Cramer–Rao Bounds for Hybrid TOA/DOA-Based Location Estimation in Sensor Networks

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***Abstract*—This letter derives the Cramer–Rao Bound (CRB) for source location estimation using joint time-/direction-of-arrival (TOA/DOA) measurements collected in a wireless sensor network setup. Each sensor is capable of measuring both TOA and DOA from a signal source, which enables it to estimate the source's location individually. Data fusion is then employed to reach a global position estimate. For both optimal measurement fusion and linear state fusion, we derive the CRBs to assess the attainable positioning accuracy, which shed light on the impact of network topology and sensor selection on localization accuracy.**

***Index Terms*—Cramer–Rao Bound (CRB), data fusion, DOA, positioning, TOA, wireless sensor networks.**

I. Introduction

SOURCE localization is an important topic in surveillance and monitoring applications of wireless sensor networks. Main techniques for location estimation are either range-based (using for instance received signal strength (RSS), time of arrival (TOA) or time difference of arrival (TDOA) measurements) or bearing-based (using for instance direction of arrival (DOA) measurements). These techniques build on the premise of simple functionalities for wireless sensor nodes, at the expense of increased coordination and costs at the network level: increased number of sensor nodes needed for triangularization, communication bandwidth consumption for extensive information exchange, and network-wide time synchronization. Alternatively, these network-level costs can be alleviated and traded off with increased sensor costs using a hybrid TOA/DOA approach, in which each sensor is equipped with an antenna array and can calibrate both the range and bearing of a source to make individual location estimation [2]. Then, sensor data fusion can be employed to improve the positioning accuracy at a network level.

Focusing on hybrid TOA/DOA estimation for positioning, this paper derives the CRBs for individual sensors and at the network level. CRB analysis has been conducted for range-only or bearing-only location methods [3]–[5]. In [1], CRBs for hybrid TOA/RSS or TDOA/RSS location estimation are discussed; therein the measurement errors enter independently in RSS and TOA estimates, rather than jointly affect both. This decoupled treatment on range and bearing estimation is not suited for the hybrid TOA/DOA scheme considered here, because we extract both the TOA and DOA information of the source from the same noise-contaminated signal received at each antenna array. The contribution of this letter is twofold: first, the CRB derivations assume a common noise model that affects the joint estimation of TOA and DOA at each local sensor; second, it investigates the CRBs of location estimates using data fusion in a network environment, which has not been presented in the literature for the hybrid TOA/DOA approach. These CRB results may provide useful guideline to sensor resource management, which we will briefly remark on in the end.

II. CRB of Hybrid TOA/DOA Estimation

Consider a far-field target source that transmits planewave and a sensor that is equipped with an N-element uniform linear array for reception. The transmission channel is assumed to be an additive white Gaussian noise (AWGN) channel with line-of-sight. Let $s(t)$ denote the known transmitted signal with unit energy $\int |s(t)|^2 dt = 1$. The received baseband signals at the N antenna elements are collected into a vector $\mathbf{r}(t) := [r_0(t), r_1(t), \ldots, r_{N-1}(t)]^T$, which takes the form

$$\mathbf{r}(t) = \mathbf{a}(\theta) A s(t-\tau) + \mathbf{n}(t). \tag{1}$$

Here, A is the distance-dependent channel gain, τ is the time delay (a.k.a. TOA) from the target to the sensor, and $\mathbf{n}(t)$ is the AWGN vector with zero mean and independent noise components, each of zero mean and power spectral density $N_0/2$. The array signature vector $\mathbf{a}(\theta)$ is given by

$$\mathbf{a}(\theta) = \left[1, e^{-j2\pi d \cos\theta/\lambda}, \ldots, e^{-j2\pi(N-1)d\cos\theta/\lambda}\right]^T \tag{2}$$

where d is the antenna element separation, λ is the wavelength of the planewave signal and θ is the DOA.

For the signal model in (1), the CRB for (τ, θ) can be shown to be the same regardless of whether A is known or unknown; hence we focus on the joint estimation of (τ, θ). In view of the AWGN in (1), the probability density function (pdf) of the received $\mathbf{r}(t)$ conditioned on τ and θ is

$$p(r; \theta, \tau) = \frac{1}{Q} \exp\left\{-\frac{1}{N_0} \int \|\mathbf{r}(t) - \mathbf{a}(\theta) A s(t-\tau)\|^2 dt\right\} \tag{3}$$

where Q is a constant. Accordingly, joint TOA/DOA maximum likelihood (ML) estimates can be found by jointly searching over (τ, θ) to maximize $p(r; \theta, \tau)$. The search boils down to 2-D sliding correlation, as follows:

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$$(\hat{\tau},\hat{\theta}) = \arg\max_{\tau,\theta} \mathrm{Re}\left\{\int \mathbf{r}^H(t)\mathbf{a}(\theta)As(t-\tau)dt\right\}. \quad (4)$$

This joint ML estimator is unbiased, and asymptotically reaches the CRB for (τ,θ) given by $\mathrm{CRB}_{\tau,\theta} = \mathbf{F}^{-1}$ [6], where $\mathbf{F}$ is the Fisher information matrix (FIM) defined as

$$\begin{aligned}\mathbf{F} &:= E\left\{[\nabla_{\tau,\theta}\ln p(r;\theta,\tau)][\nabla_{\tau,\theta}\ln p(r;\theta,\tau)]^T\right\}\\ &= \begin{pmatrix} F_{\tau\tau} & F_{\tau\theta}\\ F_{\tau\theta} & F_{\theta\theta}\end{pmatrix}.\end{aligned} \quad (5)$$

After some algebraic derivations and defining $\gamma_s := |A|^2/N_0$ as the signal-to-noise-ratio (SNR), we reach

$$F_{\tau\tau} = 8\pi^2 N\gamma_s \int |S(f)|^2 f^2 df, \quad (6)$$

$$F_{\theta\theta} = \frac{8\pi^2 d^2\gamma_s \sin^2\theta}{\lambda^2}\frac{(N-1)N(2N-1)}{6}, \quad (7)$$

$$F_{\tau\theta} = -\frac{8\pi^2 d\gamma_s \sin\theta}{\lambda}\frac{N(N-1)}{2}\int f|S(f)|^2 df. \quad (8)$$

Let $W^2 := \int 4\pi^2 f^2 |S(f)|^2 df$ denote the mean-square effective-bandwidth of the unit-energy signal $s(t)$ [6]. Note that $F_{\tau\theta}$ in (8) is zero for real-valued $s(t)$. As a result, the CRBs of TOA and DOA estimates are given respectively by

$$\mathrm{CRB}_\tau = \left[\mathbf{F}^{-1}\right]_{(1,1)} = \frac{1}{2N\gamma_s W^2}, \quad (9)$$

$$\begin{aligned}\mathrm{CRB}_\theta &= \left[\mathbf{F}^{-1}\right]_{(2,2)}\\ &= \frac{3\lambda^2}{4\pi^2 d^2\gamma_s \sin^2\theta (N-1)N(2N-1)}.\end{aligned} \quad (10)$$

Apparently, the CRB of TOA is dominantly determined by the signal bandwidth W, while that of DOA is affected by array configuration parameters N and d.

III. CRB of Source Localization in a Network

A. CRB of the Location Estimate From a Single Sensor

Let $\mathbf{u}_s := (x_s, y_s)$ and $\mathbf{u} := (x,y)$ denote the actual positions of a sensor and the source respectively, which yield the true TOA and DOA values τ and θ. The estimated TOA and DOA are $\hat{\tau} = \tau + \Delta\tau$ and $\hat{\theta} = \theta + \Delta\theta$, where $\Delta\tau$ and $\Delta\theta$ are the respective estimation errors. Given $(\hat{\tau},\hat{\theta})$, the source location estimate $\hat{\mathbf{u}} := (\hat{x},\hat{y})$ can be reached as

$$(\hat{x},\hat{y}) : \begin{cases}\hat{x} = x_s + c\hat{\tau}\cos\hat{\theta}\\ \hat{y} = y_s + c\hat{\tau}\sin\hat{\theta}\end{cases} \quad (11)$$

where c is the speed of light. The positioning errors are

$$\begin{cases}\Delta x := \hat{x} - x \simeq -c\tau\Delta\theta\sin\theta + c\Delta\tau\cos\theta\\ \Delta y := \hat{y} - y \simeq c\tau\Delta\theta\cos\theta + c\Delta\tau\sin\theta\end{cases} \quad (12)$$

where we have used the following approximations: $\Delta\theta$ and $\Delta\tau$ are small enough such that $\cos\Delta\theta \simeq 1$, $\sin\Delta\theta \simeq \theta$ and $\Delta\theta\Delta\tau \approx 0$. As a result, the location estimation errors are linearly related to TOA and DOA estimation errors, and hence the CRB for $\mathbf{u}$ can be transformed from $\mathrm{CRB}_{\tau,\theta}$ via a Jacobian matrix as follows:

$$\begin{aligned}\mathrm{CRB}_{\mathbf{u}} = \mathbf{F}_{x,y}^{-1} &= \begin{bmatrix}\frac{\partial x}{\partial\tau} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial\tau} & \frac{\partial y}{\partial\theta}\end{bmatrix}\mathrm{CRB}_{\tau,\theta}\begin{bmatrix}\frac{\partial x}{\partial\tau} & \frac{\partial x}{\partial\theta}\\ \frac{\partial y}{\partial\tau} & \frac{\partial y}{\partial\theta}\end{bmatrix}^T\\ &= \begin{bmatrix}\mathrm{CRB}_x & \mathrm{CRB}_{xy}\\ \mathrm{CRB}_{xy} & \mathrm{CRB}_y\end{bmatrix}\end{aligned} \quad (13)$$

where

$$\mathrm{CRB}_x = c^2\tau^2\sin^2\theta\ \mathrm{CRB}_\theta + c^2\cos^2\theta\ \mathrm{CRB}_\tau,$$
$$\mathrm{CRB}_y = c^2\tau^2\cos^2\theta\ \mathrm{CRB}_\theta + c^2\sin^2\theta\ \mathrm{CRB}_\tau.$$

The mean square error (MSE) of a location estimator $\hat{\mathbf{u}}$ is defined as $\mathrm{MSE}_{\hat{\mathbf{u}}} = E\{\|\hat{\mathbf{u}} - \mathbf{u}\|_2^2\} = E\{(\hat{x}-x)^2 + (\hat{y}-y)^2\}$. Its lower bound $\mathrm{MSE}_{\mathbf{u},\min}$ is thus given by

$$\begin{aligned}\mathrm{MSE}_{\mathbf{u},\min} &= \mathrm{trace}\{\mathrm{CRB}_{\mathbf{u}}\} = \mathrm{CRB}_x + \mathrm{CRB}_y\\ &= c^2\tau^2\ \mathrm{CRB}_\theta + c^2\ \mathrm{CRB}_\tau.\end{aligned} \quad (14)$$

Apparently, positioning accuracy of the hybrid TOA/DOA approach depends on not only the SNR, but also the relative position between the sensor and the source. The location estimator is most accurate when DOA is $\pi/2$, and least accurate when $\mathrm{DOA} = 0$ or π. As a result, even when the sensor is very close to the source, the position estimation error can still be large. Meanwhile, when the sensor is far away from the source, $(c\tau)^2$ is large, which considerably magnifies the role of bearing errors in the position estimation MSE. These observations motivate the use of data fusion among multiple sensors to improve localization accuracy, as discussed next.

B. CRB of Fused Location Estimate in a Sensor Network

Consider data fusion techniques for joint location estimation among a network of M sensors. In *measurement fusion*, sensors send their received signals $\mathbf{r}_i$, $i = 1,\ldots,M$, to a fusion center which makes a global estimate $\hat{\mathbf{u}} = (\hat{x},\hat{y}) = f(\mathbf{r})$, where $\mathbf{r} := (\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_M)$. In *state fusion*, sensors send their local estimates $\hat{\mathbf{u}}_i$ to the fusion center to decide on $\hat{\mathbf{u}} = g(\hat{\mathbf{u}}_1,\ldots,\hat{\mathbf{u}}_M)$, where $\hat{\mathbf{u}}_i := (\hat{x}_i,\hat{y}_i)$ is the estimate of $\mathbf{u}$ from sensor i using the hybrid TOA/DOA solution via (11). We derive the CRBs and MSEs for both fusion scenarios.

1) CRB of Measurement Fusion: Assume that the noisy signals $\{\mathbf{r}_i\}$ received at different sensors are mutually independent. The joint PDF of $\mathbf{r}$ is thus given by

$$p(\mathbf{r};x,y) = \prod_{i=1}^{M} p(\mathbf{r}_i;x,y) \quad (15)$$

where, for sensor i at $\mathbf{u}_s^{(i)} = (x_s^{(i)}, y_s^{(i)})$, $p(\mathbf{r}_i;x,y)$ can be deduced from $p(\mathbf{r}_i;\tau_i,\theta_i)$ in (3) via $x = x_s^{(i)} + c\tau_i\cos\theta_i$ and $y = y_s^{(i)} + c\tau_i\sin\theta_i$. Because $\ln p(\mathbf{r};x,y) = \sum_{i=1}^{M}\ln p(\mathbf{r}_i;x,y)$, the FIM of the log-likelihood function of (15) is given by

$$\mathbf{F}_{x,y}^{\mathrm{fusion}} := -E\begin{bmatrix}\frac{\partial^2 \ln p(\mathbf{r};x,y)}{\partial x^2} & \frac{\partial^2 \ln p(\mathbf{r};x,y)}{\partial x\partial y}\\ \frac{\partial^2 \ln p(\mathbf{r};x,y)}{\partial y\partial x} & \frac{\partial^2 \ln p(\mathbf{r};x,y)}{\partial y^2}\end{bmatrix}$$

$$= \sum_{i=1}^{M} \mathbf{F}_{x,y}^{(i)} \tag{16}$$

where the FIM for each sensor i has been derived in (13), with (τ, θ) replaced by $(\tau_i, \theta_i) = \left(\|\mathbf{u}_s^{(i)} - \mathbf{u}\|_2/c, \tan^{-1}(x - x_s^{(i)}/y - y_s^{(i)})\right), \quad \forall i$. Accordingly, the CRB for $\mathbf{u}$ in measurement fusion is

$$\mathrm{CRB}_{\mathbf{u}}^{\mathrm{fusion}} = (\mathbf{F}_{x,y}^{\mathrm{fusion}})^{-1} \tag{17}$$

which can be computed numerically for any location $\mathbf{u}$.

Correspondingly, the minimum MSE of the fused position estimate $\hat{\mathbf{u}}$ in the *measurement fusion* approach is given by

$$\mathrm{MSE}_{\mathbf{u},\min}^{\mathrm{fusion}} = \mathrm{trace}\left\{\mathrm{CRB}_{\mathbf{u}}^{\mathrm{fusion}}\right\}. \tag{18}$$

2) CRB of Linear State Fusion: To attain the CRB in (17) requires cumbersome ML search by maximizing (15) over (x, y). In practical systems, state fusion is often adopted to alleviate the computational burden. Specifically, a linear fusion rule is to make a global decision $\hat{\mathbf{u}}$ by linearly combining the local estimates $\{\hat{\mathbf{u}}_i\}_{i=1}^{M}$ of the M sensors, as follows:

$$\hat{\mathbf{u}} = g(\hat{\mathbf{u}}_1, \ldots, \hat{\mathbf{u}}_M) = \alpha_1 \hat{\mathbf{u}}_1 + \alpha_2 \hat{\mathbf{u}}_2 + \cdots + \alpha_M \hat{\mathbf{u}}_M \tag{19}$$

where $\boldsymbol{\alpha} := \{\alpha_i\}_{i=1}^{M}$ are linear weights chosen in accordance to the MSEs of the unbiased local ML estimators.

Suppose that the position estimate errors of local sensors are mutually uncorrelated. Given $\boldsymbol{\alpha}$, the CRB of the fused position estimate $\mathbf{u}$ is given by

$$\mathrm{CRB}_{\mathbf{u}}^{\mathrm{linear}}(\boldsymbol{\alpha}) = \sum_{i=1}^{M} \alpha_i^2 \mathrm{CRB}_{\mathbf{u}}^{(i)} \tag{20}$$

where $\mathrm{CRB}_{\mathbf{u}}^{(i)}$ is the CRB of $\mathbf{u}_i$ at sensor i, which has been derived in (13). Correspondingly, the minimum MSE of $\mathbf{u}$ can be computed from individual MSEs in (14) as follows:

$$\mathrm{MSE}_{\mathbf{u},\min}^{\mathrm{linear}}(\boldsymbol{\alpha}) = \mathrm{trace}\left\{\mathrm{CRB}_{\mathbf{u}}^{\mathrm{linear}}(\boldsymbol{\alpha})\right\} = \sum_{i=1}^{M} \alpha_i^2 \mathrm{MSE}_{\mathbf{u},\min}^{(i)}. \tag{21}$$

For optimal linear fusion, the linear weights $\boldsymbol{\alpha}$ are chosen to minimize the estimation MSE. Subject (21) to a normalization constraint $\sum_{i=1}^{M} \alpha_i = 1$ to avoid trivial solutions, the lowest MSE for localization using optimal linear fusion is

$$\mathrm{MSE}_{\mathbf{u},\min}^{\mathrm{linear}} = \min_{\boldsymbol{\alpha}} \mathrm{MSE}_{\mathbf{u},\min}^{\mathrm{linear}}(\boldsymbol{\alpha}) = \left(\sum_{i=1}^{M} \left(\mathrm{MSE}_{\mathbf{u},\min}^{(i)}\right)^{-1}\right)^{-1} \tag{22}$$

which corresponds to setting the optimal weights as

$$\alpha_i = \frac{\left(\mathrm{MSE}_{\mathbf{u},\min}^{(i)}\right)^{-1}}{\sum_{m=1}^{M} \left(\mathrm{MSE}_{\mathbf{u},\min}^{(m)}\right)^{-1}}, \quad i = 1, \ldots, M. \tag{23}$$

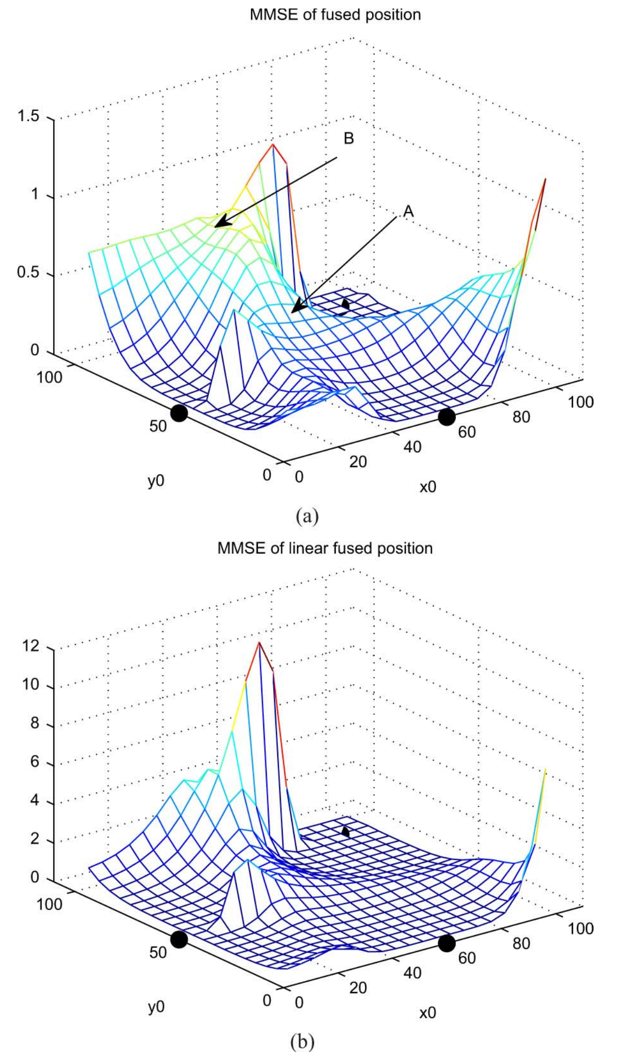

Fig. 1. Minimum MSE of the location estimate fused from three sensor measurements as a function of the source location (x, y): a) ML measurement fusion, b) Optimal linear state fusion.

IV. Performance Evaluation Results

Consider source localization in a 2-D sensing field over a $[0,\ 100] \times [0,\ 100]$ grid region. Three sensors are placed at $[60, 0]$, $[0, 50]$ and $[90, 90]$ respectively, as indicated by the solid circles on Fig. 1. Each sensor independently measures both TOA and DOA parameters, and sends data to the fusion center for either measurement fusion or state fusion. The effective signal bandwidth is $W = 10$ MHz, and each sensor has $N = 4$ antenna elements. In the presence of line of sight, the received power at a sensor obeys the typical path loss model with power attenuation proportional to r^{-n}, where $r = c\tau$ is the distance between of the source and sensor and n is the path loss exponent set to $n = 2$ in our simulations.

Fig. 1 depicts the minimum MSE of position estimates by the fused position estimates with respect to the actual source location (x, y). Evidently, the position estimate error fluctuates with the target location, indicating the impact of the relative positions between the source and sensors. The estimation MSE worsens when the DOA values at some sensor(s) are close to 0 or π, and

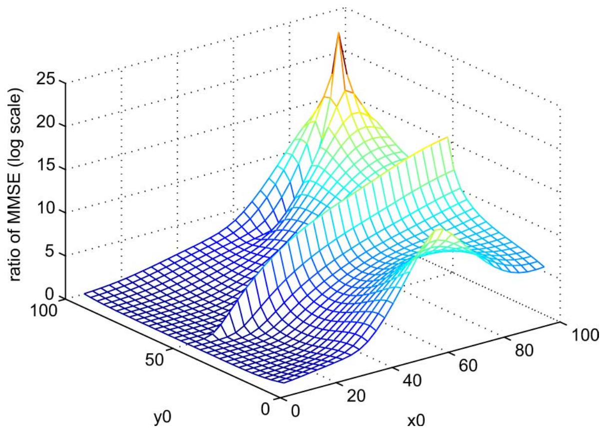

Fig. 2. Improvement of estimation MSE by multisensor data fusion over the single-sensor case (in log scale): $\log_{10}(\mathrm{MSE}_{\mathbf{u},\min}) - \log_{10}(\mathrm{MSE}^{\mathrm{fusion}}_{\mathbf{u},\min})$.

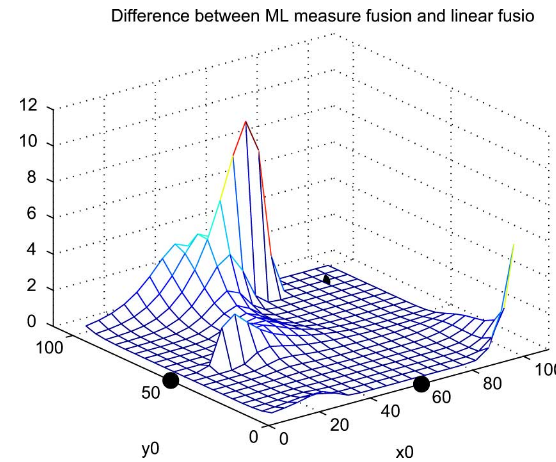

Fig. 3. Differences in minimum MSE of fused position estimates between ML measure fusion and linear fusion.

at the same time has large TOA (thus distance) from other sensors, such as the area B indicated in Fig. 1(a). Nevertheless, the positioning accuracy can be considerably improved by data fusion. For instance, a target around area A in Fig. 1(a) has DOA close to 0 with respect to the sensor at $[0, 50]$. Hence, when it is solely localized by this sensor, the positioning error can be large, as discussed in Section III-A. As the target moves away from this sensor along the same DOA line, the positioning MSE using this sensor alone is expected to deteriorate further, due to increased TOA and thus reduced receive SNR. Interestingly, the MSE depicted on Fig. 1 actually improves, thanks to the help from the other two sensors via data fusion. In general, when the DOA between a target and a sensor is close to 0 degree, the accuracy of hybrid TOA/DOA-based positioning can be improved considerably by fusing with other sensors that have more desired DOA and/or TOA from the target. Quantitatively, the MSE improvement of fusing the measurements of three sensors over a single-sensor case (one sensor at location $[0, 50]$) is depicted in Fig. 2. Evidently, a good portion of the entire sensing field benefits from data fusion. It is only when the target is within the good DOA region (close to $\pi/2$) of a nearby single sensor (small $c\tau$) that the improvement by data fusion is not substantial.

Comparison between measurement fusion and linear state fusion is delineated in Fig. 3. Throughout the sensing field, the ML measurement fusion outperforms the optimal linear state fusion. The performance gap is more evident when the source is more difficult to locate accurately, due to undesired DOA angles and/or large TOA/distances to sensors.

V. Summary

This letter derives analytical expressions of CRBs for joint TOA/DOA estimation at a single sensor site, as well as the minimum MSEs of location estimation at both a single sensor and a sensor network using either measurement fusion or state fusion techniques.

These CRBs, along with the illustrative simulation results, reveal interestingly on sensor placement and sensor selection issues in wireless sensor networks for source localization. Given fixed power and bandwidth resources, the location accuracy improves when sensors are placed or moved favorably with reference to the target source location. Meanwhile, some unfavorably placed sensors do not contribute much to the fusion performance, and thus can be turned off. The fusion gain diminishes when the number of active sensors is too large, suggesting the usefulness of sensor selection in saving network resources.

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