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On Cotree-Critical and DFS Cotree-Critical Graphs

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Abstract

We give a characterization of DFS cotree-critical graphs which is central to the linear time Kuratowski finding algorithm implemented in PIGALE (Public Implementation of a Graph Algorithm Library and Editor [2]) by the authors, and deduce a justification of a very simple algorithm for finding a Kuratowski subdivision in a DFS cotree-critical graph.

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1 Introduction

The present paper is a part of the theoretical study underlying a linear time algorithm for finding a Kuratowski subdivision in a non-planar graph ([1]; see also [7] and [9] for other algorithms). Other linear time planarity algorithms don't exhibit a Kuratowski configuration in non planar graphs, but may be used to extract one in quadratic time.

It relies on the concept of DFS cotree-critical graphs, which is a by-product of DFS based planarity testing algorithms (such as [5] and [4]). Roughly speaking, a DFS cotree-critical graph is a simple graph of minimum degree 3 having a DFS tree, such that any non-tree (i.e. cotree) edge is *critical*, in the sense that its deletion would lead to a planar graph. A first study of DFS cotree-critical graphs appeared in [3], in which it is proved that a DFS cotree-critical graph either is isomorphic to K_5 or includes a subdivision of $K_{3,3}$ and no subdivision of K_5.

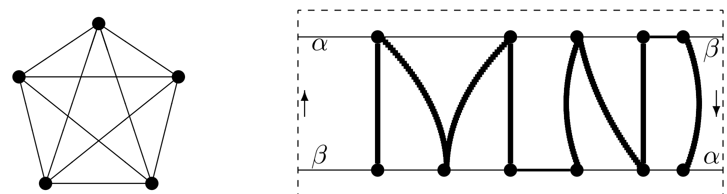

Figure 1: The DFS cotree-critical graphs are either K_5 or Möbius pseudo-ladders having all their non-critical edges (thickest) included in a single path.

The linear time Kuratowski subdivision extraction algorithm, which has been both conceived and implemented in [2] by the authors, consists in two steps: the first one correspond to the extraction of a DFS cotree-critical subgraph by a case analysis algorithm; the second one extracts a Kuratowski subdivision from the DFS cotree-critical subgraph by a very simple algorithm (see Algorithm 1), but which theoretical justification is quite complex and relies on the full characterization of DFS cotree-critical graphs that we prove in this paper: a simple graph is DFS cotree-critical if and only if it is either K_5 or a Möbius pseudo-ladder having a simple path including all the non-critical edges (see Figure 1).

The algorithm roughly works as follows: it first computes the set Crit of the critical edges of G, using the property that a tree edge is critical if and only if it belongs to a fundamental cycle of length 4 of some cotree edge to which it is not adjacent. Then, three pairwise non-adjacent non-critical edges are found to complete a Kuratowski subdivision of G isomorphic to $K_{3,3}$.

The space and time linearity of the algorithm are obvious.

Require: G is a DFS cotree-critical graph, with DFS tree Y.
Ensure: K is a Kuratowski subdivision in G.
  if G has 5 vertices **then**
    $K = G$ {G is isomorphic to K_5}
  else if G has less than 9 vertices **then**
    Extract K with any suitable method.
  else {G is a Möbius pseudo-ladder and the DFS tree is a path}
    Crit $\leftarrow E(G) \setminus Y$ {will be the set of critical edges}
    Find a vertex r incident to a single tree edge
    Compute a numbering λ of the vertices according to a traversal of the path Y starting at r, from 1 to n.
    Let e_i denote the tree edge from vertex numbered i to vertex numbered $i+1$.
    for all cotree edge $e = (u, v)$ (with $\lambda(u) < \lambda(v)$) **do**
      if $\lambda(v) - \lambda(u) = 3$ **then**
        Crit $\leftarrow$ Crit $\cup \{e_{\lambda(u)+1}\}$
      end if
    end for
    Find a tree edge $f = e_i$ with $2 < i < n - 3$ which is not in Crit.
    K has vertex set $V(G)$ and edge set Crit $\cup \{e_1, e_{n-1}, f\}$.
  end if

Algorithm 1: extracts a Kuratowski subdivision from a DFS cotree-critical graph [2].

2 Definitions and Preliminaries

For classical definitions (subgraph, induced subgraph, attachment vertices), we refer the reader to [8].

2.1 Möbius Pseudo-Ladder

A Möbius pseudo-ladder is a natural extension of Möbius ladders allowing triangles. This may be formalized by the following definition.

Definition 2.1 *Let γ be a polygon $(v_1, \ldots, v_n)$ and let $\{v_i, v_j\}$ and $\{v_k, v_l\}$ be non adjacent chords of γ. These chords are* interlaced *with respect to γ if, in circular order, one finds exactly one of $\{v_k, v_l\}$ between v_i and v_j. They are* non-interlaced*, otherwise.*

Thus, two chords of a polygon are either adjacent, or interlaced or non-interlaced.

Definition 2.2 *A* Möbius pseudo-ladder *is a non-planar simple graph, which is the union of a polygon $(v_1, \ldots, v_n)$ and chords of the polygon, such that any two non-adjacent bars are interlaced.*

With respect to such a decomposition, the chords are called bars.

A Möbius band is obtained from the projective plane by removing an open disk. Definition 2.2 means that a Möbius pseudo-ladder may be drawn in the plane as a polygon and internal chords such that any two non adjacent chords cross: consider a closed disk $\bar{\Delta}$ of the projective plane, which intersects any projective line at most twice (for instance, the disk bounded by a circle of the plane obtained by removing the line at infinity). Embed the polygon on the boundary of $\bar{\Delta}$. Then, any two projective lines determined by pairs of adjacent points intersect in $\bar{\Delta}$. Removing the interior Δ of $\bar{\Delta}$, we obtain an embedding of the Möbius pseudo ladder in a Möbius band having the polygon as its boundary (see Figure 2).

Notice that $K_{3,3}$ and K_5 are both Möbius pseudo-ladders.

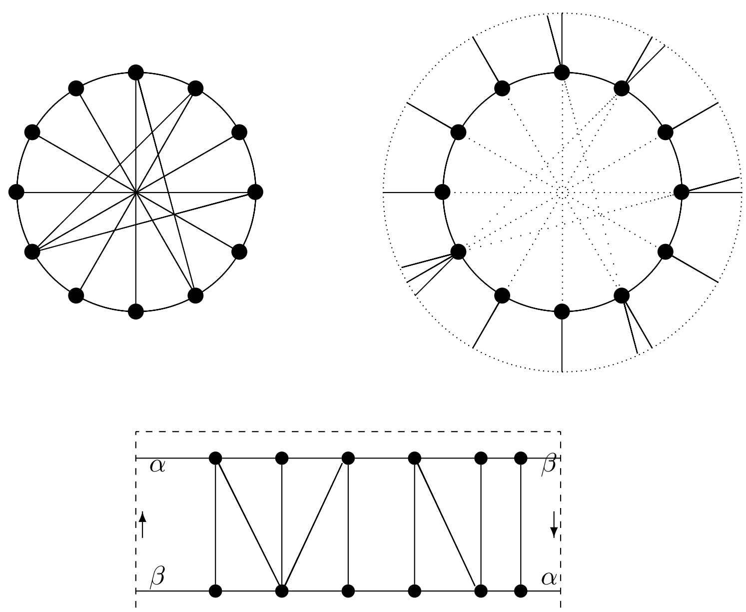

Figure 2: A Möbius pseudo-ladder on the plane, on the projective plane and on the Möbius band

2.2 Critical Edges and Cotree-Critical Graphs

Definition 2.3 *Let G be a graph. An edge $e \in E(G)$ is* critical *for G if $G - e$ is planar.*

Remark 2.1 *Let H be a subgraph of G, then any edge which is critical for G is critical for H (as $G - e$ planar implies $H - e$ planar).*

Thus, proving that an edge is non-critical for a particular subgraph of G is sufficient to prove that it is non-critical for G.

Moreover, if H is a non-planar subgraph of G, any edge in $E(G) \setminus E(H)$ is obviously non-critical for G.

Definition 2.4 *A* cotree-critical graph *is a non-planar graph G, with minimum degree* 3*, such that the set of non-critical edges of G is acyclic.*

Definition 2.5 *A* hut *is a graph obtained from a cycle $(v_1, \ldots, v_p, \ldots, v_n)$ by adding two adjacent vertices x and y, such that x is incident to $v_n, v_1, \ldots, v_p$, and y is incident to $v_p, \ldots, v_n, v_1$.*

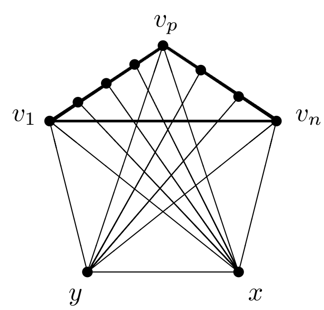

Figure 3: A hut drawn as a Möbius pseudo-ladder

We shall use the following result on cotree-critical graphs (expressed here with our terminology) later on:

Theorem 2.2 (Fraysseix, Rosenstiehl [3]) *A cotree-critical graph is either a hut or includes a subdivision of $K_{3,3}$ but no subdivision of K_5.*

2.3 Kuratowksi Subdivisions

A *Kuratowski subdivision* in a graph G is a minimal non-planar subgraph of G, that is: a non-planar subgraph K of G, such that all the edges of K are critical for K. Kuratowski proved in [6] that such minimal graphs are either subdivisions of K_5 or subdivisions of $K_{3,3}$.

If G is non-planar and if K is a Kuratowski subdivision in G, it is clear that any critical edge for G belongs to $E(K)$. This justifies a special denomination of the vertices and branches of a Kuratowski subdivision:

Definition 2.6 *Let G be a non-planar graph and let K be a Kuratowski subdivision of G. Then, a vertex is said to be a K-*vertex *(resp. a K-*subvertex*, resp. a K-*exterior vertex*) if it is a vertex of degree at least* 3 *in K (resp. a vertex of degree* 2 *in K, resp. a vertex not in K). A K-*branch *is the subdivided path of K between two K-vertices. Two K-vertices are K-*adjacent *if they are the endpoints of a K-branch. A K-branch with endpoints x and y is said to* link *x and y, and is denoted $[x, y]$. We further denote $]x, y[$ the subpath of $[x, y]$ obtained by deleting x and y.*

A K-branch is critical *for G if it includes at least one edge which is critical for G.*

2.4 Depth-First Search (DFS) Tree

Definition 2.7 *A* DFS tree *of a connected graph G, rooted at $v_0 \in V(G)$, may be recursively defined as follows: If G has no edges, the empty set is a DFS tree of G. Otherwise, let $G_1, \dots, G_k$ the connected components of $G - v_0$. Then, a DFS tree of G is the union of the DFS trees $Y_1, \dots, Y_k$ of $G_1, \dots, G_k$ rooted at $v_1, \dots, v_k$ (where $v_1, \dots, v_k$ are the neighbors of v_0 in G), and the edges $\{v_0, v_1\}, \dots, \{v_0, v_k\}$.*

Vertices of degree 1 in the tree are the terminals *of the tree.*

Definition 2.8 *A* DFS cotree-critical graph G *is a cotree-critical graph, whose non-critical edge set is a subset of a DFS tree of G.*

Lemma 2.3 *If G is k-connected ($k \geq 1$) and Y is a DFS tree of G rooted at v_0, then there exists a unique path in Y of length $k-1$ having v_0 as one of its endpoints.*

Proof: The lemma is satisfied for $k = 1$. Assume that $k > 1$ and that the lemma is true for all $k' < k$. Let v_0 be a vertex of a k-connected graph G. Then $G - v_0$ has a unique connected component H, which is $k-1$-connected. A DFS tree Y_G of G will be the union of a DFS tree Y_H of H rooted at a neighbor v_1 of v_0 and the edge $\{v_0, v_1\}$. As there exists, by induction, a unique path in Y_H of length $k-2$ having v_1 as one of its endpoints, there will exist a unique path in Y_G of length $k-1$ having v_0 as one of its endpoints. □

Corollary 2.4 *If G is 3-connected and Y is a DFS tree of G rooted at v_0, then v_0 has a unique son, and this son also has a unique son.*

Proof: As G is 3-connected, it is also 2-connected. Hence, there exists a unique tree path of length 1 and a unique tree path of length 2 having v_0 as one of its endpoints. □

Consider the orientation of a DFS tree Y of a connected graph G from its root (notice that each vertex has indegree at most 1 in Y). This orientation induces a partial order on the vertices of G, having the root of Y as a minimum. In this partial order, any two vertices which are adjacent in G are comparable (this is the usual characterization of DFS trees).

This orientation and partial order are the key to the proofs of the following two easy lemmas:

Lemma 2.5 *Let Y be a DFS tree of a graph G. Let x, y, z be three vertices of G, not belonging to the same monotone tree path. If x is a terminal of Y and x is adjacent to both y and z, then x is the root of Y.*

Proof: Assume x is not the root of Y. As y and z are adjacent to x, they are comparable with x. As x is a terminal different from the root v_0, y and y belong to the monotone tree path from v_0 to x, a contradiction. □

Lemma 2.6 *Let Y be a DFS tree of a graph G. Let x, y, z, t be four vertices of G, no three of which belong to the same tree path, and such that the tree paths from x to y and z to t intersect. Then, $\{x, y\}$ and $\{z, t\}$ cannot both be edges of G.*

Proof: Assume both $\{x, y\}$ and $\{z, t\}$ are edges of G. If x and y are adjacent, they are comparable and thus, the tree path linking them is a monotone path. Similarly, the same holds for the tree path linking z and t. As these two monotone tree paths intersect and as neither x and z belong to both paths, there exists a vertex having indegree at least 2 in the tree, a contradiction. □

3 Cotree-Critical Graphs

Lemma 3.1 *Let G be a graph and let H be the graph obtained from G by recursively deleting all the vertices of degree 1 and contracting all paths which internal vertices have degree 2 in G to single edges. Then, G is non-planar and has an acyclic set of non-critical edges if and only if H is cotree-critical.*

Proof: First notice that H is non-planar if and only if G is non-planar.

The critical edges of G that remain in H are critical edges for H, according to the commutativity of deletion, contraction of edges and deletion of isolated vertices (for $e \in E(H)$, if $G - e$ is planar so is $H - e$).

For any induced path P of G, either all the edges of P are critical for G or they are all non-critical for G. Thus, the edge of P that remains in H is critical for H if and only if at least one edge of P is critical for G. Hence, if H had a cycle of non critical edges for H, they would define a cycle of non-critical edges for G, because each (non-critical) edge for H represents a simple path of (non-critical) edges for G . Since G does not have a cycle of non-critical edges, H cannot have such a cycle either. Thus, as H has minimum degree 3, H is cotree-critical.

Conversely, assume H is cotree-critical. Adding a vertex of degree 1 does not change the status (critical/non-critical) of the other edges and cannot create a cycle of non-critical edges. Similarly, subdividing an edge creates two edges with the same status without changing the status of the other edges and hence cannot create a cycle of non-critical edges. Thus, the set of the non-critical edges of G is acyclic. □

Lemma 3.2 *Let G be a cotree-critical graph and let K be a Kuratowski subdivision of G isomorphic to $K_{3,3}$. Then, there exists in $E(G) \setminus E(K)$ no path between:*

- *two vertices (K-vertices or K-subvertices) of a same K-branch of K,*
- *two K-subvertices of K-adjacent K-branches of K.*

Proof: The two cases are shown Fig 4.

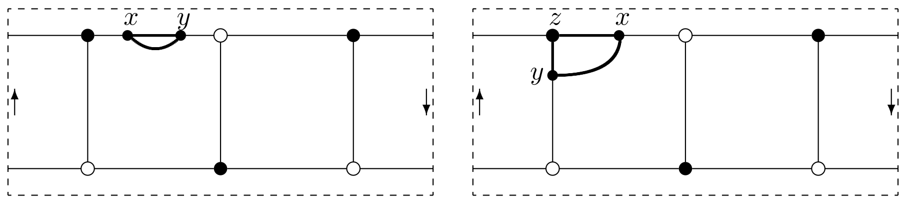

Figure 4: Forbidden paths in cotree-critical graphs (see Lemma 3.2)

If two vertices x and y (K-vertices or K-subvertices) of a same K-branch of K are joined by a path in $E(G) \setminus E(K)$, both this path and the one linking x and y in K are non-critical for G. Hence G is not cotree-critical, a contradiction.

If two K-subvertices x and y of K-adjacent K-branches of K are linked by a path in $E(G) \setminus E(K)$, this path is non-critical for G. Moreover, if z is the K-vertex adjacent to the branches including x and y, both paths from z to x and x to y are non-critical for G. Hence, G includes a non-critical cycle, a contradiction. □

We need the following definition in the proof of the next lemma:

Definition 3.1 *Let H be an induced subgraph of a graph G. The* attachment vertices *of H in G is the subset of vertices of H having a neighbor in $V(G) \setminus V(H)$.*

Lemma 3.3 *Every cotree-critical graph is* 3*-connected.*

Proof: Let G be a cotree-critical graph. Assume G has a cut-vertex v. Let H_1, H_2 be two induced subgraphs of G having v as their attachment vertex and such that H_1 is non-planar. As G has no degree 1 vertex, H_2 includes a cycle. All the edges of this cycle are non critical for G, a contradiction. Hence, G is 2-connected.

Assume G has an articulation pair $\{v, w\}$ such that there exists at least two induced subgraphs H_1, H_2 of G, different from a path, having v, w as attachment vertices. As G is non planar, we may choose H_1 in such a way that $H_1 + \{v, w\}$ is a non-planar graph (see [8], for instance). As there exists in H_2 two disjoints paths from v to w, no edge of these paths may be critical for G and H_2 hence include a cycle of non-critical edges for G, a contradiction. □

Lemma 3.4 *Let G be a cotree-critical graph and let K be a Kuratowski subdivision of G. Then, G has no K-exterior vertices, that is: $V(G) = V(K)$.*

Proof: According to Theorem 2.2, if K is a subdivision of K_5, then either $G = K$, or G is a hut, having K has a spanning subgraph. Thus, G has no K-exterior vertex in this case, and we shall assume that K is a subdivision of $K_{3,3}$.

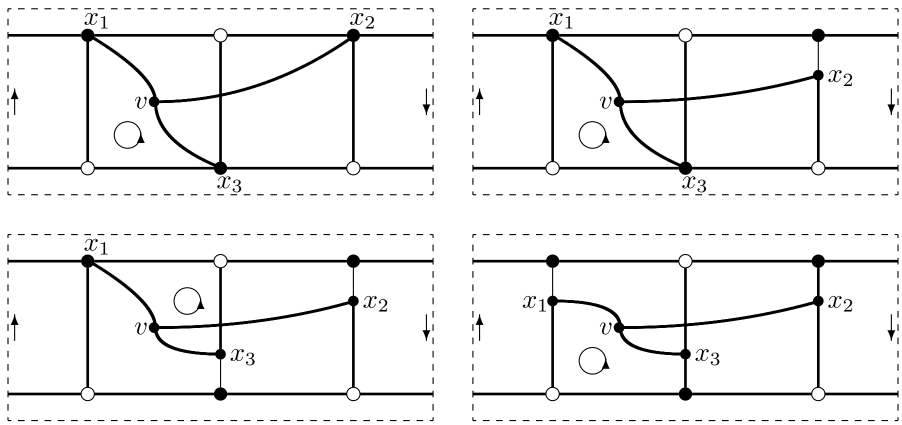

Figure 5: A cotree-critical graphs has no K-exterior vertex (see Lemma 3.4)

Assume $V(G) \setminus V(K)$ is not empty and let v be a vertex of G not in K. According to Lemma 3.3, G is 3-connected. Hence, there exists 3 disjoint paths P_1, P_2, P_3 from v to K. As $K + P_1 + P_2 + P_3$ is a non-planar subgraph of G free of vertices of degree 1, it is a subdivision of a 3-connected graph, according to Lemma 3.1 and Lemma 3.3. Thus, the vertices of attachment x_1, x_2, x_3 of P_1, P_2, P_3 in K are all different. As $K_{3,3}$ is bipartite, we may color the K-vertices of K black and white, in such a way that K-adjacent K-vertices have different colors. According to Lemma 3.2, no path in $E(G) \setminus E(H)$ may link K-vertices with different colors. Thus, we may assume no white K-vertex belong to $\{x_1, x_2, x_3\}$ and four cases may occur as shown Fig 5. All the four cases show a cycle of non-critical edges, a contradiction. □

Corollary 3.5 *If G is cotree-critical, no non-critical K-branch may be subdivided, that is: every non-critical K-branch is reduced to an edge.*

Proof: If a branch of K is non-critical for G, there exists a $K_{3,3}$ subdivision avoiding it. Hence, the branch just consists of a single edge, according to Lemma 3.3. □

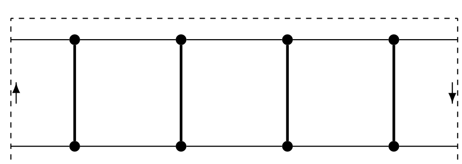

Figure 6: The 4-bars Möbius ladder M_4 (all bars are non-critical edges)

Let G be a cotree-critical graph obtained by adding an edge linking two subdivision vertices of non-adjacent edges of a subdivision of a $K_{3,3}$. This graph is unique up to isomorphism and is the Möbius ladder with 4 non-critical bars shown Figure 6.

Figure 7 shows a graph having a subdivision of a Möbius ladder with 3 bars as a subgraph, where two of the bars are not single edges.

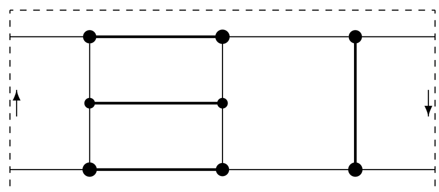

Figure 7: A graph having a subdivision of a 3-bars Möbius ladder as a subgraph (some bars are paths of critical edges)

The same way we have introduced K-vertices, K-subvertices and K-branches relative to a Kuratowski subdivision, we define M-vertices, M-subvertices and M-branches relative to a Möbius ladder subdivision.

Lemma 3.6 *Let K be a $K_{3,3}$ subdivision in a cotree-critical graph G. Not K-adjacent K-vertices of K form two classes, $\{x, y, z\}$ and $\{x', y', z'\}$, as $K_{3,3}$ is bipartite.*

If $[x, z']$ or $[x', z]$ is a critical K-branch for G, then all the edges from $]x, z[=]x, y'] \cup [y', z[$ to $]x', z'[=]x', y] \cup [y, z'[$ and the K-branch $[y, y']$ are pairwise adjacent or interlaced, with respect to the cycle (x, y', z, x', y, z').

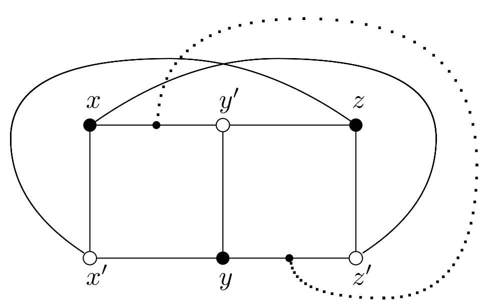

Figure 8: No edges is allowed from $]x, z[$ to $]x', z'[$ by the "outside" (see Lemma 3.6)

Proof: The union of the $K_{3,3}$ subdivision and all the edges of G incident to a vertex in $]x, z[$ and a vertex in $]x', z'[$ becomes uniquely embeddable in the plane after removal of the K-branches $[x, z']$ or $[x', z]$. Figure 8 displays the outline of a normal drawing of G in the plane which becomes plane when removing any of the K-branch $[x, z']$ or $[x', z]$. In such a drawing, given that an edge from $]x, z[$ to $]x', z'[$, if drawn outside, crosses both $[x, z']$ and $[x', z]$, all the edges from $]x, z[$ to $]x', z'[$ and the K-branch $[y, y']$ are drawn inside the cycle (x, y', z, z', y, x') without crossing and thus are adjacent or interlaced with respect to the cycle (x, y', z, x', y, z'). The result follows. □

Lemma 3.7 *If G is a cotree-critical graph having a subdivision of Möbius ladder M with 4 bars as a subgraph, then it is the union of a polygon γ and chords which are non-critical for G. Moreover the 4 bars b_1, b_2, b_3, b_4 of M are chords and any other chord is adjacent or interlaced with all of b_1, b_2, b_3, b_4 with respect to γ.*

Proof: Let G be a cotree-critical graph having a subdivision of Möbius M ladder with 4 bars b_1, b_2, b_3, b_4 as a subgraph. First notice that all the bars of the Möbius ladder are non-critical for G and that, according to Corollary 3.5, they are hence reduced to edges. According to Lemma 3.4, M covers all the vertices of G as it includes a $K_{3,3}$ and hence the polygon γ of the ladder is Hamiltonian. Thus, the remaining edges of G are non-critical chords of γ.

Let e be a chord different from b_1, b_2, b_3, b_4.

- Assume e is adjacent to none of b_1, b_2, b_3, b_4.

  Then it cannot be interlaced with less than 3 bars, according to Lemma 3.2, considering the $K_{3,3}$ induced by at least two non-interlaced bars. It cannot also be interlaced with 3 bars, according to Lemma 3.6, considering the $K_{3,3}$ induced by the 2 interlaced bars (as $\{x, x'\}, \{z, z'\}$) and one non-interlaced bar (as $\{y, y'\}$).

- Assume e is adjacent to b_1 only.

  Then it is interlaced with the 3 other bars, according to Lemma 3.2, considering the $K_{3,3}$ induced by b_1 and two non-interlaced bars.

- Assume e is adjacent to b_1 and another bar b_i.

  Assume e is not interlaced with some bar $b_j \notin \{b_1, b_i\}$ then, considering the $K_{3,3}$ induced by b_1, b_i, b_j we are led to a contradiction, according to Lemma 3.2. Thus, e is interlaced with the 2 bars to which it is not adjacent.

□

Theorem 3.8 *If G is a cotree-critical graph having a subdivision of Möbius ladder M with 4 bars as a subgraph, then it is a Möbius pseudo-ladder whose polygon γ is the set of the critical edges of G.*

Proof: According to Lemma 3.7, G is the union of a polygon γ and chords including the 4 bars of M. In order to prove that G is a Möbius pseudo-ladder, it is sufficient to prove that any two non-adjacent chords are interlaced with respect to that cycle. We choose to label the 4 bars b_1, b_2, b_3, b_4 of M according to an arbitrary traversal orientation of γ. According to Lemma 3.7, any chord e is adjacent or interlaced with all of b_1, b_2, b_3, b_4 and hence its endpoints are traversed between these of two consecutive bars $b_{\alpha(e)}, b_{\beta(e)}$ (with $\beta(e) \equiv \alpha(e) + 1 \pmod 4$), which defines functions α and β from the chords different from b_1, b_2, b_3, b_4 to $\{1, 2, 3, 4\}$.

As all the bars are interlaced pairwise and as any chord is adjacent or interlaced with all of them, we only have to consider two non-adjacent chords e, f not in $\{b_1, b_2, b_3, b_4\}$.

- Assume $\alpha(e)$ is different from $\alpha(f)$.

  Then, the edges e and f are interlaced, as the endpoints of e and f appear alternatively in a traversal of γ.

- Assume $\alpha(e)$ is equal to $\alpha(f)$.

  Let b_i, b_j be the bars such that $j \equiv \beta(e) + 1 \equiv \alpha(e) + 2 \equiv i + 3 \pmod 4$. Then, consider the $K_{3,3}$ induced by γ and the bars b_i, e, b_j. As b_i and b_j are non critical, one of the branches adjacent to both of them is critical, for otherwise a non critical cycle would exist. Hence; it follows from Lemma 3.6 that e and f are interlaced.

□

4 DFS Cotree-Critical Graphs

An interesting special case of cotree-critical graphs, the DFS cotree-critical graphs, arise when the tree may be obtained using a Depth-First Search, as it happens when computing a cotree-critical subgraph using a planarity testing algorithm. Then, the structure of the so obtained DFS cotree-critical graphs appears to be quite simple and efficient to exhibit a Kuratowski subdivision (leading to a linear time algorithm).

In this section, we first prove that any DFS cotree graph with sufficiently many vertices includes a Möbius ladder with 4 bars as a subgraph and hence are Möbius pseudo-ladders, according to Theorem 3.8. We then prove that these Möbius pseudo-ladders may be fully characterized.

Lemma 4.1 *Let G be a cotree-critical graph and let K be a Kuratowski subdivision of G isomorphic to $K_{3,3}$. Then, two K-vertices a, b which are not K-adjacent cannot be adjacent to K-subvertices on a same K-branch.*

Proof: The three possible cases are shown Figure 9; in all cases, a cycle of non-critical edges exists. □

Lemma 4.2 *Let G be a cotree-critical graph and let K be a Kuratowski subdivision of G isomorphic to $K_{3,3}$. If G has two edges interlaced as shown Figure 10, then G is not DFS cotree-critical.*

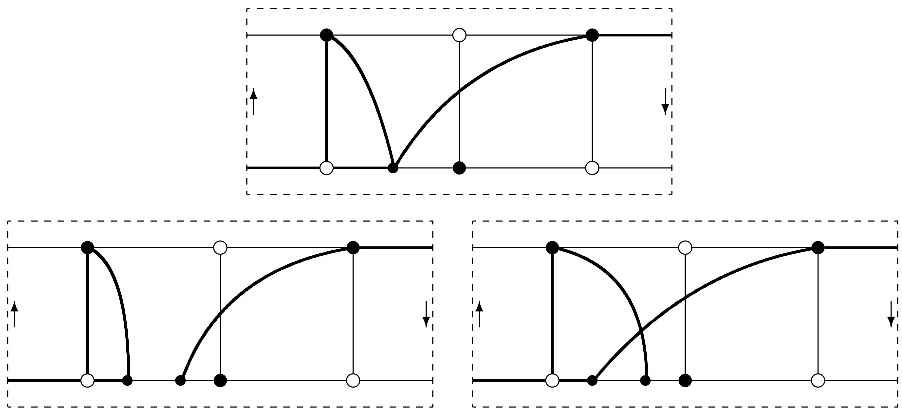

Figure 9: No two non-adjacent K-vertices may be adjacent to K-subvertices on the same K-branch (see Lemma 4.1)

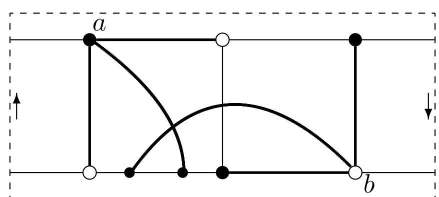

Figure 10: Case of two adjacent K-vertices adjacent to K-subvertices on the same K-branch by two interlaced edges (see Lemma 4.2)

Proof: Assume G is cotree-critical. By case analysis, one easily checks that any edge of G outside $E(K)$ is either incident to a or b. Hence, all the vertices of G incident to at most one non-critical edge is adjacent to a vertex incident with at least 3 non-critical edges (a or b). According to Corollary 2.4, the set of non-critical edges is not a subset of a DFS tree of G, so G is not DFS cotree-critical. □

Lemma 4.3 *Let G be a DFS cotree-critical graph and let K be a $K_{3,3}$ subdivision in G. Then, no two edges in $E(G) \setminus E(K)$ may be incident to the same K-vertex.*

Proof: Assume G has a subgraph formed by K and two edges e and f incident to the same K-vertex a. According to Lemma 3.4 and Lemma 3.2, K is a spanning subgraph of G and only four cases may occur, depending on the position of the endpoints of e and f different from a, as none of these may belong to a K-branch including a:

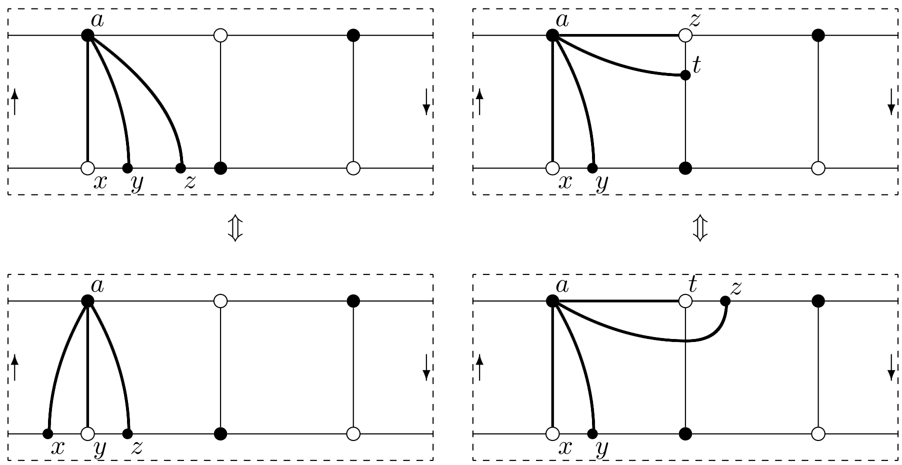

Figure 11: Cases of Lemma 4.3

- either they belong to the same K-branch,
- or they belong to two K-branches having in common a K-vertex which is not K-adjacent to a,
- or they belong to two K-branches having in common a K-vertex which is K-adjacent to a,
- or they belong to two disjoint K-branches.

By a suitable choice of the Kuratowski subdivision, the last two cases are easily reduced to the first two ones (see Fig 11).

- Consider the first case.

  Assume there exists a K-subvertex v between x and y. Then, v is not adjacent to a K-vertex different from a, according to Lemma 4.1 and Lemma 4.2. If v were adjacent to another K-subvertex w, the graph would include a Möbius ladder with 4 bars as a subgraph and, according to Theorem 3.8, would be a Möbius pseudo-ladder in which $\{a, y\}$ and $\{v, w\}$ would be non adjacent non interlaced chords, a contradiction. Thus, v may not be adjacent to a vertex different from a and we shall assume, without loss of generality, that x and y are adjacent. Similarly, we may also assume that y and z are adjacent.

  Therefore, if G is DFS cotree-critical with tree Y, y is a terminal of Y and, according to Lemma 2.5, is the root of Y, which leads to a contradiction, according to Corollary 2.4.

- Consider the second case.

  As previously, we may assume that both x, y and z, t are adjacent. G cannot be DFS cotree-critical, according to Lemma 2.6.

□

Lemma 4.4 *If G is DFS cotree-critical, includes a subdivision of $K_{3,3}$, and has at least 10 vertices, then G includes a 4-bars Möbius ladder as a subgraph.*

Proof: Let K be a $K_{3,3}$ subdivision in G.

Assume K has two K-subvertices u and v adjacent in G. According to Lemma 3.2, u and v neither belong to a same K-branch, nor to adjacent K-branches. Let $[a, a']$ (resp. $[b, b']$) be the K-branch including u (resp. v), where a is not K-adjacent to b. Let c (resp. c') be the K-vertex K-adjacent to a' and b' (resp. a and b). Then, the polygon $(c', a, u, a', c, b', v, b)$ and the chords $\{c, c'\}, \{a, b'\}, \{u, v\}$ and $\{a', b\}$ define a 4-bars Möbius ladder.

Thus, to prove the Lemma, it is sufficient to prove that if no two K-subvertices are adjacent in G, there exists another $K_{3,3}$ subdivision K' in G having two K'-subvertices adjacent in G.

As G has at least 10 vertices, there exists at least 4 K-subvertices adjacent in G to K-vertices. Let S be the set of the pairs (x, y) of K-vertices, such that there exists a K-subvertex v adjacent to x belonging to a K-branch having y as one of its endpoints. Notice that $K + \{x, v\} - \{x, y\}$ is a subdivision of $K_{3,3}$ and thus that $[x, y]$ is non-critical for G.

Assume there exists two pairs (x, y) and (y, z) in S. Let u be the vertex adjacent to x in the K-branch incident to y and let v be the vertex adjacent to y in the K-branch incident to z. Then, $K + \{x, u\} - \{x, y\}$ is a subdivision K' of $K_{3,3}$ for which $\{v, y\}$ is an edge incident to two K'-subvertices. Hence, we are done in this case.

We prove by reductio ad absurdum that the other case (no two pairs (x, y) and (y, z) belong to S) may not occur: according to Lemma 4.3, no two edges in $E(G) \setminus E(K)$ may be incident to a same K-vertex. Thus, no two pairs (x, y) and (x, z) may belong to S. Moreover, assume two pairs (x, y) and (z, y) belong to S. Then, $[x, y]$ and $[z, y]$ are non critical for G and thus not subdivided. Hence, x and z have to be adjacent to K-subvertices in the same K-branch incident to y, which contradicts Lemma 4.1. Thus, no two pairs (x, y) and (z, y) may belong to S. Then, the set $\{\{x, y\} : (x, y) \in S \text{ or } (y, x) \in S\}$ is a matching of $K_{3,3}$. As S includes at least 4 pairs and as $K_{3,3}$ has no matching of size greater than 3, we are led to a contradiction. □

Theorem 4.5 (Fraysseix, Rosenstiehl [3]) *A DFS cotree-critical graph is either isomorphic to K_5 or includes a subdivision of $K_{3,3}$ but no subdivision of K_5.*

Theorem 4.6 *Any DFS cotree-critical graph is a Möbius pseudo-ladder.*

Proof: If G is isomorphic to K_5, the result holds. Otherwise G includes a subdivision of $K_{3,3}$, according to Theorem 4.5. Then, the result is easily checked for graphs having up to 9 vertices, according to the restrictions given by Lemma 4.1 and Lemma 4.3 and, if G as at least 10 vertices, the result is a consequence of Lemma 4.4 and Theorem 3.8. □

Theorem 4.7 *A simple graph G is DFS cotree-critical if and only if it is a Möbius pseudo-ladder which non-critical edges belong to some Hamiltonian path.*

Moreover, if G is DFS cotree-critical according to a DFS tree Y and G has at least 9 vertices, then Y is a path and G is the union of a cycle of critical edges and pairwise adjacent or interlaced non critical chords.

Proof: If all the non-critical graphs belong to some simple path, the set of the non-critical edges is acyclic and the graph is cotree critical. Furthermore, as we may choose the tree including the non-critical edges as the Hamiltonian path, the graph is DFS cotree-critical.

Conversely, assume G is DFS cotree-critical. The existence of an Hamiltonian including all the non-critical edges is easily checked for graph having up to 9 vertices. Hence, assume G has at least 10 vertices. According to Theorem 4.7, G is a Möbius pseudo ladder. By a suitable choice of a Kuratowski subdivision of $K_{3,3}$, it follows from Lemma 4.3 that no vertex of G may be adjacent to more than 2 non-critical edges. Let Y be a DFS tree including all the non-critical edges. Assume Y has a vertex v of degree at least 3. Then, one of the cases shown Figure 12 occurs (as v is incident to at most 2 non-critical edges) and hence v is adjacent to a terminal w of T. According to Lemma 2.5 and Corollary 2.4, we are led to a contradiction. □

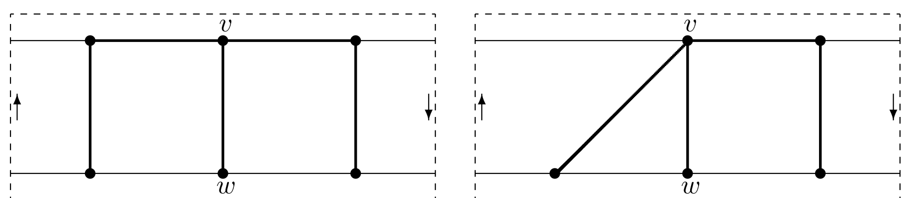

Figure 12: A vertex of degree at least 3 in the tree is adjacent to a terminal of the tree (see Theorem 4.7)

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