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Analytical Derivation of 2 × 2 MIMO Channel Capacity in Terms of Multipath Angle Spread and Signal Strength

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Abstract. The capacity of multiple-input multiple-output (MIMO) wireless communication channels is affected by the multipath angle spread and relative multiple signal strength (RMSS) at both sides of the transmitter and the receiver. In this paper, we study analytically how these two factors emerge in the MIMO capacity equation when the channel state information (CSI) is unknown at the transmitter and perfectly known at the receiver. Mathematical expression for the channel capacity is carried out for 2×2 MIMO system and two propagation paths between the base station (BS) and the mobile terminal (MS) are considered. The proposed analytical model is verified through numerical results, which show that channel capacity increases with increasing angle spread. Also, as the relative strength of multipaths becomes larger, the better channel capacity is obtained.

Keywords. Channel capacity, MIMO systems, multipath angle spread, relative multipath signal strength (RMSS), channel state information (CSI).

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1 Introduction

In recent years, multiple-input multiple-output (MIMO) systems that take advantage of rich multipath environments have attracted many researchers because of considerable channel capacity increase [1, 3], In [4], it was demonstrated that the MIMO capacity increases linearly with the number of transmit antennas M_T and receive antennas M_R. It is known that the capacity of a MIMO channel not only depends on M_T and M_R but also depends on other channel characteristics such as antenna spacing, array topology, angle spread, direction of departure (DoD) and arrival (DoA), and number of multipath signals of varying strengths [5, 6]. Many studies show that the large multipath angle spread causes improved MIMO channel capacity [7–10].

In this paper, we investigate the analytical relation between MIMO channel capacity and total angle spread of multipaths at the transmitter and the receiver incorporating in relative strengths of the multipath signals for two-input two-output system. In our analytical model, the expressions are derived under some assumptions that the receiver has perfect knowledge of the channel and equal power (EP) is allocated to each of the transmit antennas. The numerical results verify and validate our analytical MIMO channel capacity. As expected, MIMO channel capacity increases with increase in angle spread between two multipaths at transmit/receive side and the larger the relative multipath strengths results in the larger channel capacity.

The paper is organized as follows. In Section 2, we describe the exact MIMO channel model and capacity expression. Then, in Section 3, analytic derivation of MIMO capacity in terms of multipath angle spread and relative multipath signal strength is given. Section 4 presents the numerical results. Finally, in Section 5, some conclusions are given.

2 MIMO Channel Model and Capacity

To derive an analytical capacity expression of MIMO channel in terms of multipath angle spread and RMSS, MIMO channel with finite scattering is considered. This is because the finite scatterer channel model is accordant with many channel models [1]. MIMO channel can be modeled in terms of a finite number L of multipath components each of (indexed by l) has complex path attenuation β_l, the direction of departure (DoD) φ_l, the direction of arrival (DoA) ψ_l, path delay τ_l, and ω_l the phase shift caused by mobile movement (i.e., Doppler effect). It is assumed that total angle spreads at the transmitter and at the receiver for these multipath components are symbolized with $\mathbf{\Phi}$, $\mathbf{\Psi}$ respectively as depicted by Figure 1.

For an M_T element antenna at the base station (BS) and an M_R element antenna at the mobile station (MS), the channel coefficients of one of the L multipath components are described by an $M_R \times M_T$ matrix $\mathbf{H}$ of complex values. Assuming negligible path delay in the channel, MIMO

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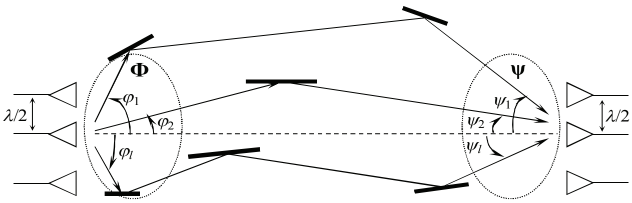

Figure 1. MIMO channel model with finite scattering.

channel model can be written as

$$\mathbf{H}(t) = \sum_{l=1}^{L} \beta_l \mathbf{a}_{R,l} \mathbf{a}_{T,l}^T e^{j\omega_l t}, \tag{1}$$

where $\beta_l = \alpha_l \exp(j\theta_l)$ is the complex attenuation caused by reflections from local scatterers. α_l is modeled as Rayleigh fading parameter, and θ_l is the complex amplitude and phase of the l-th multipath component respectively. $\mathbf{a}_{R,l}$ is $M_R \times 1$ dimensional array response vector at the receiver, and $\mathbf{a}_{T,l}$ is $M_T \times 1$ dimensional array response vector at the transmitter. For a uniform linear array (ULA), array response vector at the BS is given by

$$\mathbf{a}_{T,l} = \left[1 e^{-j\Omega_{T,l}} \cdots e^{-j(M_T-1)\Omega_{T,l}}\right]^T, \tag{2}$$

where $\Omega_{T,l} = (2\pi/\lambda) d_T \sin\varphi_l$ is the angular frequency, λ is the carrier wavelength in meters, d_T is the element spacing in the transmit antenna array, φ_l is the DoD for l-th multipath with respect to MS broadside. $\Omega_{R,l}$ is defined similarly for the receiver. The antenna element spacing at both transmitter and receiver is commonly chosen as one-half wavelengths ($\lambda/2$) so as to counteract the effect of spatial correlation and mutual coupling.

When channel state information (CSI) is unknown at the transmitter and perfectly known at the receiver, equal power (EP) is allocated to each spatial sub-channel between transmitting and receiving antennas. The resulting capacity is given by [1, 3]

$$C = \sum_{i=1}^{r} \log\left(1 + \frac{\rho}{M_T}\lambda_i\right), \tag{3}$$

where r is the number of spatial sub-channels and $\lambda_i (i = 1, 2, \ldots, r)$ is the positive eigenvalues of channel covariance matrix $\mathbf{HH}^H$, which represents the power gain, ρ denotes the signal to noise ratio (SNR).

3 Channel Capacity vs. Angle Spread & RMSS

In this study, we consider simple 2×2 MIMO system for the sake of simplicity in derivations and assume following:
Assumption 1: Two-input two-output $M_T = M_R = 2$ system is considered.
Assumption 2: A multipath channel with two multipaths ($L = 2$) is assumed.
Assumption 3: Each of propagation paths has different complex attenuations but related to each other. For instance, the complex amplitudes are $\alpha_1 = \alpha$ for the first multipath and $\alpha_2 = \xi\alpha$ for the second multipath where ξ is referred as relative multipath signal strength (RMSS).
Assumption 4: The Doppler frequencies of propagation paths are approximately the same ($\omega_1 \approx \omega_2 = \omega$).

In light of above assumptions, array response vectors at transmit and receive sides can be written as,

$$\mathbf{a}_{T,l} = \begin{bmatrix} 1 \\ e^{-j\pi \sin(\varphi + \Delta\varphi_l)} \end{bmatrix}$$
$$\mathbf{a}_{R,l} = \begin{bmatrix} 1 \\ e^{-j\pi \sin(\psi + \Delta\psi_l)} \end{bmatrix}, \tag{4}$$

where φ is the mean Direction-of-Departure (DoD) at the BS and ψ is the mean Direction-of-Arrival (DoA) at the MS, $\Delta\varphi_l$ is the angular deviation of the transmitted signal for l-th multipath, and $\Delta\psi_l$ is the angular deviation of the received signal for l-th multipath.

Defining $\gamma_l = \pi \sin(\varphi + \Delta\varphi_l) \chi_l = \pi \sin(\psi + \Delta\psi_l)$ and substituting (4) into (1), MIMO channel matrix, $\mathbf{H}$, can be expressed as

$$\mathbf{H} = \alpha e^{j\omega t} \times \begin{bmatrix} 1+\xi & e^{-j\gamma_1} + \xi e^{-j\gamma_2} \\ e^{-j\chi_1} + \xi e^{-j\chi_2} & e^{-j(\chi_1+\gamma_1)} + \xi e^{-j(\chi_2+\gamma_2)} \end{bmatrix}. \tag{5}$$

Since DoAs and DoDs of the multipaths are assumed to be uniformly distributed and symmetric around their mean angles, the following identities are used for angular deviations of the multipaths,

$$\Delta\varphi_1 = -\mathbf{\Phi}/2, \quad \Delta\varphi_2 = \mathbf{\Phi}/2 \tag{6}$$
$$\Delta\psi_1 = -\mathbf{\Psi}/2, \quad \Delta\psi_2 = \mathbf{\Psi}/2, \tag{7}$$

where $\mathbf{\Phi}$ is the angle spread of transmitted signals from the BS, and $\mathbf{\Psi}$ is the angle spread of received signals at the MS.

To simplify the analysis for the calculation of the eigenvalues of channel covariance matrix $\mathbf{R}$=$\mathbf{HH}^H$, we define some intermediate variables

$$x = \gamma_1 - \gamma_2 = -2\pi \cos\varphi \sin\mathbf{\Phi}/2 \tag{8}$$
$$y = \chi_1 - \chi_2 = -2\pi \cos\psi \sin\mathbf{\Psi}/2. \tag{9}$$

Using (5), (8), and (9), the channel covariance matrix is obtained as given by (10).

$$\mathbf{R} = |\alpha|^2 \times \begin{bmatrix} 2(1+\xi)^2 + \xi\left(e^{jx} + e^{-jx} - 2\right) & e^{j\Omega_1}\left(2+\xi+\xi e^{jx}\right) + e^{j\Omega_2}\left(\xi + 2\xi^2 + e^{-jx}\right) \\ e^{-j\Omega_1}\left(2+\xi+\xi e^{-jx}\right) + e^{-j\Omega_2}\left(\xi + 2\xi^2 + e^{jx}\right) & \left(2+2\xi^2\right) + \xi e^{-jy}\left(1+e^{-jx}\right) + \xi e^{jy}\left(1+e^{jx}\right) \end{bmatrix}. \quad (10)$$

By means of $e^{\pm ja} = \cos a \pm j \sin a$ transformation, it can be shown that the sum and product of eigenvalues (λ_1, λ_2) of $\mathbf{R}$ can be expressed in terms of x and y as follows

$$\lambda_1 + \lambda_2 = 4\left(1+\xi^2\right) + 2\xi(1 + \cos(x) + \cos(y) + \cos(x+y)) \quad (11)$$

$$\lambda_1\lambda_2 = 4\xi^2\left(1 + \cos(x)\cos(y) - 2\cos(x)\right). \quad (12)$$

Finally, substituting (11) and (12) into (3), the capacity of 2 × 2 MIMO system is given by

$$C = \log_2\left[1 + \frac{\rho}{2}(\lambda_1 + \lambda_2) + \frac{\rho^2}{4}(\lambda_1\lambda_2)\right]. \quad (13)$$

4 Numerical Results

In this section, we present numerical results of the effect of multipath angle spread and relative multipath signal strength (RMSS) on the MIMO channel capacity using our analytical derivation.

To evaluate the impact of RMSS on the capacity of MIMO channel, we assume that mean angles and angle spreads at the transmitter and receiver are the same. As clearly seen from Figure 2, as the relative strength of multipaths gets larger, the better capacity is obtained. Channel capacity also increases with increasing angle spread. The effect of angle spread is more obvious between 10 and 50 degrees range.

The plot in Figure 3 shows MIMO channel capacity results with varying angle spreads at transmit and the receive sides. The less angle spreads at the BS and MS are related, the larger the effect of angle spread on channel capacity is observed.

5 Conclusions

In this paper, the capacity of MIMO channel has been analytically derived. Two parameters that have effects on the capacity are considered. These are the multipath angle spreads at the base station (BS) and the mobile terminal (MS) and relative multipath signal strength. In our analytical method, for the sake of simplicity in derivations and to

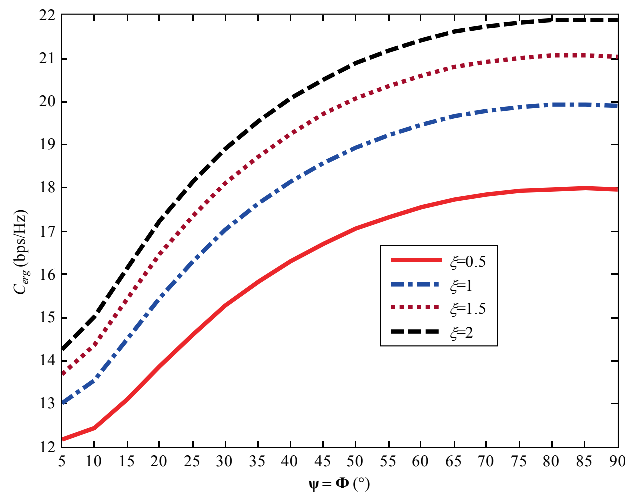

Figure 2. MIMO channel capacity variation with relative multipath signal strengths under equal angle spreads at the transmit and receive sides.

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