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Original work on **Identical Object Distribution**

Theorem 1: If there are $2n$ identical balls, they can be put in boxes A (x many) and B (y many) for $0 < x < y$ as
$\frac{2n-2}{2} = n-1$ different ways.

Proof: $2n$ ball distribution for boxes A (first term) and B (second term):

$$\left.\begin{array}{l} 1+(2n-1)=2n \\ 2+(2n-2)=2n \\ 3+(2n-3)=2n \\ . \\ . \\ . \\ (n-1)+(n+1)=2n \end{array}\right\}$$ There are $n-1$ states satisfying $0 < x < y$ condition.

Theorem 2: If there are $2n-1$ identical balls, they can be put in boxes A (x many) and B (y many) for $0 < x < y$ as
$\frac{(2n-1)-1}{2} = n-1$ different ways.

Proof: $2n-1$ ball distribution for boxes A (first term) and B (second term):

$$\left.\begin{array}{l} 1+(2n-2)=2n-1 \\ 2+(2n-3)=2n-1 \\ 3+(2n-4)=2n-1 \\ . \\ . \\ . \\ (n-1)+(n)=2n-1 \end{array}\right\}$$ There are $n-1$ states satisfying $0 < x < y$ condition.

Unification of theorem 1 and 2 gives;

Result 1: If there are n identical balls, they can be put in boxes A (x many) and B (y many) for $0 < x < y$ as b_n different ways. Where

$$b_n = \begin{cases} \frac{2n-1}{2} & n\ positive\ odd\ integer \\ \frac{2n-2}{2} & n\ positive\ even\ integer \end{cases}$$

We saw that if there are n identical balls, they can be put in boxes A (x many) and B (y many) for $0 < x < y$ as a_n different ways where

$a_1 = 0,\ a_2 = 0,\ a_3 = 1, a_4 = 1, a_5 = 2, a_6 = 2, a_7 = 3,\ a_8 = 3, a_9 = 4, a_{10} = 4, a_{11} = 5, a_{12} = 5,$

which is given by A004526 sequence of OEIS.

Question: In how many different ways can 10 identical balls be put in boxes A, B and C as x,y and z respectively, satisfying $0 < x < y < z$ condition?

Solution: $0 < x < y < z$ and $x + y + z = 10$;

A	B	C
1	2	7
1	3	6
1	4	5
2	3	5

Let's put one ball in each box. 7 remaining balls can be put in boxes B and C as $\frac{7-1}{2} = 3$ from Theorem 1.

Let's put 2 balls in each box. 4 remaining balls can be put in boxes B and C as $\frac{4-2}{2} = 1$ from Theorem 1.

Number of possible ways = 3+1=4

We can also evaluate the number of possibilities from the sequence as $a_{10} = b_7 + b_4 = 3 + 1 = 4$

Question: In how many different ways can 11 identical balls be put in boxes A, B and C as x,y and z respectively, satisfying $0 < x < y < z$ condition?
Solution: $0 < x < y < z$ and $x + y + z = 11$;

A	B	C
1	2	8
1	3	7
1	4	6
2	3	6
2	4	5

Let's put one ball in each box. 8 remaining balls can be put in boxes B and C as $\frac{8-2}{2} = 3$ from Theorem 1.

Let's put 2 balls in each box. 5 remaining balls can be put in boxes B and C as $\frac{5-1}{2} = 2$ from Theorem 1.

Number of possible ways = 3+2=5

We can also evaluate the number of possibilities from the sequence as $a_{11} = b_8 + b_5 = 3 + 2 = 5$

Question: In how many different ways can 12 identical balls be put in boxes A, B and C as x,y and z respectively, satisfying $0 < x < y < z$ condition?

Solution: $0 < x < y < z$ and $x + y + z = 12$

A	B	C
1	2	9
1	3	8
1	4	7
1	5	6
2	3	7
2	4	6
3	4	5

Let's put one ball in each box. 9 remaining balls can be put in boxes B and C as $\frac{9-1}{2}=4$ from Theorem 1.

Let's put 2 balls in each box. 6 remaining balls can be put in boxes B and C as $\frac{6-2}{2}=2$ from Theorem 1.

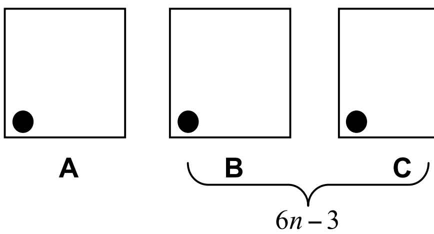
Let's put 3 balls in each box. 3 remaining balls can be put in boxes B and C as $\frac{3-1}{2}=1$ from Theorem 1.

Number of possible ways = 4+2+1=7

We can also evaluate the number of possibilities from the sequence as
$a_{12}=b_9+b_6+b_3=4+2+1=7$

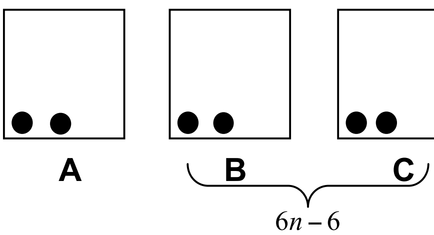
Theorem 3: If there are $6n$ identical balls, they can be put in boxes A, B and C as x, y and z respectively, satisfying $0<x<y<z$
in $3n^2-3n+1$ different ways.

Proof: Lets put one ball in each box. $6n-3$ remaining balls can be put in boxes B and C in



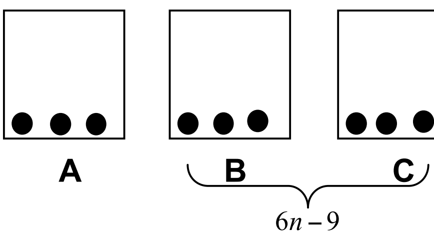
$\frac{(6n-3)-1}{2}=3n-2$ different ways

Let's put 2 balls in each box. $6n-6$ remaining balls can be put in boxes B and C in



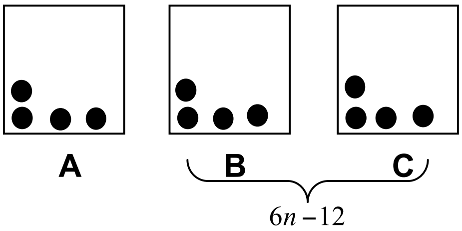
$\frac{(6n-6)-2}{2}=3n-4$ different ways.

Let's put 3 balls in each box. $6n-9$ remaining balls can be put in boxes B and C in



$\frac{(6n-9)-1}{2}=3n-5$ different ways.

Let's put 4 balls in each box. $6n-12$ remaining balls can be put in boxes B and C in



$\dfrac{(6n-12)-2}{2}=3n-7$ different ways

Let's put (2k-1) balls in each box. $6n-3.(2k-1)$ remaining balls can be put in boxes B and C in $\dfrac{[6n-3.(2k-1)]-1}{2}=3n-3k+1$ different ways.

Let's put (2k) balls in each box. $6n-6k$ remaining balls can be put in boxes B and C in $\dfrac{[6n-6k]-2}{2}=3n-3k-1$ different ways.

If we continue similarly...

Let's put (2n-2) balls in each box. $6n-3(2n-2)=6$ remaining balls can be put in boxes B and C in $\dfrac{6-2}{2}=4$ different ways.

Let's put (2n-1) balls in each box. $6n-3(2n-1)=3$ remaining balls can be put in boxes B and C in $\dfrac{3-1}{2}=2$ different ways.

Sum of the situations which we put odd number of balls in each box and distribute remaining balls:

$$\sum_{k=1}^{n}\frac{\left[6n-3.(2k-1)\right]-1}{2}=\sum_{k=1}^{n}(3n-3k+1)=3.n^2-3.\frac{n.(n+1)}{2}+n=\frac{3n^2-n}{2}$$

Sum of the situations which we put even number of balls in each box and distribute remaining balls:

$$\sum_{k=1}^{n-1}\frac{\left[6n-3.(2k)\right]-2}{2}=\sum_{k=1}^{n-1}(3n-3k-1)=3.n.(n-1)-3.\frac{(n-1).n}{2}-(n-1)=\frac{3n^2-5n+2}{2}$$

Sum of all situations: $\dfrac{3n^2-n}{2}+\dfrac{3n^2-5n+2}{2}=3n^2-3n+1$.

Theorem 4: If there are $6n-3$ identical balls, they can be put in boxes A, B and C as x, y and z respectively, satisfying $0 < x < y < z$
in $3(n-1)^2$ different ways.

Proof: From Theorem 2;

$$a_{6n} = b_{6n-3} + b_{6n-6} + \ldots + b_6 + b_3$$

$$a_{6n} = b_{6n-3} + \underbrace{b_{6n-6} + \ldots + b_6 + b_3}_{a_{6n-3}}$$

$$a_{6n} = b_{6n-3} + a_{6n-3}$$

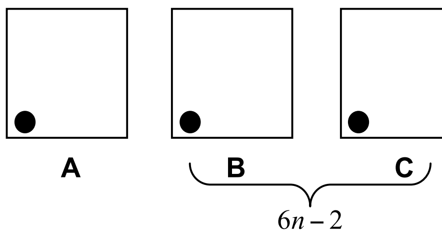
$$3n^2 - 3n + 1 = \frac{6n-3-1}{2} + a_{6n-3}$$

$$3n^2 - 3n + 1 = 3n - 2 + a_{6n-3}$$

$$a_{6n-3} = 3(n-1)^2.$$

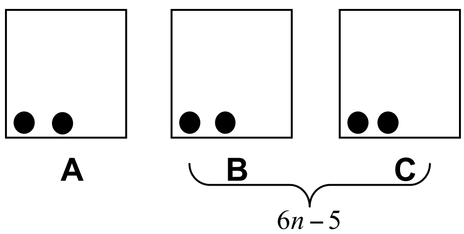
Theorem 5: If there are $6n+1$ identical balls, they can be put in boxes A, B and C as x, y and z respectively, satisfying $0 < x < y < z$
in $3n^2 - 2n$ different ways.

Proof: Lets put one ball in each box. $6n-2$ remaining balls can be put in boxes B and C in



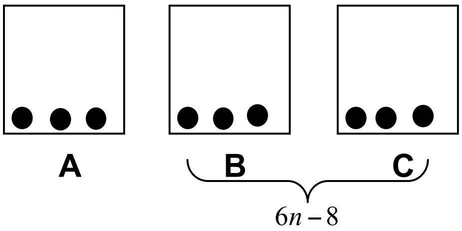
$\frac{(6n-2)-2}{2} = 3n-2$ different ways

Let's put 2 balls in each box. $6n-5$ remaining balls can be put in boxes B and C in



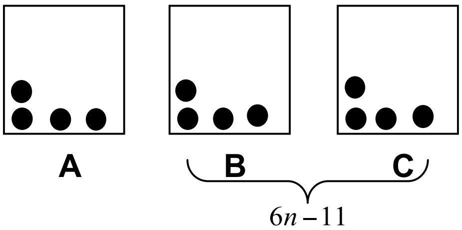
$\frac{(6n-5)-1}{2} = 3n-3$ different ways.

Let’s put 3 balls in each box. $6n-8$ remaining balls can be put in boxes B and C in



$\frac{(6n-8)-2}{2}=3n-5$ different ways.

Let’s put 4 balls in each box. $6n-11$ remaining balls can be put in boxes B and C in



$\frac{(6n-11)-1}{2}=3n-6$ different ways.

Let’s put (2k-1) balls in each box. $6n+1-3.(2k-1)$ remaining balls can be put in boxes B and C in $\frac{[6n+1-3.(2k-1)]-2}{2}=3n-3k+1$ different ways

Let’s put (2k) balls in each box. $6n+1-6k$ remaining balls can be put in boxes B and C in $\frac{[6n+1-6k]-1}{2}=3n-3k$ different ways

If we continue similarly…

Let’s put (2n-2) balls in each box. $6n+1-3(2n-2)=7$ remaining balls can be put in boxes B and C in $\frac{7-1}{2}=3$ different ways

Let’s put (2n-1) balls in each box. $6n+1-3(2n-1)=4$ remaining balls can be put in boxes B and C in $\frac{4-2}{2}=1$ different ways

Sum of the situations which we put odd number of balls in each box and distribute remaining balls:

$$\sum_{k=1}^{n}\frac{\left[6n+1-3.(2k-1)\right]-2}{2}=\sum_{k=1}^{n}(3n-3k+1)=3.n^2-3.\frac{n.(n+1)}{2}+n=\frac{3n^2-n}{2}$$

Sum of the situations which we put even number of balls in each box and distribute remaining balls:

$$\sum_{k=1}^{n-1} \frac{\left[6n+1-3.(2k)\right]-1}{2} = \sum_{k=1}^{n-1}(3n-3k) = 3.n.(n-1) - 3.\frac{(n-1).n}{2} = \frac{3n^2-3n}{2}$$

Sum of all situations : $\frac{3n^2-n}{2} + \frac{3n^2-3n}{2} = 3n^2-2n$

Theorem 6: If there are $6n-2$ identical balls, they can be put in boxes A, B and C as x, y and z respectively, satisfying $0 < x < y < z$,
in $3n^2-5n+2$ different ways.

Proof: From Theorem 2;

$$a_{6n+1} = b_{6n-2} + b_{6n-5} + ... + b_7 + b_4 .$$

$$a_{6n+1} = b_{6n-2} + \underbrace{b_{6n-5} + ... + b_7 + b_4}_{a_{6n-2}}$$

$$a_{6n+1} = b_{6n-2} + a_{6n-2}$$

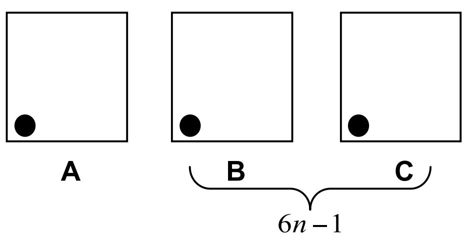
$$3n^2 - 2n = \frac{6n-2-2}{2} + a_{6n-2}$$

$$3n^2 - 2n = 3n - 2 + a_{6n-2}$$

$$a_{6n-2} = 3n^2 - 5n + 2 .$$

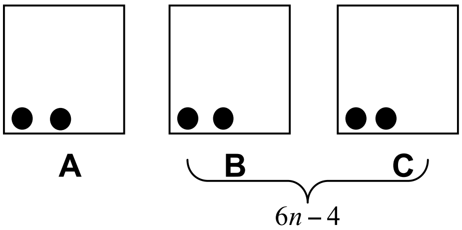
Theorem 7: If there are $6n+2$ identical balls, they can be put in boxes A, B and C as x, y and z respectively, satisfying $0 < x < y < z$
in $3n^2-n$ different ways.

Proof: Lets put one ball in each box. $6n-1$ remaining balls can be put in boxes B and C in



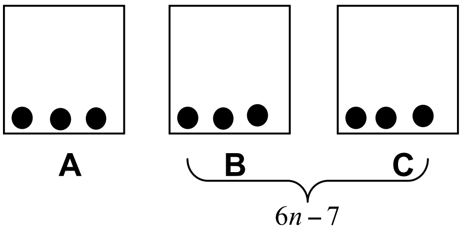
$\frac{(6n-1)-1}{2} = 3n-1$ different ways.

Let’s put 2 balls in each box. $6n-4$ remaining balls can be put in boxes B and C in



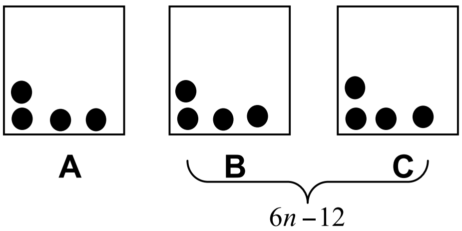
$\frac{(6n-4)-2}{2}=3n-3$ different ways.

Let’s put 3 balls in each box. $6n-7$ remaining balls can be put in boxes B and C in



$\frac{(6n-7)-1}{2}=3n-4$ different ways.

Let’s put 4 balls in each box. $6n-10$ remaining balls can be put in boxes B and C in



$\frac{(6n-10)-2}{2}=3n-6$ different ways.

Let’s put (2k-1) balls in each box. $6n+2-3.(2k-1)$ remaining balls can be put in boxes B and C in $\frac{[6n+2-3.(2k-1)]-1}{2}=3n-3k+2$ different ways.

Let’s put (2k) balls in each box. $6n+2-6k$ remaining balls can be put in boxes B and C in $\frac{[6n+2-6k]-2}{2}=3n-3k$ different ways.

If we continue similarly…

Let’s put (2n-2) balls in each box. $6n+2-3(2n-2)=8$ remaining balls can be put in boxes B and C in $\frac{8-2}{2}=3$ different ways.

Let’s put (2n-1) balls in each box. $6n+2-3(2n-1)=5$ remaining balls can be put in boxes B and C in $\frac{5-1}{2}=2$ different ways.

Sum of the situations which we put odd number of balls in each box and distribute remaining balls:

$$\sum_{k=1}^{n}\frac{\left[6n+2-3.(2k-1)\right]-1}{2}=\sum_{k=1}^{n}(3n-3k+2)=3.n^2-3.\frac{n.(n+1)}{2}+2n=\frac{3n^2+n}{2}$$

Sum of the situations which we put even number of balls in each box and distribute remaining balls:

$$\sum_{k=1}^{n-1}\frac{\left[6n+-3.(2k)\right]-2}{2}=\sum_{k=1}^{n-1}(3n-3k)=3.n.(n-1)-3.\frac{(n-1).n}{2}=\frac{3n^2-3n}{2}$$

Sum of all situations: $\frac{3n^2+n}{2}+\frac{3n^2-3n}{2}=3n^2-n$.

Theorem 8: If there are $6n-1$ identical balls, they can be put in boxes A, B and C as x, y and z respectively, satisfying $0<x<y<z$,
in $3n^2-4n+1$ different ways.

Proof: From Theorem 2;

$$a_{6n+2}=b_{6n-1}+b_{6n-4}+...+b_8+b_5$$

$$a_{6n+2}=b_{6n-1}+\underbrace{b_{6n-4}+...+b_8+b_5}_{a_{6n-1}}$$

$$a_{6n+2}=b_{6n-1}+a_{6n-1}$$

$$3n^2-n=\frac{6n-1-1}{2}+a_{6n-1}$$

$$3n^2-n=3n-1+a_{6n-1}$$

$$a_{6n-1}=3n^2-4n+1.$$

We can now conclude that;

Combining Theorem 3,4,5,6,7 and 8 provides us the following sequence. This sequence's first 5 terms are zero. This is the sequence of how many different distributions can be made with 3 boxes (A (x balls) ,B (y balls) and C (z balls) where $0 < x < y < z$) and at least 6 identical balls

$a_1 = 0$, $a_2 = 0$, $a_3 = 0$, $a_4 = 0$, $a_5 = 0$ **and for other terms;**

$$a_n = \begin{cases} 3.n^2 - 6n + 3 & n = 6k - 3 \quad k \in N^+ \\ 3.k^2 - 5.k + 2 & n = 6k - 2 \quad k \in N^+ \\ 3.k^2 - 4.k + 1 & n = 6k - 1 \quad k \in N^+ \\ 3.k^2 - 3.k + 1 & n = 6k \quad k \in N^+ \\ 3.k^2 - 2.k & n = 6k + 1 \quad k \in N^+ \\ 3.k^2 - k & n = 6k + 2 \quad k \in N^+ \end{cases}$$

Here are some terms of the sequence a_n:

n	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
a_n	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**2**	**3**	**4**	**5**	**7**	**8**	**10**	**12**	**14**	**16**	**19**	**21**	**24**

n	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**	**31**	**32**	**33**	**34**	**35**	**36**	**37**	**38**	**39**	**40**
a_n	**27**	**30**	**33**	**37**	**40**	**44**	**48**	**52**	**56**	**61**	**65**	**70**	**75**	**80**	**85**	**91**	**96**	**102**	**108**	**114**

0,0,0,0,0,1, 1, 2, 3, 4, 5, 7, 8, 10, 12, 14, 16, 19, 21, 24, 27, 30, 33, 37, 40, 44, 48, 52, 56, 61, 65, 70, 75, 80, 85, 91, 96, 102, 108, 114, 120, 127, 133, 140, 147, 154, 161, 169, 176, 184, 192, 200, 208, 217, 225, 234, 243, 252, 261, 271, 280, 290, 300, 310, 320, 331, 341

Question: In how many different ways can 16 identical balls be put in boxes A, B, C and D as x, y, z and p respectively, satisfying $0 < x < y < z < p$ condition?

Solution: $0 < x < y < z < p$ and $x + y + z + p = 16$

A	B	C	D
1	2	3	10
1	2	4	9
1	2	5	8
1	2	6	7
1	3	4	8
1	3	5	7
1	4	5	6
2	3	4	7
2	4	5	6

Let's put one ball in each box. Using Theorem 3, 12 remaining balls can be put in boxes B, C and D in

$a_{6n} = 3n^2 - 3n + 1$

$a_{12} = 3.2^2 - 3.2 + 1 = 7$ different ways

Let's put 2 balls in each box. Using Theorem 7, 8 remaining balls can be put in boxes B, C and D in

$a_{6n+2} = 3n^2 - n$

$a_8 = 3.1^2 - 1 = 2$ different ways

Total number of different distributions: 7+2 = 9

Theorem 9: : If there are n identical balls, they can be distributed as x, y, z, p in boxes A, B, C and D respectively, for $0 < x < y < z < p$, in c_n different ways. Where

(For n identical balls, their distribution as x, y, z in boxes A, B and C respectively for $0 < x < y < z$ is a_n)

$$c_n = \begin{cases} c_{4k} = c_{4k-4} + a_{4k-4} & k \in N^+ \\ c_{4k+1} = c_{4k-3} + a_{4k-3} & k \in N^+ \\ c_{4k+2} = c_{4k-2} + a_{4k-2} & k \in N^+ \\ c_{4k+3} = c_{4k-1} + a_{4k-1} & k \in N^+ \end{cases}$$

Proof:

Let's put one ball from 4k balls in each box. 4k-4 remaining balls can be put in boxes B, C and D in a_{4k-4} different ways

Let's put 2 balls from 4k balls in each box. 4k-8 remaining balls can be put in boxes B, C and D in a_{4k-8} different ways

If we continue similarly...

Let's put k-2 balls from 4k balls in each box. 8 remaining balls can be put in boxes B, C and D in a_8 different ways

Let's put k-1 balls from 4k balls in each box. 4 remaining balls can be put in boxes B, C and D in a_4 different ways

Sum of all possibilities:

$$c_{4n} = a_{4n-4} + a_{4n-8} + ... + a_8 + a_4.$$

$$c_{4n} = a_{4n-4} + \underbrace{a_{4n-8} + ... + a_8 + a_4}_{c_{4n-4}}$$

$$c_{4n} = a_{4n-4} + c_{4n-4}.$$

Let's put one ball from 4k+1 balls in each box. 4k-3 remaining balls can be put in boxes B, C and D in a_{4k-3} different ways

Let's put 2 balls from 4k+1 balls in each box. 4k-7 remaining balls can be put in boxes B, C and D in a_{4k-7} different ways

If we continue similarly…

Let’s put k-2 balls from 4k+1 balls in each box. 9 remaining balls can be put in boxes B, C and D in a_9 different ways

Let’s put k-1 balls from 4k+1balls in each box. 5 remaining balls can be put in boxes B, C and D in a_5 different ways

Let’s put k balls from 4k+1balls in each box. 1 remaining ball can be put in boxes B, C and D in a_1 different ways

Sum of all possibilities:

$$c_{4n+1} = a_{4n-3} + a_{4n-7} + ... + a_9 + a_5 + a_1 .$$

$$c_{4n+1} = a_{4n-3} + \underbrace{a_{4n-7} + ... + a_9 + a_5 + a_1}_{c_{4n-3}}$$

$$c_{4n+1} = a_{4n-3} + c_{4n-3} .$$

Let’s put one ball from 4k+2 balls in each box. 4k-2 remaining balls can be put in boxes B, C and D in a_{4k-2} different ways

Let’s put 2 balls from 4k+2 balls in each box. 4k-6 remaining balls can be put in boxes B, C and D in a_{4k-6} different ways

If we continue similarly…

Let’s put k-2 balls from 4k+2 balls in each box. 10 remaining balls can be put in boxes B, C and D in a_{10} different ways

Let’s put k-1 balls from 4k+2 balls in each box. 6 remaining balls can be put in boxes B, C and D in a_6 different ways

Let’s put k balls from 4k+2 balls in each box. 2 remaining ball can be put in boxes B, C and D in a_2 different ways

Sum of all possibilities:

$$c_{4n+2} = a_{4n-2} + a_{4n-2} + ... + a_{10} + a_6 + a_2 .$$

$$c_{4n+2} = a_{4n-2} + \underbrace{a_{4n-6} + ... + a_{10} + a_6 + a_2}_{c_{4n-2}}$$

$$c_{4n+2} = a_{4n-2} + c_{4n-2} .$$

Let's put one ball from 4k+3 balls in each box. 4k-1 remaining balls can be put in boxes B, C and D in a_{4k-1} different ways

Let's put 2 balls from 4k+3 balls in each box. 4k-5 remaining balls can be put in boxes B, C and D in a_{4k-5} different ways

If we continue similarly…

Let's put k-2 balls from 4k+3 balls in each box. 11 remaining balls can be put in boxes B, C and D in a_{11} different ways

Let's put k-1 balls from 4k+3 balls in each box. 7 remaining balls can be put in boxes B, C and D in a_7 different ways

Let's put k balls from 4k+3 balls in each box. 3 remaining ball can be put in boxes B, C and D in a_3 different ways

Sum of all possibilities:

$$c_{4n+3} = a_{4n-1} + a_{4n-1} + \ldots + a_{11} + a_7 + a_3 .$$

$$c_{4n+3} = a_{4n-1} + \underbrace{a_{4n-5} + \ldots + a_{11} + a_7 + a_3}_{c_{4n-1}}$$

$$c_{4n+3} = a_{4n-1} + c_{4n-1} .$$

We can distribute at least 10 balls to A,B,C,D boxes with $0 < x < y < z < p$ condition

Relationship between a_n and c_n:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
a_n	0	0	0	0	0	1	1	2	3	4	5	7	8	10	12	14	16	19	21	24
c_n	0	0	0	0	0	0	0	0	0	1	1	2	3	5	6	9	11	15	18	23

n	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
a_n	27	30	33	37	40	44	48	52	56	61	65	70	75	80	85	91	96	102	108	114
c_n	27	34	39	47	54	64	72	84	94	108	120	136	150	169	185	206	225	249	270	297

a_n **sequence is the sequence** A211540 on OEIS

c_n **sequence is the sequence** A266527 on OEIS

Theorem 10: If there are n identical balls, they can be put in boxes A, B, C, D and E as x, y, z, p and q respectively, satisfying $0 < x < y < z < p < q$, in d_n different ways, where

(For n identical balls, their distribution as x, y, z, p in boxes A, B, C and D respectively for $0 < x < y < z < p$ is c_n)

$$d_n = \begin{cases} d_{5k} = c_{5k-5} + d_{5k-5} & k \in N^+ \\ d_{5k+1} = c_{5k-4} + d_{5k-4} & k \in N^+ \\ d_{5k+2} = c_{5k-3} + d_{5k-3} & k \in N^+ \\ d_{5k+3} = c_{5k-2} + d_{5k-2} & k \in N^+ \\ d_{5k+4} = c_{5k-1} + d_{5k-1} & k \in N^+ \end{cases}$$

Proof:

Let's put one ball from 5k balls in each box. 5k-5 remaining balls can be put in boxes B, C, D and E in a_{5k-5} different ways

Let's put 2 balls from 5k balls in each box. 5k-10 remaining balls can be put in boxes B, C, D and E in a_{5k-10} different ways

If we continue similarly…

Let's put k-2 balls from 5k balls in each box. 10 remaining balls can be put in boxes B, C, D and E in a_{10} different ways

Let's put k-1 balls from 5k balls in each box. 5 remaining balls can be put in boxes B, C, D and E in a_5 different ways

Sum of all possibilities:

$$d_{5n} = c_{5n-5} + c_{5n-10} + ... + c_{10} + c_5 .$$

$$d_{5n} = c_{5n-5} + \underbrace{c_{5n-10} + ... + c_{10} + c_5}_{d_{5n-5}}$$

$$d_{5n} = c_{5n-5} + d_{5n-5} .$$

Similarly;

$$d_{5n+1} = c_{5n-4} + c_{5n-9} + ... + c_{11} + c_6 + c_1 .$$

$$d_{5n+1} = c_{5n-4} + \underbrace{c_{5n-9} + ... + c_{11} + c_6 + c_1}_{d_{5n-4}}$$

$$d_{5n+1} = c_{5n-4} + d_{5n-4} .$$

$$d_{5n+2} = c_{5n-3} + c_{5n-8} + \dots + c_{12} + c_7 + c_2.$$

$$d_{5n+2} = c_{5n-3} + \underbrace{c_{5n-8} + \dots + c_{12} + c_7 + c_2}_{d_{5n-3}}$$

$$d_{5n+2} = c_{5n-3} + d_{5n-3}.$$

$$d_{5n+3} = c_{5n-2} + c_{5n-7} + \dots + c_{13} + c_8 + c_3.$$

$$d_{5n+3} = c_{5n-2} + \underbrace{c_{5n-7} + \dots + c_{13} + c_8 + c_3}_{d_{5n-2}}$$

$$d_{5n+3} = c_{5n-2} + d_{5n-2}.$$

$$d_{5n+4} = c_{5n-1} + c_{5n-6} + \dots + c_{14} + c_9 + c_4.$$

$$d_{5n+4} = c_{5n-1} + \underbrace{c_{5n-6} + \dots + c_{14} + c_9 + c_4}_{d_{5n-1}}$$

$$d_{5n+4} = c_{5n-1} + d_{5n-1}.$$

We can distribute at least 15 balls to A,B,C,D,E boxes with $0 < x < y < z < p < q$ condition

Relationship between c_n and d_n

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
c_n	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**2**	**3**	**5**	**6**	**9**	**11**	**15**	**18**	**23**
d_n	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**2**	**3**	**5**	**7**

n	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
c_n	**27**	**34**	**39**	**47**	**54**	**64**	**72**	**84**	**94**	**108**	**120**	**136**	**150**	**169**	**185**	**206**	**225**	**249**	**270**	**297**
d_n	**10**	**13**	**18**	**23**	**30**	**37**	**47**	**57**	**70**	**84**	**101**	**119**	**141**	**164**	**192**	**221**	**255**	**291**	**333**	**377**

c_n **sequence is the sequence** A266527 on OEIS

d_n **sequence is the sequence** A267094 on OEIS

Theorem 11: If there are n identical balls, they can be put in boxes A, B, C, D, E and F as x, y, z, p, q, h respectively, satisfying $0 < x < y < z < p < q < h$, in e_n different ways, where

(For n identical balls, their distribution as x, y, z, p, q in boxes A, B, C, D, E respectively for $0 < x < y < z < p < q$ is d_n)

$$e_n = \begin{cases} e_{6k} = d_{6k-6} + e_{6k-6} & k \in N^+ \\ e_{6k+1} = d_{6k-5} + e_{6k-5} & k \in N^+ \\ e_{6k+2} = d_{6k-4} + e_{6k-4} & k \in N^+ \\ e_{6k+3} = d_{6k-3} + e_{6k-3} & k \in N^+ \\ e_{6k+4} = d_{6k-2} + e_{6k-2} & k \in N^+ \\ e_{6k+5} = d_{6k-1} + e_{6k-1} & k \in N^+ \end{cases}$$

Similarly,

$$e_{6n} = d_{6n-6} + d_{6n-12} + ... + d_{18} + d_{12} + d_6.$$

$$e_{6n} = d_{6n-6} + \underbrace{d_{6n-12} + ... + d_{18} + d_{12} + d_6}_{e_{6n-6}}$$

$$e_{6n} = d_{6n-6} + e_{6n-6}.$$

$$e_{6n+1} = d_{6n-5} + d_{6n-11} + ... + d_{19} + d_{13} + d_7 + d_1.$$

$$e_{6n+1} = d_{6n-5} + \underbrace{d_{6n-11} + ... + d_{19} + d_{13} + d_7 + d_1}_{e_{6n-5}}$$

$$e_{6n+1} = d_{6n-5} + e_{6n-5}.$$

$$e_{6n+2} = d_{6n-4} + d_{6n-10} + ... + d_{20} + d_{14} + d_8 + d_2.$$

$$e_{6n+2} = d_{6n-4} + \underbrace{d_{6n-10} + ... + d_{20} + d_{14} + d_8 + d_2}_{e_{6n-4}}$$

$$e_{6n+2} = d_{6n-4} + e_{6n-4}.$$

$$e_{6n+3} = d_{6n-3} + d_{6n-9} + ... + d_{21} + d_{15} + d_9 + d_3.$$

$$e_{6n+3} = d_{6n-3} + \underbrace{d_{6n-9} + ... + d_{21} + d_{15} + d_9 + d_3}_{e_{6n-3}}$$

$$e_{6n+3} = d_{6n-3} + e_{6n-3}.$$

$$e_{6n+4} = d_{6n-2} + d_{6n-8} + \ldots + d_{22} + d_{16} + d_{10} + d_4.$$

$$e_{6n+4} = d_{6n-2} + \underbrace{d_{6n-8} + \ldots + d_{22} + d_{16} + d_{10} + d_4}_{e_{6n-2}}$$

$$e_{6n+4} = d_{6n-2} + e_{6n-2}.$$

$$e_{6n+5} = d_{6n-1} + d_{6n-7} + \ldots + d_{23} + d_{17} + d_{11} + d_5.$$

$$e_{6n+5} = d_{6n-1} + \underbrace{d_{6n-7} + \ldots + d_{23} + d_{17} + d_{11} + d_5}_{e_{6n-1}}$$

$$e_{6n+5} = d_{6n-1} + e_{6n-1}.$$

We can distribute at least 21 balls to A,B,C,D,E,F boxes with $0 < x < y < z < p < q < h$ condition

Relationship between d_n and e_n ;

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
d_n	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**2**	**3**	**5**	**7**
e_n	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**

n	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
d_n	**10**	**13**	**18**	**23**	**30**	**37**	**47**	**57**	**70**	**84**	**101**	**119**	**141**	**164**	**192**	**221**	**255**	**291**	**333**	**377**
e_n	**1**	**1**	**2**	**3**	**5**	**7**	**11**	**14**	**20**	**26**	**35**	**44**	**58**	**71**	**90**	**110**	**136**	**163**	**199**	**235**

d_n **sequence is the sequence** A267094 on OEIS
e_n **sequence is the sequence** A267118 on OEIS

Theorem 12: If there are n identical balls, they can be put in boxes A, B, C, D, E, F and G as x, y, z, p, q, h, j respectively, satisfying $0 < x < y < z < p < q < h < j$, in f_n different ways, where

(For n identical balls, their distribution as x, y, z, p in boxes A, B, C, D, E, F respectively for $0 < x < y < z < p < q < h$ is e_n)

$$f_n = \begin{cases} f_{7k} = e_{7k-7} + f_{7k-7} & k \in N^+ \\ f_{7k+1} = e_{7k-6} + f_{7k-6} & k \in N^+ \\ f_{7k+2} = e_{7k-5} + f_{7k-5} & k \in N^+ \\ f_{7k+3} = e_{7k-4} + f_{7k-4} & k \in N^+ \\ f_{7k+4} = e_{7k-3} + f_{7k-3} & k \in N^+ \\ f_{7k+5} = e_{7k-2} + f_{7k-2} & k \in N^+ \\ f_{7k+6} = e_{7k-1} + f_{7k-1} & k \in N^+ \end{cases}$$

Similarly,

$f_{7n} = e_{7n-7} + e_{7n-14} + ... + e_{14} + e_7$ dir.

$$f_{7n} = e_{7n-7} + \underbrace{e_{7n-14} + ... + e_{14} + e_7}_{f_{7n-7}}$$

$$f_{7n} = e_{7n-7} + f_{7n-7}$$

$f_{7n+1} = e_{7n-6} + e_{7n-13} + ... + e_{15} + e_8 + e_1$.

$$f_{7n+1} = e_{7n-6} + \underbrace{e_{7n-13} + ... + e_{15} + e_8}_{f_{7n-6}} + e_1$$

$$f_{7n+1} = e_{7n-6} + f_{7n-6}$$

$f_{7n+2} = e_{7n-5} + e_{7n-12} + ... + e_{16} + e_9 + e_2$.

$$f_{7n+2} = e_{7n-5} + \underbrace{e_{7n-12} + ... + e_{16} + e_9 + e_2}_{f_{7n-5}}$$

$$f_{7n+2} = e_{7n-5} + f_{7n-5}$$

$$f_{7n+3} = e_{7n-4} + e_{7n-11} + \ldots + e_{17} + e_{10} + e_3 .$$

$$f_{7n+3} = e_{7n-4} + \underbrace{e_{7n-11} + \ldots + e_{17} + e_{10} + e_3}_{f_{7n-4}}$$

$$f_{7n+3} = e_{7n-4} + f_{7n-4} .$$

$$f_{7n+4} = e_{7n-3} + e_{7n-10} + \ldots + e_{18} + e_{11} + e_4 .$$

$$f_{7n+4} = e_{7n-3} + \underbrace{e_{7n-10} + \ldots + e_{18} + e_{11} + e_4}_{f_{7n-3}}$$

$$f_{7n+4} = e_{7n-3} + f_{7n-3} .$$

$$f_{7n+5} = e_{7n-2} + e_{7n-9} + \ldots + e_{19} + e_{12} + e_5 .$$

$$f_{7n+5} = e_{7n-2} + \underbrace{e_{7n-9} + \ldots + e_{19} + e_{12} + e_5}_{f_{7n-2}}$$

$$f_{7n+5} = e_{7n-2} + f_{7n-2} .$$

$$f_{7n+6} = e_{7n-1} + e_{7n-8} + \ldots + e_{20} + e_{13} + e_6 .$$

$$f_{7n+6} = e_{7n-1} + \underbrace{e_{7n-8} + \ldots + e_{20} + e_{13} + e_6}_{f_{7n-1}}$$

$$f_{7n+6} = e_{7n-1} + f_{7n-1} .$$

We can distribute at least 28 balls to A,B,C,D,E,F,G boxes with $0 < x < y < z < p < q < h < j$ condition

Relationship between e_n and f_n;

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
e_n	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
f_n	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**

n	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
e_n	**1**	**1**	**2**	**3**	**5**	**7**	**11**	**14**	**20**	**26**	**35**	**44**	**58**	**71**	**90**	**110**	**136**	**163**	**199**	**235**
f_n	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**2**	**3**	**5**	**7**	**11**	**15**	**21**	**28**	**38**	**49**	**65**

e_n **sequence is the sequence** A267118 on OEIS

f_n **sequence is the sequence** A267120 on OEIS

Theorem 13: If there are n identical balls, they can be put in boxes A, B, C, D, E, F, G and H as x, y, z, p, q, h, j, m respectively, satisfying $0 < x < y < z < p < q < h < j < m$, in g_n different ways, where

(For n identical balls, their distribution as x, y, z, p, q, h, j in boxes A, B, C, D, E, F, G respectively for $0 < x < y < z < p < q < h < j$ is f_n)

$$g_n = \begin{cases} g_{8k} = f_{8k-8} + g_{8k-8} & k \in N^+ \\ g_{8k+1} = f_{8k-7} + g_{8k-7} & k \in N^+ \\ g_{8k+2} = f_{8k-6} + g_{8k-6} & k \in N^+ \\ g_{8k+3} = f_{8k-5} + g_{8k-5} & k \in N^+ \\ g_{8k+4} = f_{8k-4} + g_{8k-4} & k \in N^+ \\ g_{8k+5} = f_{8k-3} + g_{8k-3} & k \in N^+ \\ g_{8k+6} = f_{8k-2} + g_{8k-2} & k \in N^+ \\ g_{8k+7} = f_{8k-1} + g_{8k-1} & k \in N^+ \end{cases}$$

Similarly,

$$g_{8n} = f_{8n-8} + f_{8n-16} + \ldots + f_{16} + f_8 .$$

$$g_{8n} = f_{8n-8} + \underbrace{f_{8n-16} + \ldots + f_{16} + f_8}_{g_{8n-8}}$$

$$g_{8n} = f_{8n-8} + g_{8n-8} .$$

$$g_{8n+1} = f_{8n-7} + f_{8n-15} + \ldots + f_{17} + f_9 + f_1 .$$

$$g_{8n+1} = f_{8n-7} + \underbrace{f_{8n-15} + \ldots + f_{17} + f_9 + f_1}_{g_{8n-7}}$$

$$g_{8n+1} = f_{8n-7} + g_{8n-7} .$$

$$g_{8n+2} = f_{8n-6} + f_{8n-14} + \ldots + f_{18} + f_{10} + f_2 .$$

$$g_{8n+2} = f_{8n-6} + \underbrace{f_{8n-14} + \ldots + f_{18} + f_{10} + f_2}_{g_{8n-6}}$$

$$g_{8n+2} = f_{8n-6} + g_{8n-6}.$$

$$g_{8n+3} = f_{8n-5} + f_{8n-13} + \ldots + f_{19} + f_{11} + f_3.$$

$$g_{8n+3} = f_{8n-5} + \underbrace{f_{8n-13} + \ldots + f_{19} + f_{11} + f_3}_{g_{8n-5}}$$

$$g_{8n+3} = f_{8n-5} + g_{8n-5}\ .$$

$$g_{8n+4} = f_{8n-4} + f_{8n-12} + \ldots + f_{20} + f_{12} + f_4.$$

$$g_{8n+4} = f_{8n-4} + \underbrace{f_{8n-12} + \ldots + f_{20} + f_{12} + f_4}_{g_{8n-4}}$$

$$g_{8n+4} = f_{8n-4} + g_{8n-4}\ .$$

$$g_{8n+5} = f_{8n-3} + f_{8n-11} + \ldots + f_{21} + f_{13} + f_5.$$

$$g_{8n+5} = f_{8n-3} + \underbrace{f_{8n-11} + \ldots + f_{21} + f_{13} + f_5}_{g_{8n-3}}$$

$$g_{8n+5} = f_{8n-3} + g_{8n-3}\ .$$

$$g_{8n+6} = f_{8n-2} + f_{8n-10} + \ldots + f_{22} + f_{14} + f_6.$$

$$g_{8n+6} = f_{8n-2} + \underbrace{f_{8n-10} + \ldots + f_{22} + f_{14} + f_6}_{g_{8n-2}}$$

$$g_{8n+6} = f_{8n-2} + g_{8n-2}.$$

$$g_{8n+7} = f_{8n-1} + f_{8n-9} + \ldots + f_{23} + f_{15} + f_7.$$

$$g_{8n+7} = f_{8n-1} + \underbrace{f_{8n-9} + \ldots + f_{23} + f_{15} + f_7}_{g_{8n-1}}$$

$$g_{8n+7} = f_{8n-1} + g_{8n-1}.$$

We can distribute at least 36 balls to A,B,C,D,E,F,G, H boxes with $0 < x < y < z < p < q < h < j < m$ condition

Relationship between e_n and f_n;

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
f_n	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**
g_n	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**

n	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
f_n	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**2**	**3**	**5**	**7**	**11**	**15**	**21**	**28**	**38**	**49**	**65**
g_n	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**0**	**1**	**1**	**2**	**3**	**5**

f_n **sequence is the sequence** A267120 on OEIS

g_n **sequence is the sequence** A267121 on OEIS