Article

All in Action

Arto Annila [1,2,3]

[1] Department of Physics, University of Helsinki, P.O. Box 64, FI-00014 Helsinki, Finland;
     E-Mail: arto.annila@helsinki.fi
[2] Institute of Biotechnology, University of Helsinki, P.O. Box 56, FI-00014 Helsinki, Finland
[3] Department of Biosciences, University of Helsinki, P.O. Box 56, FI-00014 Helsinki, Finland

Abstract: The principle of least action provides a holistic worldview in which Nature in its entirety and every detail is described in terms of actions. Each and every action is ultimately composed of one or multiple of the most elementary actions which relates to Planck's constant. Elements of space are closed actions, known as fermions, whereas elements of time are open actions, known as bosons. The actions span an energy landscape, the Universe, which evolves irreversibly according to the 2nd law of thermodynamics by diminishing energy density differences in least time. During evolution densely-curled actions unfold step-by-step when opening up and expelling one or multiple elementary actions to their surrounding sparser space. The energy landscape will process from one symmetry group to another until the equivalence to its dual, *i.e.*, the surrounding density has been attained. The scale-free physical portrayal of nature in terms of actions does not recognize any fundamental difference between fundamental particles and fundamental forces. Instead a plethora of particles and a diaspora of forces are perceived merely as diverse manifestations of a natural selection for various mechanisms and ways to decrease free energy in the least time.

1. Introduction

Actions integrate momenta along paths of an energy landscape [1–3]. According to the principle of least action flows of energy delineate on least-time trajectories. These geodesics can be determined when the energy landscape remains invariant under the influence of action whereas this task turns out to be intractable when the flows themselves drive the landscape in evolution. Physics, however, is able to tackle even such problematic processes with its most general concepts.

Nature can be pictured in its entirety and every detail as an energy landscape that consists of diverse densities of energy. When energy flows from a spatial density to another, the landscape will change so that continuity and conservation of energy are satisfied. Specifically, when the flows of energy circulate on closed, bound orbits, the local landscape will retain its symmetry over the conserved motional period [4]. In general, when flows spiral along open, unbound paths, the landscape will evolve irreversibly by breaking its steady-state symmetry to another in non-conserved transformations. The laws of motion can be expressed concisely as actions in classical electromagnetism [5], general relativity [6] and quantum electrodynamics [7] but it is perhaps less appreciated that also the 2nd law of thermodynamics [8] can be formulated accordingly to describe evolutionary processes.

Customarily the 2nd law is written as a differential equation, but the universal law can also be written in integral form. Then the natural law is known as the principle of least action where the integrand is kinetic energy $2K$ [1] that equals temperature T multiplied by entropy S. This identity allows us to recognize that when the system evolves along the least-action path from one state to another by consuming free energy at the maximal rate, entropy will be increasing in least time. In other words, the principle of increasing entropy given as a differential equation of motion and the principle of least action given as an integral equation of motion are equivalent imperatives [9]. The natural law for the maximal energy dispersal accounts for diverse irreversible processes that consume differences in energy density, *i.e.*, free energy [10–12]. Eventually, when all forms of free energy have been used, the system's open evolutionary path closes to the optimum orbit of a conserved stationary state. Then the energy landscape is even and can be characterized by a group of symmetry. Thus, the evolutionary equation derived from the statistical physics of open systems [13,14] offers insights as well to the motions of Hamiltonian systems where energy is conserved over the period of integration. The objective of this study is to show how some familiar forms of physics unite when nature is described in a comprehensive and self-similar manner as actions within surrounding actions.

2. The Natural Law of Maximal Energy Dispersal

The classical notion of the 2nd law of thermodynamics [8] simply says that differences in energy will level off so that the free energy minimum state will be attained. Consumption of free energy powers evolution from one state to another. Motions as flows of energy channel preferentially along the steepest descent in time ($\partial_t = \partial/\partial t$) which is equivalent to the steepest directional (*i.e.*, velocity **v**) gradient ($D = \mathbf{v} \cdot \nabla$). Along these paths the flows of energy are most voluminous and the free-energy minimum state will be attained in least time. In other words, the natural process directs itself along the resultant of forces. The statistical measure for the natural process is entropy $S = k_B \ln P$ which is the logarithmic probability and hence an additive measure [15]. Since the most voluminous flows of energy will direct along the steepest directional descents, entropy will not only be increasing but it will

be increasing in least time. This universal imperative is recognized in the maximum entropy production principle [16,17]. The maximal consumption of free energy implies also the maximum power principle [18]. Since the natural process follows the least-time path, it can be also formulated by the principle of least action.

The principle of increasing entropy as the evolutionary equation of motion is obtained from the statistical physics of open systems [13,14]:

$$d_t S = k_B\, d_t P / P = k_B L \geq 0 \,, \tag{1}$$

where the rate of entropy change is proportional by Boltzmann's factor k_B to the process generator $L = T d_t Q$ in accordance with classical thermodynamics [8]. The probability in motion $d_t P = LP$ (Equation 1) is expressed using energy densities [19] so that each energy density ϕ_j, present in indistinguishable numbers N_j of j-entities, is assigned with $\phi_j = N_j \exp(G_j/k_B T)$ where G_j is relative to the average energy $k_B T$ of the system per entity. According to the scale-free formalism [20,21] each j-entity itself is regarded as a system of diverse k-entities. Thus each population N_k is associated with $\phi_k = N_k \exp(G_k/k_B T)$. These are bound forms of energy. When energy flows from one bound form to another, energy is in a free form.

The entire system is composed of diverse systems within systems [22] (Figure 1). It is summarized by the total probability [13,14]:

$$P = \prod_{j=1} P_j = \prod_{j=1} \left(\prod_k \left(N_k e^{-\left(\Delta G_{jk} - i\Delta Q_{jk}\right)/k_B T} \right)^{g_{jk}} \Big/ g_{jk}! \right)^{N_j} \Big/ N_j! \tag{2}$$

defined in a recursive manner so that each j-entity, in indistinguishable numbers N_j, is a product ΠN_k of embedded k-entities, each distinct type available in numbers N_k. The energy difference between entities j and k is $\Delta G_{jk} = G_j - G_k$ or $g_{jk}\Delta G_{jk} = G_j - g_{jk}G_k$ when the j-entity is formed from indistinguishable (symmetrical) k-entities in degenerate numbers g_{jk}. The change in vector potential, *i.e.*, radiation ΔQ_{jk} couples to the jk-transformation orthogonal, as indicated by i, to the scalar potential difference. The notation emphasizes, for example, that when an electron falls along the electric potential gradient, it will radiate light orthogonally to its path. Likewise, electromagnetic radiation as a transverse wave will induce a varying voltage in an antenna perpendicular to the direction of propagation. The notation that explicitly distinguishes scalar and vector potentials, is of no apparent value in equilibrium thermodynamics since in the thermodynamic steady state there is no net flux of energy between the system and its surroundings. Moreover, since the statistical mechanics of closed systems aims only to reveal the stationary-state partition, there is no need to include explicitly the vector potential because at the equilibrium the net flux vanishes. However, here when describing evolutionary processes as evolving landscapes, it is useful to make the explicit distinction between bound and free forms of energy.

The probability in Equation 2 is a physical quantity, as usual in statistical mechanics. When it is multiplied by k_B:

$$\ln P = \sum_j \ln P_j \approx \sum_j N_j \left(1 - \sum_k \left(\Delta\mu_{jk} - i\Delta Q_{jk} \right) \Big/ k_B T \right) \tag{3}$$

the aforementioned additive statistical measure, entropy S for the entire system is obtained [23]. The free energy $A_{jk} = \Delta\mu_{jk} - i\Delta Q_{jk}$, also known as affinity [24], is the motive force that

directs the transforming flow d_tN_j from N_k to N_j by its scalar (chemical) potential $\Delta\mu_{jk} = \mu_j - \Sigma\mu_k = k_BT(\ln\phi_j - \Sigma\ln\phi_k)$ and vector ΔQ_{jk} potential differences. The adopted approximation $\ln N_j! \approx N_j\ln N_j - N_j$ implies that $\ln P_j$ is a sufficient statistic [25] for k_BT when the *j*-system is allowed absorb or emit quanta without a marked change in its average energy content, *i.e.*, $A_{jk}/k_BT \ll 1$. In its diverse populations N_j and in the free energy terms A_{jk} the system holds capacity $C = TdS/dT$ to resist the changes in its average energy imposed by its surroundings at a different temperature.

Figure 1. A self-similar energy level diagram describes the nested hierarchy of nature where each *j*-system (a composite solid in blue color) is regarded as a system within systems that are all being ultimately composed of multiple elementary constituents (blue solids). All systems evolve via step-by-step *jk*-transformations toward more probable states by consuming mutual energy density differences contained in chemical potentials $\Delta\mu_{jk}$ (horizontal arrows) and those in vector potentials ΔQ_{jk} (vertical wavy arrows) relative to the surroundings. During this natural process, the entities will distribute on the energy levels so that the free energy is consumed in least time. Therefore at any given time the probability distribution P_j closely outlines the maximum entropy partition of a statistical system. Exchange of entities at the same level (bow arrows) does not introduce a change in energy. Thus these reversible processes do not affect the average energy k_BT.
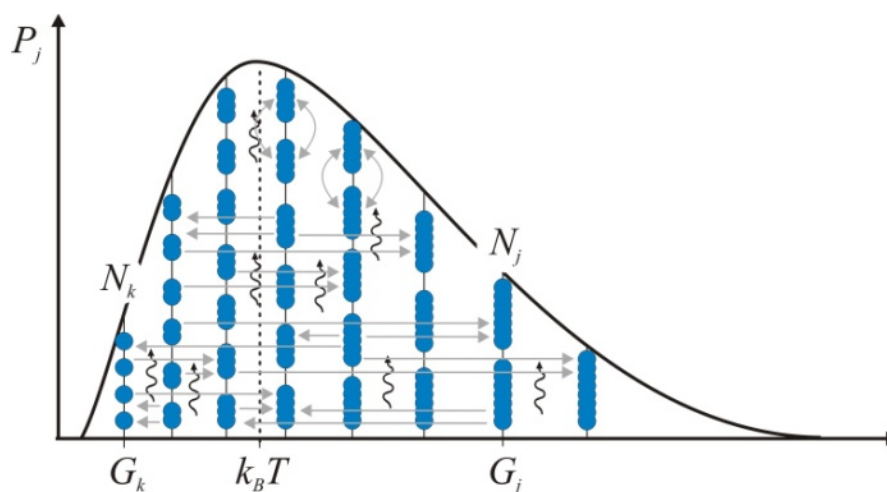
When a system is small, it does not have much capacity to resist changes that are imposed by the energy difference relative to its surroundings. Therefore such a system will undergo a dramatic change of state, *i.e.*, it will evolve abruptly. During these critical events [26], e.g., when a new *j*-species emerges or an old one goes extinct, P_j will change at once, e.g., from fifty-fifty indeterminism to full certainty. Moreover, when interactions are insufficient to establish common k_BT over a given period τ of time, the entities fail to form a system but remain as constituents that surround statistical systems at a lower level of hierarchy where interactions are more frequent and intense [27–29].

According to the natural law, given as an equation of evolution (Equation 1), a system which is higher in energy density than its surroundings, will evolve from its current state to a more probable one by displacing quanta to the sparse surroundings. Conversely, a system having lower energy density than its respective surroundings will evolve by acquiring quanta from the dense surroundings. Any two states are distinguishable from each other only when the *jk*-transforming flow is dissipative $\Delta Q_{jk} \neq 0$ [14,30].

In view of that a net non-dissipative system is stationary. The maximum entropy state is stable so that an internal perturbation δN_j away from the steady-state population N_j^{ss} will induce returning forces and opposing flows, *i.e.*, $dS(\delta N_j) < 0$ and $d_t S(\delta N_j) > 0$ [31]. In contrast any change in surroundings will compel the system to move toward a new steady state. In other words the equilibrium will shift to counteract the imposed change so that a new equilibrium will be established [32].

The evolutionary equation of motion (Equation 1) is obtained from Equation 3 by differentiating $(\partial P_j / \partial N_j)(dN_j / dt)$ [13,14]:

$$d_t \ln P = L = -\sum_{j,k} d_t N_j \frac{A_{jk}}{k_B T} \geq 0 \tag{4}$$

when the time step Δt is denoted as continuous dt. The notion of continuous motion is in accordance with entropy being a sufficient statistic for $k_B T$. It means that the steps of evolution are small in energy compared to the total energy of the system. However, the actual *jk*-transformations do advance in quantized steps of ΔQ_{jk} during Δt since each state associates with a group of symmetry [4] and a breaking one symmetry to another is a discontinuous event.

3. The Evolving Energy Landscape

The equation of motion for the evolving energy landscape is attained by multiplying Equation 4 with $k_B T$ to give continuity for the flows of energy [14,30]:

$$k_B T d_t \ln P = T d_t S = -\sum_{j,k} d_t N_j \Delta \mu_{jk} + i \sum_{j,k} d_t N_j \Delta Q_{jk} \tag{5}$$

which result from changes in the scalar and vector potentials. Using the definitions $\mu_j = \partial_{N_j} U_{jk}$ and $d_t N_j = v_j \partial_x N_j$ and $v_j = d_t x_j$ the three-term formula is transcribed to a convenient continuum approximation:

$$\sum_{j,k} d_t 2K_{jk} = -\sum_{j,k} v_j \partial_{x_j} U_{jk} + i \sum_{j,k} \partial_t Q_{jk} \tag{6}$$

where the changes in kinetic energy $2K$ and scalar U and vector Q potentials are given in the Cartesian base of space $j,k = \{x,y,z\}$ and time t. The flow equation simply says that when the irrotational potential energy $U_{jk} = -x_j m_{jk} a_k$ is consumed during dt, the power $\partial_t Q_{jk} = v_j \partial_t m_{jk} v_k = v_j \partial_t E_{jk} v_k / c^2$ is dissipated and the balance is maintained by the change in the kinetic energy $2K_{jk} = v_j m_{jk} v_k$ (Figure 2). This conversion of energy from one form to another has been conjectured already a long time ago [33,34]. Of course the change in mass (dm) is often very small, but conceptually important because it accompanies a change of state. For example, in a chemical reaction the change of mass per molecule is only a very small fraction of the mass of electron, nevertheless to recognize dm is essential because it signifies the transition from one state to another.

When the system is in a steady state, the divergence-free part of the force, *i.e.*, the net dissipation vanishes. Then the energy content of the stationary landscape spanned by the *j*- and *k*-entities is denoted by the invariant inertia $x_j m_{jk} x_k$ and by the invariant mass m_{jk}. According to $m = E/c^2$ the mass defines the *jk*-system's energy content in terms of a radiation equivalent which can be dissipated, *i.e.*, absorbed into the surrounding energy density, the free space known as the vacuum. It makes sense to give the energy content in terms of photon equivalents because heat, *i.e.*, electromagnetic radiation

is the lowest form of energy. The systemic energy is related to its surrounding radiation by the index $n_{jk} = c^2/v_j v_k$ (isotropic $v_j = v_k$). For example, energy in radiation, when it is spatially confined to a standing wave, can be given in terms of a mass equivalent. Thus the conservation of energy is respected in the transformation between the bound and free forms of energy.

> **Figure 2.** A step of evolution is a transformation from a closed action coordinated at x_k to another bound state at x_j so that the change in kinetic energy $d_t 2K$ balances the changes in the scalar $v_x \partial_x U$ and vector potentials $\partial_t Q$. The dissipation to the surroundings results from the concomitant change in mass $\Delta m = \Delta E/c^2$. In the stationary state, net fluxes vanish so that the closed least-action trajectory $d_t 2K = 0$ can be integrated over the motional period τ to yield the steady-state balance $2K + U = 0$.
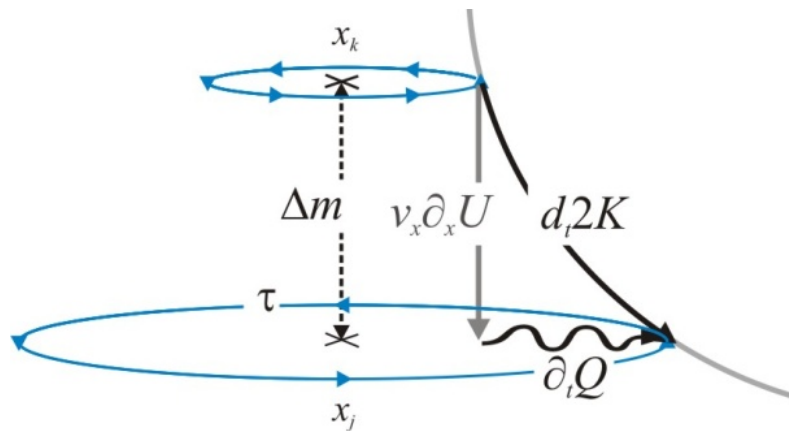
The notion of an energy landscape in motion becomes more vivid when the dissipative flow from the density at x_k toward the density at x_j during t is described to channel along an arc s_{jk}. The affine connection along a continuous curve $x = x(t)$ between the two spatial density loci spans a length $s = \int (\mathbf{F} \cdot \mathbf{v})^{1/2} dt = \int (d_t 2K)^{1/2} dt$ [35]. When the landscape is stationary, the curve is integrable. In contrast when the landscape is evolving, the flows of energy are non-integrable because their paths keep changing as the landscape is changing due to the flows themselves. At branching points where paths diverge, differentials $đ_t$ are inexact. A small flow will not perturb much a statistical system and its trajectories can be estimated rather well. However, even a small flow of energy will be sufficient to move a microscopic system substantially. Eventually the step of change from an initial state may be so dramatic that the system at a final state is beyond recognition.

The concise notation for the many motions of the differentiable landscape (Equation 6) is developed further by denoting the spatial ∂_x and temporal ∂_t gradients as a 4-vector [36,37]:

$$\partial_\mu = \left(-\partial_t / c, \nabla \right) = \left(-\partial_t / c, \partial_x, \partial_y, \partial_z \right). \tag{7}$$

when ∂_μ acts on the scalar U and vector $\mathbf{Q}$ potentials which, in turn, are given as the free energy 4-vector potential in the one-form space-time basis:

$$A_\mu = \left(-U, \mathbf{Q} \right) = \left(-U, Q_x, Q_y, Q_z \right), \tag{8}$$

the curvature in the two-form $\mathbf{F} = d\mathbf{A}$ is obtained. It is represented by the covariant antisymmetric rank 2 tensor:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & -F_x & -F_y & -F_z \\ F_x & 0 & R_z & -R_y \\ F_y & -R_z & 0 & R_x \\ F_z & R_y & -R_x & 0 \end{pmatrix} \tag{9}$$

where the translational and rotational changes in momentum are $d_t\mathbf{p} = \mathbf{F} = -\nabla U + \partial_t \mathbf{Q}/c$ and $\mathbf{R} = \nabla \times \mathbf{Q}$. When the components of scalar $\phi = U/q$ and vector $\mathbf{A} = -\mathbf{Q}/qc$ potentials are divided by charge q, $F_{\mu\nu}$ is the electromagnetic force. The change $d_t\mathbf{p}$, *i.e.*, $\mathbf{F}$ relates to the change in the angular momentum $d_t\mathbf{L}$, *i.e.*, torque $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ normalized by radius of curvature $\mathbf{r}$ (Figure 2). The continuity is preserved by the invariants $F_{\mu\nu}F^{\mu\nu} = 2(\mathbf{F}^2 - \mathbf{R}^2)$ and $*F_{\mu\nu}F^{\mu\nu} = 4\mathbf{F} \cdot \mathbf{R}$ and $*F_{\mu\nu}*F^{\mu\nu} = 2(\mathbf{F}^2 + \mathbf{R}^2)$.

Specifically, at the stationary state the exterior derivate of the system's curvature yields the conserved currents by $d*\mathbf{F} = \mathbf{J}$ so that the conserved energy density, *i.e.*, the invariant mass, orbits with phase velocity $d_t\varphi = \omega$ exactly once in a period $\tau = \omega^{-1}$ on the closed least-action path according to $\mathbf{p} \times \boldsymbol{\omega} = -\nabla U$. In the thermodynamic steady state there is no net emission from the system or net absorption to the surroundings over the motional period. Then the exterior algebra of the system's dual $d\mathbf{F} = 0$ says that the surrounding is a flat landscape with respect to the system and thus exerts no forces. In other words, the system's average energy density $k_B T$ exactly matches that of the surroundings. In the stationary state there is no net force and light propagates straight.

The surrounding vacuum cannot evolve a step further down because the electromagnetic radiation with its symmetry group U(1), the most elementary one, cannot be broken down any further. Since this form of action is already an open path, it cannot be opened any more to release photons. This is to say that the photons have no mass. Photons are the massless gauge bosons of quantum electrodynamics that communicate exchanges of energy in interactions that conserve the stationary state. These reversible flows are the familiar conserved currents along tractable trajectories that are straight geodesics of even landscapes [4]. Since the closed actions have fixed energies, they comply with the ordering relations. Therefore the steady states are countable, and any-one closed ring as a subset of affine space is an algebraic variety [38,39].

In general, the evolving energy landscape, represented by the curvature 2-form $\mathbf{F} = d\mathbf{A}$ of some principal high-symmetry bundle, is leveling off via symmetry-breaking transformations from one state to another more probable one. Bound forms of energy, *i.e.*, fermions open up to output free forms of energy, *i.e.*, bosons that dissipate to the surroundings. Since the system and its surroundings share a common interface for the flows of energy, the entropy of both the system and its surroundings will increase when mutual energy density differences decrease. The conservation of energy including both the system and its dual is respected by the differential geometry. The star operator transforms an oriented inner product density contained in an element of space, e.g., given in the Cartesian base, $*(dx \wedge dy \wedge dz) = dt$ to radiated energy density that is carried away by an element of time.

It is worth noting that continuous transformations of the Lorentz group SO(3,1) with metric signature $(+,+,+,-)$ [40] are conserved, *i.e.*, not evolutionary processes. When no symmetry is broken, no quantum is dissipated and no change of state has happened. The sesquilinear inner product defines the stationary-state unitary space with the Euclidean norm. The space is invariant under multiplication

by the complex numbers of a unit norm. In these special cases, the equation of motion describes merely a phase precession at constant energy. Since energy is constant, there is a norm and the equation for the phase precession can be solved by a unitary transformation that will remove explicit time dependence from all variables. In contrast the equation of evolution cannot be solved when a natural process has two or more alternatives paths from one state to alternative states. The difficulty in computing a non-computable [41,42] is familiar from the three-body problem [43]. Admittedly the non-Abelian gauge theory [44] accounts for directional transitions but when the integrand of action is forced to remain invariant, no net evolution will take place. Then the directional transformations formalize merely the to-and-fro flows of energy. Therefore no continuous group of transformations will account for the natural processes since in these irreversible processes from one state to another, symmetry is broken.

During evolution the landscape's curvature due to the differences in energy levels will decrease when the spatial potentials $\mu_j = \partial_{N_j} U_{jk}$ and $\mu_k = \partial_{N_k} U_{jk}$ adjust to accommodate or discard the vector potential that couples the system to its surroundings via the *jk*-transformations. The diminishing curvature of a differentiable landscape, *i.e.*, the force can be represented by a vector field gradient. The non-vanishing Lie's derivative [45] means that the change $\mathbf{v} = d_t\mathbf{x}$ in the coordinate and the change $\mathbf{F} = d_t\mathbf{p}$ in the momentum, are not collinear due to the net energy flux $\partial_t Q$ over dt to the system from the surroundings or *vice versa*. Therefore operators in $[\hat{p},\hat{x}] = -i\hbar$ do not commute by the minimum amount of action in a change of state. This uncertainty of at least one quantum in determining the state of the system is inherent in any measurement because the detection requires that at least one quantum is either emitted from the system to the detector or *vice versa* but that very quantum will change the state of the system. Consequently the systems that are undergoing changes of state contain open actions. These evolving systems cannot be ranked with respect to each other because their energy is changing.

The evolving energy landscape is represented by:

$$d_t 2K_{\mu\nu} = F^{\mu\nu} v_{\mu\nu} = \begin{pmatrix} 0 & -v_x \partial_x U & -v_y \partial_y U & -v_z \partial_z U \\ \partial_t Q_x & 0 & v_y R_z & -v_z R_y \\ \partial_t Q_y & -v_x R_z & 0 & v_z R_x \\ \partial_t Q_z & v_x R_y & -v_y R_x & 0 \end{pmatrix} \tag{10}$$

where the 4-vector velocity $v_\mu = (-c, v_x, v_y, v_z)$. The flow tensor contracts to the 0-form $d_t 2K = \Sigma d_t 2K_{\mu\nu} = -\mathbf{v} \cdot \nabla U + \partial_t Q + \mathbf{v} \times \mathbf{R}$ where the change in the kinetic energy balances the changes in the scalar potential due to matter flows as well as the changes in the vector potential due to radiation fluxes. When the system communicates with its surroundings exclusively via radiation, Equation 10 is familiar from the theory of electromagnetism [46] where the radiation $\mathbf{Q}$ at the speed of light c dissipates orthogonal to the source moving down along $-\nabla U$ at velocity $\mathbf{v}$ (Figure 2) [14]. Conversely, when the system is stationary $\partial_t \mathbf{Q} + \mathbf{v} \times \mathbf{R} = 0$, its stable orbits are governed by $\partial_t 2K + \mathbf{v} \cdot \nabla U = 0$ which is integrable to the steady-state balance $2K + U = 0$ or differentiable to the equation for standing waves.

The energy flow from the density ϕ_j, which defines its spatial locus x_j, to ϕ_k, which in turn defines x_k, is identified as the flow of time [14]. Thus the notion of time presupposes the notion of space [47]. The motion down along the spatial gradient is irreversible when emitted quanta escape forever and

reversible when quanta are reabsorbed. Emission will change the coordinate of the source relative to the sink, and absorption will change the coordinate of sink relative to the source (or *vice versa*). Conversely, when the system is stationary, so are its surroundings. It is familiar from unitary transformations that the steady phase velocity $d_t\varphi = \omega$ does not suffice to distinguish the systemic motions from the surrounding motions.

4. The Preon Action

Evolution from one stationary-state symmetry to another implies that actions are quantized because the stationary states' conserved currents are on closed orbits, and all bound curves are modular. The divisible circular group means a periodic orbit, and a rational winding number is equivalent to a mode-locked motion. Physically speaking, the state is stable when the corresponding action is modular, *i.e.*, quantized over the closed least-action path. The curve with zeros (nodes) is familiar from the Bohr's model of an atom where angular momentum $L = px = 2K\tau = n\hbar$ equals the elementary action $\hbar$ in n multiples. The kinetic energy $2K$ within the period τ is distributed on the closed orbit in modules enumerated by the principal quantum number n. Thus a step of evolution from one stationary state to another is mathematically speaking a step in the modulus of the cyclic group. When the sum of points on the ring changes, also its divisor will change. This is explicit in Equation 5 and implicit in Equation 6 where the dissipation-driven evolutionary step of N_j will in fact cause the action to step in momentum p and in length $x = vt$ from one closed orbit to another. The constant of action $\hbar$, as the absolutely least action, can be considered as a physical entity (Figure 3). The most elementary fermion is the energy density of the bound geodesic given by the geometric product $\mathbf{L} = \mathbf{px}$ [48] which has a specific handedness, usually referred to as spin $\pm\frac{1}{2}$.

Figure 3. (A) The basic element of space is the most elementary fermion, the neutrino. The confined circulation of energy exists in two chiral forms ν and ν^* corresponding to the opposite senses of circulation. The vertical bars denote the respective scalar potentials $-U = 2K$ and $U = -2K$. (B) The basic element of time is the most elementary boson, the photon. The open flow of energy exists in two forms γ and γ^* of opposite polarizations that correspond to the opposite (color-coded) phases iQ and $-iQ$ (the figures were drawn with Mathematica 7 that was appended with CurvesGraphics6 written by G. Gorni.)
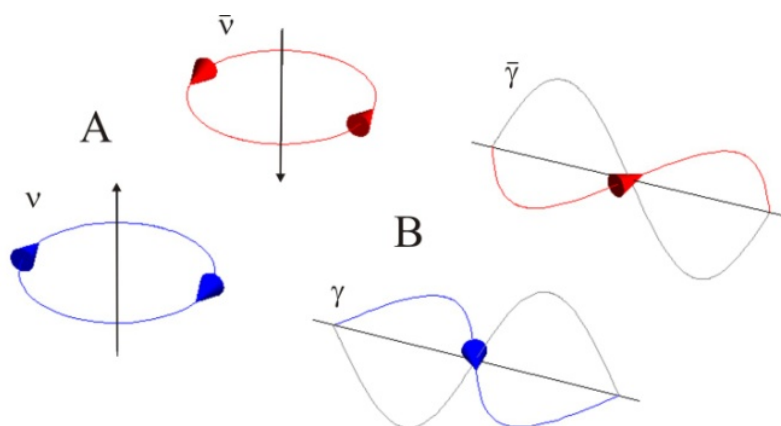
This oriented element of space is equal to the absolutely least angular momentum **L** associated with the kinetic energy within the orbital period $2K\tau = \hbar$. Thus $\hbar$ can be regarded as the fundamental element of space. The most elementary boson carries the energy along the open directed geodesic **p**v*t* which has a specific handedness, usually referred to as polarization ±1. This oriented element of time is equal to the absolutely least action $2Kt = h$ that contains the energy carried within the wave's period. This open action that equals Planck's constant can be regarded as the fundamental element of time. It takes two of these actions to reverse the polarization from +1 to −1. The first will interfere destructively (head-on) with the original handedness and the other will create the mirror hand. In the following we will refer to the fundamental oriented element of time as the photon γ and its opposite sense of polarization as the antiphoton γ^{*}. Thus the photon is considered as its own antiparticle. Accordingly, we will refer to the fundamental oriented element of space as the neutrino ν and its opposite sense of circulation as the antineutrino ν^{*}. According to the physical portrayal of nature, the constant of action is the most elementary action which is abbreviated here as the preon [49].

5. Multiple Actions

The principle of least action regards all entities of space and time, *i.e.*, diverse fermions and bosons [50] as being ultimately composed of the preon actions. Since the action is a directed path, each fermion and each boson is distinguishable from its own antiparticle which is the reversed action. An electron e^{-} is described as a least action path where preons coil to a closed torus having the electron neutrino ν_{e} chirality (Figure 4). Electron's steady-state characteristics are obtained from $d_{t}L = 0$. This resolves to a constant $2K = \int \rho \mathbf{v} \cdot \mathbf{E} dt = \int \rho \mathbf{E} \cdot d\mathbf{x} = e^{2}/4\pi\varepsilon x$ where the density ρ distributes on the torus' path length x so that the conserved quantity, known as the elementary charge $e = \rho x$, sums from the current in the field **E** according to Gauss's law. The invariant fine structure identifies by integration to the normalized constant $L/\hbar = \int 2Kdt/\hbar = e^{2}(\mu/\varepsilon)^{\frac{1}{2}}/2h = \alpha$ where the squared impedance $Z^{2} = \mu/\varepsilon = (c\varepsilon)^{-2}$, in turn, characterizes the stationary-state density that satisfies the invariant condition $\partial_{t}\phi + c^{2}\nabla \cdot \mathbf{A} = 0$ [51]. Moreover, the circulation gives rise to the magnetic moment $\mathbf{\mu}_{e} = \int \mathbf{r} \times \rho \mathbf{v} dx = e\mathbf{r} \times \mathbf{p}/m_{e} = e\mathbf{L}/m_{e}$. Its anomaly $\alpha/2\pi$ [52], *i.e.*, the excess of μ_{e} over $\mu_{B} = e\hbar/m_{e}$ results from the helical pitch. The rise of coiling contributes to $\mu_{e} = eI\omega/m_{e} = ex^{2}/t$ along the torus path length x beyond the plain multiples of $\hbar$ (as if the path were without pitch) where inertia $I = m_{e}x^{2}$ and $\mathbf{L} = I\omega$ (Figure 4).

As well, the positron e^{+} is a torus but with antineutrino ν_{e}^{*} handedness. In energy-sparse surroundings e^{+} and e^{-} interfere almost completely destructively. Only the helical-pitch modulation does not cancel. Thus, the annihilation bursts out anti-parallel rays of photons γ and γ^{*} so that each is equivalent to the comparatively low mass $m_{e} = 511$ keV/c^{2} that is characteristic of the elementary charge (Figure 4). In topological terms, the low mass means that the winding number of the elementary charge about the torroid's center is low. When dissipation is normalized by the quantum of action, the characteristic trembling frequency $\omega = 2m_{e}c^{2}/\hbar$ [53], is obtained.

The W^{-} boson that mediates the weak force is regarded as an open-ended helix of ν_{e}-chirality. The helical line is of high energy and thus in energy-sparse surroundings it is not the least-action path but decays as $W^{-} \rightarrow e^{-} + \nu_{e}^{*}$. The pitch-accumulated lag-phase $\varphi = 2\pi$ is absorbed at the torus closure by the antineutrino. Since the torus itself is a loop, the elementary charge is conserved in the decay process. Likewise, the W^{+} boson is an open-ended helix of ν_{e}^{*}-chirality that processes as

$W^+ \rightarrow e^+ + \nu_e$. The neutral Z^0 boson is also an open-ended path where a γ-linker joins two helices one having ν_e and the other ν_e^*-chirality. Energy-sparse surroundings drive the decay $Z^0 \rightarrow e^+ + e^-$. The weak bosons display extraordinary high masses in comparison to their closed form fermion-antifermion counterparts, because these open paths have comparatively little topological self-screening via intrinsic phase cancellation, *i.e.*, their winding numbers are high.

Figure 4. (A) Electron e^- is a closed least-action torus which is composed of multiple preon actions having neutrino sense of chirality. The helical pitch is seen from the roll of the arrow heads. The lag-phase due to each winding totals $\varphi = 2\pi$ around the torus. (The pitch is exaggerated for clarity). (B) Conversely, positron e^+ is a closed torus of multiple preons having antineutrino sense of chirality. (C) When the surrounding energy density is high, e^- will convert by breaking its closed chiral symmetry to the open W^- boson and electron neutrino ν_e. (D) Conversely, e^+ will convert to W^+ boson and electron antineutrino ν_e^*. The actions having opposite sense of circulations will annihilate so that streams of γ and γ^* stem only from the lag-phase modulation which accrued along the opposite helical pitches.
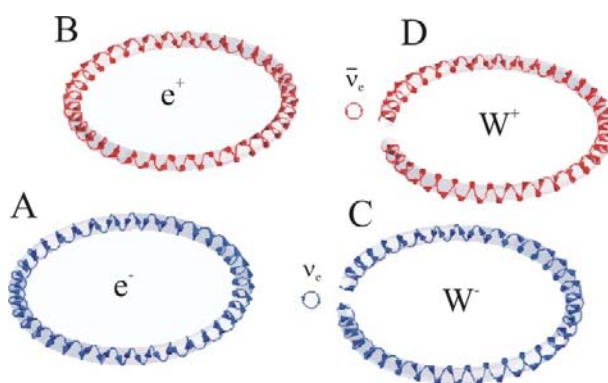

In an atomic nucleus, proton p^+ is portrayed as a least-action path where two $2/3$-helices of ν_e^*-chirality, known as up-quarks u, and one $1/3$-helix of ν_e-chirality, *i.e.*, down-quark d, join via three γ-linkers, known as gluons g. Along each u the helical path accumulates $\varphi = {}^{4\pi}/_3$ and likewise d accrues $\varphi = {}^{2\pi}/_3$ so that a gluon, as an open preon, links any two quarks at the angles that the faces of an equilateral tetrahedron make with each other (Figure 5). The front-end $\odot$ of one u links via g to the back-end $\otimes$ of the other u, and then further $\odot$-u links via g to $\otimes$-d and finally $\odot$-d links via g to $\otimes$-u to close the path. In the tripod constellation each quark as a directional element of the closed path is distinguishable from any another which is the essence of quantum chromodynamics.

Proton p^+ transforms to neutron n via electron capture where e^- breaks, when attracted to u, to W^- which makes a snug fit at u so that a partial annihilation will commence and yield d (Figure 5). Likewise, when n is free, *i.e.*, in energy-sparse surroundings, the reverse process begins when W^+ is attracted to make a snug fit at d and the partial W^+d-annihilation will yield u. The provided "wire frame models" are illustrative but perhaps puzzling since p^+ by the perimeter is longer than n, yet the mass of the neutron is slightly bigger than that of a proton. However as clarified earlier above, the mass is a measure of energy in terms of net dissipation. The topological self-screening of u in p^+ and d in n are nearly the same. The small difference is accrued from the incomplete cancellation of opposite

phases of pitch. The significant difference in proton and neutron magnetic moments, in turn, is understood to stem from the substantial differences in the oriented areas that are closed by the respective currents. The wire frames (Figure 5) allow an easy imagination of various natural process such as a pion decay π^+: $ud^* \to W^+ \to e^+ + \nu_e^*$, where the ud^*-ring opens so that d^* (oppositely wound d) will resettle via the high-mass intermediate W^+ integrally to the low-mass e^+-torus.

Figure 5. (A) Proton p^+ is a closed circulation where each up-quark u (red) is $^2/_3$ of torus of ν_e^*-chirality and the down-quark d (blue) is $^1/_3$ of torus of ν_e-chirality. The lag-phase $\varphi = {}^{4\pi}/_3$ accrued along each u and $\varphi = {}^{2\pi}/_3$ along d due to the helical pitch, define the relative angles of quarks, *i.e.*, symmetry of the closure linked by gluons g (black arrows). (B) Electron capture $p^+ + e^- + \nu_e^* \to n$ is intermediated by W^- which as an open action will initiate a partial annihilation at an exposed end of u (blow-up) that will yield d. (C) The resulting neutron n is a three-gluon-linked closed action udd.
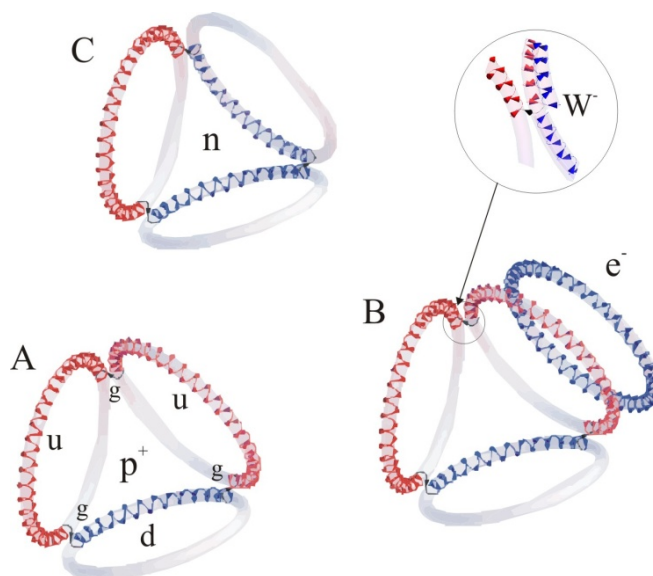
On the other hand the proton decay as a putative process $p^+ \to e^+ + \pi_0$ [54] is seen in terms of actions so that the X-boson (balanced by $e^+ + d^*$) would be attracted to p^+ to make a snug fit along u-$\otimes$-g-$\odot$-u. However, such an annihilation process is unlikely to take place in low-density surroundings (Equation 1) because it would yield disjoint d and d^* which are high-energy by-products. The process does not yield the anticipated pion π_0 which would indeed have only moderate mass since in the meson, the circulations of linked quark and antiquark pair cancel apart from the lag-phase accrued along their opposite pitches. Thus, the violation of color confinement would require extraordinarily high-energy surroundings to provide the asymptotic freedom [55] for quark-gluon plasma, or the violation of baryon number conservation would require a blazing transformation [56,57].

Heavier fermions are viewed as excited strings of the aforementioned ground-state actions. An excitation lifts the elementary closed symmetry SU(2) so that an action at a higher harmonic (bending) mode has more to dissipate and thus manifests itself as a heavier particle that tends to decay back to ground state. Altogether the collective modulations of the preon multiples are known in the Standard Model as flavors of leptons, quarks and mesons.

6. Fundamental Forces

According to the principle of least action a difference of energy of any kind is a force that will be leveled off in the least time. Consequently no interaction is fundamentally any different from any other, when everything is viewed as being ultimately composed of multiple preon actions. Energy dispersal only manifests itself differently at different densities of energy. High-density actions are highly curved paths. A strong force is required to open these closed actions, *i.e.*, to turn a fermion to a boson. In this sense for every fermion there is a corresponding boson which is the notion of supersymmetry. At high nuclear densities quarks integrate seamlessly via gluons to a closed path. Internal reflections at density boundaries, familiar from optics, presumably confined as well the very high density of the nascent Universe. Likewise, a weak force is sufficient to open lepton curvatures to weak bosons and photons.

The Coulomb interaction, in turn, manifests itself between actions whose circulations generate charges. In the thermodynamic steady state the bosons form standing waves between the oppositely charged fermions. For example, at intermediate densities a bound action known as the hydrogen atom results from exchange of γ and γ^* between p^+ and e^- (Figure 6).

Figure 6. (A) Electromagnetic interaction arises from the density difference between charged fermions and their surroundings. (B) Sparse surroundings cause attraction by accepting full wavelength modules $|\infty|$ of photons from the bound actions that pair opposite charges q^- and q^+. For clarity only the action along the shortest electromagnetic field line (dashed) is decorated with density modules. Beyond the bound pair of charges energy density propagates at the speed $c^2 = 1/\mu_0\varepsilon_0$. The energy density remains finite also long the dipole axis where the anti-phase waves do not couple to the antenna. (C) Like charges generate alike-polarized, open actions that cannot pair to form a bound action. (D) Therefore in sparse surroundings repulsion remains the mechanism to dilute the density between q^+ and q^+.
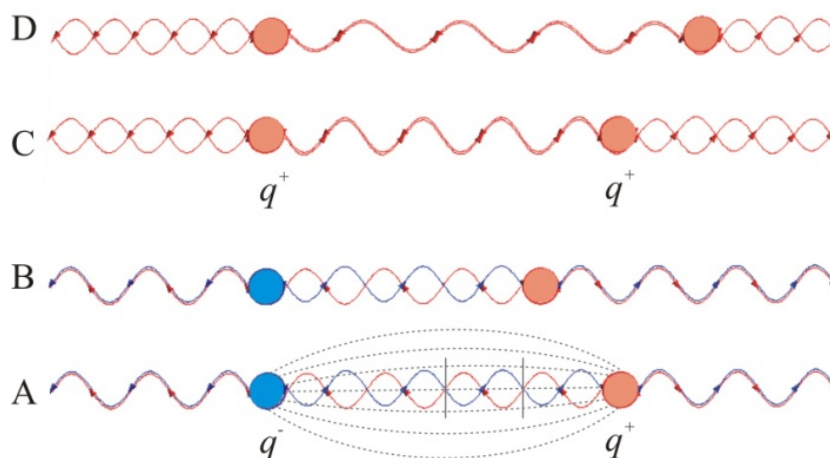

When the surrounding density increases, photons from the surroundings will integrate in the modular paths along the field lines so that these will extend up to an excited state. Conversely, when the surrounding density settles down again, the sparse surroundings will accept the discarded

full-wavelength modules of photons as the system returns to the ground state. In contrast, like charges, which generate fields of like polarization, cannot support standing waves between them to form a stable state. In the sparse surroundings the density between them will dilute by repulsion.

Moreover, it is important to realize that the density due to the four-potential (Equation 8) exists beyond net neutral bodies although the resultant electromagnetic field of anti-phase waves, *i.e.*, the force experienced by charges, vanishes. Interference effects of the non-vanishing density manifest themselves in double-slit experiments [14].

The gravitational interaction, just as other interactions, can be understood to result from density differences. Two net neutral bodies are attracted when their sparse surroundings, characterized by $\mu_o \varepsilon_o = c^{-2}$ and $\mu_o / \varepsilon_o = Z^2$, accept actions that are released in transitions from one standing-wave orbit to another (Figure 7).

Figure 7. (A) Gravitational interaction arises from the energy density difference between net neutral bodies and their surroundings. (B) Sparse surroundings attract by accepting doubly paired density modules $\|\infty\|$ of gravitons from the bound actions that pair the bodies along gravitational lines. (C) Irradiative combustion of high-density actions is a powerful mechanism to diminish the density difference between a body and its surroundings. (D) The burning star gives away density, hence the conservation of energy (Equation 6) requires for a planet to advance its perimeter from one modular orbit to another.
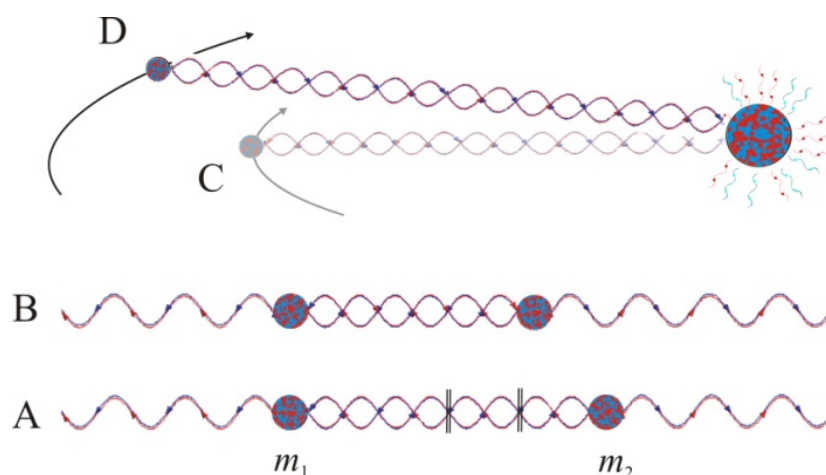
The gravitational force is the difference in density between two celestial bodies and their surrounding density just as it is the difference between the density in a microscopy cavity and its surrounding density. A cavity, tiny or enormous, cannot accommodate all the standing modes propagating in the universal surroundings [58]. The fundamental modulus of a standing gravitational wave is a pair of bosons with opposite phases. The expression of a standing wave as a sum of oppositely propagating waves subject to boundary conditions [59] is familiar from the derivation of Planck's law of radiation [60]. Since a modulus of density wave between net neutral bodies is without longitudinal vector character, a traceless symmetric rank-2 tensor (*cf.* electromagnetic wave) describes transverse modes of this particle, known as the graviton. It takes four elementary actions to reverse the polarization of graviton from +2 to −2. Two preons will interfere destructively with the pair with original handedness and the other two will create the mirror-handed pair.

The familiar form of the conserved potential U for two net neutral bodies displaced by $\mathbf{r}_{12}$ is obtained, as before, from the steady-state condition, *i.e.*, the least-action which is equivalent to the steady-state balance $2K + U = 0$ as follows:

$$d_t\mathbf{L} = d_t\mathbf{p} \times \mathbf{r} = 2K = \omega^2 m_1 r_1^2 = \omega^2 \frac{m_1 m_2 r_{12}^2}{M} = \frac{R^3}{Mt^2}\frac{m_1 m_2}{r_{12}} = \frac{1}{2\pi\rho t^2}\frac{m_1 m_2}{r_{12}} = G\frac{m_1 m_2}{r_{12}} = -U \quad (11)$$

where the angular momentum $L = I\omega$ depends on inertia $I = \Sigma m_i r_i r_i = \Sigma m_i m_j r_{ij}^2/\Sigma m_i = \Sigma m_i m_j r_{ij}^2/M$ and period $\tau^2 = 1/\omega^2 = t^2 r^3/R^3$. The inertia, when normalized by the total mass M of the Universe, can be seen as an expression of the principle $2\pi G\rho t^2 = 1$ where the mass density ρ is within the radius $R = ct$ [61], so that any given density is coupled to every other density. The numerous (decay) paths to disperse energy span the affine energy landscape, the universal buoyancy where the density flows level off any density difference in the least time. Thus the cosmological principle, *i.e.*, homogeneity at the largest scale can be seen as a consequence of a natural selection for the maximal dispersal.

According to the 2nd law of thermodynamics a system may emit quanta when its surroundings are lower in energy density. The ultimate sink is the universal surrounding free space. Today, at $t = 13.7$ billion years, the very low average mass density $\rho \approx 9.9 \times 10^{-27}$ kg/m^3 [62,63] corresponds to a tiny curvature over a titanic radius $R = ct$ [64–66] and sets the gravitational constant $G = 1/2\pi\rho t^2$ to its current value. The minute but non-negligible acceleration $a_t = GM/R^2 = 1/\varepsilon_o\mu_o R = c^2/R = c/t = cH$ drives the on-going expansion at the rate H [67,68] that is changing as $d_t H = -H^2 = -2\pi\rho G$. On the basis of ρ or equivalently of H the dilution factor $n = L/\hbar$ is on the order of 10^{120}. This factor is the well-known discrepancy between reality and the calculated vacuum energy density (as if no expansion had taken place). At any given time the total mass of the Universe in its dissipative equivalent $Mc^2 = Ma_t R = R\partial_r U = -U$ matches the scalar potential. This innate equivalence is known also as the zero-energy principle [6]. The balance is also reflected in the quantization of the electromagnetic spectrum. The cosmic microwave background radiation profiles according to the Planck's law a discrete quasi-stationary entropy partition of fermions (Figure 1). The unfolding Universe is stepping from one mode to lower and lower harmonics [69] along its least action path toward the perfect, *i.e.*, torsion-free flatness $d_t L = \tau = 0$. The acceleration, which is proportional to the winding number $\theta \propto R^{-2}$ of those folded actions, will be gradually limiting toward zero. When all densities have vanished, no differences, *i.e.*, forces will exist either.

The Universe has today evolved to span numerous levels of hierarchy but there is no fundamental distinction between the diverse forms of energy, e.g., light, quarks, atoms, molecules, beings, planets, solar systems and galaxies, since they all are ultimately composed of preons. The vast variation in energy densities is reflected in the relative strengths of interactions that extend over some 38 orders of magnitude but there is no profound distinction between strong, weak, electromagnetic and gravitational interactions. All forces as density differences between the preons in diverse fermionic varieties are communicated by the preons in various bosonic forms. At any level of nature's hierarchy, e.g., from neutrino via electron to an atom, a closed least action affords at most three dimensions of space which are usually interpreted as the degrees of freedom. One axis is orthogonal to the plane of the other two where the least action curve exists and closes to a cycle. Indeed it makes sense to see space at higher and higher magnifications as being folded to densely curved actions, but not to imagine the degrees of freedom as an abstract concept without physical correspondence. In turn, an open action

associates with the degree of freedom that span the dimension interpreted as time. When the system evolves from one spatial state to another, open actions carry the flow of energy that is experienced as the flow of time. Its direction is due to diminishing density differences. At a thermodynamic steady state in- and outgoing bosons are equally abundant. When the two populations of polarizations form a standing wave, motions are fully reversible. There is no time-dependence but then time is a bidirectional parameter with two degrees of freedom. Indeed it makes sense to see time when broken down to finer and finer steps, as a quantized energy transduction process but not to regard irreversibility and reversibility as abstract notions without physical correspondence.

7. The Mass Gap

Evolution is basically an energy dispersal process that advances from one state to another by spontaneous symmetry breaking. Although it is obvious, it is important to note that any transformation from a closed orbit to an open curve is discontinuous. In other words, the action is either closed or open but not anything in between since no change of symmetry is indivisible. Eventually, when evolution arrives at the state of next most elementary symmetry, the closed preon opens up and drains altogether as an open preon (Figure 8).

The winding number of the most elementary non-commutative ring is ±1 whereas for an open wave it is zero. Thus the closed preon associated with the SU(2) symmetry has the least mass $m_\nu > 0$, *i.e.*, the conserved quantity whereas the open preon associated with U(1) is without mass $m_\gamma = 0$. The most elementary symmetry cannot be broken any further. For that reason there is no primitive root of unity, *i.e.*, a standing-wave solution corresponding to a mass. It follows that there is a mass gap which is a finite difference in energy between the lowest bound state and the free, open state.

It is common to denote by a vacuum vector Ω the eigenstate of Hamiltonian H with zero energy $H\Omega = 0$, but this notion is abstract without physical correspondence. When there is no energy density, there is no state either. The surrounding free space of the Universe is not empty but the energy density known as the vacuum amplitude, in some theories referred to as the Higgs field, is in balance with its fermion systems (Figure 1). This balance between radiation and matter is apparent from the cosmic microwave spectrum that matches the black-body spectrum. The conservation of energy requires that at any given step during the universal energy dispersal process, the change in kinetic energy density, deemed as continuous in $d_t(\rho v^2) = -\mathbf{v}\cdot\nabla u - \varepsilon_o c^2 \nabla\cdot(\mathbf{E}\times\mathbf{B})$, balances the change in the scalar potential density u due to the average density ρ of matter with the change in the electromagnetic radiation density. The vector $\mu_o^{-1}\mathbf{E}\times\mathbf{B}$ [46] embodies energy of free space which is unique up to a phase and invariant under the Poincaré group in accordance with Maxwell equation $d\mathbf{F} = 0$ in the absence of charges and currents.

The Lie derivative is conveniently continuous in evaluating the change of one vector field $(\partial_t Q/c)$ along the flow of another vector field $(\mathbf{v}\cdot\nabla U)$. The smooth, differentiable manifold, such as the gauge group U(1)×SU(2)×SU(3) of the Standard Model, is admittedly suitable for mathematical manipulation, but it would be appropriate only for a stationary system because it fails to describe the discontinuous event of breaking symmetry.

Figure 8. Level diagram depicts an array of actions (colored) that are classified according to unitary groups of symmetry. Each stationary action equals the kinetic energy $2K$ integrated over the period τ along a directed path. It associates via Noether's theorem with a conserved quantity m. A mass gap $\Delta m \neq 0$ exists between the closed action of the most elementary fermions, the neutrinos defined by the next lowest symmetry group U(2) and the open action of the propagating photons defined by the most elementary group U(1). The radiated energy $\Delta E = \Delta mc^2$ in the transformation from U(2) down to U(1) amounts from the photons that disperse in the surrounding free space. The energy density of the free space is non-zero as is apparent from the finite speed of light. Only one of the two chiral forms of U(2) fermions can be detected via emissive transformations to U(1) whereas a detection by absorption would transform the U(2) particle to another one of a higher symmetry group.
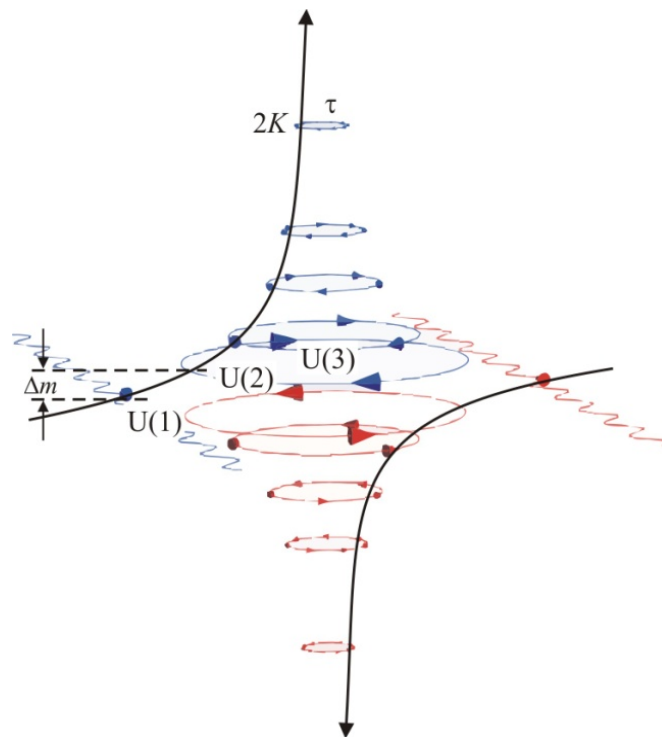
The step from one stationary-state symmetry to another takes at least one quantum of action from the system to its surroundings. When the quantized action is understood as the basic constituent of nature, no renormalization is required to escape from singularities that trouble theories which are based on the energy concept [70]. Thus, it is not productive to maintain that the Lagrangian would be invariant and that evolution would be a continuous group of transformations. Accordingly, it is inconsistent to insist on the existence of a theory which would both comply with a gauge group via renormalization, and also display a mass gap [71].

Theorem: For any-one compact simple gauge group G, there exists no quantum Yang-Mills theory on R^4 that has a mass gap.

Proof: The Yang-Mills Lagrangian:

$$L = \frac{1}{4g^2} \int Tr\, \mathbf{F} \wedge *\mathbf{F} \tag{12}$$

where the two-form **F** denotes curvature via a gauge-covariant extension of the exterior derivative from the one-form **A** of the G gauge connection and g^2 is the determinant of the metric tensor of the spacetime. It follows that G governs the symmetry of the action, which is the integral of *L*. Since the quadratic form that is constructed from **F** and its Hodge dual ∗**F** on the Lie algebra of G on R^4 is invariant, so also *L* is invariant and so is its integral action invariant. The invariant action relates via the Noether's theorem to a quantity *m* whose value is conserved. It then follows that a difference between any two values m_G and m_H of a conserved quantity cannot be compared within any-one theory based only on a single gauge group. Specifically, in a Yang–Mills theory that is based exclusively on one gauge group G describes only a single, invariant state. When the theory is based on U(1), it describes the vacuum state (the free space). When the theory is based on another Lie group, it describes another state. The difference $\Delta m = m_H - m_G \neq 0$ in the conserved quantity between U(1) and the next lowest symmetry group is referred to as the mass gap. However, the gap is not defined, *i.e.*, it does not exist within any given theory based exclusively on a single gauge group.

Corollary: For any-two distinct gauge groups G and H, there exists a theory on R^4 that has a mass gap.

Proof: Each gauge group relates the symmetry of action as the invariant integral of Lagrangian on R^4 via the Noether's theorem to a quantity *m* whose value is conserved. Conversely, the two distinct gauge groups relate two distinct symmetries of invariant actions to two different values m_G and m_H of the conserved quantity. Thus there exists a difference $\Delta m = m_H - m_G \neq 0$ in the conserved quantity, *i.e.*, the mass gap, between the vacuum state energy E_G governed by the most elementary G = U(1) symmetry group and the lowest excited state energy E_H governed by the next most elementary symmetry group H.

The resolution of the Yang-Mills existence and the mass gap, while trivial in its proof, is revealing. The focus is not on energy but on action which is quantized for each state according to the corresponding gauge symmetry (Figure 8). Therefore, no matter how small the energy *E* of an excited state will become over an increasing time *t*, the product $Et = n\hbar$ remains invariant holding at least one ($n = 1$) multiple of the quantized action. Likewise, no matter how far the range **x** of a field will be extending with diminishing momentum **p**, their product **px** remains invariant so that associated observables remain countable. In other words, a theory based on action is self-similar but does not require renormalization since it is not troubled by infinities and singularities related to energy and time as well as to momentum and length. Yet, it may appear for some to be unconvincing to claim, as above, that SU(2) which is by determinant isomorphic to U(1), does have a distinct property, *i.e.*, mass. However, a unitary group U(*n*) is non-Abelian for $n > 1$ whereas U(1) is Abelian. The sense of circulation distinguishes the closed action from a point, and also from its open form because when the chiral, closed action is mapped on the polarized open action, the two open termini are distinct from the integrally closed and connected path. To bridge a topological inequivalence requires a dissipative transformation process. The mass gap is between the photon and the neutrino. Moreover, the Euler-Lagrange integral of a bound trajectory is well-defined, *i.e.*, countable whereas Maupertuis' action as an open path is ambiguous, *i.e.*, uncountable. The ambiguity is reflected in the assignment of a value to the free energy of the vacuum. Instead the surrounding vacuum characteristics $c^2 = 1/\varepsilon_0\mu_0$ are defined via the closure according to the Stokes theorem.

8. Algebraic and Non-algebraic Varieties

In the universal energy landscape a system as a variety composed of varieties is in relation to all other varieties via flows of energy. The energetic status of a system can be quantified by probability:

$$P = \prod_j P_j = \prod_j \left(\prod_k \left(e^{-\left(\Delta U_{jk} - i\Delta Q_{jk} \right) t / n\hbar} \right)^{N_j} \bigg/ N_j! \right) \tag{13}$$

where actions are quantized $L = 2Kt = k_B Tt = n\hbar$ (see Equation 2). The stationary state ($d_t P = 0$) measure $\ln P = \Sigma N_j$ is a sum over all closed graphs, as is familiar from index theory [72] and from the linked cluster theorem for reversible, *i.e.*, non-propagating processes [73].

It is in the objective of physics to predict by way of calculation. However, as has been emphasized all along, natural processes are in general non-computable. Computation is ultimately ranking, and intuitively it is impossible to rank a quantity such as an eigenvalue that is changing during the computation. Since an evolutionary step means a change in the modulus of an action, a descending sequence eventually attains the preon ring (infima) in some finite surroundings. Conversely, an ascending sequence attains some maximum number of modules (suprema) at a congruence which corresponds to the thermodynamic steady state in some finite surroundings. The ordering ensures that any bounded space is compact. The stationary state's to-and-fro flows of energy are the affine connections that establish the ordering relations between conserved actions. These equivalence relations identify the countable subspaces, known as algebraic varieties, from which new spaces can in turn be constructed algebraically. The stationary state varieties form an orbit space. Specifically any two varieties are equivalent if and only if they are on the same orbit. Conversely, the finiteness properties ensure that when the orbit space, an *n*-dimensional oriented and closed manifold *M*, is divided in two, the system's group of *k* varieties and to the surrounding's group of *n-k* varieties are duals $H^k(M) > H^{n-k}(M)$. The Poincaré duality theorem states that for *M* the kth cohomology group of *M* is isomorphic to the $(n-k)^{\text{th}}$ homology group of *M*, for all integers *k*.

The correspondence between the system and its surroundings is stated by the Hodge conjecture so that de Rham cohomology classes are algebraic, *i.e.*, they are sums of Poincaré duals of the homology classes of subvarieties. The conjecture says in terms of physics that the stationary systems can be algebraically constructed from subsystems that are duals of the stationary surroundings' subsystems. Since at the conserved stationary state there are no open actions, all Hodge classes are generated by the Hodge classes of divisors where a divisor on an algebraic curve is a formal sum of its closed points on the basis of the Lefschetz theorem on (1,1)-classes. All closed curves with zeros (nodes) are modular in accordance with all rational elliptic curves being modular [74]. Since an algebraic variety is the set of solutions of a system of polynomial equations (elliptic curves), an algebraic number field has a norm and at least one zero. It can be considered as a geometric object of the affine manifold [75]. Since every non-constant single-variable polynomial with complex coefficients has at least one complex root, the field of complex numbers is algebraically closed. Moreover, Hilbert's Nullstellensatz relates ideals of polynomial rings to subsets of affine space.

The stationary energy landscape with its closed, oriented and unitary structure is a Kähler manifold *M* that can be decomposed to its cohomology with complex coefficients, corresponding to the scalar and vector potentials, so that $H^n(M, \mathbf{X}) = \oplus H^{p,q}(M)$ where $p + q = n$ and $H^{p,q}(M)$ is the subgroup of

cohomology classes that are represented by harmonic forms of type (p, q). It is instructive to consider the (twice) differentiable and connected M, because an n-form ω, *i.e.*, the Lagrangian, can be paired by integration, *i.e.*, via the action $\int \omega = \langle \omega, [M] \rangle$ with the homology class $[M]$, *i.e.*, the fundamental class whose the top relative homology group is infinite cyclic $H^n(M, Z) > Z$. The action $\int \omega$ over M depends only on the cohomology class of ω. The physical insight to the mathematical problem of non-countability is consistent with the fact that the Hodge conjecture holds for sufficiently general and simple Abelian varieties such as for products of elliptic curves. Moreover, the stationary least action is consistent with the combination of two theorems of Lefschetz that prove the Hodge conjecture true when the manifold has dimension at most three which is required for the stability of the free-energy minimum varieties.

Although the Hodge conjecture complies with conservation and continuity, it is ambiguous in how the duality came about. It is in essence asking an evolutionary question, which cohomology classes in $H^{k,k}(M)$ form from the complex subvarieties? Of course, it has been found mathematically that there are also non-algebraic varieties. For example, when the variety has complex multiplication by an imaginary quadratic field, then the Hodge class is not generated by products of divisor classes. These troublesome varieties correspond to the open actions. They are without norm, hence non-modular and indivisible. The n-form ω of a non-modular curve over a finite field is non-intergrable because it is without bounds. Although the natural processes terminate at the irreducible open preon, the path is open because one photon after another leaves the system. The manifold keeps contracting, mathematically without a bound, but physically the landscape ceases to exist when the last fermion opens up and transforms to boson that leaves forever. Also singularities associated with the squared operators, e.g., the square of an exterior derivate, are troublesome abstractions. At a singular point the algebraic variety is not flat which means it is non-stationary and uncountable. Thus the proof of the conjecture that Hodge cycles are rational linear combinations of algebraic cycles, hinges on excluding non-integrable classes, *i.e.*, those without divisors. This depends on the definition of manifold.

Conjecture: Let M be a projective complex manifold. Then every Hodge class on M is a linear combination with rational coefficients of the cohomology classes of complex subvarieties of M.

Proof: A divisor on an algebraic curve is a formal sum of its closed points. It follows that if M holds any Hodge class that cannot be generated by the Hodge classes of divisors, then the conjecture must be false. The definition of a projective complex manifold as a submanifold of a complex projective space determined by the zeros of a set of homogeneous polynomials, excludes any indivisible polynomial without roots [76]. Since transition functions between coordinate charts are by definition holomorphic functions, also singular points are excluded from M. The coordinate chart is a map $\phi\colon U \to V$ from an open set in M to an open set in R^n of the dimension of M is one-to-one. The homeomorphism excludes any open curve, *i.e.*, any polynomial without at least one root. Thus it follows that Hodge cycles are rational linear combinations of algebraic cycles on a projective complex manifold. Since the projective complex manifold has no class without a divisor, the conjecture is true.

9. Discussion

The value of a natural principle is in comprehending complex as well as simple phenomena in the same basic terms. The principle of least action came first into sight by Fermat, when addressing the particular least-time propagation of light and later by Maupertuis when rationalizing the general

least-time flows of any form of energy. However, Euler and especially Lagrange found mathematical reasons to narrow the general formulation to computable cases, *i.e.*, to bound and stationary systems. Ever since physics has mostly trotted along reductionist and deterministic tracks nevertheless, there is nothing in the character of physics that would exclude to pursue holistic and non-deterministic prospects.

Physics, just as any other discipline, is beguiled by ambiguity in its central concepts, most notably space and time. According to the principle of least action space at any level of natural hierarchy is embodied in closed, stationary actions that evolve by the act of time which, in turn, is embodied in quanta of open actions. Density differences diminish when dense systems are discarding quanta to their sparse surroundings and as well when sparse systems are acquiring quanta from their dense surroundings. Curiously though, when there are alternative paths for the flows, the natural processes are non-deterministic because the flows by the very fact of flowing affect the driving density differences. This non-computable character of nature may not please our desire to predict but we had better acknowledge it and get acquainted with its basis. Also we need to recognize that an observation is a causal connection via energy transduction process from an object to an observer where indeterminism in outcomes arises from the relative phases of transmitter and receiver motions [14,30,77].

The worldview provided by the least-time principle recognizes no fundamental distinction between fundamental particles and fundamental forces but regards all structures and processes as having emerged, proliferated and eventually been extinguished in leveling off density differences. According to the 2nd law of thermodynamics the flows of energy will search and naturally select [78] mechanisms inanimate and animate so that flows will level off energy differences in the least time [79–82]. In other words, evolution keeps redirecting its course along the resultant of forces. Although it is natural to search for symmetry, as it relates to a free-energy minimum state, the mathematical beauty as such cannot tell us how it came about and how it will break down. Closed currents as manifestations of symmetry and topological invariance give rise to conserved quantities, most notably to mass which is ultimately valued in open-action equivalents that are accommodated in the thinning of surrounding space—the physical vacuum [83].

The principle of least action accounts for everything by counting preons. This holistic picture of nature is not new [84–86] nonetheless the tenet may appear to some as naive, but in return it provides insight to parameters and processes also beyond those analyzed here. Plethora of particles and diaspora of fundamental forces are seen as merely manifestations of energy dispersal not as ultimate causes or consequences. The on-going universal unfolding of strings of actions accelerates with the increasing radius of curvature, just as earlier, when primordial high densities, presumably similar those in present flavors of elementary particles, began to thin out from resonant confinements closed by internal reflections. The chirality consensus in actions that settled in at baryogenesis is seen as to have been an effective means to disperse energy just as the chirality standard of natural amino acids that established itself at prebiotic genesis as an effective means to facilitate energy dispersal [87,88]. Diversity in nature is astounding but not arbitrary. We recognize rules and regularities. Scale-independent power-laws, log-normal distributions and tree-like structures that branch out in logistic manner as well as spirals that wind out in logarithmic manner are found in diverse disciplines. The universal spatial patterns as well as ubiquitous time courses of sigmoid growth, bifurcations and chaos are ascribed to

the decrease of free energy in the least time [89]. Moreover, the physical portrayal of nature by diverse actions in multiples of $\hbar$ is tangible in illustrating mathematical problems related to intractability [41,90,91], emergence, change in modularity [92–95], symmetry breaking and division in duals which all are troubled by the problem of the uncountable.

Varied writings of Pierre Louis Moreau de Maupertuis reveal that he was outwardly stimulated in applying the principle of least action to decipher puzzles and phenomena in diverse disciplines [96]. Although Maupertuis' formulation failed to meet the integrability condition that was insisted upon by his rigorous contemporaries and has been required ever since on diverse occasions, it took time to understand the underlying reason that distinguishes reversible from irreversible and tractability from intractability and that discerns space from time. Undoubtedly the revision of M[aupertuis']-theory presented here does not cohere with current classiness but calls for revival of intuition in the quest of unity.

Acknowledgments

References

1. De Maupertuis, P.L.M. Les Loix du mouvement et du repos déduites d'un principe metaphysique. *Hist. Acad. Roy. Sci. Belles Lettres Berlin* **1746**, *1*, 267–294.
2. Euler, L. *Methodus Inveniendi Lineas Curvas Maximi Minimive Proprietate Gaudentes, sive Solutio Problematis Isoperimetrici Lattissimo Sensu Accepti*; Marcum-Michaelem Bousquet & Socios: Lausanne, Switzerland, 1744; pp. 1–224.
3. Lagrange, J.L. *Analytical Mechanics*; Kluwer Academic: Dordrecht, The Netherlands, 1997; Chapter II, pp. 311–573; translated by Boissonnade, A.; Vagliente, V.N.
4. Noether, E. Invariant variation problem. *Transp. Theory Stat. Phys.* **1971**, *1*, 183–207; translated by Tavel, M.A.
5. Wheeler, J.A.; Feynman, R.P. Classical electrodynamics in terms of direct interparticle action. *Rev. Mod. Phys.* **1949**, *21*, 425–433.
6. Feynman, R.P.; Morinigo, F.B.; Wagner, W.G.; Hatfield, B. *Feynman Lectures on Gravitation*; Addison-Wesley: Reading, MA, USA, 1995; pp. 9–10.
7. Feynman, R.P. *Quantum Electrodynamics*; W.A. Benjamin: New York, NY, USA, 1961; pp. 3–33.
8. Carnot, S. *Reflections on the Motive Power of Heat and on Machines Fitted to Develop This Power*; Waverly Press, Inc: Baltimore, MD, USA, 1943; pp. 1–103; translated by Thurston, R.H.
9. Kaila, V.R.I.; Annila, A. Natural selection for least action. *Proc. R. Soc. A* **2008**, *464*, 3055–3070.
10. Grönholm, T.; Annila, A. Natural distribution. *Math. Biosci.* **2007**, *210*, 659–667.
11. Mäkelä, T.; Annila, A. Natural patterns of energy dispersal. *Phys. Life. Rev.* **2010**, doi:10.1016/j.plrev.2010.10.001.
12. Annila, A.; Salthe, S. Economies evolve by energy dispersal. *Entropy* **2009**, *11*, 606–633.
13. Sharma, V.; Annila, A. Natural process—Natural selection. *Biophys. Chem.* **2007**, *127*, 123–128.

14. Tuisku, P.; Pernu, T.K.; Annila, A. In the light of time. *Proc. R. Soc. A*. **2009**, *465*, 1173–1198.
15. Boltzmann, L. *Theoretical Physics and Philosophical Problems*; Reidel: Dordrecht, The Netherlands, 1974; partially translated by McGuinness, B.
16. Jaynes, E.T. Information theory and statistical mechanics. *Phys. Rev.* **1957**, *106*, 620–630.
17. Jaynes, E.T. *Probability Theory. The Logic of Science*; Cambridge University Press: Cambridge, UK, 2003; pp. 1101–1123.
18. Lotka, A.J. Natural selection as a physical principle. *Proc. Natl. Acad. Sci.* **1922**, *8*, 151–154.
19. Gibbs, J.W. *The Scientific Papers of J. Willard Gibbs*; Ox Bow Press: Woodbridge, CT, USA, 1993–1994; Chapter II, pp. 100–114.
20. Wyld, H.W. Formulation of the theory of turbulence in an incompressible fluid. *Ann. Phys.* **1961**, *14*, 143–165.
21. Keldysh, L.V. Diagram technique for nonequilibrium processes. *Sov. Phys. JETP* **1965**, *20*, 1018–1026.
22. Salthe, S.N. *Evolving Hierarchical Systems: Their Structure and Representation*; Columbia University Press: New York, NY, USA, 1985; pp. 4–11.
23. Karnani, M.; Pääkkönen, K.; Annila, A. The physical character of information. *Proc. R. Soc. A*. **2009**, *465*, 2155–2175.
24. De Donder, T. *Thermodynamic Theory of Affinity: A Book of Principles*; Oxford University Press: Oxford, UK, 1936; pp. 18–40.
25. Kullback, S. *Information Theory and Statistics*; Wiley: New York, NY, USA, 1959; pp. 18–25.
26. Bak, P. *How Nature Works: The Science of Self-organized Criticality*; Copernicus: New York, NY, USA, 1996; pp. 1–212.
27. Salthe, S.N. Summary of the principles of hierarchy theory. *Gen. Syst. Bull.* **2002**, *31*, 13–17.
28. Salthe, S.N. The natural philosophy of work. *Entropy* **2007**, *9*, 83–99.
29. Annila, A.; Kuismanen, E. Natural hierarchy emerges from energy dispersal. *Biosystems* **2008**, *95*, 227–233.
30. Annila, A. The 2nd law of thermodynamics delineates dispersal of energy. *Int. Rev. Phys*. **2010**, *4*, 29–34.
31. Kondepudi, D.; Prigogine, I. *Modern Thermodynamics*; Wiley: New York, NY, USA, 1998; pp. 411–420.
32. Atkins, P.W.; de Paula, J. *Physical Chemistry*; Oxford University Press: New York, NY, USA, 2006; pp. 169–175.
33. Du Châtelet, G.E. *Institutions de Physique*; Prault: Paris, France, 1740; pp. 1–450.
34. 'sGravesande, W.J. *Physices Elementa Mathematica, Experimentis Confirmata, Sive Introductio ad Philosophiam Newtonianam*; Van der Aa: Leiden, The Netherlands, 1720.
35. Lavenda, B.H. *Nonequilibrium Statistical Thermodynamics*; John Wiley & Sons: New York, NY, USA, 1985; pp. 1–13.
36. Flanders, H. *Differential Forms with Applications to the Physical Sciences*; Dover Publications: Mineola, NY, USA, 1989; pp. 44–48.
37. Spivak, M. *Calculus on Manifolds*; W. A. Benjamin: Menlo Park, CA, USA, 1965; pp. 15–45.
38. Birkhoff, G.; Mac Lane, S. *A Brief Survey of Modern Algebra*; Macmillan: New York, NY, USA, 1965; pp. 193–196.

39. Wallace, D.A.R. *Groups, Rings and Fields*; Springer-Verlag: Berlin, Germany, 1998; pp. 145–190.

40. Taylor, E.F.; Wheeler, J.A. *Spacetime Physics*; Freeman: San Francico, CA, USA, 1992; pp. 36–42.

41. Sipser, M. *Introduction to the Theory of Computation*; Pws Publishing: New York, NY, USA, 2001; pp. 241–276.

42. Annila, A. Physical portrayal of computational complexity. **2009**. Available online: http://arxiv.org/ftp/arxiv/ papers/0906/0906.1084.pdf (accessed on 29 October 2010).

43. Poincaré, J.H. Sur le problème des trois corps et les équations de la dynamique. Divergence des séries de M. Lindstedt. *Acta Math.* **1890**, *13*, 1–270.

44. Connes, A. *Noncommutative Geometry (Géométrie non Commutative)*; Academic Press: San Diego, CA, USA, 1994; pp. 356–375.

45. Szekeres, P. *A Course in Modern Mathematical Physics*; Cambridge University Press: Cambridge, UK, 2004; pp. 436–538.

46. Poynting, J.H. *Collected Scientific Papers*; Cambridge University Press: London, UK, 1920; pp. 175–223.

47. Barbour, J. *The End of Time: The Next Revolution in Our Understanding of the Universe*; Oxford University Press: New York, NY, USA, 1999; p. 19.

48. Hestenes, D.; Sobczyk, G. *Clifford Algebra to Geometric Calculus. A Unified Language for Mathematics and Physics*; Reidel: Dordrecht, The Netherlands, 1984; pp. 1–43.

49. Pati, J.C.; Salam, A. Lepton number as the fourth 'color'. *Phys. Rev. D* **1974**, *10*, 275–289.

50. Griffiths D.J. *Introduction to Elementary Particles*; John Wiley & Sons: New York, NY, USA, 1987; p. 42.

51. Lorenz, L. On the identity of the vibrations of light with electrical currents. *Philos. Mag.* **1867**, *34*, 287–301.

52. Peskin, M.E.; Schroeder, D.V. *An Introduction to Quantum Field Theory*; Addison-Wesley: Reading, MA, USA, 1995; pp. 189–198.

53. Schrödinger, E. *Über die kräftefreie Bewegung in der Relativistischen Quantenmechanik. Sitz. Preuss. Akad. Wiss. Phys.-Math. Kl.* **1930**, *24*, 418–428.

54. Langacker, P. Grand unified theories and proton decay. *Phys. Rep.* **1981**, *72*, 185–385.

55. Pokorski, S. *Gauge Field Theories*; Cambridge University Press: Cambridge, UK, 1987; pp. 188–194.

56. Kaiser, C.R.; Hannikainen, D.C. Pair annihilation and radio emission from galactic jet sources: The case of Nova Muscae. *MNRAS* **2002**, *330*, 225–231.

57. Hawking, S.W. Black hole explosions. *Nature* **1974**, *248*, 30–31.

58. Casimir, H.B.G.; Polder, D. The influence of retardation on the London-van der Waals forces. *Phys. Rev.* **1948**, *73*, 360–372.

59. Crawford, F.S. *Waves in Berkeley Physics Course* 3; McGraw-Hill: New York, NY, USA, 1968; pp. 49–72.

60. Alonso, M.; Finn, E.J. *Fundamental University Physics* 3; Addison-Wesley: Reading, MA, USA, 1983; pp. 531–532.

61. Bondi, H.; Samuel, J. The Lense-Thirring Effect and Mach's Principle. **1996**. Available online: http://arxiv.org/ PS_cache/gr-qc/pdf/9607/9607009v1.pdf (accessed on 29 October 2010).

62. Bennett, C.L.; Halpern, M.; Hinshaw, G.; Jarosik, N.; Kogut1, A.; Limon, M.; Meyer, S.S.; Page,

L.; Sperge, D.N.; Tucker, G.S.; *et al*. First-year Wilkinson Microwave Anisotropy Probe (WMAP) observations: Preliminary maps and basic results. *Astrophys. J. Suppl. Series* **2003**, *148*, 1–27.

63. Unsöld, A.; Baschek, B. *The New Cosmos, an Introduction to Astronomy and Astrophysics*; Springer-Verlag: New York, NY, USA, 2002; pp. 482–486.

64. Sciama, D.W. On the origin of inertia. *MNRAS* **1953**, *113*, 34–42.

65. Dicke, R.H. Dirac's cosmology and Mach's principle. *Nature* **1961**, *192*, 440–441.

66. Haas, A. An attempt to a purely theoretical derivation of the mass of the universe. *Phys. Rev.* **1936**, *49*, 411–412.

67. Lemaître, G. Un univers homogène de masse constante et de rayon croissant rendant compte de la vitesse radiale des nébuleuses extra-galactiques. *Ann. Soc. Sci. Brux.* **1927**, *47*, 49–56.

68. Hubble, E. A relation between distance and radial velocity among extra-galactic nebulae. *Proc. Natl. Acad. Sci. U. S. A.* **1929**, *15*, 168–173.

69. Lehto, A. On the Planck scale and properties of matter. *Nonlinear Dynamics* **2009**, *55*, 279–298.

70. Ryder, L. *Quantum Field Theory*; Cambridge University Press: Cambridge, UK, 1996; pp. 308–389.

71. Quantum Yang-Mills Theory. Available online: http://www.claymath.org/millennium/Yang-Mills_Theory/yangmills.pdf (accessed on 29 October 2010).

72. Strogatz, S.H. *Nonlinear Dynamics and Chaos with Applications to Physics, Biology, Chemistry and Engineering*; Westview: Cambridge, MA, USA, 2000; pp. 174–180.

73. Mattuck, R.D. *A Guide to Feynman Diagrams in the Many-body Problem*; McGraw-Hill: New York, NY, USA, 1976; pp. 101–117.

74. Silverman, J.H.; Tate, J. *Rational Points on Elliptic Curves*; Springer-Verlag: New York, NY, USA, 1992; pp. 105–144.

75. Eisenbud, D. *Commutative Algebra with a View toward Algebraic Geometry*; Springer-Verlag: New York, NY, USA, 1999; pp. 31–37.

76. Kodaira, K. *Complex Manifolds and Deformation of Complex Structures*; Springer: New York, NY, USA, 1986; pp. 39–59.

77. Annila, A.; Kallio-Tamminen, T. Tangled in entanglements. **2010**. Available online: http://arxiv.org/ftp/arxiv/ papers/1006/1006.0463.pdf (accessed on 29 October 2010).

78. Darwin, C. *On the Origin of Species*; John Murray: London, UK, 1859; pp. 80–130.

79. Jaakkola, S.; El-Showk, S.; Annila, A. The driving force behind genomic diversity. **2008**. Available online: http://arxiv.org/ftp/arxiv/papers/0807/0807.0892.pdf (accessed on 29 October 2010).

80. Würtz, P.; Annila, A. Roots of diversity relations. *J. Biophys.* **2008**, doi:10.1155/2008/654672.

81. Karnani, M.; Annila, A. Gaia again. *Biosystems* **2009**, *95*, 82–87.

82. Würtz, P.; Annila, A. Ecological succession as an energy dispersal process. *Biosystems* **2010**, *100*, 70–78.

83. Shipov, G.I. *A Theory of Physical Vacuum, A New Paradigm*; Zao Gart: Moscow, Russia, 1998; p. 312.

84. Eddington, A.S. Preliminary note on the masses of the electron, the proton and the Universe. *Proc. Camb. Phil. Soc.* **1931**, *27*, 15.

85. Dirac, P.A.M. A new basis for cosmology. *Proc. R. Soc. A* **1938**, *165*, 199–208.

86. Georgiev, G.; Georgiev, I. The least action and the metric of an organized system. *Open Syst. Inform. Dynam.* **2002**, *9*, 371–380.

87. Annila, A.; Annila, E. Why did life emerge? *Int. J. Astrobiol.* **2008**, *7*, 293–300.

88. Jaakkola, S.; Sharma, V.; Annila, A. Cause of chirality consensus. *Curr. Chem. Biol.* **2008**, *2*, 53–58.

89. Annila, A.; Salthe, S. Physical foundations of evolutionary theory. *J. Non-Equil. Thermodyn.* **2010**, *35*, 301–321.

90. Sharma, V.; Kaila, V.R.I.; Annila, A. Protein folding as an evolutionary process. *Physica A* **2009**, *388*, 851–862.

91. The P *versusversus* NP problem. Available online: http://www.claymath.org/millennium/ P_vs_NP/pvsnp.pdf (accessed on 29 October 2010).

92. Annila, A. Space, time and machines. **2009**. Available online: http://arxiv.org/ftp/arxiv/ papers/0910/0910.2629.pdf (accessed on 29 October 2010).

93. The Riemann Hypothesis. Available online: http://www.claymath.org/millennium/ Riemann_Hypothesis/riemann.pdf (accessed on 29 October 2010).

94. Existence and smoothness of the Navier-Stokes equation. Available online: http://www.claymath.org/millennium/Navier-Stokes_Equations/navierstokes.pdf (accessed on 29 October 2010).

95. The Birch and Swinnerton-Dyer Conjecture. Available online: http://www.claymath.org/ millennium/Birch_and_Swinnerton-Dyer_Conjecture/birchswin.pdf (accessed on 29 October 2010).

96. Beeson, D. *Maupertuis: An Intellectual Biography*; Voltaire Foundation: Oxford, UK, 1992; pp. 1–304.