Ensuring Economic Fairness in Wide-Area Control for Power Systems via Game Theory

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***Abstract*— Wide-area control helps in suppression of inter-area oscillations in electric power systems, but potentially requires a substantial investment into the communication network needed to exchange state information among the power companies. To provide companies with incentives to subsidize the inter-area communication links, a fair cost-allocation method based on the theory of cooperative network-formation games (NFG) is developed. The Nash Bargaining Solution (NBS) is utilized to fairly allocate the inter-area communication cost to the companies, which act as game-players. First, the wide-area control problem is formulated using the state-feedback-based LQR minimization approach, and the social inter-area cost is computed using a sparsity-promoting algorithm. Second, the disagreement point, which determines the maximum cost each area is willing to pay, is computed. This selfish cost is proportional to the energy an area saves by utilizing inter-area feedback and is derived from the Nash Equilibria (NEs) of two noncooperative NFGs, with and without inter-area feedback, respectively. Finally, the social cost is divided optimally among the companies, with all players benefiting from cooperation. The proposed cost allocation is illustrated for the Australian 50-bus power system example.**

***Index Terms*— Wide-Area Control, Power Systems, Cost Allocation, Nash Equilibrium, Sparsity promotion, Nash Bargaining Solution, Network Formation Games, Cyber-physical Systems.**

I. Introduction

Wide-area control of power systems requires real-time feedback of massive volumes of sensor data from one operating region of the grid to controllers located at other regions [1]. Over the past few years several researchers have started investigating such control designs using robust control methods [2], [3], adaptive control [4], and LQR-based optimal control [5]. One of the foremost requirements for wide-area control is the need for a highly robust communication system that works in sync with the control functionalities. The envisioned architecture of wide-area communication for the US grid, often referred to as the North American Synchrophasor Initiative Network or NASPI-net [6], involves Phasor Measurement Units (PMU) inside the operating boundary of utility companies to send real-time data to local controllers via a local-area network, and to remote controllers over a secure wide-area network. Each area is equipped with its own dedicated phasor gateway, which routes the incoming PMU data-streams to the respective controllers. Installation of communication links for transporting feedback data from PMUs to controllers means a significant financial investment by the utilities. A few recent papers such as [5], [7] have proposed economically lucrative wide-area control designs by sparsifying the number of communication links between PMUs and controllers. However, the question of fair allocation of installation costs of wide-area power networks still remains open.

This paper considers the task of fair allocation of communication costs in the wide-area control networks for power systems. Wide-area control network design is formulated as an LQR state-feedback problem. The costs of constructing this network are allocated to participating power companies via the Nash Bargaining Solution (NBS) based on cooperative network-formation game (NFG) theory [8]. First, the social cost is computed using the sparsity-promoting LQR optimization algorithm in [5]. This design results in a feedback gain matrix with reduced number of inter-area feedback links. Second, the disagreement point is computed, where each area specifies the maximum costs it is willing to pay. These selfish costs increase with the amount of energy each company saves by participating in the inter-area network. If an area's need for inter-area feedback is relatively small, it might not have to contribute, especially when a sparse social network is desired. To compute these energy savings, we utilize the Nash Equilibria (NEs) of two noncooperative linear-quadratic games with state feedback [9].

In [10] we presented some preliminary results to address this problem by modeling an area's selfish cost as hypothetical cost of global inter-area feedback that satisfies that area's energy objective. However, this approach involved heuristic cost adjustment due to incompatibility of areas' selfish networks. In contrast, in this paper, we use the energy savings afforded by wide-area control to construct the disagreement point, thus measuring the worth of inter-area feedback from each area's point of view – a sounder objective than that in [10]. Moreover, the proposed approach to computing the disagreement point using a NE of a noncooperative solution [8] results in a compatible network that corresponds to a single feedback matrix and, thus, does not require heuristic adjustment. The first noncooperative game used for computing the disagreement point models a decentralized control scenario where the companies compete to minimize their individual energies in the absence of inter-area feedback and thus do not incur any wide-area communication cost. We develop a novel iterative method to compute this NE. On the other hand, the second game is coupled since companies provide state feedback to each other

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while competing to satisfy their selfish energy objectives. To compute this NE, we adopt the weakly-coupled system assumption [11]. Weakly coupled interconnected system has been studied in [12] for the decomposition and aggregation of power networks. We employ the algorithm in [13], which was shown to converge to a unique NE for a sufficiently small coupling parameter ϵ. The energy each area saves by participating in the inter-area feedback is computed from the NEs of these two games, and the disagreement point is formed by dividing the overall network cost of the coupled game proportionally to these savings. Finally, NBS cost allocation is performed. We employ the Australian 50-bus power system divided into 4 areas to illustrate the proposed method and provide insights on the relationship between the allocated cost and companies' feedback requirements.

The remainder of the paper is organized as follows. In Section II we present the model of a multi-machine power system network and formulate wide-area control design as a sparsity-promoting LQR problem. In Section III we describe the NBS cost allocation. Numerical results for the Australian 50-bus system divided into 4 areas are presented in Section IV, and Section V concludes the paper.

II. PROBLEM STATEMENT

A. Power System Model

Consider a power system with n generators divided into r coherent areas, and define the corresponding sets of generator indices as:

$$\begin{aligned} s_1 = (1,2,...,n_1) \Rightarrow &\quad \text{belongs to area } 1.\\ s_2 = (n_1+1, n_1+2, ..., n_1+n_2) \Rightarrow &\quad \text{belongs to area } 2.\\ \cdots \Rightarrow &\quad \cdots\\ s_r = (n_1+n_2+...+n_{r-1}+1, ..., n) \Rightarrow &\quad \text{belongs to area } r. \end{aligned}$$

where n_i is the number of generators in area i for $i = 1, ..., r$. Let the number of states for each generator j be m_j for $j = 1, ...n$. We arrange the states in tuples for each generator i, and denote them by $\mathcal{X}_i$. So $\mathcal{X}_i \in \mathbb{R}^{m_i \times 1}$ for $i = 1, ..., n$. The controller inputs for the i^{th} generator form $\mathcal{U}_i \in \mathbb{R}^{p_i \times 1}$ for $i = 1, ..., n$, where p_i is the number of control inputs for generator i. We write the small-signal model as

$$\begin{aligned} \left[\begin{array}{c} \left.\begin{matrix}\dot{\mathcal{X}}_1(t)\\ \vdots\\ \dot{\mathcal{X}}_{n_1}(t)\end{matrix}\right\} := \dot{\boldsymbol{x}}_1(t)\\ \hline \left.\begin{matrix}\dot{\mathcal{X}}_{n_1+1}(t)\\ \vdots\\ \dot{\mathcal{X}}_{n_1+n_2}(t)\end{matrix}\right\} := \dot{\boldsymbol{x}}_2(t)\\ \hline \vdots\\ \hline \left.\begin{matrix}\dot{\mathcal{X}}_{n-n_r+1}(t)\\ \vdots\\ \dot{\mathcal{X}}_{n}(t)\end{matrix}\right\} := \dot{\boldsymbol{x}}_r(t) \end{array}\right] &= \begin{bmatrix} \boldsymbol{A}_{11} & \boldsymbol{A}_{12} & \cdots & \boldsymbol{A}_{1r}\\ \boldsymbol{A}_{21} & \boldsymbol{A}_{22} & \cdots & \boldsymbol{A}_{2r}\\ \vdots & & & \\ \boldsymbol{A}_{r1} & \boldsymbol{A}_{r2} & \cdots & \boldsymbol{A}_{rr} \end{bmatrix} \left[\begin{array}{c} \mathcal{X}_1(t)\\ \vdots\\ \mathcal{X}_{n_1}(t)\\ \hline \mathcal{X}_{n_1+1}(t)\\ \vdots\\ \mathcal{X}_{n_1+n_2}(t)\\ \hline \vdots\\ \hline \mathcal{X}_{n-n_r+1}(t)\\ \vdots\\ \mathcal{X}_{n}(t) \end{array}\right] \\ &+ \begin{bmatrix} \boldsymbol{B}_{11} & \boldsymbol{B}_{12} & \cdots & \boldsymbol{B}_{1r}\\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22} & \cdots & \boldsymbol{B}_{2r}\\ \vdots & & & \\ \boldsymbol{B}_{r1} & \boldsymbol{B}_{r2} & \cdots & \boldsymbol{B}_{rr} \end{bmatrix} \left[\begin{array}{c} \left.\begin{matrix}\mathcal{U}_1(t)\\ \vdots\\ \mathcal{U}_{n_1}(t)\end{matrix}\right\} := \boldsymbol{u}_1(t)\\ \hline \left.\begin{matrix}\mathcal{U}_{n_1+1}(t)\\ \vdots\\ \mathcal{U}_{n_1+n_2}(t)\end{matrix}\right\} := \boldsymbol{u}_2(t)\\ \hline \vdots\\ \hline \left.\begin{matrix}\mathcal{U}_{n-n_r+1}(t)\\ \vdots\\ \mathcal{U}_{n}(t)\end{matrix}\right\} := \boldsymbol{u}_r(t) \end{array}\right] + \tilde{\boldsymbol{B}} d(t) \end{aligned} \tag{1}$$

where for area i: $\boldsymbol{x}_i(t) \in \mathbb{R}^{M_i \times 1}$ is the vector of states, with $M_i = \sum_{j=n_1+...+n_{i-1}+1}^{n_1+...+n_i} m_j$; $\boldsymbol{u}_i(t) \in \mathbb{R}^{N_i \times 1}$ is the vector of control inputs, with $N_i = \sum_{j=n_1+...+n_{i-1}+1}^{n_1+...+n_i} p_j$; $d(t)$ is a scalar impulsive disturbance input entering the electro-mechanical swing dynamics of any generator, while $\tilde{\boldsymbol{B}} \in \mathbb{R}^{s \times 1}$ is an indicator vector whose entries are all zero except for the one corresponding to the acceleration equation of the generator at which $d(t)$ enters. Stacking the vector states together, the network model can be compactly written as

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \tilde{\boldsymbol{B}}\boldsymbol{d}(t). \tag{2}$$

where $\boldsymbol{x}(t) \in \mathbb{R}^{s \times 1}$, $\boldsymbol{u}(t) \in \mathbb{R}^{q \times 1}$, with $s := \sum_{i=1}^{r} M_i$, and $q := \sum_{i=1}^{r} N_i$.

In eq.(1) we consider the generic case where each generator may have different number of states and control inputs. However, if a traditional 3$^{\text{rd}}$ order swing and excitation system model is used for the synchronous generators [4], then $m_i = 3$ and $p_i = 1$ for all $i = 1, ..., n$. In that case, each generator has 3 states: the generator phase angle (radians) δ_i, the generator rotor velocity (rad/sec) ω_i, and the quadrature-axis internal emf E_i. Following [4], it can be shown that the linearized small-signal dynamic model of the networked power system can be expressed in the Kron-reduced form as

$$\begin{bmatrix} \Delta\dot{\boldsymbol{\delta}}\\ \boldsymbol{M}\Delta\dot{\boldsymbol{\omega}}\\ \boldsymbol{T}\Delta\dot{\boldsymbol{E}} \end{bmatrix} = \underbrace{\begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} & \boldsymbol{0}\\ -\boldsymbol{L} & -\boldsymbol{D} & -\boldsymbol{P}\\ \boldsymbol{K} & \boldsymbol{0} & \boldsymbol{J} \end{bmatrix}}_{\mathcal{A}} \begin{bmatrix} \Delta\boldsymbol{\delta}\\ \Delta\boldsymbol{\omega}\\ \Delta\boldsymbol{E} \end{bmatrix} + \underbrace{\begin{bmatrix} \boldsymbol{0} & \boldsymbol{0}\\ \boldsymbol{I} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix}}_{\mathcal{B}} \begin{bmatrix} \Delta\boldsymbol{P}_m\\ \Delta\boldsymbol{E}_F \end{bmatrix} \tag{3}$$

where $\Delta\boldsymbol{\delta} = \text{col}(\Delta\delta_1, ..., \Delta\delta_n)$, $\Delta\boldsymbol{\omega} = \text{col}(\Delta\omega_1, ..., \Delta\omega_n)$, $\Delta\boldsymbol{E} = \text{col}(\Delta E_1, ..., \Delta E_n)$, $\Delta\boldsymbol{P}_m = \text{col}(\Delta P_{m1}, ..., \Delta P_{mn})$. $\Delta\boldsymbol{E}_F = \text{col}(\Delta E_{F1}, ..., \Delta E_{Fn})$, respectively, represent the small-signal changes in phase angle, frequency, excitation voltage, mechanical power input, and excitation voltage input. $\boldsymbol{M} = \text{diag}(M_1, ..M_n)$ represents the generator inertias, while $\boldsymbol{T} = \text{diag}(\tau_1, ..\tau_n)$ represents the excitation time constants. The expressions for the various matrices on the RHS can be found in [4]. Equation (3) serves as the primary model for wide-area control indicating how the designable control inputs $\Delta\boldsymbol{P}_m$ and $\Delta\boldsymbol{E}_F$ enter the system dynamics. The turbine mechanical power $\Delta\boldsymbol{P}_m$, however, typically has a much lower bandwidth than needed for oscillation damping. Therefore, for all practical wide-area control designs, $\Delta\boldsymbol{P}_m$ is treated as zero, and $\Delta\boldsymbol{E}_F$ is designed via PMU data feedback. The $\boldsymbol{A}$ matrix for the general case in (1) can be obtained by permuting matrix $\mathcal{A}$ in (3) in terms of the sets $s_1, s_2, ..., s_r$ defined at the beginning of this section.

B. Wide-area control (WAC) via LQR

We formulate the wide-area control problem as a LQR problem for its small-signal states in (3). The objective function is chosen so that all generators arrive at a consensus in their small-signal changes in phase angles and frequencies, as dictated by the physical topology of the network, reflected in $\boldsymbol{L}$. However, since $\boldsymbol{L}$ is in the Kron-reduced form, the topology contained in $\boldsymbol{L}$ is an all-to-all graph. Therefore, the objective function is chosen as

$$E_{\text{states}} = \quad \Delta\boldsymbol{\delta}^T \bar{\mathcal{L}} \Delta\boldsymbol{\delta} + \Delta\boldsymbol{\omega}^T \bar{\mathcal{L}} \Delta\boldsymbol{\omega} + \Delta\boldsymbol{E}^T \Delta\boldsymbol{E} \tag{4}$$

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If more detailed models of synchronous generators are used, then the last term in (4) may be replaced by $(\boldsymbol{x}^-)^T\boldsymbol{x}^-$, where $\boldsymbol{x}^-$ contains all states except the electro-mechanical states $\Delta\boldsymbol{\delta}$ and $\Delta\boldsymbol{\omega}$. The matrix $\bar{\mathcal{L}}$ is constructed as

$$\bar{\mathcal{L}} = n\boldsymbol{I} - \boldsymbol{1}_n \cdot \boldsymbol{1}_n^T. \tag{5}$$

Physically, this means that the first two terms in the objective function (4) are in the consensus form

$$\begin{aligned}\Delta\boldsymbol{\delta}^T\bar{\mathcal{L}}\Delta\boldsymbol{\delta} &= \sum_{i=1}^{n}\sum_{j>i}^{n}(\Delta\delta_i - \Delta\delta_j)^2. \\ \Delta\boldsymbol{\omega}^T\bar{\mathcal{L}}\Delta\boldsymbol{\omega} &= \sum_{i=1}^{n}\sum_{j>i}^{n}(\Delta\omega_i - \Delta\omega_j)^2. \end{aligned}\tag{6}$$

Next, we express (4) as

$$\begin{aligned} E_{\text{states}} &= \begin{bmatrix}\Delta\boldsymbol{\delta}\\ \Delta\boldsymbol{\omega}\\ \Delta\boldsymbol{E}\end{bmatrix}^T \underbrace{\begin{bmatrix}\bar{\mathcal{L}} & & \\ & \bar{\mathcal{L}} & \\ & & \boldsymbol{I}\end{bmatrix}}_{\boldsymbol{Q}'} \begin{bmatrix}\Delta\boldsymbol{\delta}\\ \Delta\boldsymbol{\omega}\\ \Delta\boldsymbol{E}\end{bmatrix} \\ &= \boldsymbol{x}^T(\mathcal{P}^T\boldsymbol{Q}'\mathcal{P})\boldsymbol{x} \\ &= \boldsymbol{x}^T\boldsymbol{Q}\boldsymbol{x} \end{aligned}\tag{7}$$

where $\mathcal{P}$ is a permutation matrix defined in the appendix, and $\boldsymbol{x}$ is the state vector defined in (2).

The wide-area control (WAC) problem for the small-signal model (2) can be stated as:

$$\begin{aligned} \min_{\boldsymbol{K}} \ & \int_{t=0}^{\infty}[\boldsymbol{x}(t)^T\boldsymbol{Q}\boldsymbol{x}(t) + \boldsymbol{u}(t)^T\boldsymbol{R}\boldsymbol{u}(t)]dt \\ \text{s.t. } & \boldsymbol{u}(t) = -\boldsymbol{K}x(t) \\ & \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \tilde{\boldsymbol{B}}\boldsymbol{d}(t). \end{aligned}\tag{8}$$

where $\boldsymbol{Q}$ is as designed in (7). Throughout the paper we consider $\boldsymbol{R}$ to be the identity matrix so that the energy of every controller has the same weight. The solution of (8) has the form

$$\boldsymbol{K} = \begin{bmatrix}\boldsymbol{K}_{11} & \boldsymbol{K}_{12} & \cdots & \boldsymbol{K}_{1r}\\ \boldsymbol{K}_{21} & \boldsymbol{K}_{22} & \cdots & \boldsymbol{K}_{2r}\\ & & \vdots & \\ \boldsymbol{K}_{r1} & \boldsymbol{K}_{r2} & \cdots & \boldsymbol{K}_{rr}\end{bmatrix}\tag{9}$$

In (9), each block $\boldsymbol{K}_{ij} \in \mathbb{R}^{N_i\times M_i}$ with $i,j = 1,...,r$ represents the block of feedback gains from area j to area i. Thus the diagonal and off-diagonal blocks represent *intra-area* and *inter-area* feedback, respectively. This cost can be reduced significantly if the number of inter-area links can be reduced, i.e., if the off-diagonal blocks of $\boldsymbol{K}$ contain as many zero elements as possible. We next augment the wide-area control problem (8) with such a sparse solution for $\boldsymbol{K}$.

C. Sparsity-Promoting Wide-Area Control

The solution of the traditional WAC problem requires a very dense communication network, which has high communication cost. To characterize the communication cost, we neglect the cost of intra-area (local) links and assign the same fixed cost to all inter-area PMU-to-controller links. This cost can be reduced by employing sparse off-diagonal blocks of $\boldsymbol{K}$, for example by utilizing the *sparsity-promoting* LQR minimization algorithms proposed in [7], [14]. The sparsity-promoting wide-area control (SPWAC) objective can then be stated as follows:

Given a nonnegative *sparsity parameter* γ, solve

$$\begin{aligned} \min_{\boldsymbol{K}} \ & \underbrace{\int_{t=0}^{\infty}[\boldsymbol{x}(t)^T\boldsymbol{Q}\boldsymbol{x}(t) + \boldsymbol{u}(t)^T\boldsymbol{R}\boldsymbol{u}(t)]dt}_{e(\boldsymbol{K})} + \gamma v(\boldsymbol{K}) \\ \text{s.t. } & \boldsymbol{u}(t) = -\boldsymbol{K}x(t) \\ & \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \tilde{\boldsymbol{B}}\boldsymbol{d}(t). \end{aligned}\tag{10}$$

where $e(\boldsymbol{K})$ is the energy term, and $v(\boldsymbol{K})$ is the communication cost approximated by the weighted l_1-norm

$$v(\boldsymbol{K}) = \sum_{i=1}^{q}\sum_{j=1}^{s} w_{ij}|k_{ij}| \tag{11}$$

where k_{ij} is the element in row i, column j of the feedback matrix $\boldsymbol{K}$, and w_{ij} is the nonnegative weight of this cost component. Since we aim to reduce only the inter-area costs, we can assign the weights w_{ij} so that

$$\begin{cases} w_{ij} = 1, & k_{ij} \text{ is inter-area link} \\ w_{ij} = 0, & k_{ij} \text{ is intra-area link} \end{cases}\tag{12}$$

When $\gamma = 0$, the SPWAC problem reduces to the traditional WAC problem. As γ increases, a sparser feedback gain solution emerges. Note that the l_1 norm (11) is a proxy for the cardinality metric of matrix $\boldsymbol{K}$ given by the number of non-zero elements in the off-diagonal blocks of $\boldsymbol{K}$, i.e.,

$$\text{card}_{\text{off}}(\boldsymbol{K}) = \sum_{i,j=1, i\neq j}^{r} \|\text{vec}\boldsymbol{K}_{ij}\|_0 \tag{13}$$

where vec operator converts a matrix to a column vector, and $\|\cdot\|_0$ is the vector l_0 norm [15].

III. Cost Allocation using Nash Bargaining Solution

We model inter-area communication cost allocation for SPWAC as a cooperative game. The Nash bargaining solution (NBS) is employed to divide this cost fairly among the areas. First, we review the NBS concept. Then its application to SPWAC cost allocation is presented.

A. NBS for cooperative games

In a cooperative game with transferable cost, the players bargain before the game is played and arrive at a binding agreement that specifies cost allocation among the players at the completion of the game. The Shapely value solution is often used to allocate costs to players in bargaining games. While it provides a unique and fair cost allocation [16], its major drawbacks include a requirement to define the characteristic function for every subset of the grand coalition and very large computation time for networks with many players [8]. Another approach to cost allocation in cooperative games is the NBS. It has been described for a two person general-sum game in [16]. In [8], NBS has been extended to multiple players, applied to cooperative network formation games, and was shown to require much lower computational load than the Shapely value solution.

The NBS proceeds in three steps. First, let v_{soc} denote the cost of a social optimization solution, where r players cooperate to find the lowest system cost overall. Then, a *disagreement point* of the cooperative game is defined as

$$\mathbf{v} = (v_1, v_2 ..., v_r) \tag{14}$$

where v_i is the maximum cost that the i^{th} player is willing to pay. Finally, the cost v_{soc} is split among the players. In the proposed formulation, the allocated costs α_i must satisfy [8]

$$\begin{aligned} & \max_{\alpha_i} \prod_{i=1}^{r} (v_i - \alpha_i) \\ \text{s.t.} \quad & \sum_{i=1}^{r} \alpha_i = v_{\text{soc}} \\ & 0 \le \alpha_i \le v_i, \quad \forall i = 1, ..., r. \end{aligned} \tag{15}$$

The last inequality is required to ensure nonnegative costs and successful cooperation. From [8], when the v_i's are sorted in the descending order, the allocated costs are

$$\alpha_i = \begin{cases} v_i - \frac{\sum_{i=1}^{m} v_i - v_{\text{soc}}}{m} & , i = 1, ..., m \\ 0 & , i = m+1, ..., r \end{cases} \tag{16}$$

where m is the largest index that satisfies

$$\frac{1}{m-1} \left(\sum_{i=1}^{m-1} v_i - v_{\text{soc}} \right) < v_m \tag{17}$$

B. Cost Allocation in SPWAC using NBS

We model cost allocation in SPWAC (10) using a cooperative NFG with transferable cost. The players are areas $\{1, 2, ..., r\}$, which cooperate to form links from PMUs to controllers to achieve desired energy performance and cost efficiency.

The proposed *SPWAC cost allocation algorithm* proceeds in 3 steps.

1) Social solution

We first compute the communication cost v_{soc} of social optimization. In this case the optimal feedback matrix $\boldsymbol{K}_{\text{soc}}(\gamma)$ satisfies (10) for a given sparsity parameter γ. Then the optimal social energy is

$$E_{soc}(\gamma) \quad := \quad e(\boldsymbol{K}_{\text{soc}}(\gamma)) \tag{18}$$

and the cost of social optimization is

$$v_{\text{soc}} = C_{\text{soc}}(\gamma) \quad := \quad \text{card}_{\text{off}}(\boldsymbol{K}_{\text{soc}}(\gamma)) \tag{19}$$

This cost definition accounts only for the inter-area communication cost (13) for a fixed value of γ.

2) The disagreement point

i) *Selfish energy optimization via a noncooperative game:*

In the selfish dynamics, each area aims to maximize its energy savings when investing in SPWAC. To quantify each area's objective function, we define the selfish energy

$$\begin{aligned} E_i &= \int_0^\infty \left(\boldsymbol{x}(t)^{\mathrm{T}} \boldsymbol{Q}_i \boldsymbol{x}(t) + \boldsymbol{u}_i(t)^{\mathrm{T}} \boldsymbol{R}_i \boldsymbol{u}_i(t) \right) dt, \\ i &= 1, ..., r \end{aligned} \tag{20}$$

where $\boldsymbol{Q}_i \in \mathbb{R}^{s \times s} \ge 0$, $\boldsymbol{R}_i \in \mathbb{R}^{N_i \times N_i} > 0$ are the design matrices for the selfish LQR objective. Similarly to the cooperative case, we choose $\boldsymbol{R}_i = \boldsymbol{I}_{N_i \times N_i}$. In the noncooperative case, each area aims to optimize only its own intra-area energy expressed in the consensus form:

$$\begin{aligned} \boldsymbol{x}_i^T \boldsymbol{Q}_{ii} \boldsymbol{x}_i \quad &:= \quad \sum_{k \in s_i} \sum_{\substack{j \in s_i \\ j > k}} (\Delta\delta_k - \Delta\delta_j)^2 \\ & \quad + \sum_{k \in s_i} \sum_{\substack{j \in s_i \\ j > k}} (\Delta\omega_k - \Delta\omega_j)^2 + \sum_{k \in s_i} \Delta E_k^2 \\ = \boldsymbol{x}^T \boldsymbol{Q}_i \boldsymbol{x} \quad &:= \quad \Delta\boldsymbol{\delta}^T \mathcal{L}_i \Delta\boldsymbol{\delta} + \Delta\boldsymbol{\omega}^T \mathcal{L}_i \Delta\boldsymbol{\omega} + \Delta\boldsymbol{E}^T \mathcal{H}_i \Delta\boldsymbol{E}, \\ & \quad i = 1, ..., r \end{aligned} \tag{21}$$

and

$$\mathcal{L}_i = \text{diag}(\boldsymbol{L}_1^i, \boldsymbol{L}_2^i, ..., \boldsymbol{L}_r^i) \tag{22}$$

$$\mathcal{H}_i = \text{diag}(\boldsymbol{H}_1^i, \boldsymbol{H}_2^i, ..., \boldsymbol{H}_r^i) \tag{23}$$

where

$$\boldsymbol{L}_j^i = \begin{cases} n_i \boldsymbol{I}_{n_i \times n_i} - \boldsymbol{1}_{n_i} \boldsymbol{1}_{n_i}^T & , j = i \\ \boldsymbol{0}_{n_j \times n_j} & , j \neq i \end{cases} \tag{24}$$

$$\boldsymbol{H}_j^i = \begin{cases} \boldsymbol{I}_{(M_j - 2n_j) \times (M_j - 2n_j)} & , j = i \\ \boldsymbol{0}_{(M_j - 2n_j) \times (M_j - 2n_j)} & , j \neq i \end{cases} \tag{25}$$

Note, again, if detailed models of generators are used then the last term in (21) should be replaced by $(\boldsymbol{x}^-)^T \mathcal{H}_i \boldsymbol{x}^-$ where $\boldsymbol{x}^-$ represents all the non-electromechanical states. We refer to the matrix $\boldsymbol{Q}_i$ that satisfies (21) as $\boldsymbol{Q}_i^{\text{intra}}$. While minimizing (20) using $\boldsymbol{Q}_i^{\text{intra}}$ satisfies the design objective of each area, it does not take into account the energy of the inter-area oscillations. We quantify each area's contribution to the inter-area energy as follows. In (6), we collect the difference terms that only account for the generators in area i and attribute $1/2$ of this energy to area i. Thus, the area i's share of the inter-area energy in terms of the phase angle is

$$\begin{aligned} & \frac{1}{2} \sum_{\substack{k=1,...n, \\ k \in s_i}} \sum_{\substack{j=1,...,n, \\ j \notin s_i}} (\Delta\delta_k - \Delta\delta_j)^2 \\ & \qquad = \Delta\boldsymbol{\delta}^T \mathcal{L}_i^{\text{sh}} \Delta\boldsymbol{\delta} \end{aligned} \tag{26}$$

where

$$\mathcal{L}_i^{\text{sh}} = \frac{n - 2n_i}{2} \mathcal{I}_i + \frac{n_i}{2} \boldsymbol{I} - \mathcal{I}_i \boldsymbol{1}_n \boldsymbol{1}_n^T (\boldsymbol{I} - \mathcal{I}_i) \tag{27}$$

$$\mathcal{I}_i = \text{diag}(\boldsymbol{I}_1^i, \boldsymbol{I}_2^i, ..., \boldsymbol{I}_r^i) \tag{28}$$

$$\boldsymbol{I}_j^i = \begin{cases} \boldsymbol{I}_{n_i \times n_i} & , j = i \\ \boldsymbol{0}_{n_j \times n_j} & , j \neq i \end{cases} \tag{29}$$

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The combined intra-area and inter-area energy associated with area i is computed using

$$\begin{aligned} \boldsymbol{x}^T \boldsymbol{Q}_i^{\mathrm{sh}} \boldsymbol{x} &= \Delta\boldsymbol{\delta}^T(\mathcal{L}_i + \mathcal{L}_i^{\mathrm{sh}})\Delta\boldsymbol{\delta} \\ &\quad + \Delta\boldsymbol{\omega}^T(\mathcal{L}_i + \mathcal{L}_i^{\mathrm{sh}})\Delta\boldsymbol{\omega} + (\boldsymbol{x}^-)^T \mathcal{H}_i \boldsymbol{x}^-, \\ i &= 1, ..., r \end{aligned} \tag{30}$$

The resulting combined energy

$$\begin{aligned} E_i^{\mathrm{sh}} &= \int_0^\infty \left(\boldsymbol{x}(t)^{\mathrm{T}} \boldsymbol{Q}_i^{\mathrm{sh}} \boldsymbol{x}(t) + \boldsymbol{u}_i(t)^{\mathrm{T}} \boldsymbol{R}_i \boldsymbol{u}_i(t) \right) dt, \\ i &= 1, ..., r \end{aligned} \tag{31}$$

Note that the matrix $\boldsymbol{Q}_i^{\mathrm{intra}}$ is employed for selfish optimization while $\boldsymbol{Q}_i^{\mathrm{sh}}$ is used to determine the share of the i^{th} area in the overall system energy after selfish optimization is carried out. The factor of $1/2$ in (26) indicates that inter-area power transfer terms associated with generators in two different areas are split equally among these areas. Other power marketing mechanisms and selfish design objectives will be considered in our future research.

To find the disagreement point, we consider two noncooperative games. In both games, the i^{th} area's strategic variable is $\boldsymbol{u}_i(t)$, and its objective function is E_i in (20). The strategy set $(\boldsymbol{u}_1^*(t), \boldsymbol{u}_2^*(t), \ldots, \boldsymbol{u}_r^*(t))$ is a Nash Equilibrium if [9], [13]

$$\begin{aligned} E_i(\boldsymbol{u}_i^*(t), \boldsymbol{u}_{-i}^*(t)) \leq E_i(\boldsymbol{u}_i(t), \boldsymbol{u}_{-i}^*(t)), \qquad t = [0, \infty) \\ \forall i = 1, \ldots, r \end{aligned} \tag{32}$$

where $\boldsymbol{u}_{-i}(t) := (\boldsymbol{u}_1(t), \ldots, \boldsymbol{u}_{i-1}(t), \boldsymbol{u}_{i+1}(t), \ldots, \boldsymbol{u}_r(t))$ is the vector of strategies formed by all players except for player i. These noncooperative games are discussed next.

ii) *Decoupled noncooperative game:*

In the first game dynamics, each area employs only intra-area feedback and is not allowed to utilize feedback from any other area. Thus $\boldsymbol{u}_i(t) = -\boldsymbol{K}_{ii}\boldsymbol{x}_i(t)$, where $\boldsymbol{K}_{ii}$ is the diagonal feedback block of area i in (9), i.e., the feedback is decoupled among the areas. For linear static feedback, it can be shown that the NE criterion (32) can be restated as

$$E_i(\boldsymbol{K}_{ii}^*, \boldsymbol{K}_{-i}^*) \quad \leq \quad E_i(\boldsymbol{K}_{ii}, \boldsymbol{K}_{-i}^*), \quad i = 1, ..., r \tag{33}$$

where the tuple $\boldsymbol{K}_{-i} := (\boldsymbol{K}_{11}, ..., \boldsymbol{K}_{(i-1)(i-1)}, \boldsymbol{K}_{(i+1)(i+1)}, ..., \boldsymbol{K}_{rr})$ represents the strategies for all other areas, and

$$\begin{aligned} E_i(\boldsymbol{K}_{ii}, \boldsymbol{K}_{-i}) = \int_0^\infty \left(\boldsymbol{x}(t)^{\mathrm{T}} \boldsymbol{Q}_i \boldsymbol{x}(t) + \boldsymbol{u}_i(t)^{\mathrm{T}} \boldsymbol{R}_i \boldsymbol{u}_i(t) \right) dt, \\ \text{s.t.} \quad \boldsymbol{u}_i(t) = -\boldsymbol{K}_{ii}\boldsymbol{x}_i(t) \\ \dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \tilde{\boldsymbol{B}}\boldsymbol{d}(t). \end{aligned} \tag{34}$$

We refer to the Nash strategies $(\boldsymbol{K}_{11}^*, ..., \boldsymbol{K}_{rr}^*)$ in (33) as *Decoupled Nash Equilibrium (DNE).*

To find the DNE, we utilize the idea of the infinitesimal gradient ascend (IGA) dynamics [17]. In IGA, each player iteratively updates its strategy according to the gradient descent direction of its objective function projected onto the constraint. The function $E_i(\boldsymbol{K}_{ii}, \boldsymbol{K}_{-i})$ is differentiable in each element of the matrix $\boldsymbol{K}_{ii}$. Thus, we define its gradient with respect to $\boldsymbol{K}_{ii}$

$$\nabla_{\boldsymbol{K}_{ii}} E_i(\boldsymbol{K}_{ii}, \boldsymbol{K}_{-i}) = (g_{j,k}^i)_{N_i \times M_i} \tag{35}$$

and

$$g_{j,k}^i = \frac{\partial}{\partial \boldsymbol{K}_{ii}(j,k)} E_i(\boldsymbol{K}_{ii}, \boldsymbol{K}_{-i}) \tag{36}$$

where $\boldsymbol{K}_{ii}(j,k)$ is the element in row j, column k in matrix $\boldsymbol{K}_{ii}$. Each area i updates its strategy according to the following rule

$$\begin{aligned} \boldsymbol{K}_{ii}(k+1) &= \boldsymbol{K}_{ii}(k) - \eta_k \cdot \nabla_{\boldsymbol{K}_{ii}} E_i(\boldsymbol{K}_{ii}, \boldsymbol{K}_{-i}) \\ i &= 1, ..., r \end{aligned} \tag{37}$$

where $k = 0, 1, 2, ...$ denotes the iteration index, and η_k is the stepsize in iteration k, which is infinitesimally small in [17]. It's easy to show that the DNE (33) is achieved if the gradient $\nabla_{\boldsymbol{K}_{ii}} E_i(\boldsymbol{K}_{ii}^*, \boldsymbol{K}_{-i}) = \boldsymbol{0}$ for all $i = 1, ..., r$.

In our implementation, we replace the gradient in (37) with the Newton direction [18] to speed up convergence, and select η_k by Armijo line search [19]. We employ the stopping criteria $||\nabla_{\boldsymbol{K}_{ii}} E_i(\boldsymbol{K}_{ii}, \boldsymbol{K}_{-i})|| < 10^{-2}, \forall i = 1, ..., r$. Extensive simulations demonstrate convergence to a unique equilibrium of the players' strategies (or at least of their energies). The convergence and robustness properties of this algorithm will be further investigated in our future work.

We define the combined energy of area i (31) associated with DNE as

$$E_i^{\mathrm{D}} = E_i^{\mathrm{sh}}(\mathrm{DNE}) \tag{38}$$

The terms E_i^{D} represent the energies areas achieve in the decentralized case, i.e. in the absence of cooperation and inter-area feedback, while employing their individual objectives. The total energy of this decentralized implementation is

$$\tilde{E}^{\mathrm{D}} := \sum_{i=1}^{r} E_i^{\mathrm{D}}. \tag{39}$$

iii) *Coupled noncooperative game:*

Second, the areas abandon the decentralized design above and invest in WAC while still optimizing their individual energies. To determine the energy savings associated with this investment, we construct a noncooperative game where the areas can send feedback to each other. In this case, we refer to (32) as Coupled Nash Equilibrium (CNE). The resulting combined energy of area i in (31) is denoted

$$E_i^{\mathrm{C}} := E_i^{\mathrm{sh}}(\mathrm{CNE}) \tag{40}$$

We combine the areas' energies in (40) to compute the total energy of the noncooperative wide-area control network

$$\tilde{E}^{\mathrm{C}} := \sum_{i=1}^{r} E_i^{\mathrm{C}} \tag{41}$$

The Nash strategies $\boldsymbol{u}_i^*(t)$ in eq.(32) can be determined by solving the cross-coupled algebraic Riccati equations

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(CARE) [13].

$$\boldsymbol{P}_i\left(\boldsymbol{A}-\sum_{j=1}^{r}\boldsymbol{S}_j\boldsymbol{P}_j\right)+\left(\boldsymbol{A}-\sum_{j=1}^{r}\boldsymbol{S}_j\boldsymbol{P}_j\right)^T\boldsymbol{P}_i+\boldsymbol{P}_i\boldsymbol{S}_i\boldsymbol{P}_i+\boldsymbol{Q}_i=\boldsymbol{0},\quad \forall i=1,...,r \tag{42}$$

where

$$\boldsymbol{B}_i := \begin{bmatrix}\boldsymbol{B}_{1i}\\ \vdots \\ \boldsymbol{B}_{ri}\end{bmatrix} \tag{43}$$

$$\begin{aligned}\boldsymbol{S}_i &:= \boldsymbol{B}_i\boldsymbol{R}_i^{-1}\boldsymbol{B}_i^T\\ \boldsymbol{u}_i^*(t) &= -\boldsymbol{R}_i^{-1}\boldsymbol{B}_i^T\boldsymbol{P}_i\boldsymbol{x}(t).\end{aligned} \tag{44}$$

From [11], [13], unique solution to CARE exists under the assumption of sufficiently weak coupling among the areas, i.e., the matrix $\boldsymbol{A}$ in (1) can be expressed as

$$\boldsymbol{A}=\boldsymbol{A}_0+\epsilon\boldsymbol{A}'(\epsilon) \tag{45}$$

where $0<\epsilon<<1$ is a small parameter, $\boldsymbol{A}_0$ is a block diagonal matrix representing intra-area coupling, and $\boldsymbol{A}'(\epsilon)\in\mathbb{R}^{s\times s}$ is a matrix whose elements are continuous functions of ϵ, and $\boldsymbol{A}_0$ and $\boldsymbol{A}'(\epsilon)$ have comparable norms [12]. Similarly, weak coupling holds for matrix $\boldsymbol{B}$

$$\boldsymbol{B}=\boldsymbol{B}_0+\epsilon\boldsymbol{B}'(\epsilon). \tag{46}$$

We adopt this assumption and utilize an iterative algorithm in Theorem 2 of [13] to solve CARE and find the unique CNE. Moreover, to assure that the problem is well-posed, we assume that the triples $(\boldsymbol{A}_{ii},\boldsymbol{B}_{ii},\sqrt{\boldsymbol{Q}_{ii}}), i=1,...,r$ are stabilizable and detectable, where $\boldsymbol{A}_{ii}$ and $\boldsymbol{B}_{ii}$ are defined in (1) and $\boldsymbol{Q}_{ii}$ is the i^{th} diagonal block of $\boldsymbol{Q}_i$ in (21).

iv) *The selfish costs:*

The difference between the energies of the coupled (41) and and decentralized (39) designs for each area indicates the energy saved due to utilization of inter-area feedback in the absence of cooperation. The communication cost associated with this feedback is given by

$$C_{\text{CNE}}=\text{card}_{\text{off}}(\boldsymbol{K}_{\text{CNE}}) \tag{47}$$

where $\boldsymbol{K}_{\text{CNE}}$ is the optimal feedback gain matrix that satisfies (32) for coupled Nash strategies. The selfish costs v_i are obtained by dividing of C_{CNE} proportionally to the areas' energy savings, which results in the disagreement point

$$\begin{aligned}v_i &= \frac{E_i^{\text{D}}-E_i^{\text{C}}}{\tilde{E}_{\text{D}}-\tilde{E}_{\text{C}}}\cdot C_{\text{CNE}}.\\ i &= 1,...,r.\end{aligned} \tag{48}$$

3) Cost allocation

Finally, the allocated costs are computed as in (16). Note that for $\gamma=0$, $v_{\text{soc}}=C_{\text{CNE}}$ since both are dense feedback costs. As γ increases, the $v_{\text{soc}}(\gamma)$ sequence decreases [5]. Thus, $C_{\text{CNE}}\geq C_{\text{soc}}$ in (19) for all γ values. It follows that

$$\sum_{i=1}^{n}v_i>v_{\text{soc}} \tag{49}$$

which guarantees successful bargaining [8]. Moreover, $v_i=\alpha_i$ for $\gamma=0$, and if $v_i\geq v_j$, then $\alpha_i\geq\alpha_j$ $(i\neq j)$, i.e., the more an area gains from inter-area feedback while pursuing its selfish energy objective, the more it has to pay. Finally, we require $\alpha_i\geq 0$ since all areas must cooperate to ensure suppression of inter-area oscillations, and thus should not require additional incentives in the form of reimbursement [10].

IV. Numerical Results

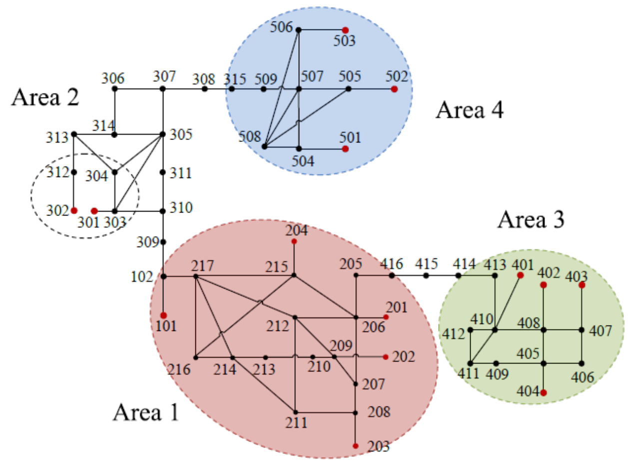
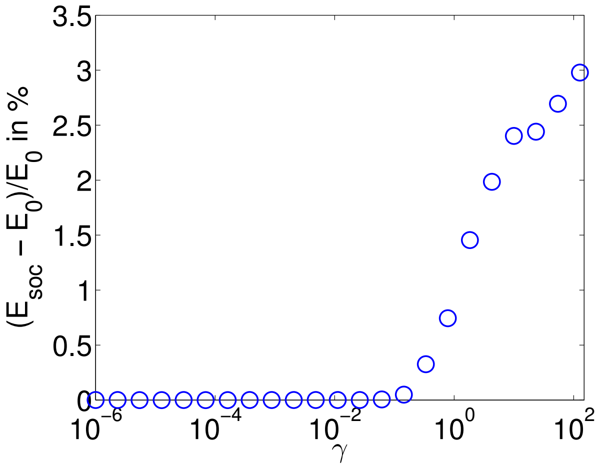


Fig. 1: The Australian 50-bus system [20].

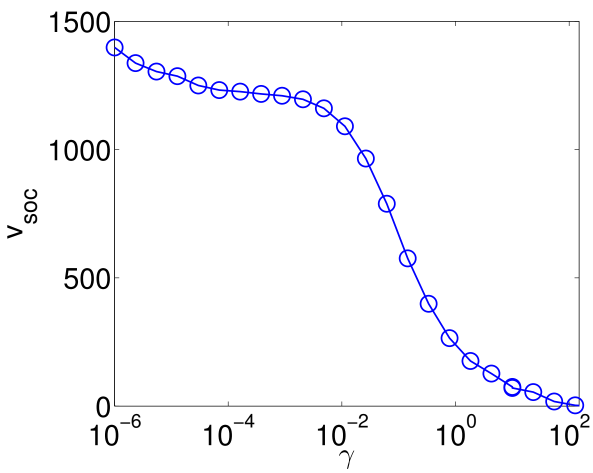
We validate our results using a 50-bus power system, as shown in Fig. 1. This power system model consists of 14 synchronous generators, divided into 4 coherent areas, and is a moderately accurate representation of the power grid in south-eastern Australia [20]. The area distribution is shown in different colors in Fig. 1, with the red dots denoting generator buses. Generators 1 to 5 belong to Area 1, generator 6 and 7 belong to area 2, generator 8 to 11 belong to area 3, and generator 12 to 14 belong to area 4. The model considers detailed representation of the generator dynamics, with each machine modeled by 14 states. However, since we are primarily interested in the electro-mechanical states, we perform an initial round of model reduction using singular perturbation to eliminate the non-electromechanical states, which have very low participation in the wide-area swing dynamics. We design a sparsity promoting LQR controller for this model following (10), and apply the proposed NBS approach of cost allocation for the underlying communication network.

Fig. 2 shows the social energy (18) and the communication cost v_{soc} (19). We observe that the inter-area communication cost decreases and the global social energy increases, respectfully, with the sparsity-promoting parameter γ, which reflects the trade-off between the communication cost and the energy performance [5]. Fig. 3 shows the allocated communication costs for areas 1 to 4, and Fig. 4 illustrates the proportional allocated costs. In this example, all areas initially contribute to the overall communication cost, but $\alpha_2=0$ for most of the γ values, while $\alpha_1=0$ for $\gamma>0.06$. Thus, these areas have low feedback needs. On the other hand, $\alpha_4>0$ for most of the γ range, while α_3 is the largest cost for all γ

values. We have observed that as γ increases, the sparsity-promoting algorithm eliminates all cyber links except those leading to controllers in areas 3 and 4, with links leading to area 3 dominating the sparsest networks. This is consistent with Fig. 3 and 4 results, which indicate that areas 3 and 4 require inter-area feedback the most, and thus pay the most for it. Finally, note from Fig. 2 that the energies remain very low for $\gamma \leq 0.06$ while v_{soc} decreases significantly in this range. Thus, cost-efficient SPWAC operating scenarios corresponds to $\gamma = 0.06$. For this γ value area 1, 3, and 4 experience significant cost savings due to cooperation without compromising energy performance.



(a) Social energy vs γ. $E_0 = E_{\text{soc}}$ for dense feedback ($\gamma = 0$).



(b) Social communication cost v_{soc} vs γ.

Fig. 2: Communication cost and global energy of social optimization for SPWAC [5].

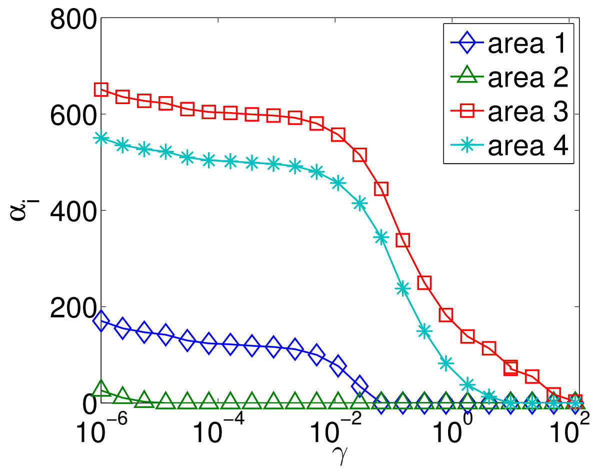


Fig. 3: Allocated communication costs α_i vs γ for area 1-4; the selfish costs v_i are approximated by α_i values for $\gamma = 10^{-6}$.

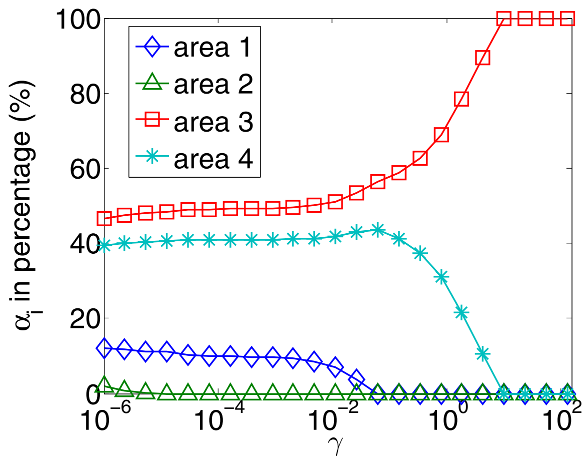


Fig. 4: Proportional allocated communication costs vs γ.

V. Conclusions

A novel game-theoretic strategy by which expenditure costs for wide-area communication in a power system can be fairly distributed among various utility companies was investigated. The allocated costs were computed for a 50-bus power system model divided into 4 areas. Future work will focus on selecting appropriate selfish energy objectives and power marketing mechanisms as well as robustness and convergence of proposed game-theoretic algorithms for cyber-physical systems.

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APPENDIX
THE PERMUTATION MATRIX $\mathcal{P}$

The permutation matrix $\mathcal{P}$ in (7) is

$$\mathcal{P} = \begin{bmatrix} \mathcal{P}_1 \\ \hline \mathcal{P}_2 \end{bmatrix} \tag{50}$$

where

$$\begin{aligned} \mathcal{P}_2 &= \mathrm{diag}(\mathcal{T}_1, \mathcal{T}_2, ..., \mathcal{T}_n). && (51) \\ \mathcal{T}_i &= \begin{bmatrix} \mathbf{0}_{(m_i-2)\times 2} & \boldsymbol{I}_{(m_i-2)\times(m_i-2)} \end{bmatrix} && (52) \\ \mathcal{P}_1 &= (p_{ij})_{2n\times s} && (53) \end{aligned}$$

and

$$\begin{aligned} p_{ij} &= \begin{cases} \delta_{j,k_i} & , 1 \le j \le n \\ \delta_{j,k_i+1} & , n+1 \le j \le 2n \end{cases} \\ k_i &= 1 + \sum_{k=1}^{i-1} m_k. && (54) \end{aligned}$$