Recent Developments in Boolean Networks Control

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Abstract

The aim of this survey paper is to provide the state of the art of the research on control and optimal control of Boolean Control Networks (BNCs), under the assumption that all the state variables are accessible and hence available for feedback. Necessary and sufficient conditions for stabilizability to a limit cycle or to an equilibrium point are given. Additionally, it is shown that when such conditions are satisfied, stabilization can always be achieved by means of state-feedback. Analogous results are obtained for the safe control problem, that is investigated for the first time in this survey. Finite and infinite horizon optimal control are subsequently considered, and solution algorithms are provided, based on suitable adaptations of the Riccati difference and algebraic equations. Finally, an appropriate definition of the cost function allows to restate and to solve both stabilization and safe control as infinite horizon optimal control problems.

Keywords

Boolean Control Networks, Algebraic Representation, Controllability, Stabilization, Optimal control, Safe control.

1 Introduction

Recent times have seen an increasing interest in Boolean control networks (BCNs), witnessed by the rising number of contributions dealing with either the theoretical or the practical aspects of this subject. This renewed interest is strongly motivated by the large number of physical processes whose logical/qualitative behavior can be conveniently described by means of this class of models. Indeed, in a number of contexts, ranging from biology (Sridharan, Layek, Datta, & Venkatraj, 2012), to game theory (Cheng, 2014; Thunberg, Ogren, & Hu, 2011), to multi-agent systems and consensus problems (Green, Leishman, & Sadedin, 2007; Lou & Hong, 2010), to smart homes (Kabir, Hoque, Koo, & Yang, 2014), the describing variables display only two operation levels (on/off, high/low, 1/0, ...), and the status of each of them is related to the statuses of the others by means of logical functions (combinations of “and”, “or” and “negation” operators). The application area where BCNs have proved to be more successful, however, is gene regulatory networks (Kauffman, 1969; Shmulevich, Dougherty, Kim, & Zhang, 2002). Indeed, genes may be regarded as binary devices that can be either active or inactive. Also, genes can be activated or inhibited, and this action can be modeled by resorting to external Boolean inputs.

Another research area where BCNs could prove their effectiveness is in the modeling and control of hybrid systems. Typically, logic relationships among the various subsystems involved in their description are formalized by means of discrete-event dynamic systems (DEDS), as for instance Petri nets (Koutsoukos & Antsaklis, 1999; Xu, Li, & Li, 1996). As BCNs represent a powerful tool to formalize dynamic systems whose describing variables update according to logical functions, they can surely compete and possible outperform DEDS in this role, thus benefitting various application areas where hybrid systems are used, e.g. intelligent manufacturing.

The algebraic representation recently introduced by D. Cheng and co-authors has allowed to cast BCNs into the framework of linear state models (operating on canonical vectors) (Cheng, 2009; Cheng & Qi, 2010a, 2010b; Cheng, Qi, & Li, 2011). This new set-up has proved to be highly beneficial for the research in the field, since it has allowed to derive matrix based characterizations for a number of properties of BCNs, and hence has suggested new approaches to the solution of several control problems for these networks. To mention a few, stability, stabilizability, controllability (Cheng & Liu, 2009; Cheng, Qi, Li, & Liu, 2011; Fornasini & Valcher, 2013b; Laschov & Margaliot, 2012; F. Li & Sun, 2012), disturbance decoupling (Cheng, 2011), observability (Fornasini & Valcher, 2013a; Kobayashi & Imura, 2009), Kalman decomposition (Zou & Zhu, 2015), fault detection (Fornasini & Valcher, 2015), and optimal control (Zhao, Li, & Cheng, 2011; Fornasini & Valcher, 2014b; Laschov & Margaliot, 2011a, 2011b), have been successfully investigated by referring to the algebraic representations of BCNs.

The aim of this survey paper is to overview some recent results regarding the control and optimal control of Boolean Control Networks (BNCs), under the assumption that all the state variables are accessible and hence available for feedback.

In detail, in Section 2 we review the algebraic representation of BCNs and recall the definitions of reachability and controllability, as well as their algebraic characterizations. This section is based on (Cheng, Qi, & Li, 2011; Laschov & Margaliot, 2012) and earlier works of D. Cheng and co-authors.

Section 3 addresses the problem of stabilizing a BCN to an equilibrium point and more generally to a limit cycle. The first problem was investigated in (Cheng & Liu, 2009; Cheng, Qi, & Li, 2010; Cheng, Qi, Li, & Liu, 2011; F. Li & Sun, 2012), and the latter in (Fornasini & Valcher, 2013b). State feedback stabilization techniques for BCNs have been successfully applied to some special gene regulatory networks. For instance, they have been used to control the dynamics of the lac Operon in the bacterium Escherichia Coli in (R. Li, Yang, & Chu, 2013) (see also (Bof, Fornasini, & Valcher, 2015; H. Li & Wang, 2013), where output feedback techniques have been applied).

The safe control problem investigated in Section 4 is, at the best of our knowledge, original, and can be regarded as a sort of stabilization to a given subset of states. In detail, the set of all states is partitioned into safe and unsafe states, and the target is to control the state evolution so that it always reaches the safe set, and once entered the safe set it can steadily remain there.

In many situations, a control problem is characterized not only by a target but also by a cost to achieve the target. When so, the problem can be naturally stated as a (either finite or infinite horizon) optimal control problem. This issue has been recently addressed in a few contributions. Specifically, in (Zhao et al., 2011) (see also Chapter 15 in (Cheng, Qi, & Li, 2011)) the problem of finding the input sequence that maximizes, on the infinite horizon, an average payoff that weights both the state and the input at every time $t \in \mathbb{Z}_+$, was investigated. Also, (Laschov & Margaliot, 2011b) and (Laschov & Margaliot, 2011a) considered the optimal control problem over a finite horizon, but restricted the analysis to the case when the payoff function only depends on the state of the BCN at the end of the control interval. The optimal solution is obtained by resorting to the maximum principle, and has the structure of a time varying state feedback law. In Sections 5 and 6 we report in concise form some of the results obtained in (Fornasini & Valcher, 2014b). A more comprehensive analysis of the optimal control problem, including comparisons with the aforementioned results obtained by Cheng and co-authors or Margaliot and co-authors, can be found in (Fornasini & Valcher, 2014b). Just to mention an interesting motivating example, the constrained intervention in a mammalian cell-cycle network (Faryabi, Vahedi, Chamberland, Datta, & Dougherty, 2008; Faure, Naldi, Chaouiya, & Thieffry, 2006; Hochma, Margaliot, Fornasini, & Valcher, 2013) has been formalized in (Fornasini & Valcher, 2014b) as an optimal control problem for BCNs.

Finally, in Section 7, we show that an appropriate definition of the cost function allows to restate and to solve both stabilization and safe control as infinite horizon optimal control problems.

A drawback of the algebraic state representation of BCNs is its computational complexity, as it converts a BCN with n state-variables and m input variables, into a state model of size 2^n, with 2^m inputs. Consequently, any algorithm based on these representations has an exponential time-complexity. However, Akutsu et. al. (Akutsu, Hayashida, Ching, & Ng, 2007) have already proved that finding control strategies for arbitrary BCNs is in general NP-hard. So, the computational complexity of the algorithms available to solve these problems seems to be intrinsic of the problems, and independent of the model adopted to describe the BCNs.

Nonetheless, when the graphs associated with the BCNs are sufficiently sparse (as it typically happens with large gene regulatory networks), the performances of the branch and bound algorithms proposed to solve such control problems are extremely good .

Notation. $\mathbb{Z}_+$ denotes the set of nonnegative integers. Given two integers $k, n \in \mathbb{Z}_+$, with $k \leq n$, by the symbol $[k, n]$ we denote the set of integers $\{k, k+1, \ldots, n\}$. We consider Boolean vectors and matrices, taking values in $\mathcal{B} := \{0, 1\}$, with the usual Boolean operations.

δ_k^i denotes the ith canonical vector of size k, $\mathcal{L}_k$ the set of all k-dimensional canonical vectors, and $\mathcal{L}_{k\times n} \subset \mathcal{B}^{k\times n}$ the set of all $k \times n$ matrices whose columns are canonical vectors of size k. Any matrix $L \in \mathcal{L}_{k\times n}$ can be represented as a row whose entries are canonical vectors in $\mathcal{L}_k$, namely $L = \begin{bmatrix} \delta_k^{i_1} & \delta_k^{i_2} & \ldots & \delta_k^{i_n} \end{bmatrix}$, for suitable indices $i_1, i_2, \ldots, i_n \in [1, k]$. The k-dimensional vector with all entries equal to 1, is denoted by $\mathbf{1}_k$. The (ℓ, j)th entry of a matrix L is denoted by $[L]_{\ell,j}$, while the ℓth entry of a vector $\mathbf{v}$ is $[\mathbf{v}]_\ell$. The ith column of a matrix L is $\mathrm{col}_i(L)$.

There is a bijective correspondence between Boolean variables $X \in \mathcal{B}$ and vectors $\mathbf{x} \in \mathcal{L}_2$, defined by the relationship

$$\mathbf{x} = \begin{bmatrix} X \\ \bar{X} \end{bmatrix}.$$

We introduce the <u>(left) semi-tensor product</u> $\ltimes$ between matrices (in particular, vectors) as follows (Cheng, Qi, & Li, 2011; Laschov & Margaliot, 2012; H. Li & Wang, 2012): given $L_1 \in \mathbb{R}^{r_1\times c_1}$ and $L_2 \in \mathbb{R}^{r_2\times c_2}$ (in particular, $L_1 \in \mathcal{L}_{r_1\times c_1}$ and $L_2 \in \mathcal{L}_{r_2\times c_2}$), we set

$$L_1 \ltimes L_2 := (L_1 \otimes I_{T/c_1})(L_2 \otimes I_{T/r_2}), \qquad T := \text{l.c.m.}\{c_1, r_2\},$$

where l.c.m. denotes the least common multiple. The semi-tensor product represents an extension of the standard matrix product, by this meaning that if $c_1 = r_2$, then $L_1 \ltimes L_2 = L_1 L_2$. Note that if $\mathbf{x}_1 \in \mathcal{L}_{r_1}$ and $\mathbf{x}_2 \in \mathcal{L}_{r_2}$, then $\mathbf{x}_1 \ltimes \mathbf{x}_2 \in \mathcal{L}_{r_1 r_2}$. For the various properties of the semi-tensor product we refer to (Cheng, Qi, & Li, 2011). By resorting to the semi-tensor product, we can extend the previous correspondence to a bijective correspondence between $\mathcal{B}^n$ and $\mathcal{L}_{2^n}$. This is possible in the following way: given $X = \begin{bmatrix} X_1 & X_2 & \ldots & X_n \end{bmatrix}^\top \in \mathcal{B}^n$, set

$$\mathbf{x} := \begin{bmatrix} X_1 \\ \bar{X}_1 \end{bmatrix} \ltimes \begin{bmatrix} X_2 \\ \bar{X}_2 \end{bmatrix} \ltimes \cdots \ltimes \begin{bmatrix} X_n \\ \bar{X}_n \end{bmatrix} = \begin{bmatrix} X_1 X_2 \ldots X_{n-1} X_n \\ X_1 X_2 \ldots X_{n-1} \bar{X}_n \\ X_1 X_2 \ldots \bar{X}_{n-1} X_n \\ \vdots \\ \bar{X}_1 \bar{X}_2 \ldots \bar{X}_{n-1} \bar{X}_n \end{bmatrix}.$$

2 Preliminaries

A Boolean control network is described by the following equation

$$X(t+1) = f(X(t), U(t)), \qquad t \in \mathbb{Z}_+, \tag{1}$$

where $X(t)$ and $U(t)$ denote the n-dimensional state variable and the m-dimensional input at time t, taking values in $\mathcal{B}^n$ and $\mathcal{B}^m$, respectively. f is a (logic) function, i.e. $f : \mathcal{B}^n \times \mathcal{B}^m \to \mathcal{B}^n$. By resorting to the semi-tensor product $\ltimes$, state and input Boolean variables can be represented as canonical vectors in $\mathcal{L}_N,\ N := 2^n$, and $\mathcal{L}_M,\ M := 2^m$, respectively, and the BCN (1) satisfies (Cheng, Qi, & Li, 2011) the following algebraic representation:

$$\mathbf{x}(t+1) = L \ltimes \mathbf{u}(t) \ltimes \mathbf{x}(t), \qquad t \in \mathbb{Z}_+, \tag{2}$$

where $\mathbf{x}(t) \in \mathcal{L}_N$ and $\mathbf{u}(t) \in \mathcal{L}_M$. $L \in \mathcal{L}_{N \times NM}$ is a matrix whose columns are canonical vectors of size N. For every choice of the input variable at time t, namely for every $\mathbf{u}(t) = \delta_M^k, k \in [1, M]$, $L \ltimes \mathbf{u}(t) =: L_k$ is a matrix in $\mathcal{L}_{N \times N}$, and we refer to the Boolean network (BN)

$$\mathbf{x}(t+1) = L_k \mathbf{x}(t), \qquad t \in \mathbb{Z}_+, \tag{3}$$

as to the kth subsystem of the BCN. Note that the matrix L can be expressed in terms of the matrices L_k as:

$$L = \begin{bmatrix} L_1 & L_2 & \dots & L_M \end{bmatrix}.$$

Definition 1. (Cheng, Qi, & Li, 2011) *Given a BCN (2), we say that $\mathbf{x}_f = \delta_N^i$ is reachable from $\mathbf{x}_0 = \delta_N^j$ if there exists $\tau \in \mathbb{Z}_+$ and an input $\mathbf{u}(t), t \in [0, \tau - 1]$, that leads the state trajectory from $\mathbf{x}(0) = \mathbf{x}_0$ to $\mathbf{x}(\tau) = \mathbf{x}_f$. The BCN is controllable if $\mathbf{x}_f$ is reachable from $\mathbf{x}_0$, for every choice of $\mathbf{x}_0, \mathbf{x}_f \in \mathcal{L}_N$.*

A state $\mathbf{x}_f = \delta_N^i$ is reachable from $\mathbf{x}_0 = \delta_N^j$ if and only if (Cheng, Qi, & Li, 2011) there exists $\tau \in \mathbb{Z}_+$ such that the Boolean sum of the matrices $L_k, k \in [1, M]$, namely

$$L_{tot} := \bigvee_{k=1}^{M} L_k,$$

satisfies $[L_{tot}^{\tau}]_{ij} = 1$. Consequently, by the theory of positive matrices (Brualdi & Ryser, 1991), the

BCN is controllable if and only if L_{tot} is an irreducible matrix, or, equivalently, the Boolean matrix

$$\mathbb{L} := \bigvee_{i=0}^{N-1} (L_{tot})^i \tag{4}$$

has all unitary entries. In the sequel, we will denote the set of states reachable from $\mathbf{x}_0$ as $\mathcal{R}(\mathbf{x}_0)$.

3 Stabilization

The first natural target of control is stabilization. Since a BCN is not a linear system and there is no equivalent of the zero state, we first need to understand what is a meaningful definition of stabilization in this context. The most intuitive extension of the concept of stabilization to the zero state is stabilization to some fixed state (an <u>equilibrium point</u>), namely regulation to a constant value. In turn, this can be seen as a special case of stabilization to a <u>limit cycle</u>, namely tracking of a periodic state trajectory.

Definition 2. *A BCN (2) is <u>stabilizable</u> to the elementary cycle $\mathcal{C} = (\delta_N^{i_1}, \delta_N^{i_2}, \dots, \delta_N^{i_k})$, where $\delta_N^{i_h} \neq \delta_N^{i_k}$, for $h \neq k$, if for every $\mathbf{x}(0) \in \mathcal{L}_N$ there exist $\mathbf{u}(t), t \in \mathbb{Z}_+$, and $\tau \in \mathbb{Z}_+$ such that $\mathbf{x}(t) = \delta_N^{i_j}$ for every $t \geq \tau$, where $j \in [1, k]$ and $j \equiv (t - \tau + 1) \bmod k$.*

Stabilization to an elementary limit cycle is characterized in the following proposition.

Proposition 1. *(Fornasini & Valcher, 2013b) A BCN (2) is stabilizable to the elementary cycle $\mathcal{C} = (\delta_N^{i_1}, \delta_N^{i_2}, \dots, \delta_N^{i_k})$, where $\delta_N^{i_h} \neq \delta_N^{i_k}$, for $h \neq k$, if and only if the following two conditions hold*

1) *$\delta_N^{i_1}$ is reachable from every initial state $\mathbf{x}(0)$, which amounts to saying that*

$$\delta_N^{i_1} \in \mathcal{R}^* := \bigcap_{\mathbf{x}(0) \in \mathcal{L}_N} \mathcal{R}(\mathbf{x}(0));$$

2) *for every $(i_\ell, i_{\ell+1}), \ell \in [1, k]$, (with $i_{k+1} = i_1$) there exists $\delta_M^{j_\ell}$ such that $\delta_N^{i_{\ell+1}} = L \ltimes \delta_M^{j_\ell} \ltimes \delta_N^{i_\ell} = L_{j_\ell} \delta_N^{i_\ell}$.*

In the special case of equilibrium points we have the following result.

Corollary 1. *(Fornasini & Valcher, 2013b; R. Li et al., 2013) A BCN (2) is stabilizable to the state $\mathbf{x}_e := \delta_N^i \in \mathcal{L}_N$ if and only if the following two conditions hold*

1) $\mathbf{x}_e$ *is reachable from every initial state* $\mathbf{x}(0)$*, i.e.,* $\mathbf{x}_e \in \mathcal{R}^*$*;*

2) $\mathbf{x}_e$ *is an equilibrium point of the* k*th subsystem (3), for some* $k \in [1, M]$*, namely there exists* δ_M^k *such that* $\delta_N^i = L \ltimes \delta_M^k \ltimes \delta_N^i = L_k \delta_N^i$*.*

To understand what are the states to which we may stabilize a BCN (2), we can proceed as follows: we first determine the set $\mathcal{R}^*$ of all states that are reachable from every initial state. This amounts to checking which rows of $\mathbb{L}$ have all unitary entries. Then, we identify the set $\mathcal{X}_e$ of all equilibrium points of the various subsystems: this amounts to determine all the vectors $\delta_N^i, i \in [1, N]$ such that there exists $j \in [1, M]$ for which $\mathrm{col}_i(L_j) = \delta_N^i$. Clearly, the BCN will be stabilizable to all vectors $\delta_N^i \in \mathcal{R}^* \cap \mathcal{X}_e$.

A similar characterization can be obtained for the set of all possible limit cycles to which the BCN can be stabilized. Again, we first consider the set $\mathcal{R}^*$ of all states that can be reached from every initial state, and then consider all cycles involving only elements of $\mathcal{R}^*$.

Up to now the stabilization problem to some cyclic trajectory (in particular, to an equilbrium point) has been addressed by assuming that at every time instant $t \in \mathbb{Z}_+$ the input variable $\mathbf{u}(t)$ can be freely chosen in $\mathcal{L}_M$. A quite interesting fact is that the stabilization problem can be solved by means of a time-invariant state feedback law, by this meaning that at every time instant t the input $\mathbf{u}(t)$ can be expressed as $\mathbf{u}(t) = K\mathbf{x}(t)$, for some matrix $K = \begin{bmatrix} \delta_M^{k_1} & \delta_M^{k_2} & \dots & \delta_M^{k_N} \end{bmatrix} \in \mathcal{L}_{M \times N}$. When so, the BCN is transformed into a BN:

$$\mathbf{x}(t) = L^{(K)}\mathbf{x}(t), \qquad t \in \mathbb{Z}_+,$$

$$L^{(K)} := \begin{bmatrix} \mathrm{col}_1(L_{k_1}) & \mathrm{col}_2(L_{k_2}) & \dots & \mathrm{col}_N(L_{k_N}) \end{bmatrix},$$

having the cycle $\mathcal{C}$ as its unique attractive set.

Proposition 2. *(Fornasini & Valcher, 2013b) If a BCN (2) is stabilizable to some elementary cycle* $\mathcal{C} = (\delta_N^{i_1}, \delta_N^{i_2}, \dots, \delta_N^{i_k})$*, then it is stabilizable by means of a state feedback law.*

The proof of the previous result is constructive and provides all matrices that implement a feedback law stabilizing the BCN in minimum time to the given cycle $\mathcal{C}$. Suppose, first, that the BCN is stabilizable to $\mathcal{C}$, namely conditions 1) and 2) of Proposition 1 hold. By condition 2), we know that, for every $\mathbf{x} = \delta_N^{i_\ell}, \ell \in [1, k]$, there exists $\mathbf{u} = \delta_M^{j_\ell}, j_\ell \in [1, M]$, such that $\delta_N^{i_{\ell+1}} = L \ltimes \delta_M^{j_\ell} \ltimes \delta_N^{i_\ell} = L \ltimes \mathbf{u} \ltimes \mathbf{x}$ (where $i_{k+1} = i_1$). By imposing $\mathbf{u} = \delta_M^{j_\ell} = K\delta_N^{i_\ell} = K\mathbf{x}, \forall\, \ell \in [1, k]$, we actually impose that $\mathrm{col}_{i_\ell}(K) := \delta_M^{j_\ell}, \forall\, \ell \in [1, k]$.

Let $\mathcal{S}_t, t \in \mathbb{Z}_+$, denote the set of all states $\delta_N^i, i \in [1, N]$, whose minimum distance from the cycle $\mathcal{C}$ is t, by this meaning that the length of the shortest path from the state δ_N^i to any state of $\mathcal{C}$ is t. Clearly, $\mathcal{S}_0 = \mathcal{C}$, and $\mathcal{S}_{t+1} \neq \emptyset$ implies $\mathcal{S}_t \neq \emptyset$. On the other hand, for every $t > N - k$, $\mathcal{S}_t = \emptyset$. Finally, by assumption 1), $\mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2 \cup \cdots \cup \mathcal{S}_{N-k} = \mathcal{L}_N$, and all sets $\mathcal{S}_t$ are disjoint. Since for every $\mathbf{x} = \delta_N^i \in \mathcal{S}_{t+1}$ there exists $\mathbf{u} = \delta_M^j$ such that $L \ltimes \delta_M^j \ltimes \delta_N^i \in \mathcal{S}_t$, it is easy to see that by assuming $\mathrm{col}_i(K) := \delta_M^j$, for every $\delta_N^i \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \cdots \cup \mathcal{S}_{N-k}$, we assign all the remaining columns of K. Therefore, the feedback law $\mathbf{u}(t) = K\mathbf{x}(t)$ allows to converge to $\mathcal{C}$ and to remain therein.

An independent proof for the case of an equilibrium point has been given in (R. Li et al., 2013).

Remark 1. *The state-feedback laws proposed in (Fornasini & Valcher, 2013b; R. Li et al., 2013) are minimal time. If we drop this requirement, we can obtain a larger set of state-feedback laws. In particular, when solving the stabilization problem by means of an output feedback law, the underlying state-feedback law need not be minimal time (Bof et al., 2015).*

On the other hand, a time-varying state feedback law does not represent an interesting option for this control problem. Indeed, it does not allow to achieve stabilization when a stabilizing time-invariant law does not exist, and it does not offer any advantage in terms of performances when it exists.

Example 1. *Consider a BCN (2), with $n = 3$ and $m = 1$, and suppose that*

$$\begin{aligned} L_1 &:= L \ltimes \delta_2^1 = \begin{bmatrix} \delta_8^2 & \delta_8^3 & \delta_8^4 & \delta_8^5 & \delta_8^2 & \delta_8^7 & \delta_8^8 & \delta_8^3 \end{bmatrix}, \\ L_2 &:= L \ltimes \delta_2^2 = \begin{bmatrix} \delta_8^5 & \delta_8^1 & \delta_8^6 & \delta_8^2 & \delta_8^4 & \delta_8^7 & \delta_8^8 & \delta_8^8 \end{bmatrix}. \end{aligned}$$

A possible limit cycle is $\mathcal{C} = (\delta_8^2, \delta_8^3, \delta_8^4)$. The transition from δ_8^2 to δ_8^3 and from δ_8^3 to δ_8^4 is associated with $\mathbf{u} = \delta_2^1$ (and hence with L_1),while the transition from δ_8^4 to δ_8^2 is associated with $\mathbf{u} = \delta_2^2$ (with L_2). Accordingly we have that $K\delta_4^2 = \delta_2^1, K\delta_4^3 = \delta_2^1, K\delta_4^4 = \delta_2^2$. If we consider now the states that have distance t from $\mathcal{C}$, we find $\mathcal{S}_1 = \{\delta_8^1, \delta_8^5, \delta_8^8\}$; $\mathcal{S}_2 = \{\delta_8^7\}$; $\mathcal{S}_3 = \{\delta_8^6\}$. Keeping in mind what are the input values that induce the shortest paths from each of these states to $\mathcal{C}$, we obtain as possible feedback matrices all matrices

$$K = \begin{bmatrix} \delta_2^1 & \delta_2^1 & \delta_2^1 & \delta_2^2 & * & * & * & \delta_2^1 \end{bmatrix},$$

where $*$ denotes columns that can be either δ_2^1 or δ_2^2.

4 Safe control

When dealing with a physical system whose logical functioning can be modeled by means of a BCN, it may happen that a subset of all possible states of the BCN is regarded as unsafe. We denote the set of all unsafe states by the symbol X_u and accordingly think of the complementary set as the set of safe states X_s. In this case, the natural problem arises:

Given a BCN (2), whose set of states $\mathcal{L}_N$ is partitioned into a set of unsafe states X_u and a set of safe states X_s, is it possible to control the system so that every state trajectory that stems from a safe state remains in X_s and every trajectory that originates in an unsafe state leaves X_u in a finite number of steps and then remains in X_s?

The mathematical translation into the language of the BCNs of the safe control problem is rather straightforward: we look for conditions ensuring that

- for every $\mathbf{x}_0 \in X_s$ there exists $\mathbf{u}(t), t \in \mathbb{Z}_+$, such that $\mathbf{x}(t) \in X_s$ for every $t \in \mathbb{Z}_+$;
- for every $\mathbf{x}_0 \in X_u$ there exist $\mathbf{u}(t), t \in \mathbb{Z}_+$, and $T > 0$ such that $\mathbf{x}(t) \in X_s$ for every $t \in \mathbb{Z}_+, t \geq T$.

This easily leads to a characterization of the problem solvability.

Proposition 3. *Given a BCN (2) and a set of unsafe states X_u, the safe control problem is solvable if and only if the following two conditions hold*

1) *for every $\mathbf{x} \in X_s = \mathcal{L}_N \setminus X_u$ there exists $\mathbf{u} \in \mathcal{L}_M$ such that $L \ltimes \mathbf{u} \ltimes \mathbf{x} \in X_s$;*
2) *the set X_s is reachable from every $\mathbf{x} \in X_u$, which amounts to saying that for every $\mathbf{x} \in X_u$ there exists $\bar{\mathbf{x}} \in X_s$ such that $\bar{\mathbf{x}}$ is reachable from $\mathbf{x}$.*

Proof. It is easily seen that condition 1) is equivalent to saying that every trajectory starting in X_s can be kept, by resorting to a suitable input, within X_s. On the other hand, it is easily seen that the possibility of eventually leading any trajectory stemming from X_u into X_s is equivalent to the reachability of X_s from every state of X_u, and the first condition ensures that once the trajectory enters X_s it will be able to remain therein. □

A mathematical characterization of problem solvability in terms of the matrices L_{tot} and $\mathbb{L}$ we introduced in Section 2 can be immediately provided.

Proposition 4. *Given a BCN (2) and a set of unsafe states X_u, suppose without loss of generality that the BCN states are relabelled so that $X_u = \{\delta_N^i : i \in [k+1, N]\}$ while $X_s = \{\delta_N^i : i \in [1, k]\}$. Then the safe control problem is solvable if and only if the following two conditions hold*

1) for every $j \in [1, k]$ there exists $i \in [1, k]$ such that $[L_{tot}]_{ij} = 1$;

2) for every $j \in [k+1, N]$ there exists $i \in [1, k]$ such that $[\mathbb{L}]_{ij} = 1$.

To conclude, we prove that also the safe control problem can always be solved by resorting to a state feedback law.

Proposition 5. *If the safe control problem is solvable for a BCN (2), then it is solvable by means of a state feedback law.*

Proof. If the safe control problem is solvable, then conditions 1) and 2) of Proposition 3 hold. We want to make use of these two conditions to define the columns of K, one by one. We first consider the indices $i \in [1, N]$ such that $\delta_N^i \in X_s$. By condition 1), we know that, for every $\mathbf{x} = \delta_N^i \in X_s$ there exists $\mathbf{u} = \delta_M^j, j \in [1, M]$, such that $L \ltimes \delta_M^j \ltimes \delta_N^i = L \ltimes \mathbf{u} \ltimes \mathbf{x} \in X_s$, and hence it is sufficient to impose

$$\mathbf{u} = \delta_M^j = K\delta_N^i = K\mathbf{x},$$

which amounts to imposing $\mathrm{col}_i(K) := \delta_M^j$.

Let $\mathcal{S}_t, t \in \mathbb{Z}_+$, denote the set of all states $\delta_N^i \in X_u$ whose minimum distance from the set X_s is t, by this meaning that the length of the shortest path from the state δ_N^i to any state of X_s is t. Clearly, $\mathcal{S}_0 = X_s$, and $\mathcal{S}_{t+1} \neq \emptyset$ implies $\mathcal{S}_t \neq \emptyset$. On the other hand, for every $t > |X_u|$, $\mathcal{S}_t = \emptyset$. Finally, by assumption 2),

$$\mathcal{S}_0 \cup \mathcal{S}_1 \cup \mathcal{S}_2 \cup \cdots \cup \mathcal{S}_{|X_u|} = \mathcal{L}_N,$$

and all sets $\mathcal{S}_t$ are disjoint. Since for every $\mathbf{x} = \delta_N^i \in \mathcal{S}_{t+1}$ there exists $\mathbf{u} = \delta_M^j$ such that $L \ltimes \delta_M^j \ltimes \delta_N^i \in \mathcal{S}_t$, it is easy to see that by assuming $\mathrm{col}_i(K) := \delta_M^j$, for every $\delta_N^i \in \mathcal{S}_1 \cup \mathcal{S}_2 \cup \cdots \cup \mathcal{S}_{|X_u|} = X_u$, we assign all the remaining columns of K. Therefore, the feedback law $\mathbf{u}(t) = K\mathbf{x}(t)$ allows to solve the safe control problem. □

5 Finite horizon optimal control of BCNs

The interest in the optimal control problem for BCNs arises primarily, but not exclusively, from two research areas. On the one hand, interesting applications to game theory have been illustrated in

(Cheng, Qi, & Li, 2011; Zhao et al., 2011). On the other hand, the results presented in (Ching et al., 2009; Liu, 2013; Liu, Guo, & Zhou, 2010; Pal, Datta, & Dougherty, 2006; Yang, Wai-Ki, Nam-Kiu, & Ho-Yin, 2010) provide evidence of the fact that optimal control problems naturally arise when dealing with biological systems in general, and genetic networks in particular, and BCNs often represent a very convenient set-up where to investigate these problems. In particular, the constrained intervention in a mammalian cell-cycle network (Faryabi et al., 2008; Faure et al., 2006; Hochma et al., 2013) can be naturally stated as an optimal control problem.

We first introduce the finite horizon optimal control problem. If we assume that the initial state is given and the final state is potentially arbitrary, this problem can be stated in general form as follows:

Given the BCN (2), with initial state $\mathbf{x}(0) = \mathbf{x}_0 \in \mathcal{L}_N$, *determine an input sequence that minimizes the cost function:*

$$J_T(\mathbf{x}_0, \mathbf{u}(\cdot)) = \mathcal{Q}_f(\mathbf{x}(T)) + \sum_{t=0}^{T-1} \mathcal{Q}(\mathbf{u}(t), \mathbf{x}(t)), \tag{5}$$

where $\mathcal{Q}_f(\cdot)$ *is any function defined on* $\mathcal{L}_N$, *and* $\mathcal{Q}(\cdot, \cdot)$ *is any function defined on* $\mathcal{L}_M \times \mathcal{L}_N$.

The cost function (5) weights the BCN state at every time instant: the final state is weighted by a special function $\mathcal{Q}_f(\cdot)$, while the state at every intermediate instant t is weighted, together with the input value at the same time, by the function $\mathcal{Q}(\cdot, \cdot)$.

In (Fornasini & Valcher, 2014b), we have shown that, by taking advantage of the properties of the semi-tensor product and of the fact that the state and input vectors are always canonical vectors, every cost function described as in (5) can be equivalently expressed as a linear cost function of the form

$$J_T(\mathbf{x}_0, \mathbf{u}(\cdot)) = \mathbf{c}_f^\top \mathbf{x}(T) + \sum_{t=0}^{T-1} \mathbf{c}^\top \ltimes \mathbf{u}(t) \ltimes \mathbf{x}(t), \tag{6}$$

where $\mathbf{c}_f \in \mathbb{R}^N$ and $\mathbf{c} \in \mathbb{R}^{NM}$ are nonnegative vectors. We assume

$$\mathbf{c}^\top = \left[\mathbf{c}_1^\top \;\middle|\; \dots \;\middle|\; \mathbf{c}_M^\top \right], \qquad \mathbf{c}_i \in \mathbb{R}^N.$$

In order to solve this problem, we adopt a technique that is similar to the square completion technique adopted to solve the quadratic optimal control problem for linear, time-invariant, discrete-time state space models. However, since our cost function is "linear", the terms we will add are in turn "linear" functions of the state and input vectors. We observe that for every choice of a family of

N-dimensional real vectors $\mathbf{m}(t), t \in [0,T]$, and every state trajectory $\mathbf{x}(t), t \in [0,T]$, of the BCN, one has

$$0 = \sum_{t=0}^{T-1} [\mathbf{m}(t+1)^\top \mathbf{x}(t+1) - \mathbf{m}(t)^\top \mathbf{x}(t)] + \mathbf{m}(0)^\top \mathbf{x}(0) - \mathbf{m}(T)^\top \mathbf{x}(T).$$

Consequently, the cost function (6) can be equivalently written as

$$\begin{aligned} J_T(\mathbf{x}_0, \mathbf{u}(\cdot)) &= \mathbf{m}(0)^\top \mathbf{x}(0) + [\mathbf{c}_f - \mathbf{m}(T)]^\top \mathbf{x}(T) + \sum_{t=0}^{T-1} \mathbf{c}^\top \ltimes \mathbf{u}(t) \ltimes \mathbf{x}(t) \\ &+ \sum_{t=0}^{T-1} [\mathbf{m}(t+1)^\top \mathbf{x}(t+1) - \mathbf{m}(t)^\top \mathbf{x}(t)]. \end{aligned}$$

By making use of the state update equation of the BCN (2) and of the fact that, for every choice of $\mathbf{u}(t) \in \mathcal{L}_M$, one has

$$\mathbf{m}(t)^\top \mathbf{x}(t) = \begin{bmatrix} \mathbf{m}(t)^\top & \mathbf{m}(t)^\top & \dots & \mathbf{m}(t)^\top \end{bmatrix} \ltimes \mathbf{u}(t) \ltimes \mathbf{x}(t),$$

we get

$$\begin{aligned} J_T(\mathbf{x}_0, \mathbf{u}(\cdot)) &= \mathbf{m}(0)^\top \mathbf{x}(0) + [\mathbf{c}_f - \mathbf{m}(T)]^\top \mathbf{x}(T) \\ &+ \sum_{t=0}^{T-1} \left(\mathbf{c}^\top + \mathbf{m}(t+1)^\top L - \begin{bmatrix} \mathbf{m}(t)^\top & \dots & \mathbf{m}(t)^\top \end{bmatrix} \right) \ltimes \mathbf{u}(t) \ltimes \mathbf{x}(t). \end{aligned}$$

Now, since the values of the vectors $\mathbf{m}(t), t \in [0,T]$, do not affect the value of the index, we choose them according to the following **algorithm**:

- [Initialization] Set $\mathbf{m}(T) := \mathbf{c}_f$;
- [Recursion] For $t = T-1, T-2, \dots, 1, 0$, the jth entry of the vector $\mathbf{m}(t)$ is chosen according to the recursive rule:

$$[\mathbf{m}(t)]_j := \min_{i \in [1,M]} \Big([\mathbf{c}_i]_j + [\mathbf{m}(t+1)^\top L_i]_j \Big), \ \forall\, j \in [1,N]. \tag{7}$$

We notice that, by the previous algorithm, for every $t \in [0, T-1]$ the vector

$$\begin{aligned}
\mathbf{w}(t)^\top &:= \begin{bmatrix} \mathbf{w}_1(t)^\top & \mathbf{w}_2(t)^\top & \dots & \mathbf{w}_M(t)^\top \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{c}_1^\top & \big| & \dots & \big| & \mathbf{c}_M^\top \end{bmatrix} + \mathbf{m}(t+1)^\top \begin{bmatrix} L_1 & L_2 & \dots & L_M \end{bmatrix} \\
&- \begin{bmatrix} \mathbf{m}(t)^\top & \mathbf{m}(t)^\top & \dots & \mathbf{m}(t)^\top \end{bmatrix}
\end{aligned}$$

is nonnegative and satisfies the following condition: for every $j \in [1, N]$ there exists $i \in [1, M]$ such that $[\mathbf{w}_i(t)]_j = 0$. As a result, the index

$$J_T(\mathbf{x}_0, \mathbf{u}(\cdot)) = \mathbf{m}(0)^\top \mathbf{x}(0) + \sum_{t=0}^{T-1} \begin{bmatrix} \mathbf{w}_1(t)^\top & \mathbf{w}_2(t)^\top & \dots & \mathbf{w}_M(t)^\top \end{bmatrix} \ltimes \mathbf{u}(t) \ltimes \mathbf{x}(t)$$

is minimized by the input sequence $\mathbf{u}(t), t \in [0, T-1]$, that is obtained according to this rule:

$$\mathbf{x}(t) = \delta_N^j \qquad \Rightarrow \qquad \mathbf{u}(t) = \delta_M^{i^*(j,t)},$$

where[1]

$$i^*(j,t) = \arg \min_{i \in [1,M]} \Big([\mathbf{c}_i]_j + [\mathbf{m}(t+1)^\top L_i]_j \Big).$$

In this way,

$$\begin{bmatrix} \mathbf{w}_1^\top(t) & \dots & \mathbf{w}_M^\top(t) \end{bmatrix} \ltimes \mathbf{u}(t) \ltimes \mathbf{x}(t) = 0, \ \forall\, t \in [0, T-1],$$

and by the nonnegativity of the vector $\mathbf{w}(t)$, this is the minimum possible value that this term can take.

So, we have proved the following result.

Theorem 1. *Given the BCN (2), with initial state $\mathbf{x}(0) = \mathbf{x}_0 \in \mathcal{L}_N$, the minimum value of the cost function (6) is $J_T^*(\mathbf{x}_0) := \min_{\mathbf{u}(\cdot)} J_T(\mathbf{x}_0, \mathbf{u}(\cdot)) = \mathbf{m}(0)^\top \mathbf{x}(0)$, where $\mathbf{m}(0)$ is obtained according to the previous algorithm. Moreover, the optimal control input can be implemented by means of a time-varying state feedback law. Actually, the optimal input can be expressed as*

$$\mathbf{u}(t) = K(t)\mathbf{x}(t),$$

[1]Note that the index that minimizes the function is not necessarily unique: so there is not necessarily a unique optimal input.

where the state feedback matrix is

$$K(t) = \begin{bmatrix} \delta_M^{i^*(1,t)} & \delta_M^{i^*(2,t)} & \dots & \delta_M^{i^*(N,t)} \end{bmatrix}.$$

Remark 2. *Equation (7) can be viewed as the equivalent for BCNs of the difference Riccati equation for standard discrete-time linear systems with a quadratic cost function. The updating algorithm, however, is based on a linear functional instead of a quadratic one, due to the structure of the cost function and of the state updating equation.*

Example 2. *Consider the BCN (2) and suppose that* $N = 8$, $M = 2$ *and*

$$\begin{aligned} L_1 &:= L \ltimes \delta_2^1 = \begin{bmatrix} \delta_8^4 & \delta_8^5 & \delta_8^4 & \delta_8^5 & \delta_8^6 & \delta_8^7 & \delta_8^8 & \delta_8^7 \end{bmatrix}, \\ L_2 &:= L \ltimes \delta_2^2 = \begin{bmatrix} \delta_8^2 & \delta_8^4 & \delta_8^1 & \delta_8^7 & \delta_8^6 & \delta_8^5 & \delta_8^6 & \delta_8^6 \end{bmatrix}. \end{aligned}$$

We consider the problem of minimizing the cost function (6) for $T = 4$, *by assuming*

$$\mathbf{c}_f = \begin{bmatrix} 1 & 1 & 1 & 2 & 1 & 10 & 0 & 0 \end{bmatrix}^\top, \qquad \mathbf{c} = \begin{bmatrix} \mathbf{1}_8^\top & \mathbf{0}_8^\top \end{bmatrix}^\top,$$

and initial condition $\mathbf{x}(0) = \delta_8^1$.

It is worth noticing that the input $\mathbf{u}(t) = \delta_2^2$ *has zero cost. So, one would be tempted to just assume* $\mathbf{u}(t) = \delta_2^2$ *for every* $t \in [0, 3]$. *This way, however,* $\mathbf{x}(4)$ *would be equal to* δ_8^6, *which is the "most expensive" final state. So, we proceed according to the algorithm:*

- $\mathbf{m}(4) = \mathbf{c}_f = \begin{bmatrix} 1 & 1 & 1 & 2 & 1 & 10 & 0 & 0 \end{bmatrix}^\top$;
- $\mathbf{m}(3) = \begin{bmatrix} 1 & 2 & 1 & 0 & 10 & 1 & 1 & 1 \end{bmatrix}^\top$ *and* $K(3) = \begin{bmatrix} \delta_2^2 & \delta_2^2 & \delta_2^2 & \delta_2^2 & \delta_2^2 & \delta_2^2 & \delta_2^1 & \delta_2^1 \end{bmatrix}$;
- $\mathbf{m}(2) = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 2 & 1 & 1 \end{bmatrix}^\top$ *and* $K(2) = \begin{bmatrix} \delta_2^1 & \delta_2^2 & \delta_2^2 & \delta_2^2 & \delta_2^2 & \delta_2^1 & \delta_2^2 & \delta_2^2 \end{bmatrix}$;
- $\mathbf{m}(1) = \begin{bmatrix} 0 & 1 & 1 & 1 & 2 & 1 & 2 & 2 \end{bmatrix}^\top$ *and* $K(1) = \begin{bmatrix} \delta_2^2 & \delta_2^2 & \delta_2^2 & \delta_2^2 & \delta_2^2 & \delta_2^2 & \delta_2^2 & \delta_2^2 \end{bmatrix}$;
- $\mathbf{m}(0) = \begin{bmatrix} 1 & 1 & 0 & 2 & 1 & 2 & 1 & 1 \end{bmatrix}^\top$ *and* $K(0) = \begin{bmatrix} \delta_2^2 & \delta_2^2 & \delta_2^2 & \delta_2^2 & \delta_2^2 & \delta_2^2 & \delta_2^2 & \delta_2^2 \end{bmatrix}$.

As a consequence, $J_4^*(\delta_8^1) = \min_{\mathbf{u}(\cdot)} J_4(\delta_8^1, \mathbf{u}(\cdot)) = \mathbf{m}(0)^\top \delta_8^1 = 1$. *An optimal input sequence is* $\mathbf{u}^*(0) = \mathbf{u}^*(1) = \mathbf{u}^*(2) = \delta_2^2, \mathbf{u}^*(3) = \delta_2^1$, *and it corresponds to the state-trajectory* $\mathbf{x}^*(0) = \delta_8^1$, $\mathbf{x}^*(1) = \delta_8^2$, $\mathbf{x}^*(2) = \delta_8^4$, $\mathbf{x}^*(3) = \delta_8^7$, $\mathbf{x}^*(4) = \delta_8^8$. ♠

6 Infinite horizon optimal control problem

The natural extension of the previous optimal control problem to the infinite horizon case can be stated as follows:

Given the BCN (2), with initial state $\mathbf{x}(0) = \mathbf{x}_0 \in \mathcal{L}_N$*, determine an input sequence that minimizes the quadratic cost function:*

$$J(\mathbf{x}_0, \mathbf{u}(\cdot)) = \sum_{t=0}^{+\infty} \mathbf{c}^\top \ltimes \mathbf{u}(t) \ltimes \mathbf{x}(t), \tag{8}$$

where $\mathbf{c} \in \mathbb{R}^{NM}$. As in the finite horizon case, we assume that the vector $\mathbf{c}$ is nonnegative.

We first remark that the only possibility of obtaining a finite value for the optimum index

$$J^*(\mathbf{x}_0) := \min_{\mathbf{u}(\cdot)} J(\mathbf{x}_0, \mathbf{u}(\cdot)),$$

is represented by the existence of a periodic state-input trajectory $(\mathbf{x}(t), \mathbf{u}(t))_{t\in\mathbb{Z}_+}$ of zero cost, that can be "reached" from $\mathbf{x}_0$. This amounts to saying that there must be $T > 0$, $\tau \geq 0$ and $\mathbf{u}(t), t \in \mathbb{Z}_+$, such that

$$(\mathbf{x}(t), \mathbf{u}(t)) = (\mathbf{x}(t+T), \mathbf{u}(t+T)),\ \forall\, t \in \mathbb{Z}_+, t \geq \tau, \tag{9}$$

and

$$\mathbf{c}^\top \ltimes \mathbf{u}(t) \ltimes \mathbf{x}(t) = 0,\ \forall\, t \in \mathbb{Z}_+, t \geq \tau. \tag{10}$$

Condition (10) implies, in particular, that the optimal control problem has a finite solution only if the vector $\mathbf{c}$ has some zero entry.

Proposition 6. *(Fornasini & Valcher, 2014b) The minimum value* $J^*(\mathbf{x}_0)$ *of the infinite horizon cost function (8) is finite for every choice of the initial state* $\mathbf{x}_0 \in \mathcal{L}_N$ *if and only if for every* $\mathbf{x}_0$ *there exists an input sequence that makes the resulting state-input trajectory both periodic and zero-cost starting from some time instant* $\tau \geq 0$.

The previous characterization essentially requires to perform two checks on the BCN: (1) to verify the existence of zero-cost periodic state-input trajectories, and (2) to check that every initial state $\mathbf{x}_0$ can reach (at least) one of the states belonging to any such periodic trajectory[2]. So, we now look into

[2] In the following, we will informally talk about zero-cost periodic state trajectory by this meaning the projection of a zero-cost periodic state-input trajectory over the state component.

these two issues. (1) How can we check whether zero-cost periodic state-input trajectories exist? A simple test can be performed in the following way. Let $C_i^{(0)}$ be the $N \times N$ matrix, whose columns are obtained from the cost vector $\mathbf{c}_i^\top$ according to the following rule

$$\mathrm{col}_j(C_i^{(0)}) := \begin{cases} \delta_N^j, & \text{if } [\mathbf{c}_i]_j = 0; \\ 0_N, & \text{otherwise}; \end{cases} \qquad j \in [1, N].$$

It is easily seen that $L_i C_i^{(0)}$ is obtained from L_i by simply replacing with zero columns the columns corresponding to state transitions (driven by the input value $\mathbf{u} = \delta_M^i$) of positive cost. Consequently,

$$L^{(0)} := (L_1 C_1^{(0)}) \vee (L_2 C_2^{(0)}) \vee \ldots \vee (L_M C_M^{(0)})$$

is the Boolean matrix representing all the state transitions that can be achieved at zero cost, provided that a suitable input is selected. In other words, $[L^{(0)}]_{h,j} = 1$ if and only if there exists $i \in [1, M]$ such that

$$\delta_N^h = L \ltimes \delta_M^i \ltimes \delta_N^j \qquad \text{and} \qquad \mathbf{c}^\top \ltimes \delta_M^i \ltimes \delta_N^j = 0.$$

So, it is clear that a zero-cost periodic state (and hence state-input) trajectory exists if and only if the digraph $\mathcal{D}(L^{(0)})$ has at least one cycle or, equivalently, $L^{(0)}$ is not nilpotent.

(2) Is it possible from every initial state to reach at least one of the (states belonging to) periodic zero-cost state trajectories? If $L^{(0)}$ is not nilpotent, then $(L^{(0)})^N \neq 0$. Consequently, we may introduce the set

$$H := \{h \in [1, N] : (\delta_N^h)^\top (L^{(0)})^N \neq 0\} \tag{11}$$

of the indices of the nonzero rows in $(L^{(0)})^N$. Elementary graph theory allows us to say that $h \in H$ if and only if the state δ_N^h belongs to one of these periodic zero-cost state trajectories. From every initial state $\mathbf{x}_0 = \delta_N^j \in \mathcal{L}_N$ it is possible to reach some state $\delta_N^h, h \in H$, if and only if for every $j \in [1, N]$ the jth column of L_{tot}^N has at least one nonzero entry indexed by H.

The previous comments immediately suggest a way to obtain the minimum cost $J^*(\mathbf{x}_0)$ for every $\mathbf{x}_0 = \delta_N^j \in \mathcal{L}_N$. Clearly, $J^*(\delta_N^h) = 0$ for every $h \in H$. On the other hand, for every state $\delta_N^j, j \notin H$, it is sufficient to determine the minimum cost state-input trajectory $(\mathbf{x}(t), \mathbf{u}(t))_{t \in \mathbb{Z}_+}$ starting from $\mathbf{x}(0)$ and reaching some state $\delta_N^h, h \in H$, in a finite number (at most $N-1$) of steps.

Remark 3. *It is worthwhile noticing that*

$$H \subseteq H^* := \{h \in [1, N] : J^*(\delta_N^h) = 0\},$$

but the two sets do not necessarily coincide, as there may be states δ_N^j *that access at zero cost some states* $\delta_N^h, h \in H$*, but* $j \notin H$*.*

Under the previous assumptions, and by making use of the results regarding the finite horizon optimal control problem, we can derive the expression of the optimal cost corresponding to any given initial condition, and show that the optimal solution can be expressed as a time-invariant state-feedback.

We first note that, when T is sufficiently large, the finite vector sequence $\{\mathbf{m}(t)\}_{t=T,T-1,\dots,2,1,0}$, generated by the algorithm described in Section 5, starting from the initial condition $\mathbf{m}(T) = 0$, converges in a finite number of steps to a nonnegative vector $\mathbf{m}^*$.

Lemma 1. *The finite vector sequence* $\{\mathbf{m}(t)\}_{t=T,T-1,\dots,2,1,0}$*, generated by the algorithm:*

$$\begin{aligned} \mathbf{m}(T) &= 0; \\ [\mathbf{m}(t)]_j &= \min_{i\in[1,M]} [\mathbf{c}_i^\top + \mathbf{m}(t+1)^\top L_i]_j, \\ &\qquad\qquad j \in [1, N], t = T-1, T-2, \dots, 1, 0, \end{aligned}$$

satisfies

$$0 = \mathbf{m}(T) \le \mathbf{m}(T-1) \le \cdots \le \mathbf{m}(1) \le \mathbf{m}(0). \tag{12}$$

Moreover, for T *sufficiently large, there exists* $\Delta \ge 0$ *such that*

$$\mathbf{m}^* := \mathbf{m}(T-\Delta) = \mathbf{m}(T-\tau), \qquad \forall\, \tau \in [\Delta, T]. \tag{13}$$

As an immediate consequence of the previous lemma, we can claim what follows:

Theorem 2. *Assume that the set* H *defined in (11) is not empty, and for every choice of* $\mathbf{x}_0$ *there exists at least one state* $\mathbf{x}_f = \delta_N^h, h \in H$*, reachable from* $\mathbf{x}_0$*. Then*

1. *there exists* $\bar{T} \ge 0$ *such that, for every* $\mathbf{x}_0$*,*

$$J_T^*(\mathbf{x}_0) = J_{\bar{T}}^*(\mathbf{x}_0) = (\mathbf{m}^*)^\top \mathbf{x}_0, \qquad \forall\, T \ge \bar{T},$$

and therefore

$$J^*(\mathbf{x}_0) = \min_{\mathbf{u}(\cdot)} \sum_{t=0}^{+\infty} \mathbf{c}^\top \ltimes \mathbf{u}(t) \ltimes \mathbf{x}(t) = (\mathbf{m}^*)^\top \mathbf{x}_0.$$

2. $\mathbf{m}^*$ *is obtained through the algorithm of Section 5, by assuming* $\mathbf{c}_f = 0$*, and is a fixed point of the algorithm, namely a solution of the family of equations:*

$$[\mathbf{m}^*]_j = \min_{i\in[1,M]} [\mathbf{c}_i^\top + (\mathbf{m}^*)^\top L_i]_j, \qquad j \in [1, N], \tag{14}$$

that represent the equivalent, for BCNs, of the algebraic Riccati equation for linear systems.

3. *Upon defining*

$$i^*(j) := \arg \min_{i\in[1,M]} [\mathbf{c}_i^\top + (\mathbf{m}^*)^\top L_i]_j,$$

the optimal control input can be implemented by means of the time-invariant state-feedback law:

$$\mathbf{u}(t) = K\mathbf{x}(t),$$

where the (not necessarily unique) feedback matrix is

$$K = \begin{bmatrix} \delta_M^{i^*(1)} & \delta_M^{i^*(2)} & \dots & \delta_M^{i^*(N)} \end{bmatrix}.$$

For all the computational issues related to the solution of the infinite horizon optimal control problem we refer the interested reader to the paper (Fornasini & Valcher, 2014b).

7 Stabilization and safe control as optimal control problems

We have seen in Section 3 that if a BCN (2) is stabilizable to an elementary cycle $\mathcal{C}$, and in particular to an equilibrium point $\mathbf{x}_e$, then stabilization is achievable by means of a static state-feedback law. We want to show that the same result can be obtained by casting this problem into the optimal control set-up, and by resorting to the results of the previous section.

Assume $\mathcal{C} = (\delta_N^{i_1}, \delta_N^{i_2}, \dots, \delta_N^{i_k})$, where $\delta_N^{i_h} \neq \delta_N^{i_k}$, for $h \neq k$, and set

$$V(\delta_N^{i_h}) := \begin{cases} \{j \in [1, M] : \delta_N^{i_{h+1}} = L \ltimes \delta_M^j \ltimes \delta_N^{i_h}\}, & \text{if } h \in [1, k-1]; \\ \{j \in [1, M] : \delta_N^{i_1} = L \ltimes \delta_M^j \ltimes \delta_N^{i_k}\}, & \text{if } h = k. \end{cases}$$

Introduce the cost vector $\mathbf{c}^\top := \begin{bmatrix} \mathbf{c}_1^\top & \mathbf{c}_2^\top & \dots & \mathbf{c}_M^\top \end{bmatrix}$, with

$$[\mathbf{c}_j]_i = \begin{cases} 0, & \text{if } \exists\, h \in [1,k] \text{ s.t. } i = i_h \text{ and } j \in V(\delta_N^{i_h}); \\ 1, & \text{otherwise.} \end{cases} \tag{15}$$

We can now provide the following result, whose simpler version for an equilibrium point has been given in (Fornasini & Valcher, 2014a).

Theorem 3. *Given $\mathcal{C} = (\delta_N^{i_1}, \delta_N^{i_2}, \dots, \delta_N^{i_k})$, the BCN (2) is stabilizable to $\mathcal{C}$ if and only if $J^*(\mathbf{x}_0) = \min_{\mathbf{u}(\cdot)} J(\mathbf{x}_0, \mathbf{u}(\cdot)) = \min_{\mathbf{u}(\cdot)} \sum_{t=0}^{+\infty} \mathbf{c}^\top \ltimes \mathbf{u}(t) \ltimes \mathbf{x}(t)$, with $\mathbf{c}$ given in (15), is finite for each $\mathbf{x}_0 \in \mathcal{L}_N$.*

Proof. If the BCN is stabilizable to $\mathcal{C}$ then, by Proposition 1 point 1), for every $\mathbf{x}_0$ there exists $\tau \in \mathbb{Z}_+$ and an input sequence $\tilde{\mathbf{u}}(t), t \in [0, \tau-1]$, that drives the BCN state, say $\tilde{\mathbf{x}}(t)$, to $\delta_N^{i_1}$ at time τ. On the other hand, by Proposition 1 point 2), if such input sequence satisfies for $t \geq \tau$ the following conditions:

$$\tilde{\mathbf{u}}(t) = \begin{cases} \delta_M^j, \exists\, j \in V(\delta_N^{i_1}), & \text{if } t-\tau+1 \equiv 0 \bmod k; \\ \delta_M^j, \exists\, j \in V(\delta_N^{i_2}), & \text{if } t-\tau+1 \equiv 1 \bmod k; \\ \dots & \dots \\ \delta_M^j, \exists\, j \in V(\delta_N^{i_k}), & \text{if } t-\tau+1 \equiv k-1 \bmod k \end{cases}$$

then $\mathbf{c}^\top \ltimes \tilde{\mathbf{u}}(t) \ltimes \tilde{\mathbf{x}}(t) = 0$ for every $t \geq \tau$. We therefore have $J^*(\mathbf{x}_0) \leq \sum_{t=0}^{\tau-1} \mathbf{c}^\top \ltimes \tilde{\mathbf{u}}(t) \ltimes \tilde{\mathbf{x}}(t) < +\infty$.

Conversely, if $J^*(\mathbf{x}_0) < +\infty$ for every $\mathbf{x}_0 \in \mathcal{L}_N$, there exists $\tau \in \mathbb{Z}_+$ such that $\mathbf{c}^\top \ltimes \mathbf{u}(t) \ltimes \mathbf{x}(t) = 0,\ \forall\, t \geq \tau$. By the way the vector $\mathbf{c}$ has been defined, this implies, in particular, that the state trajectory from τ onward moves along the cycle $\mathcal{C}$, and hence the stabilization problem is solved. $\square$

This result allows to reduce the solution of the stabilization problem to the solution of an infinite-horizon optimal control problem. This provides an alternative proof of the fact that when the stabilization problem is solvable, then it can be solved by means of a time-invariant state-feedback law. Note that $J^*(\mathbf{x}_0)$ will always be equal to the length of the shortest path from $\mathbf{x}_0$ to $\mathcal{C}$.

Similarly to what we just did for the stabilization problem, we can translate also the safe control problem into an infinite-horizon optimal control problem. Assume, w.l.o.g., $X_s = \{\delta_N^i, i \in [1,k]\}$, and $X_u = \{\delta_N^i, i \in [k+1, N]\}$. Introduce the cost vector $\mathbf{c}^\top := \begin{bmatrix} \mathbf{c}_1^\top & \mathbf{c}_2^\top & \dots & \mathbf{c}_M^\top \end{bmatrix}$, with

$$[\mathbf{c}_j]_i = \begin{cases} 0, & \text{if } \delta_N^i \in X_s \text{ and } L \ltimes \delta_M^j \ltimes \delta_N^i \in X_s; \\ 1, & \text{otherwise.} \end{cases} \tag{16}$$

We can now provide the following result, whose proof can be derived along the same lines as the previous proof.

Theorem 4. *Given $X_s = \{\delta_N^i, i \in [1, k]\}$ and $X_u = \{\delta_N^i, i \in [k+1, N]\}$, the safe control problem is solvable for the BCN (2) if and only if $J^*(\mathbf{x}_0) = \min_{\mathbf{u}(\cdot)} J(\mathbf{x}_0, \mathbf{u}(\cdot)) = \min_{\mathbf{u}(\cdot)} \sum_{t=0}^{+\infty} \mathbf{c}^\top \ltimes \mathbf{u}(t) \ltimes \mathbf{x}(t)$, with $\mathbf{c}$ given in (16), is finite for each $\mathbf{x}_0 \in \mathcal{L}_N$.*

8 Conclusions

In this paper we have investigated a number of fundamental control issues for BCNs. In detail, after having introduced the concepts of reachability and controllability, we have addressed the stabilization to a limit cycle (in particular, to an equilibrium point) and the safe control of a BCN. In both cases, we have proved that when the problem is solvable it can be solved by means of a time-invariant state-feedback law.

Finite and infinite horizon control problems have been posed and solved. In both cases, by resorting to a technique that is similar to the square completion typically adopted in linear quadratic optimal control, we have been able to derive an algorithm that leads to the optimal solution and allows to determine a state-feedback law implementing the optimal control.

To conclude, we have shown that also stabilization to a limit cycle and safe control can be stated and solved as optimal control problems.

Future research directions include exploring under what conditions the aforementioned issues may be solved by resorting to output feedback techniques. Some results about output feedback stabilization have appeared in (Bof et al., 2015) and (H. Li & Wang, 2013), but a complete solution is still missing and represents a challenging open problem.

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