filippov solution of $\mathcal{G}$,

$$\int_{t_0}^{t} s_k(u(\sigma), y(\sigma))\mathrm{d}\sigma = 0, \tag{44}$$

which implies that $\mathcal{G}$ is weakly lossless with respect to the set-valued supply rate map $\mathcal{S}_{\mathrm{r}} : U \times Y \to \mathcal{B}(\mathbb{R})$.

Alternatively, if (12) is satisfied as an equality for some storage functions $V_{\mathrm{s}_i}(\cdot) \in \mathcal{V}_{\mathrm{s}}(x),\ i \in \{1, \ldots, r\}$, and all Filippov solutions of $\mathcal{G}$, then (44) holds for some $s_k(u, y) \in \mathcal{S}_{\mathrm{r}},\ k \in \{1, \ldots, r\}$, and hence, $\mathcal{G}$ is strongly lossless with respect to the set-valued supply rate map $\mathcal{S}_{\mathrm{r}} : U \times Y \to \mathcal{B}(\mathbb{R})$. □

4. Extended Kalman–Yakubovich–Popov conditions

In this section, we show that dissipativeness, exponential dissipativeness, and losslessness of discontinuous nonlinear affine dynamical systems $\mathcal{G}$ of the form

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t), \qquad x(t_0) = x_0, \quad \text{a.e. } t \geq t_0, \tag{45}$$
$$y(t) = h(x(t)) + J(x(t))u(t), \tag{46}$$

where $x(t) \in \mathcal{D} \subseteq \mathbb{R}^n$, $\mathcal{D}$ is an open set with $0 \in \mathcal{D}$, $u(t) \in U \subseteq \mathbb{R}^m$, $y(t) \in Y \subseteq \mathbb{R}^l$, $f : \mathcal{D} \to \mathbb{R}^n$, $G : \mathcal{D} \to \mathbb{R}^{n\times m}$, $h : \mathcal{D} \to Y$, and $J : \mathcal{D} \to \mathbb{R}^{l\times m}$, can be characterized in terms of the system functions $f(\cdot), G(\cdot),\ h(\cdot)$, and $J(\cdot)$. We assume that $f(\cdot), G(\cdot),\ h(\cdot)$, and $J(\cdot)$ are Lebesgue measurable and locally essentially bounded. Since the nonlinear maps characterizing the system dynamics (45) and (46) are assumed to be locally essentially bounded, it follows from the classical existence theorem for differential inclusions [22, p. 97] that a Filippov solution to (45) exists, and hence, (45) and (46) can be represented as (1)–(3).

Note that (45) and (46) include piecewise continuous dynamical systems as well as switched dynamical systems as special cases. For example, if $f(\cdot), G(\cdot),\ h(\cdot)$, and $J(\cdot)$ are piecewise continuous, then (45) and (46) can be represented as a differential inclusion involving Filippov set-valued maps of piecewise-continuous vector fields given by $\mathcal{F}(x) = \mathcal{K}[f](x) = \overline{\mathrm{co}}\{\lim_{i\to\infty} f(x_i) : x_i \to x,\ x_i \notin \mathcal{S}_f\}$, where $\mathcal{S}_f$ has measure zero and denotes the set of points where f is discontinuous [21], and similarly for $G(\cdot), h(\cdot)$, and $J(\cdot)$. Here, we assume that $\mathcal{F}$ has at least one equilibrium point so that, without loss of generality, $0 \in \mathcal{F}(0)$.

For the next set of results, we concentrate on weakly dissipative (resp., lossless) systems so that single-valued supply rate and storage function maps suffice. In addition, we consider the special case of dissipative systems with supply rate maps consisting of quadratic supply rates [2,3]. Specifically, set $\mathcal{D} = \mathbb{R}^n$, $U = \mathbb{R}^m$, $Y = \mathbb{R}^l$, let $Q \in \mathbb{S}^l$, $R \in \mathbb{S}^m$, and $S \in \mathbb{R}^{l\times m}$ be given, and assume $\{s(u, y)\} = \{y^{\mathrm{T}}Qy + 2y^{\mathrm{T}}Su + u^{\mathrm{T}}Ru\}$, where $\mathbb{S}^q$ denotes the set of $q \times q$ symmetric matrices. Furthermore, we assume that there exists a function $\kappa : \mathbb{R}^l \to \mathbb{R}^m$ such that $\kappa(0) = 0$ and $s(\kappa(y), y) < 0,\ y \neq 0$. Next, define

$$\mathcal{L}_{\mathfrak{G}} V_s(x) \triangleq \{q \in \mathbb{R}^{1\times m} : \text{ there exists } v \in \mathfrak{G}(x) \text{ such that } p^{\mathrm{T}} v = q \text{ for all } p^{\mathrm{T}} \in \partial V_s(x)\},$$

where $\mathfrak{G}(x) \triangleq \bigcap_{\delta>0}\bigcap_{\mu(\mathcal{S})=0} \overline{\mathrm{co}}\{G(\mathcal{B}_\delta(x)) \setminus \mathcal{S}\},\ x \in \mathbb{R}^n$, and $\bigcap_{\mu(\mathcal{S})=0}$ denotes the intersection over all sets $\mathcal{S}$ of Lebesgue measure zero. For notational convenience, in the remainder of the paper we write $\mathcal{L}_f V_s(x)$ and $\mathcal{L}_G V_s(x)$ for the sets $\mathcal{L}_{\mathcal{F}} V_s(x)$ and $\mathcal{L}_{\mathfrak{G}} V_s(x)$, respectively. Furthermore, we assume that the smallest available storage map of $\mathcal{G}$ is locally Lipschitz continuous and regular. Finally, for the results of this section, we assume that the set $\mathcal{L}_G V_s(x)$ is single-valued[6] for almost all $x \in \mathbb{R}^n$ modulo $\mathcal{L}_G V_s(x) \neq \varnothing$.

Theorem 4.1. *Let $Q \in \mathbb{S}^l$, $S \in \mathbb{R}^{l\times m}$, $R \in \mathbb{S}^m$, and let $\mathcal{G}$ be weakly zero-state observable and weakly completely reachable. If there exist functions $V_s : \mathbb{R}^n \to \mathbb{R}$, $\ell : \mathbb{R}^n \to \mathbb{R}^p$, and $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ and a scalar $\varepsilon > 0$ (resp., $\varepsilon = 0$) such that $V_s(\cdot)$ is locally Lipschitz continuous, regular, and positive definite, $V_s(0) = 0$, and, for almost all $x \in \mathbb{R}^n$,*

$$0 = \min \mathcal{L}_f V_s(x) + \varepsilon V_s(x) - h^{\mathrm{T}}(x)Qh(x) + \ell^{\mathrm{T}}(x)\ell(x), \tag{47}$$

$$0 = \frac{1}{2}\mathcal{L}_G V_s(x) - h^{\mathrm{T}}(x)(QJ(x) + S) + \ell^{\mathrm{T}}(x)\mathcal{W}(x), \tag{48}$$

$$0 = R + S^{\mathrm{T}}J(x) + J^{\mathrm{T}}(x)S + J^{\mathrm{T}}(x)QJ(x) - \mathcal{W}^{\mathrm{T}}(x)\mathcal{W}(x), \tag{49}$$

$$[\ell(x) + \mathcal{W}(x)u]^{\mathrm{T}}[\ell(x) + \mathcal{W}(x)u] \geq \max \mathcal{L}_f V_s(x) - \min \mathcal{L}_f V_s(x), \quad u \in \mathbb{R}^m, \tag{50}$$

then $\mathcal{G}$ is weakly exponentially dissipative (resp., weakly dissipative) with respect to the supply rate map $\{s(u, y)\} = \{y^{\mathrm{T}}Qy + 2y^{\mathrm{T}}Su + u^{\mathrm{T}}Ru\}$. Conversely, if $\mathcal{G}$ is weakly exponentially dissipative (resp., weakly dissipative) with respect to the supply rate map $\{s(u, y)\}$, then there exist functions $V_s : \mathbb{R}^n \to \mathbb{R}$, $\ell : \mathbb{R}^n \to \mathbb{R}^p$, and $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ and a scalar $\varepsilon > 0$ (resp., $\varepsilon = 0$) such that $V_s(\cdot)$ is locally Lipschitz continuous, regular, and positive definite, $V_s(0) = 0$, and, for almost all $x \in \mathbb{R}^n$, (47)–(49) hold.

Proof. First, suppose that there exist functions $V_s : \mathbb{R}^n \to \mathbb{R}$, $\ell : \mathbb{R}^n \to \mathbb{R}^p$, and $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ and a scalar $\varepsilon > 0$ such that $V_s(\cdot)$ is locally Lipschitz continuous, regular, and positive definite, and (47)–(50) are satisfied. Then, for every admissible input $u(t) \in \mathbb{R}^m,\ t \geq 0$, it follows from (47)–(50) that

$$\begin{aligned}
\int_{t_1}^{t_2} e^{\varepsilon t} s(u(t), y(t))\mathrm{d}t &= \int_{t_1}^{t_2} e^{\varepsilon t}\left[y^{\mathrm{T}}(t)Qy(t) + 2y^{\mathrm{T}}(t)Su(t) + u^{\mathrm{T}}(t)Ru(t)\right]\mathrm{d}t \\
&= \int_{t_1}^{t_2} e^{\varepsilon t}\left[h^{\mathrm{T}}(x(t))Qh(x(t)) + 2h^{\mathrm{T}}(x(t))(S + QJ(x(t)))u(t)\right. \\
&\quad \left. + u^{\mathrm{T}}(t)(J^{\mathrm{T}}(x(t))QJ(x(t)) + S^{\mathrm{T}}J(x(t)) + J^{\mathrm{T}}(x(t))S + R)u(t)\right]\mathrm{d}t \\
&= \int_{t_1}^{t_2} e^{\varepsilon t}\left[\min \mathcal{L}_f V_s(x(t)) + \varepsilon V_s(x(t)) + \mathcal{L}_G V_s(x(t))u(t) + \ell^{\mathrm{T}}(x(t))\ell(x(t))\right. \\
&\quad \left. + 2\ell^{\mathrm{T}}(x(t))\mathcal{W}(x(t))u(t) + u^{\mathrm{T}}(t)\mathcal{W}^{\mathrm{T}}(x(t))\mathcal{W}(x(t))u(t)\right]\mathrm{d}t \\
&= \int_{t_1}^{t_2} e^{\varepsilon t}\left[\min \mathcal{L}_f V_s(x(t)) + \mathcal{L}_G V_s(x(t))u(t) + \varepsilon V_s(x(t))\right. \\
&\quad \left. + [\ell(x(t)) + \mathcal{W}(x(t))u(t)]^{\mathrm{T}}[\ell(x(t)) + \mathcal{W}(x(t))u(t)]\right]\mathrm{d}t \\
&\geq \int_{t_1}^{t_2} e^{\varepsilon t}\left[\max \mathcal{L}_f V_s(x(t)) + \mathcal{L}_G V_s(x(t))u(t) + \varepsilon V_s(x(t))\right]\mathrm{d}t,
\end{aligned} \tag{51}$$

where $x(t),\ t \geq 0$, satisfies (45).

Next, using the sum rule for computing the generalized gradient of a locally Lipschitz continuous function [21] it follows that

$$\mathcal{L}_{f+Gu} V_s(x) \subseteq \mathcal{L}_f V_s(x) + \mathcal{L}_{Gu} V_s(x)$$

6 The assumption that $\mathcal{L}_G V_s(x)$ is single-valued is necessary for obtaining Kalman–Yakubovich–Popov conditions for (45) and (46) with Lebesgue measurable and locally essentially bounded system functions $f(\cdot)$, $G(\cdot)$, $h(\cdot)$, and $J(\cdot)$, and with locally Lipschitz continuous storage functions $V_s(\cdot)$. Specifically, as will be seen in the proof of Theorem 4.1, the requirement that there exists $\overline{z} \in \mathcal{L}_G V_s(x)$ (resp., $\underline{z} \in \mathcal{L}_G V_s(x)$) such that, for all $u \in \mathbb{R}^m$, $\max[\mathcal{L}_G V_s(x)u] = \overline{z}u$ (resp., $\min[\mathcal{L}_G V_s(x)u] = \underline{z}u$) used in the proof of Theorem 4.1 holds if and only if $\mathcal{L}_G V_s(x)$ is a singleton. To see this, let $q, r \in \mathcal{L}_G V_s(x)$, with $q \neq r$, and assume, *ad absurdum*, the required $\overline{z}$ exists. Then, either $q - \overline{z} \neq 0$ or $r - \overline{z} \neq 0$. Assume $q - \overline{z} \neq 0$ and let $u^{\mathrm{T}} = q - \overline{z}$. Then, $qu - \overline{z}u = (q - \overline{z})u = (q - \overline{z})(q - \overline{z})^{\mathrm{T}} = \|q - \overline{z}\|_2^2 > 0$. Hence, $qu > \overline{z}u$, which leads to a contradiction. A similar construction shows the result for $\underline{z} \in \mathcal{L}_G V_s(x)$.

for almost all $x \in \mathbb{R}^n$. Now, it follows from Lemma 3.1 that $\frac{\mathrm{d}}{\mathrm{d}t}V_{\mathrm{s}}(x(t)) \in \mathcal{L}_{f+Gu}V_{\mathrm{s}}(x(t)) \subseteq \mathcal{L}_f V_{\mathrm{s}}(x(t)) + \mathcal{L}_{Gu}V_{\mathrm{s}}(x(t))$ for almost all $t \geq 0$. Hence,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}V_{\mathrm{s}}(x(t)) &\leq \max \mathcal{L}_{f+Gu}V_{\mathrm{s}}(x(t)) \\ &\leq \max\left[\mathcal{L}_f V_{\mathrm{s}}(x(t)) + \mathcal{L}_G V_{\mathrm{s}}(x(t))u(t)\right] \\ &= \max \mathcal{L}_f V_{\mathrm{s}}(x(t)) + \mathcal{L}_G V_{\mathrm{s}}(x(t))u(t), \quad \text{a.e. } t \geq 0,\ u(\cdot) \in \mathcal{U}. \end{aligned} \tag{52}$$

Next, note that

$$e^{\varepsilon t}V_{\mathrm{s}}(x(t)) = e^{\varepsilon t_0}V_{\mathrm{s}}(x(t_0)) + \int_{t_0}^{t} \frac{\mathrm{d}}{\mathrm{d}\sigma}(e^{\varepsilon\sigma}V_{\mathrm{s}}(x(\sigma)))\mathrm{d}\sigma, \tag{53}$$

where the integral in (53) is the Lebesgue integral.

Using (52) and (53), it follows from (51) that

$$\begin{aligned}\int_{t_1}^{t_2} e^{\varepsilon t}s(u(t), y(t))\mathrm{d}t &\geq \int_{t_1}^{t_2} e^{\varepsilon t}\left[\frac{\mathrm{d}}{\mathrm{d}t}V_{\mathrm{s}}(x(t)) + \varepsilon V_{\mathrm{s}}(x(t))\right]\mathrm{d}t \\ &= \int_{t_1}^{t_2} \frac{\mathrm{d}}{\mathrm{d}t}(e^{\varepsilon t}V_{\mathrm{s}}(x(t)))\mathrm{d}t \\ &= e^{\varepsilon t_2}V_{\mathrm{s}}(x(t_2)) - e^{\varepsilon t_1}V_{\mathrm{s}}(x(t_1)), \quad \text{a.e. } t \geq 0,\ u(\cdot) \in \mathcal{U}.\end{aligned}$$

The assertion now follows from Definition 3.3.

Conversely, suppose that $\mathcal{G}$ is weakly exponentially dissipative with respect to the supply rate map $\{s(u, y)\}$. Now, it follows from Theorem 3.1 that the smallest available storage map $V_{\mathrm{as}}(x)$ of $\mathcal{G}$ is finite for all $x \in \mathbb{R}^n$, $V_{\mathrm{as}}(0) = 0$, and

$$e^{\varepsilon t_2}V_{\mathrm{as}}(x(t_2)) \leq e^{\varepsilon t_1}V_{\mathrm{as}}(x(t_1)) + \int_{t_1}^{t_2} e^{\varepsilon t}s(u(t), y(t))\mathrm{d}t \tag{54}$$

for almost all $t_2 \geq t_1$ and $u(\cdot) \in \mathcal{U}$. Dividing (54) by $t_2 - t_1$ and letting $t_2 \to t_1$ it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}V_{\mathrm{as}}(x(t)) + \varepsilon V_{\mathrm{as}}(x(t)) \leq s(u(t), y(t)), \quad \text{a.e. } t \geq 0, \tag{55}$$

where $x(t),\ t \geq 0$, is a solution satisfying (45) and $\frac{\mathrm{d}}{\mathrm{d}t}V_{\mathrm{as}}(x(t)) = \limsup_{h\to 0^+}[V_{\mathrm{as}}(x(t+h)) - V_{\mathrm{as}}(x(t))]/h$. Now, with $t = 0$, it follows from (55) that

$$\frac{\mathrm{d}}{\mathrm{d}t}V_{\mathrm{as}}(x_0) + \varepsilon V_{\mathrm{as}}(x_0) \leq s(u, y(0)), \quad u \in \mathbb{R}^m.$$

Next, let $d : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ be such that

$$d(x, u) \triangleq -\frac{\mathrm{d}}{\mathrm{d}t}V_{\mathrm{as}}(x) - \varepsilon V_{\mathrm{as}}(x) + s(u, y). \tag{56}$$

Now, it follows from (55) that $d(x, u) \geq 0, x \in \mathbb{R}^n,\ u \in \mathbb{R}^m$. Since $\frac{\mathrm{d}}{\mathrm{d}t}V_{\mathrm{as}}(x) \in \mathcal{L}_f V_{\mathrm{as}}(x) + \mathcal{L}_{Gu}V_{\mathrm{as}}(x)$ for almost all $x \in \mathbb{R}^n$, it follows that

$$\frac{\mathrm{d}}{\mathrm{d}t}V_{\mathrm{as}}(x) \geq \min \mathcal{L}_f V_{\mathrm{as}}(x) + \mathcal{L}_G V_{\mathrm{as}}(x)u, \quad \text{a.e. } x \in \mathbb{R}^n,\ u \in \mathbb{R}^m, \tag{57}$$

and hence, it follows from (56) and (57) that

$$-[\min \mathcal{L}_f V_{\mathrm{as}}(x) + \mathcal{L}_G V_{\mathrm{as}}(x)u + \varepsilon V_{\mathrm{as}}(x)] + s(u, h(x) + J(x)u) \geq d(x, u) \geq 0, \quad \text{a.e. } x \in \mathbb{R}^n,\ u \in \mathbb{R}^m. \tag{58}$$

Since the left-hand side of (58) is quadratic in u, there exist functions $\ell : \mathbb{R}^n \to \mathbb{R}^p$ and $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ such that

$$\begin{aligned}[\ell(x) + \mathcal{W}(x)u]^{\mathrm{T}}[\ell(x) + \mathcal{W}(x)u] &= -[\min \mathcal{L}_f V_{\mathrm{as}}(x) + \mathcal{L}_G V_{\mathrm{as}}(x)u + \varepsilon V_{\mathrm{as}}(x)] + s(u, h(x) + J(x)u) \\ &= -[\min \mathcal{L}_f V_{\mathrm{as}}(x) + \mathcal{L}_G V_{\mathrm{as}}(x)u + \varepsilon V_{\mathrm{as}}(x)] + [h(x) + J(x)u]^{\mathrm{T}} \\ &\quad \times Q[h(x) + J(x)u] + 2[h(x) + J(x)u]^{\mathrm{T}}Su + u^{\mathrm{T}}Ru.\end{aligned}$$

Now, equating coefficients of equal powers yields (47)–(49) with $V_{\mathrm{s}}(x) = V_{\mathrm{as}}(x)$ and with the positive definiteness of $V_{\mathrm{s}}(x),\ x \in \mathbb{R}^n$, following from Theorem 3.2.

Finally, the proof for the weakly dissipative case follows by using an identical construction with $\varepsilon = 0$. □

Remark 4.1. Note that if $\mathcal{W}^{\mathrm{T}}(x)\mathcal{W}(x)$ is invertible for all $x \in \mathbb{R}^n$, then inequality (50) can be equivalently written as

$$[\ell(x) - \mathcal{W}(x)(\mathcal{W}^{\mathrm{T}}(x)\mathcal{W}(x))^{-1}\mathcal{W}^{\mathrm{T}}(x)\ell(x)]^{\mathrm{T}}[\ell(x) - \mathcal{W}(x)(\mathcal{W}^{\mathrm{T}}(x)\mathcal{W}(x))^{-1}\mathcal{W}^{\mathrm{T}}(x)\ell(x)] \geq \max \mathcal{L}_f V_{\mathrm{s}}(x) - \min \mathcal{L}_f V_{\mathrm{s}}(x), \quad x \in \mathbb{R}^n, \tag{59}$$

which is free of $u \in \mathbb{R}^m$. This follows from the fact that (50) holds if and only if

$$\min_u [\ell(x) + \mathcal{W}(x)u]^{\mathrm{T}}[\ell(x) + \mathcal{W}(x)u] \geq \max \mathcal{L}_f V_{\mathrm{s}}(x) - \min \mathcal{L}_f V_{\mathrm{s}}(x), \quad x \in \mathbb{R}^n, \tag{60}$$

holds. A similar expression to (59) involving generalized inverses also holds in the case where $\mathcal{W}^{\mathrm{T}}(x)\mathcal{W}(x)$ is singular for some $x \in \mathbb{R}^n$.

The following result gives sufficient conditions for weak dissipativity and weak exponential dissipativity of $\mathcal{G}$ based on $\max \mathcal{L}_f V_{\mathrm{s}}(\cdot)$.

Theorem 4.2. *Let $Q \in \mathbb{S}^l$, $S \in \mathbb{R}^{l\times m}$, $R \in \mathbb{S}^m$, and let $\mathcal{G}$ be weakly zero-state observable and weakly completely reachable. If there exist functions $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$, $\ell : \mathbb{R}^n \to \mathbb{R}^p$, and $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ and a scalar $\varepsilon > 0$ (resp., $\varepsilon = 0$) such that $V_{\mathrm{s}}(\cdot)$ is locally Lipschitz continuous, regular, and positive definite, $V_{\mathrm{s}}(0) = 0$, and, for almost all $x \in \mathbb{R}^n$,*

$$0 = \max \mathcal{L}_f V_{\mathrm{s}}(x) + \varepsilon V_{\mathrm{s}}(x) - h^{\mathrm{T}}(x)Qh(x) + \ell^{\mathrm{T}}(x)\ell(x), \tag{61}$$

$$0 = \frac{1}{2}\mathcal{L}_G V_{\mathrm{s}}(x) - h^{\mathrm{T}}(x)(QJ(x) + S) + \ell^{\mathrm{T}}(x)\mathcal{W}(x), \tag{62}$$

$$0 = R + S^{\mathrm{T}}J(x) + J^{\mathrm{T}}(x)S + J^{\mathrm{T}}(x)QJ(x) - \mathcal{W}^{\mathrm{T}}(x)\mathcal{W}(x), \tag{63}$$

then $\mathcal{G}$ is weakly exponentially dissipative (resp., weakly dissipative) with respect to the supply rate map $\{s(u, y)\} = \{y^{\mathrm{T}}Qy + 2y^{\mathrm{T}}Su + u^{\mathrm{T}}Ru\}$.

Proof. Suppose that there exist functions $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$, $\ell : \mathbb{R}^n \to \mathbb{R}^p$, and $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ and a scalar $\varepsilon > 0$ such that $V_{\mathrm{s}}(\cdot)$ is locally Lipschitz continuous, regular, and positive definite, and (61)–(63) are satisfied. Then, for every admissible input $u(\cdot) \in \mathcal{U}$, it follows from (61)–(63) and (52) that

$$\begin{aligned}
\int_{t_1}^{t_2} e^{\varepsilon t} s(u(t), y(t))\mathrm{d}t &= \int_{t_1}^{t_2} e^{\varepsilon t}\left[y^{\mathrm{T}}(t)Qy(t) + 2y^{\mathrm{T}}(t)Su(t) + u^{\mathrm{T}}(t)Ru(t)\right]\mathrm{d}t \\
&= \int_{t_1}^{t_2} e^{\varepsilon t}\left[h^{\mathrm{T}}(x(t))Qh(x(t)) + 2h^{\mathrm{T}}(x(t))(S + QJ(x(t)))u(t)\right. \\
&\quad \left. + u^{\mathrm{T}}(t)(J^{\mathrm{T}}(x(t))QJ(x(t)) + S^{\mathrm{T}}J(x(t)) + J^{\mathrm{T}}(x(t))S + R)u(t)\right]\mathrm{d}t \\
&= \int_{t_1}^{t_2} e^{\varepsilon t}\left[\max \mathcal{L}_f V_{\mathrm{s}}(x(t)) + \mathcal{L}_G V_{\mathrm{s}}(x(t))u(t) + \varepsilon V_{\mathrm{s}}(x(t))\right. \\
&\quad \left. + [\ell(x(t)) + \mathcal{W}(x(t))u(t)]^{\mathrm{T}}[\ell(x(t)) + \mathcal{W}(x(t))u(t)]\right]\mathrm{d}t \\
&\geq \int_{t_1}^{t_2} e^{\varepsilon t}\left[\max \mathcal{L}_f V_{\mathrm{s}}(x(t)) + \mathcal{L}_G V_{\mathrm{s}}(x(t))u(t) + \varepsilon V_{\mathrm{s}}(x(t))\right]\mathrm{d}t \\
&\geq \int_{t_1}^{t_2} e^{\varepsilon t}\left[\frac{\mathrm{d}}{\mathrm{d}t}V_{\mathrm{s}}(x(t)) + \varepsilon V_{\mathrm{s}}(x(t))\right]\mathrm{d}t \\
&= e^{\varepsilon t_2}V_{\mathrm{s}}(x(t_2)) - e^{\varepsilon t_1}V_{\mathrm{s}}(x(t_1)), \quad \text{a.e. } t \geq 0,
\end{aligned}$$

where $x(t)$, $t \geq t_0$, is a solution satisfying (45). The result is now immediate from Definition 3.3. The proof for the weak dissipative case follows an identical construction by setting $\varepsilon = 0$. □

Finally, we provide necessary and sufficient conditions for the case where $\mathcal{G}$ given by (45) and (46) is weakly lossless with respect to the supply rate map $\{s(u, y)\} = \{y^{\mathrm{T}}Qy + 2y^{\mathrm{T}}Su + u^{\mathrm{T}}Ru\}$.

Theorem 4.3. *Let $Q \in \mathbb{S}^l$, $S \in \mathbb{R}^{l\times m}$, $R \in \mathbb{S}^m$, and let $\mathcal{G}$ be weakly zero-state observable and weakly completely reachable. If there exists a function $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$ such that $V_{\mathrm{s}}(\cdot)$ is locally Lipschitz continuous, regular and positive definite, $V_{\mathrm{s}}(0) = 0$, and, for almost all $x \in \mathbb{R}^n$,*

$$0 = \min \mathcal{L}_f V_{\mathrm{s}}(x) - h^{\mathrm{T}}(x)Qh(x), \tag{64}$$

$$0 = \frac{1}{2}\mathcal{L}_G V_{\mathrm{s}}(x) - h^{\mathrm{T}}(x)(QJ(x) + S), \tag{65}$$

$$0 = R + S^{\mathrm{T}}J(x) + J^{\mathrm{T}}(x)S + J^{\mathrm{T}}(x)QJ(x), \tag{66}$$

$$\max \mathcal{L}_f V_{\mathrm{s}}(x) = \min \mathcal{L}_f V_{\mathrm{s}}(x), \tag{67}$$

then $\mathcal{G}$ is weakly lossless with respect to the supply rate map $\{s(u, y)\} = \{y^{\mathrm{T}}Qy + 2y^{\mathrm{T}}Su + u^{\mathrm{T}}Ru\}$. Conversely, if $\mathcal{G}$ is weakly lossless with respect to the supply rate map $\{s(u, y)\}$, then there exists function $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$ such that $V_{\mathrm{s}}(\cdot)$ is locally Lipschitz continuous, regular, and positive definite, $V_{\mathrm{s}}(0) = 0$, and, for almost all $x \in \mathbb{R}^n$, (64)–(66) hold.

Proof. The proof is analogous to the proof of Theorem 4.1 and, hence, is omitted. □

Next, we provide several definitions of nonlinear discontinuous dynamical systems which are dissipative or exponentially dissipative with respect to supply rate maps of a specific form.

Definition 4.1. A differential inclusion $\mathcal{G}$ of the form (1)–(3) with $m = l$ is *weakly* (resp., *strongly*) *passive* if $\mathcal{G}$ is weakly (resp., strongly) dissipative with respect to the set-valued supply rate map $\mathcal{S}_{\mathrm{r}}$ consisting of supply rates of the form $s(u, y) = 2u^{\mathrm{T}}y$.

Definition 4.2. A differential inclusion $\mathcal{G}$ of the form (1)–(3) is *weakly* (resp., *strongly*) *nonexpansive* if $\mathcal{G}$ is weakly (resp., strongly) dissipative with respect to the set-valued supply rate map $\mathcal{S}_{\mathrm{r}}$ consisting of supply rates of the form $s(u, y) = \gamma^2 u^{\mathrm{T}}u - y^{\mathrm{T}}y$, where $\gamma > 0$ is given.

Definition 4.3. A differential inclusion $\mathcal{G}$ of the form (1)–(3) with $m = l$ is *weakly* (resp., *strongly*) exponentially *passive* if $\mathcal{G}$ is weakly (resp., strongly) exponentially dissipative with respect to the set-valued supply rate map $\mathcal{S}_{\mathrm{r}}$ consisting of supply rates of the form $s(u, y) = 2u^{\mathrm{T}}y$.

Definition 4.4. A differential inclusion $\mathcal{G}$ of the form (1)–(3) is *weakly* (resp., *strongly*) *exponentially nonexpansive* if $\mathcal{G}$ is weakly (resp., strongly) exponentially dissipative with respect to the set-valued supply rate map $\mathcal{S}_{\mathrm{r}}$ consisting of supply rates of the form $s(u, y) = \gamma^2 u^{\mathrm{T}}u - y^{\mathrm{T}}y$, where $\gamma > 0$ is given.

The following results present the nonlinear versions of the Kalman–Yakubovich–Popov *strict positive real lemma* (resp., *positive real lemma*) and *strict bounded real lemma* (resp., *bounded real lemma*) for weakly exponentially passive (resp., weakly passive) and weakly exponentially nonexpansive (resp., weakly nonexpansive) discontinuous systems, respectively.

Corollary 4.1. *Let $\mathcal{G}$ be weakly zero-state observable and weakly completely reachable. If there exist functions $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$, $\ell : \mathbb{R}^n \to \mathbb{R}^p$, and $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ and a scalar $\varepsilon > 0$ (resp., $\varepsilon = 0$) such that $V_{\mathrm{s}}(\cdot)$ is locally Lipschitz continuous, regular, and positive definite, $V_{\mathrm{s}}(0) = 0$, and, for almost all $x \in \mathbb{R}^n$,*

$$0 = \min \mathcal{L}_f V_{\mathrm{s}}(x) + \varepsilon V_{\mathrm{s}}(x) + \ell^{\mathrm{T}}(x)\ell(x), \tag{68}$$

$$0 = \frac{1}{2}\mathcal{L}_G V_{\mathrm{s}}(x) - h^{\mathrm{T}}(x) + \ell^{\mathrm{T}}(x)\mathcal{W}(x), \tag{69}$$

$$0 = J(x) + J^{\mathrm{T}}(x) - \mathcal{W}^{\mathrm{T}}(x)\mathcal{W}(x), \tag{70}$$

$$[\ell(x) + \mathcal{W}(x)u]^{\mathrm{T}}[\ell(x) + \mathcal{W}(x)u] \geq \max \mathcal{L}_f V_{\mathrm{s}}(x) - \min \mathcal{L}_f V_{\mathrm{s}}(x), \quad u \in \mathbb{R}^m, \tag{71}$$

then $\mathcal{G}$ is weakly exponentially passive (resp., weakly passive). Conversely, if $\mathcal{G}$ is weakly exponentially passive (resp., weakly passive), then there exist functions $V_{\mathrm{s}} : \mathbb{R}^n \to \mathbb{R}$, $\ell : \mathbb{R}^n \to \mathbb{R}^p$, and $\mathcal{W} : \mathbb{R}^n \to \mathbb{R}^{p\times m}$ and a scalar $\varepsilon > 0$ (resp., $\varepsilon = 0$) such that $V_{\mathrm{s}}(\cdot)$ is locally Lipschitz continuous, regular, and positive definite, $V_{\mathrm{s}}(0) = 0$, and, for almost all $x \in \mathbb{R}^n$, (68)–(70) hold.

Proof. The result is a direct consequence of Theorem 4.1 with $l = m$, $Q = 0$, $S = I_m$, and $R = 0$. Specifically, with $\kappa(y) = -y$ it follows that $s(\kappa(y), y) = -2y^{\mathrm{T}}y < 0,\ y \neq 0$, so that all the assumptions of Theorem 4.1 are satisfied. □

Example 4.1. Consider the harmonic oscillator $\mathcal{G}$ with Coulomb friction given by [19]

$$m\ddot{x}(t) + b\,\mathrm{sign}(\dot{x}(t)) + kx(t) = u(t), \qquad x(0) = x_0, \qquad \dot{x}(0) = \dot{x}_0, \quad \text{a.e. } t \geq 0, \tag{72}$$

$$y(t) = \frac{1}{2}\dot{x}(t), \tag{73}$$

or, equivalently,

$$\begin{bmatrix}\dot{x}_1(t)\\ \dot{x}_2(t)\end{bmatrix} = \begin{bmatrix} x_2(t) \\ -\frac{k}{m}x_1(t) - \frac{b}{m}\,\mathrm{sign}(x_2(t))\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{m}\end{bmatrix}u(t), \qquad \begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix} = \begin{bmatrix}x_{10}\\ x_{20}\end{bmatrix}, \quad \text{a.e. } t \geq 0, \tag{74}$$

$$y(t) = \frac{1}{2}x_2(t), \tag{75}$$

where $m, b, k > 0$. Next, consider the continuously differentiable storage function $V_s(x) = \frac{1}{2}kx_1^2 + \frac{1}{2}mx_2^2$ and note that, for almost all $x \in \mathbb{R}^2$, $\mathcal{L}_f V_s(x) = \{-b|x_2|\}$ and $\mathcal{L}_G V_s(x) = \{x_2\}$, which implies that $\min \mathcal{L}_f V_s(x) = \max \mathcal{L}_f V_s(x) = -b|x_2|$. Now, with $\ell(x) = \pm\sqrt{b|x_2|}$ and $\mathcal{W}(x) = 0$, (68)–(71) are satisfied. Hence, it follows from Corollary 4.1 that $\mathcal{G}$ is weakly passive. □

Example 4.2. Consider a controlled smooth oscillator with nonsmooth friction and uncertain coefficients given in [25] represented by the differential inclusion $\mathcal{G}$ given by

$$\dot{x}(t) \in \mathcal{K}[f](x(t)) + Gu(t), \qquad x(0) = x_0, \quad \text{a.e. } t \geq 0, \tag{76}$$

$$y(t) = \frac{1}{2}x_2(t), \tag{77}$$

where $G = [0, 1]^{\mathrm{T}}$ and $\mathcal{K}[f] : \mathbb{R}^2 \to \mathcal{B}(\mathbb{R}^2)$ is given by

$$\mathcal{K}[f](x) \triangleq \begin{cases} [-2x_2 - 1, -x_2 - 1] \times \{x_1\}, & (x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0, \\ \{-x_2 - \operatorname{sign}(x_1)\} \times \{x_1\}, & (x_1, x_2) \in \mathbb{R}^2 \setminus \{(0, x_2) : x_2 \in \mathbb{R}\} \cup \{(x_1, x_2) : x_1 > 0, x_2 > 0\}, \\ [-2x_2 - 1, -x_2 + 1] \times \{0\}, & (x_1, x_2) \in \mathbb{R}^2 : x_2 > 0, x_1 = 0, \\ [-x_2 - 1, -x_2 + 1] \times \{0\}, & (x_1, x_2) \in \mathbb{R}^2 : x_2 < 0, x_1 = 0, \\ [-1, 1] \times \{0\}, & (x_1, x_2) = (0, 0). \end{cases}$$

Next, consider the continuously differentiable storage function $V_s(x) = \frac{1}{2}(x_1^2 + x_2^2)$ and note that for almost all $x \in \mathbb{R}^2$,

$$\mathcal{L}_f V_s(x) = \begin{cases} \{[-1, 0]x_1x_2 - x_1\}, & (x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0, \\ \{-|x_1|\}, & (x_1, x_2) \in \mathbb{R}^2 \setminus \{(0, x_2) : x_2 \in \mathbb{R}\} \cup \{(x_1, x_2) : x_1 > 0, x_2 > 0\}, \\ \{0\}, & (x_1, x_2) \in \mathbb{R}^2 : x_2 \neq 0, x_1 = 0, \\ \{0\}, & (x_1, x_2) = (0, 0), \end{cases}$$

$$\mathcal{L}_G V_s(x) = \{x_2\},$$

which implies that $\max \mathcal{L}_f V_s(x) = 0$ and $\min \mathcal{L}_f V_s(x) = -|x_1|$ for almost all $x \in \mathbb{R}^2$. Now, it follows from (68)–(71) that

$$0 = -|x_1| + \ell^2(x), \tag{78}$$

$$0 = \frac{1}{2}x_2 - \frac{1}{2}x_2 + \ell(x)\mathcal{W}(x), \tag{79}$$

$$0 = \mathcal{W}^2(x), \tag{80}$$

$$|x_1| \leq [\ell(x) + \mathcal{W}(x)u]^2, \quad u \in \mathbb{R}. \tag{81}$$

Hence, with $\ell(x) = \pm\sqrt{|x_1|}$ and $\mathcal{W}(x) = 0$, it follows from Corollary 4.1 that $\mathcal{G}$ is weakly passive. □

Example 4.3. Consider a controlled nonsmooth harmonic oscillator with nonsmooth friction and nonsmooth output given by [25]

$$\dot{x}(t) = f(x(t)) + Gu(t), \qquad x(0) = x_0, \quad \text{a.e. } t \geq 0, \tag{82}$$

$$y(t) = \frac{1}{2}\operatorname{sign}(x_2(t)), \tag{83}$$

where $f(x) = [-\operatorname{sign}(x_2) - \frac{1}{2}\operatorname{sign}(x_1),\ \operatorname{sign}(x_1)]^{\mathrm{T}}$ and $G = [0, 1]^{\mathrm{T}}$. Next, consider the locally Lipschitz continuous storage function $V_s(x) = |x_1| + |x_2|$ and note that

$$\partial V_s(x_1, x_2) = \begin{cases} \{\operatorname{sign}(x_1)\} \times \{\operatorname{sign}(x_2)\}, & (x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_2 \neq 0, \\ \{\operatorname{sign}(x_1)\} \times [-1, 1], & (x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_2 = 0, \\ [-1, 1] \times \{\operatorname{sign}(x_2)\}, & (x_1, x_2) \in \mathbb{R}^2 : x_2 \neq 0, x_1 = 0, \\ \overline{\mathrm{co}}\{(1, 1), (-1, 1), (-1, -1), (1, -1)\}, & (x_1, x_2) = (0, 0). \end{cases}$$

Hence,

$$\mathcal{L}_f V_s(x_1, x_2) = \begin{cases} \left\{-\frac{1}{2}\right\}, & (x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_2 \neq 0, \\ \varnothing, & (x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_2 = 0, \\ \varnothing, & (x_1, x_2) \in \mathbb{R}^2 : x_2 \neq 0, x_1 = 0, \\ \{0\}, & (x_1, x_2) = (0, 0), \end{cases}$$

$$\mathcal{L}_G V_s(x_1,x_2)=\begin{cases}\{\operatorname{sign}(x_2)\}, & (x_1,x_2)\in\mathbb{R}^2: x_1\neq 0, x_2\neq 0,\\ \varnothing, & (x_1,x_2)\in\mathbb{R}^2: x_1\neq 0, x_2=0,\\ \{\operatorname{sign}(x_2)\}, & (x_1,x_2)\in\mathbb{R}^2: x_2\neq 0, x_1=0,\\ \{0\}, & (x_1,x_2)=(0,0),\end{cases}$$

which implies that $\max \mathcal{L}_f V_s(x)=0$, $\min \mathcal{L}_f V_s(x)=-\frac{1}{2}$, and $\mathcal{L}_G V_s(x)=\{\operatorname{sign}(x_2)\}$ for almost all $x\in\mathbb{R}^2$. Now, it follows from (68)–(71) that

$$0=-\frac{1}{2}+\ell^2(x), \tag{84}$$

$$0=\frac{1}{2}\operatorname{sign}(x_2)-\frac{1}{2}\operatorname{sign}(x_2)+\ell(x)\mathcal{W}(x), \tag{85}$$

$$0=\mathcal{W}^2(x), \tag{86}$$

$$\frac{1}{2}\leq[\ell(x)+\mathcal{W}(x)u]^2,\quad u\in\mathbb{R}. \tag{87}$$

Hence, with $\ell(x)=\pm\sqrt{\frac{1}{2}}$ and $\mathcal{W}(x)=0$, it follows from Corollary 4.1 that $\mathcal{G}$ is weakly passive. □

Corollary 4.2. *Let $Q\in\mathbb{S}^l$, $S\in\mathbb{R}^{l\times m}$, $R\in\mathbb{S}^m$, and let $\mathcal{G}$ be weakly zero-state observable and weakly completely reachable. If there exist functions $V_s:\mathbb{R}^n\to\mathbb{R}$, $\ell:\mathbb{R}^n\to\mathbb{R}^p$, and $\mathcal{W}:\mathbb{R}^n\to\mathbb{R}^{p\times m}$ and a scalar $\varepsilon>0$ (resp., $\varepsilon=0$) such that $V_s(\cdot)$ is locally Lipschitz continuous, regular, and positive definite, $V_s(0)=0$, and, for almost all $x\in\mathbb{R}^n$,*

$$0=\min \mathcal{L}_f V_s(x)+\varepsilon V_s(x)+h^{\mathrm{T}}(x)h(x)+\ell^{\mathrm{T}}(x)\ell(x), \tag{88}$$

$$0=\frac{1}{2}\mathcal{L}_G V_s(x)+h^{\mathrm{T}}(x)J(x)+\ell^{\mathrm{T}}(x)\mathcal{W}(x), \tag{89}$$

$$0=\gamma^2 I_m-J^{\mathrm{T}}(x)J(x)-\mathcal{W}^{\mathrm{T}}(x)\mathcal{W}(x), \tag{90}$$

$$[\ell(x)+\mathcal{W}(x)u]^{\mathrm{T}}[\ell(x)+\mathcal{W}(x)u]\geq\max \mathcal{L}_f V_s(x)-\min \mathcal{L}_f V_s(x),\quad u\in\mathbb{R}^m, \tag{91}$$

where $\gamma>0$, then $\mathcal{G}$ is weakly exponentially nonexpansive (resp., weakly nonexpansive). Conversely, if $\mathcal{G}$ is weakly exponentially nonexpansive (resp., weakly nonexpansive), then there exist functions $V_s:\mathbb{R}^n\to\mathbb{R}$, $\ell:\mathbb{R}^n\to\mathbb{R}^p$, and $\mathcal{W}:\mathbb{R}^n\to\mathbb{R}^{p\times m}$ and a scalar $\varepsilon>0$ (resp., $\varepsilon=0$) such that $V_s(\cdot)$ is locally Lipschitz continuous, regular, and positive definite, $V_s(0)=0$, and, for almost all $x\in\mathbb{R}^n$, (88)–(90) hold.

Proof. The result is a direct consequence of Theorem 4.1 with $Q=-I_l$, $S=0$, and $R=\gamma^2 I_m$. Specifically, with $\kappa(y)=-\frac{1}{2\gamma}y$ it follows that $s(\kappa(y),y)=-\frac{3}{4}y^{\mathrm{T}}y<0$, $y\neq 0$, so that all the assumptions of Theorem 4.1 are satisfied. □

Example 4.4. Consider the controlled dynamical system $\mathcal{G}$ given by

$$\dot{x}(t)=f(x(t))+G(x(t))u(t),\qquad x(0)=x_0,\quad \text{a.e. } t\geq 0, \tag{92}$$

$$y(t)=x(t), \tag{93}$$

where $x(t)=[x_1(t),x_2(t)]^{\mathrm{T}}$, $u(t)=[u_1(t),u_2(t)]^{\mathrm{T}}$,

$$f(x)=\begin{bmatrix}|x_1|(-x_1+|x_2|)\\ x_2(-x_1-|x_2|)\end{bmatrix},\qquad G(x)=\begin{bmatrix}|x_1| & 0\\ 0 & x_2\end{bmatrix}.$$

Next, consider the locally Lipschitz continuous storage function $V_s(x)=2|x_1|+2|x_2|$ and note that

$$\partial V_s(x_1,x_2)=\begin{cases}\{2\operatorname{sign}(x_1)\}\times\{2\operatorname{sign}(x_2)\}, & (x_1,x_2)\in\mathbb{R}^2: x_1\neq 0, x_2\neq 0,\\ \{2\operatorname{sign}(x_1)\}\times[-2,2], & (x_1,x_2)\in\mathbb{R}^2: x_1\neq 0, x_2=0,\\ [-2,2]\times\{2\operatorname{sign}(x_2)\}, & (x_1,x_2)\in\mathbb{R}^2: x_2\neq 0, x_1=0,\\ \overline{\operatorname{co}}\{(1,1),(-1,1),(-1,-1),(1,-1)\}, & (x_1,x_2)=(0,0).\end{cases}$$

Hence,

$$\mathcal{L}_f V_s(x_1,x_2)=\begin{cases}\{-2x_1^2-2x_2^2\}, & (x_1,x_2)\in\mathbb{R}^2: x_1\neq 0, x_2\neq 0,\\ \{-2x_1^2\}, & (x_1,x_2)\in\mathbb{R}^2: x_1\neq 0, x_2=0,\\ \{-2x_2^2\}, & (x_1,x_2)\in\mathbb{R}^2: x_2\neq 0, x_1=0,\\ \{0\}, & (x_1,x_2)=(0,0),\end{cases}$$

$$\mathcal{L}_G V_s(x_1, x_2) = \begin{cases} \{[2x_1, 2|x_2|]\}, & (x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_2 \neq 0, \\ \{[2x_1, 0]\}, & (x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, x_2 = 0, \\ \{[0, 2|x_2|]\}, & (x_1, x_2) \in \mathbb{R}^2 : x_2 \neq 0, x_1 = 0, \\ \{[0, 0]\}, & (x_1, x_2) = (0, 0), \end{cases}$$

which implies that $\min \mathcal{L}_f V_s(x) = \max \mathcal{L}_f V_s(x) = -2x_1^2 - 2x_2^2$ and $\mathcal{L}_G V_s(x) = \{[2x_1, 2|x_2|]\}$ for almost all $x \in \mathbb{R}^2$. Now, it follows from (88)–(91) that

$$0 = -2x_1^2 - 2x_2^2 + x_1^2 + x_2^2 + \ell^{\mathrm{T}}(x)\ell(x), \tag{94}$$

$$0 = \frac{1}{2}[2x_1, 2|x_2|] + \ell^{\mathrm{T}}(x)\mathcal{W}(x), \tag{95}$$

$$0 = \gamma^2 I_2 - \mathcal{W}^{\mathrm{T}}(x)\mathcal{W}(x), \tag{96}$$

$$0 \leq [\ell(x) + \mathcal{W}(x)u]^{\mathrm{T}} [\ell(x) + \mathcal{W}(x)u], \quad u \in \mathbb{R}^2. \tag{97}$$

Hence, with $\gamma = 1$, $\ell(x) = -[x_1, |x_2|]^{\mathrm{T}}$, and $\mathcal{W}(x) = I_2$, it follows from Corollary 4.2 that $\mathcal{G}$ is weakly nonexpansive. □

The following stability theorems are needed for the next result and the results of the next section. In addressing the stability properties of a Filippov solution of a discontinuous dynamical system the usual stability definitions are valid [3,29,30]. Here, we state the stability theorems for only the local case; the global stability theorems are similar except for the additional assumption of properness on the Lyapunov function and nonrestricting the domain of analysis. For the remainder of the paper, the adjective "weak" is used in reference to a stability property when the stability property is satisfied by at least one Filippov solution starting from every initial condition in $\mathcal{D}$, whereas "strong" is used when the stability property is satisfied by all Filippov solutions starting from every initial condition in $\mathcal{D}$.

Theorem 4.4 ([25,30]). *Consider the differential inclusion $\mathcal{G}$ given by (4). Let x_e be an equilibrium point of $\mathcal{G}$ and let $\mathcal{D} \subseteq \mathbb{R}^n$ be an open and connected set with $x_e \in \mathcal{D}$. If $V : \mathcal{D} \to \mathbb{R}$ is a positive definite, locally Lipschitz continuous, and regular function such that $\max \mathcal{L}_f V(x) \leq 0$ (resp., $\max \mathcal{L}_f V(x) < 0$, $x \neq x_e$) for almost all $x \in \mathcal{D}$ such that $\mathcal{L}_f V(x) \neq \varnothing$, then x_e is strongly Lyapunov (resp., strongly asymptotically) stable. If, in addition, $\max \mathcal{L}_f V(x) \leq -\varepsilon < 0$ for almost all $x \in \mathcal{D}, x \neq x_e$, such that $\mathcal{L}_f V(x) \neq \varnothing$, then x_e is strongly finite-time stable. Finally, if there exist scalars $\alpha, \beta, \gamma > 0$ and $p \geq 1$, such that $V : \mathcal{D} \to \mathbb{R}$ satisfies $\alpha\|x - x_e\|^p \leq V(x) \leq \beta\|x - x_e\|^p$ and $\max \mathcal{L}_f V(x) \leq -\gamma\|x - x_e\|^p$ for almost all $x \in \mathcal{D}$, $x \neq x_e$, such that $\mathcal{L}_f V(x) \neq \varnothing$, then x_e is strongly exponentially stable.*

The following definitions are needed for the statement of the next result. We say a set $\mathcal{M}$ is *weakly positively invariant* (resp., *strongly positively invariant*) with respect to (4) if, for every $x_0 \in \mathcal{M}$, $\mathcal{M}$ contains a right maximal solution (resp., all right maximal solutions) of (4) [25].

Theorem 4.5 ([25,30]). *Consider the differential inclusion $\mathcal{G}$ given by (4). Let x_e be an equilibrium point of $\mathcal{G}$, let $\mathcal{D} \subseteq \mathbb{R}^n$ be an open strongly positively invariant set with respect to (4) such that $x_e \in \mathcal{D}$, and let $V : \mathcal{D} \to \mathbb{R}$ be locally Lipschitz continuous and regular on $\mathcal{D}$. Assume that, for every $x \in \mathcal{D}$ and every Filippov solution $\psi(\cdot)$ satisfying $\psi(t_0) = x$, there exists a compact subset $\mathcal{D}_c$ of $\mathcal{D}$ containing $\psi(t)$ for all $t \geq 0$. Furthermore, assume that $\max \mathcal{L}_f V(x) \leq 0$ for almost all $x \in \mathcal{D}$ such that $\mathcal{L}_f V(x) \neq \varnothing$. Finally, define $\mathcal{R} \triangleq \{x \in \mathcal{D} : 0 \in \mathcal{L}_f V(x)\}$ and let $\mathcal{M}$ be the largest weakly positively invariant subset of $\overline{\mathcal{R}} \cap \mathcal{D}$. If $x(t_0) \in \mathcal{D}_c$, then $x(t) \to \mathcal{M}$ as $t \to \infty$. If, alternatively, $\mathcal{R}$ contains no invariant set other than $\{x_e\}$, then the solution $x(t) \equiv x_e$ of $\mathcal{G}$ is strongly asymptotically stable for all $x_0 \in \mathcal{D}_c$.*

In light of the above definitions and theorems the following result is immediate.

Proposition 4.1. *Consider the differential inclusion $\mathcal{G}$ given by (1)–(3). Then the following statements hold:*

(i) *If $\mathcal{G}$ is strongly passive with a locally Lipschitz continuous, regular, and positive definite storage function $V_s(\cdot)$, then the zero solution $x(t) \equiv 0$ of the undisturbed ($u(t) \equiv 0$) system $\mathcal{G}$ is strongly Lyapunov stable.*
(ii) *If $\mathcal{G}$ is strongly exponentially passive with a locally Lipschitz continuous, regular, and positive definite storage function $V_s(\cdot)$, then the zero solution $x(t) \equiv 0$ of the undisturbed ($u(t) \equiv 0$) system $\mathcal{G}$ is strongly asymptotically stable.*
(iii) *If $\mathcal{G}$ is strongly zero-state observable and strongly nonexpansive with locally Lipschitz continuous, regular, and positive definite storage function $V_s(\cdot)$, then the zero solution $x(t) \equiv 0$ of the undisturbed ($u(t) \equiv 0$) system $\mathcal{G}$ is strongly asymptotically stable.*
(iv) *If $\mathcal{G}$ is strongly exponentially nonexpansive with a locally Lipschitz continuous, regular, and positive definite storage function $V_s(\cdot)$, then the zero solution $x(t) \equiv 0$ of the undisturbed ($u(t) \equiv 0$) system $\mathcal{G}$ is strongly asymptotically stable.*

Proof. Statements (i)–(iv) are immediate and follow from (13)–(15) using Lyapunov and invariant set stability arguments given by Theorems 4.4 and 4.5, respectively. □

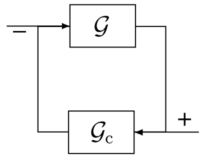

Fig. 1. Feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$.

5. Stability of feedback interconnections of dissipative discontinuous dynamical systems

In this section, we consider feedback interconnections of dissipative discontinuous dynamical systems. Specifically, using the notions of dissipativity and exponential dissipativity for discontinuous dynamical systems, with appropriate set valued storage maps and set-valued supply rate maps, we construct (not necessarily smooth) Lyapunov functions for interconnected discontinuous dynamical systems by appropriately combining the set-valued storage maps for each subsystem.

We begin by considering the nonlinear discontinuous dynamical system $\mathcal{G}$ given by

$$\dot{x}(t) = f(x(t)) + G(x(t))u(t), \qquad x(0) = x_0, \quad \text{a.e. } t \geq 0, \tag{98}$$

$$y(t) = h(x(t)) + J(x(t))u(t), \tag{99}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^l$, $f : \mathbb{R}^n \to \mathbb{R}^n$, $G : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, $h : \mathbb{R}^n \to \mathbb{R}^l$, and $J : \mathbb{R}^n \to \mathbb{R}^{l\times m}$, with the nonlinear feedback discontinuous system $\mathcal{G}_c$ given by

$$\dot{x}_c(t) = f_c(x_c(t)) + G_c(u_c(t), x_c(t))u_c(t), \qquad x_c(0) = x_{c0}, \quad \text{a.e. } t \geq 0, \tag{100}$$

$$y_c(t) = h_c(u_c(t), x_c(t)) + J_c(u_c(t), x_c(t))u_c(t), \tag{101}$$

where $x_c \in \mathbb{R}^{n_c}$, $u_c \in \mathbb{R}^l$, $y_c \in \mathbb{R}^m$, $f_c : \mathbb{R}^{n_c} \to \mathbb{R}^{n_c}$, $G_c : \mathbb{R}^l \times \mathbb{R}^{n_c} \to \mathbb{R}^{n_c\times l}$, $h_c : \mathbb{R}^l \times \mathbb{R}^{n_c} \to \mathbb{R}^m$, and $J_c : \mathbb{R}^l \times \mathbb{R}^{n_c} \to \mathbb{R}^{m\times l}$. We assume that $f(\cdot)$, $G(\cdot)$, $h(\cdot)$, $J(\cdot)$, $f_c(\cdot)$, $G_c(\cdot)$, $h_c(\cdot,\cdot)$, and $J_c(\cdot,\cdot)$ are Lebesgue measurable and locally essentially bounded, (100) and (101) has at least one equilibrium point, and the required properties for the existence of solutions of the feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ are satisfied. Note that with the negative feedback interconnection given by Fig. 1, $u_c = y$ and $y_c = -u$. We assume that the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is well posed, that is, $\det[I_m + J_c(y, x_c)J(x)] \neq 0$ for all y, x, and x_c.

The following results give sufficient conditions for Lyapunov, asymptotic, and exponential stability of the feedback interconnection given by Fig. 1. For the remainder of the paper, we assume that the forward path $\mathcal{G}$ and the feedback path $\mathcal{G}_c$ in Fig. 1 are strongly dissipative systems that admit single-valued supply rate maps and storage maps. This assumption holds when the closed-loop system (98)–(101) admits a unique solution and is only made for notational convenience. Similar stability results hold for the more general case wherein $\mathcal{G}$ and $\mathcal{G}_c$ admit set-valued storage and set-valued supply rate maps. Finally, we also note that the obtained stability results also hold for the case where $\mathcal{G}$ and $\mathcal{G}_c$ are weakly dissipative. In this case, however, the set-valued Lie derivative operator should be replaced with the upper right Dini directional derivative in the proofs of the stability theorems.

Theorem 5.1. *Consider the closed-loop system consisting of the nonlinear discontinuous dynamical systems $\mathcal{G}$ and $\mathcal{G}_c$ with input–output pairs (u, y) and (u_c, y_c), respectively, and with $u_c = y$ and $y_c = -u$. Assume $\mathcal{G}$ and $\mathcal{G}_c$ are strongly zero-state observable, strongly completely reachable, and strongly dissipative with respect to the supply rate maps $\{s(u, y)\}$ and $\{s_c(u_c, y_c)\}$ and with locally Lipschitz continuous, regular, and radially unbounded storage functions $V_s(\cdot)$ and $V_{sc}(\cdot)$, respectively, such that $V_s(0) = 0$ and $V_{sc}(0) = 0$. Furthermore, assume there exists a scalar $\sigma > 0$ such that $s(u, y) + \sigma s_c(u_c, y_c) \leq 0$, for all $u \in \mathbb{R}^m$, $y \in \mathbb{R}^l$, $u_c \in \mathbb{R}^l$, $y_c \in \mathbb{R}^m$ such that $u_c = y$ and $y_c = -u$. Then the following statements hold:*

(i) *The negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is strongly Lyapunov stable.*
(ii) *If $\mathcal{G}_c$ is strongly exponentially dissipative with respect to supply rate map $\{s_c(u_c, y_c)\}$ and* $\operatorname{rank}[G_c(u_c, 0)] = m$, $u_c \in \mathbb{R}^l$, *then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is globally strongly asymptotically stable.*
(iii) *If $\mathcal{G}$ and $\mathcal{G}_c$ are strongly exponentially dissipative with respect to supply rate maps $\{s(u, y)\}$ and $\{s_c(u_c, y_c)\}$, respectively, and $V_s(\cdot)$ and $V_{sc}(\cdot)$ are such that there exist constants α, α_c, β, and $\beta_c > 0$ such that*

$$\alpha\|x\|^2 \leq V_s(x) \leq \beta\|x\|^2, \quad x \in \mathbb{R}^n, \tag{102}$$

$$\alpha_c\|x_c\|^2 \leq V_{sc}(x_c) \leq \beta_c\|x_c\|^2, \quad x_c \in \mathbb{R}^{n_c}, \tag{103}$$

then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is globally strongly exponentially stable.

Proof. (i) Note that the closed-loop dynamics of the feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is given by

$$\begin{bmatrix}\dot{x}(t)\\ \dot{x}_c(t)\end{bmatrix} = \begin{bmatrix} f_1(x(t), x_c(t))\\ f_2(x(t), x_c(t))\end{bmatrix} \triangleq \tilde{f}(x(t), x_c(t)), \qquad \begin{bmatrix}x(t_0)\\ x_c(t_0)\end{bmatrix} = \begin{bmatrix}x_0\\ x_{c0}\end{bmatrix}, \quad \text{a.e. } t \geq t_0. \tag{104}$$

Now, consider the Lyapunov function candidate $V(x,x_c) = V_s(x) + \sigma V_{sc}(x_c)$. Since $\mathcal{L}_{\tilde{f}} V(x,x_c) \subseteq \mathcal{L}_{\tilde{f}} V_s(x) + \sigma \mathcal{L}_{\tilde{f}} V_{sc}(x_c)$ for almost all $(x,x_c) \in \mathbb{R}^n \times \mathbb{R}^{n_c}$, it follows that

$$\begin{aligned}\max \mathcal{L}_{\tilde{f}} V(x,x_c) &\le \max\{\mathcal{L}_{f_1} V_s(x) + \sigma \mathcal{L}_{f_2} V_{sc}(x_c)\}\\ &\le \max \mathcal{L}_{f_1} V_s(x) + \sigma \max \mathcal{L}_{f_2} V_{sc}(x_c).\end{aligned}$$

Next, since $s(u,y) + \sigma s_c(u_c,y_c) \le 0$, for all $u \in \mathbb{R}^m, y \in \mathbb{R}^l,\ u_c \in \mathbb{R}^l,\ y_c \in \mathbb{R}^m$, $\frac{\mathrm{d}}{\mathrm{d}t} V_s(x(t)) \in \mathcal{L}_{f_1} V_s(x(t))$, a.e. $t \ge 0$, and $\frac{\mathrm{d}}{\mathrm{d}t} V_{sc}(x_c(t)) \in \mathcal{L}_{f_2} V_{sc}(x_c(t))$, a.e. $t \ge 0$, there exist $u', y',\ u_c'$ and y_c' such that

$$\max \mathcal{L}_{\tilde{f}} V(x,x_c) \le \max \mathcal{L}_{f_1} V_s(x) + \sigma \max \mathcal{L}_{f_2} V_{sc}(x_c) \le s(u',y') + \sigma s_c(u_c',y_c') \le 0$$

for almost all $x \in \mathbb{R}^n$ and $x_c \in \mathbb{R}^{n_c}$. Now, it follows from Theorem 4.4 that the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is strongly Lyapunov stable.

(ii) If $\mathcal{G}_c$ is strongly exponentially dissipative it follows that there exist $u', y',\ u_c'$ and y_c' and a scalar $\varepsilon_c > 0$ such that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} V(x,x_c) &\le \max \mathcal{L}_{\tilde{f}} V(x,x_c)\\ &\le \max \mathcal{L}_{f_1} V_s(x) + \sigma \max \mathcal{L}_{f_2} V_{sc}(x_c)\\ &\le -\sigma\varepsilon_c V_{sc}(x_c) + s(u',y') + \sigma s_c(u_c',y_c')\\ &\le -\sigma\varepsilon_c V_{sc}(x_c), \quad \text{a.e. } (x,x_c) \in \mathbb{R}^n \times \mathbb{R}^{n_c}.\end{aligned}$$

Now, let $\mathcal{R} \triangleq \{(x,x_c) \in \mathbb{R}^n \times \mathbb{R}^{n_c} : \frac{\mathrm{d}}{\mathrm{d}t} V(x,x_c) = 0 \in \mathcal{L}_{\tilde{f}} V(x,x_c)\}$ and, since $V_{sc}(x_c)$ is positive definite, note that $\frac{\mathrm{d}}{\mathrm{d}t} V(x,x_c) = 0$ if and only if $x_c = 0$. Now, since $\operatorname{rank}[G_c(u_c,0)] = m, u_c \in \mathbb{R}^l$, it follows that on every invariant set $\mathcal{M}$ contained in $\mathcal{R}$, $u_c(t) = y(t) \equiv 0$, and hence, by (101), $u(t) \equiv 0$ so that $\dot{x}(t) = f(x(t))$. Now, since $\mathcal{G}$ is strongly zero-state observable it follows that $\mathcal{M} = \{(0,0)\}$ is the largest strongly positively invariant set contained in $\mathcal{R}$. Hence, it follows from Theorem 4.5 that $\operatorname{dist}(\psi(t),\mathcal{M}) \to 0$ as $t \to \infty$ for all Filippov solutions $\psi(\cdot)$ of (104). Now, global strong asymptotic stability of the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ follows from the fact that $V_s(\cdot)$ and $V_{sc}(\cdot)$ are, by assumption, radially unbounded.

(iii) Finally, if $\mathcal{G}$ and $\mathcal{G}_c$ are strongly exponentially dissipative it follows that there exist $u', y',\ u_c'$ and y_c', and scalars $\varepsilon > 0$ and $\varepsilon_c > 0$ such that

$$\begin{aligned}\max \mathcal{L}_{\tilde{f}} V(x,x_c) &\le \max \mathcal{L}_{f_1} V_s(x) + \sigma \max \mathcal{L}_{f_2} V_{sc}(x_c)\\ &\le -\varepsilon V_s(x) - \sigma\varepsilon_c V_{sc}(x_c) + s(u',y') + \sigma s_c(u_c',y_c')\\ &\le -\min\{\varepsilon,\varepsilon_c\} V(x,x_c), \quad (x,x_c) \in \mathbb{R}^n \times \mathbb{R}^{n_c}.\end{aligned}$$

Hence, it follows from Theorem 4.4 that the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is globally strongly exponentially stable. □

The next result presents Lyapunov, asymptotic, and exponential stability of dissipative discontinuous feedback systems with supply rate maps consisting of quadratic supply rates.

Theorem 5.2. *Let $Q \in \mathbb{S}^l, S \in \mathbb{R}^{l\times m},\ R \in \mathbb{S}^m, Q_c \in \mathbb{S}^m, S_c \in \mathbb{R}^{m\times l}$, and $S_c \in \mathbb{S}^l$. Consider the closed-loop system consisting of the nonlinear discontinuous dynamical systems $\mathcal{G}$ given by (98) and (99) and $\mathcal{G}_c$ given by (100) and (101), and assume $\mathcal{G}$ and $\mathcal{G}_c$ are strongly zero-state observable. Furthermore, assume $\mathcal{G}$ is strongly dissipative with respect to the supply rate map $\{s(u,y)\} = \{y^{\mathrm{T}}Qy + 2y^{\mathrm{T}}Su + u^{\mathrm{T}}Ru\}$ and has a locally Lipschitz continuous, regular, and radially unbounded storage function $V_s(\cdot)$, and $\mathcal{G}_c$ is strongly dissipative with respect to the supply rate map $\{s_c(u_c,y_c)\} = \{y_c^{\mathrm{T}}Q_c y_c + 2y_c^{\mathrm{T}}S_c u_c + u_c^{\mathrm{T}}R_c u_c\}$ and has a locally Lipschitz continuous, regular, and radially unbounded storage function $V_{sc}(\cdot)$. Finally, assume there exists $\sigma > 0$ such that*

$$\hat{Q} \triangleq \begin{bmatrix} Q + \sigma R_c & -S + \sigma S_c^{\mathrm{T}} \\ -S^{\mathrm{T}} + \sigma S_c & R + \sigma Q_c \end{bmatrix} \le 0. \tag{105}$$

Then the following statements hold:

(i) *The negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is strongly Lyapunov stable.*
(ii) *If $\mathcal{G}_c$ is strongly exponentially dissipative with respect to supply rate map $\{s_c(u_c,y_c)\}$ and $\operatorname{rank}[G_c(u_c,0)] = m, u_c \in \mathbb{R}^l$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is globally strongly asymptotically stable.*
(iii) *If $\mathcal{G}$ and $\mathcal{G}_c$ are strongly exponentially dissipative with respect to supply rate maps $\{s(u,y)\}$ and $\{s_c(u_c,y_c)\}$ and there exist constants α, β, α_c, and $\beta_c > 0$ such that (102) and (103) hold, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is globally strongly exponentially stable.*
(iv) *If $\hat{Q} < 0$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is globally strongly asymptotically stable.*

Proof. Statements (i)–(iii) are a direct consequence of Theorem 5.1 by noting that

$$s(u, y) + \sigma s_c(u_c, y_c) = \begin{bmatrix} y \\ y_c \end{bmatrix}^T \hat{Q} \begin{bmatrix} y \\ y_c \end{bmatrix},$$

and hence, $s(u, y) + \sigma s_c(u_c, y_c) \le 0$.

To show (iv) consider the Lyapunov function candidate $V(x, x_c) = V_s(x) + \sigma V_{sc}(x_c)$. Now, since $\mathcal{G}$ and $\mathcal{G}_c$ are strongly dissipative it follows that there exist u', y', u'_c and y'_c with $u'_c = y'$ and $y'_c = -u'$ such that

$$\begin{aligned}\frac{d}{dt}V(x, x_c) &\le \max \mathcal{L}_{\tilde{f}} V(x, x_c) \\ &\le \max \mathcal{L}_{f_1} V_s(x) + \sigma \max \mathcal{L}_{f_2} V_{sc}(x_c) \\ &\le s(u, y) + \sigma s_c(u_c, y_c) \\ &= y^T Q y + 2y^T S u + u^T R u + \sigma(y_c^T Q_c y_c + 2y_c^T S_c u_c + u_c^T R_c u_c) \\ &= \begin{bmatrix} y \\ y_c \end{bmatrix}^T \hat{Q} \begin{bmatrix} y \\ y_c \end{bmatrix} \\ &\le 0, \quad \text{a.e. } (x, x_c) \in \mathbb{R}^n \times \mathbb{R}^{n_c},\end{aligned}$$

which implies that the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is strongly Lyapunov stable. Next, let $\mathcal{R} \triangleq \{(x, x_c) \in \mathbb{R}^n \times \mathbb{R}^{n_c} : \frac{d}{dt}V(x, x_c) = 0 \in \mathcal{L}_{\tilde{f}} V(x, x_c)\}$ and note that $\frac{d}{dt}V(x, x_c) = 0$ if and only if $(y, y_c) = (0, 0)$. Now, since $\mathcal{G}$ and $\mathcal{G}_c$ are strongly zero-state observable it follows that $\mathcal{M} = \{(0, 0)\}$ is the largest strongly positively invariant set contained in $\mathcal{R}$. Hence, it follows from Theorem 4.5 that $\text{dist}(\psi(t), \mathcal{M}) \to 0$ as $t \to \infty$ for all Filippov solutions $\psi(\cdot)$ of (104). Finally, global strong asymptotic stability follows from the fact that $V_s(\cdot)$ and $V_{sc}(\cdot)$ are, by assumption, radially unbounded, and hence, $V(x, x_c) \to \infty$ as $\|(x, x_c)\| \to \infty$. □

The following corollary is a direct consequence of Theorem 5.2. Note that if a nonlinear discontinuous dynamical system $\mathcal{G}$ is strongly dissipative with respect to a supply rate map $\{s(u, y)\} = \{u^T y - \varepsilon u^T u - \hat{\varepsilon} y^T y\}$, where $\varepsilon, \hat{\varepsilon} \ge 0$, then with $\kappa(y) = ky$, where $k \in \mathbb{R}$ is such that $k(1 - \varepsilon k) < \hat{\varepsilon}$, $s(u, y) = [k(1 - \varepsilon k) - \hat{\varepsilon}]y^T y < 0$, $y \ne 0$. Hence, if $\mathcal{G}$ is strongly zero-state observable it follows from Theorem 3.2 that all storage functions of $\mathcal{G}$ are positive definite.

Corollary 5.1. *Consider the closed-loop system consisting of the nonlinear discontinuous dynamical systems $\mathcal{G}$ given by (98) and (99) and $\mathcal{G}_c$ given by (100) and (101), and assume $\mathcal{G}$ and $\mathcal{G}_c$ are strongly zero-state observable. Then the following statements hold:*

(i) *If $\mathcal{G}$ is strongly passive, $\mathcal{G}_c$ is strongly exponentially passive, and $\text{rank}[G_c(u_c, 0)] = m$, $u_c \in \mathbb{R}^l$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is strongly asymptotically stable.*

(ii) *If $\mathcal{G}$ and $\mathcal{G}_c$ are strongly exponentially passive with storage functions $V_s(\cdot)$ and $V_{sc}(\cdot)$, respectively, such that (102) and (103) hold, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is strongly exponentially stable.*

(iii) *If $\mathcal{G}$ is strongly nonexpansive with gain $\gamma > 0$, $\mathcal{G}_c$ is strongly exponentially nonexpansive with gain $\gamma_c > 0$, $\text{rank}[G_c(u_c, 0)] = m$, $u_c \in \mathbb{R}^l$, and $\gamma\gamma_c \le 1$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is strongly asymptotically stable.*

(iv) *If $\mathcal{G}$ and $\mathcal{G}_c$ are strongly exponentially nonexpansive with storage functions $V_s(\cdot)$ and $V_{sc}(\cdot)$, respectively, such that (102) and (103) hold, and with gains $\gamma > 0$ and $\gamma_c > 0$, respectively, such that $\gamma\gamma_c \le 1$, then the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is strongly exponentially stable.*

Proof. The proof is a direct consequence of Theorem 5.2. Specifically, (i) and (ii) follow from Theorem 5.2 with $Q = Q_c = 0$, $S = S_c = I_m$, and $R = R_c = 0$, whereas (iii) and (iv) follow from Theorem 5.2 with $Q = -I_l$, $S = 0$, $R = \gamma^2 I_m$, $Q_c = -I_{l_c}$, $S_c = 0$, and $R_c = \gamma_c^2 I_{m_c}$. □

Example 5.1. Consider the nonlinear mechanical system $\mathcal{G}$ with a discontinuous spring force given by

$$\ddot{x}(t) + \text{sign}(x(t)) = u(t), \qquad x(0) = x_0, \qquad \dot{x}(0) = \dot{x}_0, \quad \text{a.e. } t \ge 0, \tag{106}$$

$$y(t) = \frac{1}{2}\dot{x}(t), \tag{107}$$

or, equivalently,

$$\dot{x}_1(t) = x_2(t), \qquad x_1(0) = x_{10}, \quad \text{a.e. } t \ge 0, \tag{108}$$

$$\dot{x}_2(t) = -\text{sign}(x_1(t)) + u(t), \qquad x_2(0) = x_{20}, \tag{109}$$

$$y(t) = \frac{1}{2}x_2(t), \tag{110}$$

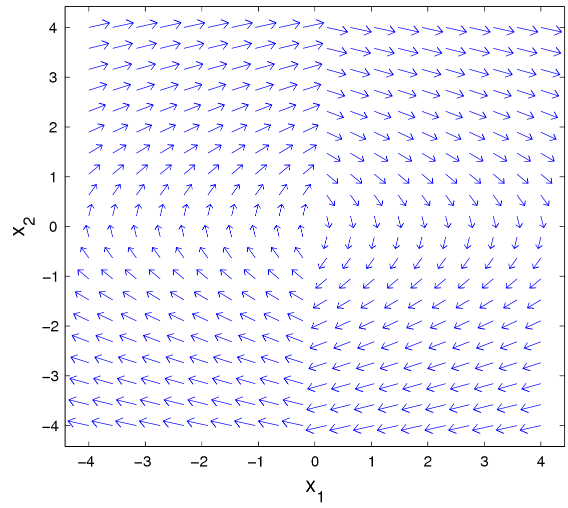

Fig. 2. Phase portrait of the nonsmooth harmonic oscillator.

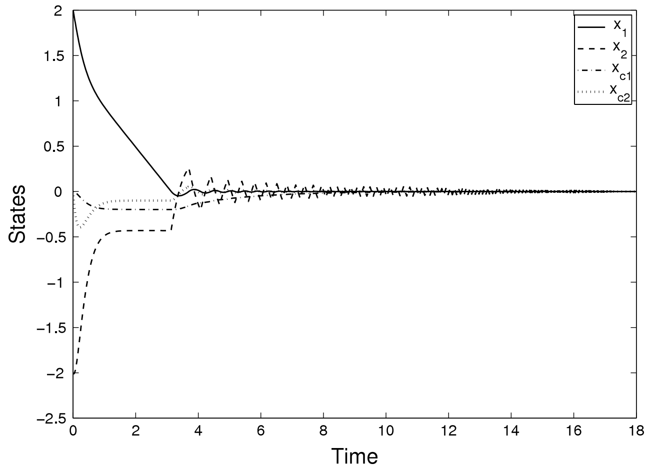

Fig. 3. State trajectories of the closed-loop system versus time for the full-order controller.

and the continuous nonlinear second-order dynamic controller $\mathcal{G}_c$ given by

$$\dot{x}_{c1}(t) = -\frac{1}{2}x_{c1}(t) - x_{c2}(t), \qquad x_{c1}(0) = x_{c10}, \quad t \geq 0, \tag{111}$$

$$\dot{x}_{c2}(t) = -10x_{c1}^3(t) - 10x_{c2}(t) + 5u_c(t), \qquad x_{c2}(0) = x_{c20}, \tag{112}$$

$$y_c(t) = 10x_{c2}(t). \tag{113}$$

Furthermore, consider the feedback interconnection of (108)–(113) given by $u = -y_c$ and $u_c = y$. Next, let $V_s(x) = |x_1| + \frac{1}{2}x_2^2$ and note that, for almost all $x \in \mathbb{R}^2$,

$$\partial V_s(x_1, x_2) = \begin{cases} \{\operatorname{sign}(x_1)\} \times \{x_2\}, & (x_1, x_2) \in \mathbb{R}^2 : x_1 \neq 0, \\ [-1, 1] \times \{x_2\}, & (x_1, x_2) \in \mathbb{R}^2 : x_1 = 0. \end{cases}$$

Hence, $\mathcal{L}_f V_s(x_1, x_2) = \{0\}$ and $\mathcal{L}_G V_s(x_1, x_2) = \{x_2\}$, which implies that $\min \mathcal{L}_f V_s(x) = \max \mathcal{L}_f V_s(x) = 0$ for almost all $x \in \mathbb{R}^2$. Now, with $Q = 0, S = 1$, and $R = 0$, (64)–(67) are satisfied. Hence, it follows from Theorem 4.3 that $\mathcal{G}$ is weakly lossless with respect to the supply rate map $\{2yu\}$.

Next, note that with $V_{sc}(x_c) = 10x_{c1}^4 + 2x_{c2}^2$, $\varepsilon \in (0, 2]$, $\ell(x_c) = \pm\sqrt{10x_{c1}^4(2-\varepsilon) + 2x_{c2}^2(20-\varepsilon)}$, and $\mathcal{W}(x_c) \equiv 0$, it follows from Corollary 4.1 that $\mathcal{G}_c$ is exponentially passive. Furthermore, $\operatorname{rank}[G_c(u_c, 0)] = 1$, $u_c \in \mathbb{R}$. Now, it follows from (ii) of Theorem 5.2 that the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is globally asymptotically stable. Fig. 2 shows the phase portrait of the open-loop ($u(t) \equiv 0$) nonsmooth harmonic oscillator, whereas Fig. 3 shows the state trajectories of the closed-loop system versus time for $x(0) = [2, -2]^{\mathrm{T}}$ and $x_c(0) = 0$.

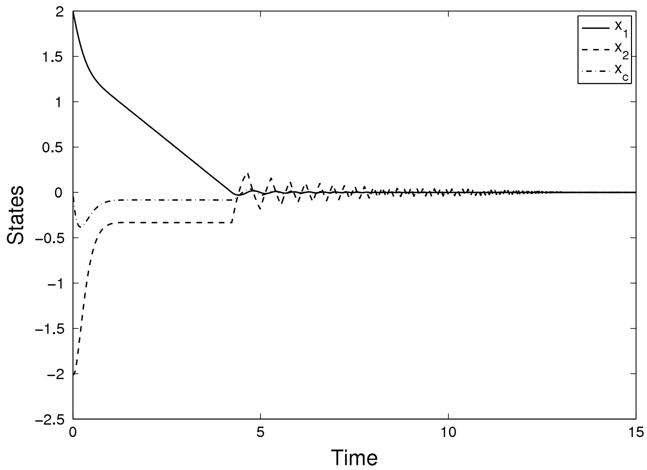

Fig. 4. State trajectories of the closed-loop system versus time for the reduced-order controller.

Alternatively we consider the reduced-order dynamic controller $\mathcal{G}_c$ given by

$$\dot{x}_c(t) = -10x_c(t) + 20u_c(t), \qquad x_c(0) = x_{c0}, \quad t \geq 0, \tag{114}$$

$$y_c(t) = 12x_c(t). \tag{115}$$

Note that with $V_{sc}(x_c) = \frac{3}{5}x_c^2$, $\varepsilon = 20$, $\ell(x_c) \equiv 0$, and $\mathcal{W}(x_c) \equiv 0$, it follows from Corollary 4.1 that $\mathcal{G}_c$ is exponentially passive. Moreover, $\operatorname{rank}[G_c(u_c, 0)] = 1$, $u_c \in \mathbb{R}$. Hence, it follows from (ii) of Theorem 5.2 that the negative feedback interconnection of $\mathcal{G}$ and $\mathcal{G}_c$ is globally asymptotically stable. Fig. 4 shows the state trajectories of the closed-loop system versus time for $x(0) = [2, -2]^T$ and $x_c(0) = 0$. □

6. Conclusion

In this paper, we extended the notion of dissipativity theory for continuous dynamical systems with continuously differentiable flows to discontinuous dynamical systems characterized by Filippov set-valued maps. Specifically, using set-valued supply rate maps and set-valued storage maps, dissipativity properties for discontinuous dynamical systems were developed. Furthermore, extended Kalman–Yakubovich–Popov conditions in terms of the discontinuous system dynamics for characterizing dissipativity via generalized Clarke gradients of locally Lipschitz continuous storage functions for discontinuous systems were developed. In addition, using the concepts of dissipativity for discontinuous dynamical systems with appropriate set-valued storage maps and set-valued supply rate maps, general stability criteria for feedback interconnections of discontinuous dynamical systems were given. Future extensions will focus on using these results to develop control design protocols for dynamical networks with switching topologies involving state-dependent communication links for addressing information link failures and communication dropouts.

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