Hopf bifurcation on a sphere

Rachel Sigrist

December 14, 2010

Abstract

Using the general theory of Hopf bifurcation with symmetry we study here the example where the group of symmetries is $\mathbf{O(3)}$, the rotations and reflections of a sphere. We make some amendments to previously published lists of $\mathbf{C}$-axial isotropy subgroups of $\mathbf{O(3)} \times S^1$ and list the isotropy subgroups with four-dimensional fixed-point subspaces. We then study the particular example where $\mathbf{O(3)} \times S^1$ acts on the space $V_3 \oplus V_3$ where V_3 is the space of spherical harmonics of degree three. We find that in this case there are six $\mathbf{C}$-axial isotropy subgroups of $\mathbf{O(3)} \times S^1$. The equivariant Hopf theorem guarantees the existence of periodic solutions with each of these symmetries in $\mathbf{O(3)} \times S^1$ equivariant differential equations. Three of the solutions are found to be standing waves and the other three are travelling waves. We compute conditions for each of these solution branches to be stable and by restricting the $\mathbf{O(3)} \times S^1$ equivariant differential equations to four-dimensional invariant subspaces we are able to find additional periodic and quasiperiodic solutions.

1 Introduction

Bifurcations from states with spherical symmetry occur in several physical and biological systems. For example, bifurcations to stationary patterns occur in Rayleigh–Bénard convection in a spherical shell [3, 4, 25]. If the fluid within the spherical shell is subjected to a magnetic field (and is electrically conducting) or concentration gradient in addition to the temperature gradient then it is possible for a bifurcation to oscillating solutions to occur. Examples of such oscillating convection can be found in [6, 19, 20]. Convection within a spherical shell has applications including continental drift driven by convection currents in the Earth's mantle and also convection within the Sun where the strong magnetic field has an influence on the convective motion..

Another physical example of a bifurcation from a spherically symmetric state is the buckling of a sphere or spherical shell under external uniform pressure. This has applications including the evolution of a gas bubble in a liquid [18, 24].

Both stationary and Hopf bifurcation can occur in reaction–diffusion systems on a sphere, as discussed in [27]. Stationary patterns resulting from reaction–diffusion systems on a sphere are considered in [28] and a specific example of a reaction–diffusion system which undergoes a Hopf bifurcation is discussed in [7, 8, 29].

Biological examples of bifurcations from states with spherical symmetry include a spherical ball of cells developing into an asymmetric shape. It is possible for this ball of cells to be an embryo as in [27] or a solid tumour as in [5].

Using equivariant bifurcation theory it is possible to consider the behaviours of each of the example systems above by using only the spherical symmetry of the system. In this paper we will study the case of a Hopf bifurcation from a state with spherical, or $\mathbf{O(3)}$, symmetry without any reference to the details of any particular system. We will use the general theory of Hopf bifurcation in the presence of a group of symmetries Γ developed by Golubitsky and Stewart [10] and Golubitsky *et al* [12]. This theory is often referred to as the equivariant singularity theory. An overview of the results which we will require from the work of Golubitsky and Stewart [10] and Golubitsky *et al* [12] is given in section 2 of this paper. These results allow us to prove the existence of branches of periodic solutions of equivariant differential equations with the symmetries of certain subgroups of the group $\Gamma \times S^1$. These subgroups are the $\mathbf{C}$-axial subgroups of $\Gamma \times S^1$—the isotropy subgroups with two-dimensional fixed-point subspaces.

For the case where $\Gamma = \mathbf{O(3)}$ and the representation of $\mathbf{O(3)} \times S^1$ is on the direct product $V_\ell \oplus V_\ell$, where V_ℓ is the space of spherical harmonics of degree ℓ on which $\mathbf{O(3)}$ acts absolutely irreducibly, the $\mathbf{C}$-axial subgroups were first listed by Golubitsky and Stewart in [10]. One error in this list was corrected in Golubitsky *et al* [12, Chapter XVIII section 5], however a small number of other errors remain. In section 3 of the present work we correct these errors and present an amended list of the $\mathbf{C}$-axial subgroups, giving reasons why the changes to Table 5.1 of [12, Chapter XVIII section 5] are required. We also compute the isotropy subgroups of $\mathbf{O(3)} \times S^1$ with four-dimensional fixed-point subspaces. It is possible, under certain conditions, for solutions to equivariant differential equations with these symmetry groups to exist. Indeed, if the isotropy subgroup is maximal then the existence of a solution with these symmetries is guaranteed.

Remark 1.1. We should mention that although we use the equivariant singularity theory to study the Hopf bifurcations with symmetry in this paper, this is not the only possible treatment. The equivariant degree based method (see [2] and [17]) also allows such problems to be studied without some of the assumptions required by the equivariant singularity theory.

The specific example of a Hopf bifurcation with $\mathbf{O(3)}$ symmetry where the representation of $\mathbf{O(3)} \times S^1$ is on the space $V_2 \oplus V_2$ has been studied previously. Iooss and Rossi [16] found five primary branches of periodic solutions using analytical methods and extensive calculations. These findings were confirmed by Haaf *et al* [14], where they realised V_2 as the set of symmetric traceless 3×3 matrices and used equivariant group theoretic methods to find the five primary solution branches. Both papers investigate the stability of the five solution branches and find that it is necessary to consider fifth order terms in the normal form of the equivariant differential equations to fully determine the stability of all five solution branches.

In sections 4 and 5 of this work we will consider the primary branches of periodic solutions which are guaranteed to exist for the representation of $\mathbf{O(3)} \times S^1$ on the space $V_3 \oplus V_3$. We compute the normal form of the equivariant differential equations for this representation to cubic order and find that this is sufficient to determine the direction of branching and stability of each of the primary solution branches. Finally in section 6 we investigate solutions of the equivariant differential equations with less symmetry whose existence is not guaranteed by the equivariant Hopf theorem. To find such solutions we study the dynamics in the restriction of the equivariant differential equations to four-dimensional invariant subspaces.

2 Background

In this section we state without proof the main results required for this work. This includes some general theory of Hopf bifurcation with symmetry and background on the representations and subgroups of $\mathbf{O}(\mathbf{3})$. Further details including proofs can be found in [10] and [12].

2.1 General theory of Hopf bifurcation with symmetry

Consider the system of ordinary differential equations

$$\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = f(\mathbf{z}, \lambda), \tag{1}$$

where $\mathbf{z} \in \mathbf{R}^n$, $\lambda \in \mathbf{R}$ is a bifurcation parameter and $f : \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ is a smooth vector field which is equivariant under the action of a symmetry group Γ, i.e.

$$\gamma f(\mathbf{z}, \lambda) = f(\gamma \mathbf{z}, \lambda) \qquad \forall \gamma \in \Gamma. \tag{2}$$

Suppose that there is a trivial equilibrium solution $f(\mathbf{0}, \lambda) \equiv \mathbf{0}$ which is Γ-invariant and undergoes a Hopf bifurcation at $(\mathbf{z}, \lambda) = (\mathbf{0}, 0)$. Since points $\mathbf{z}$ on the same group orbit,

$$\Gamma \mathbf{z} = \{\gamma \mathbf{z} : \gamma \in \Gamma\}, \tag{3}$$

have the same stability properties, at the Hopf bifurcation there will be multiple pairs of eigenvalues crossing the imaginary axis. Assume that (1) is already reduced to the centre subspace so that $\mathbf{z} \in \mathbf{R}^{2p} \cong \mathbf{C}^p$ and the Jacobian $(\mathrm{d}f)_{(\mathbf{0},0)}$ has purely imaginary eigenvalues. For this to occur, the imaginary eigenspace must be Γ-simple, which is the case when the representation W of Γ is given by $W \cong V \oplus V$ where V is an absolutely irreducible representation of Γ. In suitable coordinates and rescaling time if necessary we then have that

$$J = (\mathrm{d}f)_{(\mathbf{0},0)} = \begin{bmatrix} 0 & I_p \\ -I_p & 0 \end{bmatrix} \tag{4}$$

and the eigenvalues of $(\mathrm{d}f)_{(\mathbf{0},\lambda)}$ are $\sigma(\lambda) \pm \mathrm{i}\rho(\lambda)$, each of multiplicity p, where $\sigma(0) = 0$ and $\rho(0) = 1$. We assume further that the eigenvalues cross the imaginary axis with nonzero speed, i.e.

$$\sigma'(0) = \left.\frac{\mathrm{d}\sigma}{\mathrm{d}\lambda}\right|_{\lambda=0} \neq 0. \tag{5}$$

Near the Hopf bifurcation we expect to find branches of periodic solutions. The isotropy subgroup of $\mathbf{z} \in \mathbf{R}^n$ is defined as

$$\Sigma_{\mathbf{z}} = \{\gamma \in \Gamma \ : \ \gamma \mathbf{z} = \mathbf{z}\} \subset \Gamma. \tag{6}$$

Let $\mathbf{z}(t)$ be a periodic solution of (1) with period 2π. A symmetry of $\mathbf{z}(t)$ is an element $(\gamma, \psi) \in \Gamma \times S^1$ such that

$$(\gamma, \psi) \cdot \mathbf{z}(t) := \gamma \mathbf{z}(t + \psi) = \mathbf{z}(t), \qquad \forall t.$$

Here S^1 is the circle group of phase shifts acting on the space of 2π periodic functions. We say that (γ, ψ) is a spatiotemporal symmetry. The isotropy subgroup of $\mathbf{z}(t)$ is then

$$\Sigma_{\mathbf{z}(t)} = \{(\gamma, \psi) \in \Gamma \times S^1 \ : \ \gamma \mathbf{z}(t + \psi) = \mathbf{z}(t)\} \subset \Gamma \times S^1$$

where the phase shift $\psi \in S^1$ acts on $\mathbf{z} \in \mathbf{C}^p$ by

$$\psi \cdot \mathbf{z} = \mathrm{e}^{\mathrm{i}\psi}\mathbf{z}. \tag{7}$$

Hence symmetry groups of periodic orbits are isotropy subgroups of $\Gamma \times S^1$ in representations of Γ on $V \oplus V$. If $\Sigma \subset \Gamma \times S^1$ is an isotropy subgroup then the fixed-point subspace of Σ is

$$\mathrm{Fix}(\Sigma) = \{\mathbf{z} \in \mathbf{C}^p \ : \ \sigma\mathbf{z} = \mathbf{z}, \quad \forall \sigma \in \Sigma\} \tag{8}$$

and if $\dim \mathrm{Fix}(\Sigma) = 2$ then we say that Σ is **C**-axial.

The following theorem guarantees the existence of periodic solutions of (1) with certain symmetry groups.

Theorem 2.1 (Equivariant Hopf Theorem). *Assume that system* (1) *satisfies the conditions* (4) *and* (5) *stated above and let Σ be a* **C***-axial subgroup of $\Gamma \times S^1$. Then there exists a unique branch of small amplitude periodic solutions to* (1) *with period near 2π and symmetry group Σ.*

For a proof of this theorem see [10] or [12, Chapter XVI section 4]. It is possible for the condition $\dim \mathrm{Fix}(\Sigma) = 2$ in this theorem to be weakened to Σ being a maximal isotropy subgroup of $\Gamma \times S^1$:

Theorem 2.2 (Fiedler [9]). *Assume that system* (1) *satisfies the conditions* (4) *and* (5) *stated above and suppose that Σ is a maximal isotropy subgroup of $\Gamma \times S^1$. Then there exist small amplitude periodic solutions to (1) with period near 2π, having Σ as their group of symmetries.*

Recall that an isotropy subgroup $\Sigma \subset \Gamma \times S^1$ is maximal if there does not exist an isotropy subgroup Δ of $\Gamma \times S^1$ satisfying $\Sigma \subsetneq \Delta \subsetneq \Gamma \times S^1$.

Suppose that $\mathbf{z}(t)$ is any periodic solution of (1). To compute the stability of $\mathbf{z}(t)$ we use a Birkhoff normal form of $f(\mathbf{z}, \lambda)$: by a suitable coordinate change, up to any given order k, the vector field f can be made to commute with $\Gamma \times S^1$. Suppose that the vector field f in (1) is in Birkhoff normal form. Then it is possible to perform a Liapunov-Schmidt reduction on (1) such that the reduced equation g has the form

$$g(\mathbf{z}, \lambda, \tau) = f(\mathbf{z}, \lambda) - (1 + \tau)J\mathbf{z} \tag{9}$$

where τ is the period-scaling parameter (see [12, Chapter XVI, Theorem 10.1]). Let $\mathbf{z}(t)$ be a periodic solution of (1) with isotropy Σ and let $(\mathbf{z}_0, \lambda_0, \tau_0)$ be the corresponding solution to $g = 0$. Then by [12, Chapter XVI Corollary 10.2], $\mathbf{z}(t)$ is orbitally stable if the $n - \dim\Gamma + \dim\Sigma - 1$ eigenvalues of $(\mathrm{d}g)_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ which are not forced to be zero by the action of $\Gamma \times S^1$ have negative real parts.

When $\dim \mathrm{Fix}(\Sigma) = 2$, the assumption that f is in Birkhoff normal form implies that we can apply the standard Hopf theorem to (1) restricted to $\mathrm{Fix}(\Sigma) \times \mathbb{R}$. In this case exchange of stability occurs at the bifurcation point so that if the trivial steady-state solution $\mathbf{z} = \mathbf{0}$ is stable subcritically, then a subcritical branch of periodic solutions with isotropy subgroups Σ is unstable. Supercritical branches may be either stable or unstable depending on the signs of the real parts of the eigenvalues on the complement of $\mathrm{Fix}(\Sigma)$.

The results above rely on the fact that the vector field f is in Birkhoff normal form. However, there is no change of coordinates which puts f in Birkhoff normal form to all orders. When studying the stability of the periodic solutions of (1) with **C**-axial symmetry we use the kth order truncation of f which commutes with $\Gamma \times S^1$. In doing so we ignore higher order terms which do not commute necessarily with S^1 and that can change the dynamics and also possibly the stability of the periodic solutions. Assume that

$$f(\mathbf{z},\lambda) = \tilde{f}(\mathbf{z},\lambda) + o(\|\mathbf{z}\|^k) \tag{10}$$

where $\tilde{f}$ commutes with $\Gamma \times S^1$ but the perturbation $o(\|\mathbf{z}\|^k)$ commutes only with Γ. Here $h(z) = o(\|\mathbf{z}\|^k)$ means that $h(z)/\|\mathbf{z}\|^k \to 0$ as $\|\mathbf{z}\| \to 0$. There is a result which shows that, provided k is large enough, the stability results above remain true for f of the form (10). In order to state this result we require the following definition.

Definition 2.3. Suppose that $\dim \mathrm{Fix}(\Sigma) = 2$. Then Σ has *p-determined stability* if all eigenvalues of

$$(\mathrm{d}\tilde{g})_{(\mathbf{z}_0,\lambda_0,\tau_0)} = (\mathrm{d}\tilde{f})_{(\mathbf{z}_0,\lambda_0)} - (1+\tau_0)J,$$

other than those forced to be zero by Σ, have the form

$$\xi_j = \alpha_j a^{m_j} + o(a^{m_j}),$$

where $\mathbf{z}(t)$ is a branch of periodic solutions to $\dot{\mathbf{z}} = \tilde{f}(\mathbf{z},\lambda)$ with symmetry Σ, $a = \|\mathbf{z}(t)\|$ and α_j is a **C**-valued function of the Taylor coefficients of terms of degree $\leq p$ in $\tilde{f}$.

We say that $\tilde{f}$ is *nondegenerate for* Σ if all α_j have non-zero real parts. Suppose that the hypotheses of the equivariant Hopf theorem hold, and that the isotropy subgroup $\Sigma \subset \Gamma \times S^1$ has p-determined stability. Let $k \geq p$ and assume that $\tilde{f}$ is nondegenerate for Σ. Then for λ sufficiently near $\mathbf{0}$, the stabilities of a periodic solution of (1) with isotropy subgroup Σ are given by the same expressions in the coefficients of f as those that determine the stability of a solution of the truncated Birkhoff normal form $\dot{\mathbf{z}} = \tilde{f}(\mathbf{z},\lambda)$ with isotropy subgroup Σ (see [12, Chapter XVI Theorem 11.2]). Hence we can use the kth order Taylor series of f which commutes with $\Gamma \times S^1$ to compute the stability of a periodic solution with isotropy subgroup Σ whose existence is guaranteed by the equivariant Hopf theorem, as long as $k \geq p$ when Σ has p-determined stability.

In this paper we will use the equivariant Hopf theorem to find branches of periodic solutions at a Hopf bifurcation with $\mathbf{O(3)}$ symmetry. To do this we need to compute the isotropy subgroups of $\mathbf{O(3)} \times S^1$ for representations $V \oplus V$, where V is an absolutely irreducible representation of $\mathbf{O(3)}$. We now discuss the representations and subgroups of $\mathbf{O(3)}$.

2.2 Representations and Subgroups of O(3)

The orthogonal group $\mathbf{O(3)}$ consists of all 3×3 matrices A satisfying $A^{-1} = A^T$. These matrices have $\det(A) = \pm 1$. Algebraically

$$\mathbf{O(3)} = \mathbf{SO(3)} \times \mathbb{Z}_2^c,$$

where $\mathbf{SO(3)}$ is the group of all rotations of the sphere, i.e. $A \in \mathbf{O(3)}$ with $\det(A) = 1$, and $\mathbb{Z}_2^c = \{-I, I\}$, where the element $-I$ is inversion in the centre of the sphere. If a point on the surface of the sphere is given in spherical polar coordinates by (θ,ϕ) then the action of the element $-I$ on this point is

$$(\theta,\phi) \to (\pi-\theta, \pi+\phi) \qquad \text{where} \qquad 0 \leq \theta \leq \pi \quad \text{and} \quad 0 \leq \phi < 2\pi.$$

For each irreducible representation of $\mathbf{SO(3)}$ there are two irreducible representations of $\mathbf{O(3)}$, where the element $-I$ either acts as plus or minus the identity, giving rise to the plus and minus representations of $\mathbf{O(3)}$ respectively. The group $\mathbf{SO(3)}$ has precisely one irreducible representation in each odd dimension $2\ell+1$ for $\ell \geq 0$, denoted by V_ℓ, where V_ℓ is the space of spherical harmonics of degree ℓ. The spherical harmonics are functions of a point (θ,ϕ) on the surface of a sphere and are given by

$$Y_\ell^m(\theta,\phi) = (-1)^m \left(\frac{(2\ell+1)}{4\pi} \frac{(\ell-m)!}{(\ell+m)!} \right)^{1/2} \mathcal{P}_\ell^m(\cos\theta) \mathrm{e}^{\mathrm{i}m\phi} \tag{11}$$

for $-\ell \leq m \leq \ell$ and where

$$\mathcal{P}_\ell^m(x) = \frac{(1-x^2)^{m/2}}{2^\ell\ \ell!} \frac{\mathrm{d}^{\ell+m}}{\mathrm{d}x^{\ell+m}} (x^2-1)^\ell$$

is the associated Legendre function. The spherical harmonics satisfy

$$Y_\ell^{-m}(\theta,\phi) = (-1)^m \overline{Y_\ell^m}(\theta,\phi),$$

where the bar denotes complex conjugate.

The natural representation of $\mathbf{O(3)}$ on V_ℓ is defined to be the plus representation, where $-I$ acts as the identity, if ℓ is even and the minus representation, where $-I$ acts as minus the identity, if ℓ is odd.

Throughout this paper we assume that the representation of $\mathbf{O(3)}$ is given by $V_\ell \oplus V_\ell$ so that a Hopf bifurcation can occur. A vector $\mathbf{x} \in V_\ell \oplus V_\ell$ can be written as

$$\mathbf{x} = \sum_{m=-\ell}^{\ell} \left(z_m Y_\ell^m(\theta,\phi) + \overline{z}_m \overline{Y_\ell^m}(\theta,\phi) \right).$$

The action of $\mathbf{O(3)}$ on $\mathbf{x} \in V_\ell \oplus V_\ell$ is therefore determined by its action on

$$\mathbf{z} = (z_{-\ell}, z_{-\ell+1}, \ldots, z_\ell) \in \mathbf{C}^{2\ell+1}.$$

We next discuss briefly the subgroups of $\mathbf{O(3)}$. The conjugacy classes of subgroups of $\mathbf{O(3)}$ fall into three classes:

Class I Subgroups of $\mathbf{SO(3)}$. These include the planar subgroups—$\mathbf{O(2)}$, $\mathbf{SO(2)}$, $\mathbf{D}_m$ for $m \geq 2$ and $\mathbb{Z}_m$ for $m \geq 1$—and the exceptional subgroups $\mathbb{I}$, $\mathbb{O}$ and $\mathbb{T}$ of rotations of an icosahedron, octahedron and tetrahedron respectively.

Class II Subgroups of $\mathbf{O(3)}$ which contain $-I$. These subgroups are of the form $J \times \mathbb{Z}_2^c$, where J is a subgroup of $\mathbf{SO(3)}$.

Class III Subgroups not in $\mathbf{SO(3)}$ and not containing $-I$. By [12, Chapter XIII section 9(a)] these subgroups are $\mathbf{O(2)}^-$, $\mathbb{O}^-$, $\mathbf{D}_{2m}^d$, $\mathbf{D}_m^z$ and $\mathbb{Z}_{2m}^-$.

Further details on these subgroups, including containment relations between the subgroups can be found in [12, Chapter XIII].

3 Isotropy subgroups of $\mathbf{O(3)} \times \mathbf{S^1}$

To use the equivariant Hopf theorem to find branches of periodic solutions near a Hopf bifurcation with $\mathbf{O(3)}$ symmetry we need to compute the $\mathbf{C}$-axial subgroups of $\mathbf{O(3)} \times S^1$ for representations $V_\ell \oplus V_\ell$, where V_ℓ is an irreducible representation of $\mathbf{O(3)}$. This was first done by Golubitsky and Stewart in [10]. One error in this list was corrected in Golubitsky *et al* [12, Chapter XVIII section 5], however we have found it necessary to make further amendments to their results. Here we will outline the method used by Golubitsky *et al* [12] to compute the $\mathbf{C}$-axial subgroups and give the results of our computations, pointing out how and why they differ from the previously accepted results.

In addition we use the same method to compute the isotropy subgroups, Σ of $\mathbf{O(3)} \times S^1$ with four-dimensional fixed-point subspaces. If Σ is maximal then Theorem 2.2 guarantees the existence of a branch of periodic solutions of (1) with Σ symmetry. If Σ is not maximal then it may still possible for a solution to (1) with Σ symmetry to exist, depending on the values of coefficients in the Taylor expansion of the vector field f.

3.1 Method for computing isotropy subgroups of $\mathbf{O(3)} \times S^1$

Depending on the representation of $\mathbf{O(3)} \times S^1$ there are two types of subgroup which could be isotropy subgroups. Throughout this paper we take the action of $\psi \in S^1$ on $\mathbf{z} \in \mathbf{C}^{2\ell+1}$ to be as in (7). This means that for any subgroup of $\mathbf{O(3)} \times S^1$ of the form $K \times S^1$ where K is a subgroup of $\mathbf{O(3)}$, $\mathrm{Fix}(K \times S^1) = \{\mathbf{0}\}$. Hence the only isotropy subgroup of this type is $\mathbf{O(3)} \times S^1$ which is the isotropy subgroup of the stationary trivial solution $\mathbf{z} = \mathbf{0}$ of (1).

Every other isotropy subgroup Σ of $\mathbf{O(3)} \times S^1$ is a twisted subgroup

$$H^\alpha = \{(h, \alpha(h)) \ : \ h \in H\},$$

where H is a subgroup of $\mathbf{O(3)}$ and $\alpha : H \to S^1$ is a homomorphism. Let K denote the group of spatial symmetries of a solution $\mathbf{z}(t)$ of (1), i.e.

$$K = \{\gamma \in \mathbf{O(3)} : \gamma\mathbf{z}(t) = \mathbf{z}(t) \ \ \forall t\},$$

then $K = H^\alpha \cap \mathbf{O(3)} = \ker(\alpha)$ and hence K is a normal subgroup of H. In order to list the twisted subgroups of $\mathbf{O(3)} \times S^1$ we use the following steps of Golubitsky *et al* [12, Chapter XVI section 7]

1. For each conjugacy class of subgroups of $\mathbf{O(3)}$ choose a representative H.
2. Find all normal subgroups $K \subset H$ such that $H/K \cong S^1, \mathbb{Z}_n$ or $\mathbb{1}$.
3. Choose one representative of each conjugacy class of K's giving a list of pairs (H, K).
4. Find all possible homomorphisms α for each pair by listing the automorphisms of H/K not including those that are induced by conjugation by elements $\gamma \in N_{\mathbf{O(3)}}(H) = \{\gamma \in \mathbf{O(3)} : \gamma H \gamma^{-1} = H\}$.

We should mention that an alternative method for computing the twisted subgroups Σ of $\Gamma \times S^1$ with $\dim \mathrm{Fix}(\Sigma) = 2$ was given by Golubitsky and Stewart in [13]. Although this method requires less computation, the reasons for some of the steps in the procedure are less intuitive than the method of [12, Chapter XVI section 7] outlined above.

Carrying out the procedure outlined in the steps above gives a complete list of twisted subgroups of $\mathbf{O(3)} \times S^1$ up to conjugacy. To determine in which representations these twisted subgroups can be isotropy subgroups we can first make use of Lemma 15.1 of [10]. This implies that for a twisted subgroup H^α to be an isotropy subgroup of $\mathbf{O(3)} \times S^1$, H must be a class II subgroup of $\mathbf{O(3)}$. Furthermore for any value of ℓ, in the plus representation on $V_\ell \oplus V_\ell$, the element $-I$ acts as the identity and therefore the element $(-I, 0) \in \mathbf{O(3)} \times S^1$ must lie in every isotropy subgroup. This means that H and $K = \ker(\alpha)$ must both be class II subgroups of $\mathbf{O(3)}$. In the minus representation, $-I$ acts as minus the identity and since a time shift symmetry $\psi \in S^1$ acts as in (7) the time shift by $\psi = \pi$ acts as -1. Hence $(-I, \pi) \in \mathbf{O(3)} \times S^1$ acts as the identity and must therefore be contained in every isotropy subgroup. This means that H must be a class II subgroup and K must be either a class I or class III subgroup of $\mathbf{O(3)}$.

Considering now only those twisted subgroups which can be isotropy subgroups by the above remark, one can compute the formula for the dimension of the fixed point subspace of each twisted subgroup. To do this one can use for example Proposition 8.3 of [12] and the formulae for the dimensions of the fixed-point subspaces of the subgroups of $\mathbf{O(3)}$ given by Ihrig and Golubitsky [15]. One can then use this information to compile a list of the values of ℓ for which each of the twisted subgroups of $\mathbf{O(3)} \times S^1$ have a two dimensional fixed-point subspace in each representation on $V_\ell \oplus V_\ell$. Full details of our computations of these subgroups can be found in [26] and the resulting list of twisted subgroups with two-dimensional fixed-point subspaces is given in [26, Table 4.2].

The next task is to determine which of the twisted subgroups H^α which have a two-dimensional fixed-point subspace are in fact isotropy subgroups. By [10, Lemma 15.2] such an H^α is an isotropy subgroup if H^α is a maximal in $\mathbf{O(3)} \times S^1$, i.e. if there is no twisted subgroup L^ϕ such that $H^\alpha \subsetneq L^\phi \subsetneq \mathbf{O(3)} \times S^1$, or if whenever $H^\alpha \subset L^\phi$, the fixed-point subspace of L^ϕ has dimension less than 2 (hence 0).

Remark 3.1. This is a special case of a result called the chain criterion (see [23, Appendix A]) which (for our purposes) says that any twisted subgroup $H^\alpha \neq \mathbf{O(3)} \times S^1$ is an isotropy subgroup of $\mathbf{O(3)} \times S^1$ if $\dim \mathrm{Fix}(H^\alpha) > 0$ and for each strictly larger group L^ϕ,

$$\dim \mathrm{Fix}(L^\phi) < \dim \mathrm{Fix}(H^\alpha). \tag{12}$$

In order to compute the $\mathbf{C}$-axial isotropy subgroups of $\mathbf{O(3)} \times S^1$ we must now decide when $H^\alpha \subset L^\phi$. This occurs if and only if $H \subset L$ and ϕ extends α. This implies that $\ker(\alpha) \subset \ker(\phi)$. In the majority cases it is easy to determine if one twisted subgroup lies inside another, however in other cases it is not so obvious and this is where we have noticed some errors in the computations of Golubitsky and Stewart [10] and Golubitsky *et al* [12].

3.2 C-axial isotropy subgroups

The results of our computations of the $\mathbf{C}$-axial isotropy subgroups of $\mathbf{O(3)} \times S^1$ using the method outlined above are given in Table 1.

The differences between Table 1 and the results of Golubitsky *et al* given in Table 5.1 of [12, Chapter XVIII] are as follows:

Table 1: The **C**-axial subgroups of $\mathbf{O(3)} \times S^1$ for the representations $V_\ell \oplus V_\ell$. The last two columns give the values of ℓ for which the subgroups are isotropy subgroups. Here $H = J \times \mathbb{Z}_2^c$.

J	K	$\alpha(H)$	Plus representation	Minus representation
$\mathbf{O(2)}$	$\mathbf{O(2)} \times \mathbb{Z}_2^c$	$\mathbb{1}$	Even ℓ	
$\mathbf{O(2)}$	$\mathbf{O(2)}$	$\mathbb{Z}_2$		Even ℓ
$\mathbf{O(2)}$	$\mathbf{SO(2)} \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$	Odd ℓ	
$\mathbf{O(2)}$	$\mathbf{O(2)}^-$	$\mathbb{Z}_2$		Odd ℓ
$\mathbf{SO(2)}$	$\mathbb{Z}_n \times \mathbb{Z}_2^c$	S^1	All ℓ	
$\mathbf{SO(2)}$	$\mathbb{Z}_{2n}^-$	S^1		All ℓ
$\mathbb{I}$	$\mathbb{I} \times \mathbb{Z}_2^c$	$\mathbb{1}$	6, 10, 12, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28, 31, 32, 33, 34, 35, 37, 38, 39, 41, 43, 44, 47, 49, 53, 59.	
$\mathbb{I}$	$\mathbb{I}$	$\mathbb{Z}_2$		6, 10, 12, 15, 16, 18, 20, 21, 22, 24, 25, 26, 27, 28, 31, 32, 33, 34, 35, 37, 38, 39, 41, 43, 44, 47, 49, 53, 59.
$\mathbb{O}$	$\mathbb{O} \times \mathbb{Z}_2^c$	$\mathbb{1}$	4, 6, 8, 9, 10, 13, 14, 15, 17, 19, 23.	
$\mathbb{O}$	$\mathbb{O}$	$\mathbb{Z}_2$		4, 6, 8, 9, 10, 13, 14, 15, 17, 19, 23.
$\mathbb{O}$	$\mathbb{T} \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$	3, 6, 7, 9, 10, 11, 12, 13, 14, 16, 17, 20.	
$\mathbb{O}$	$\mathbb{O}^-$	$\mathbb{Z}_2$		3, 6, 7, 9, 10, 11, 12, 13, 14, 16, 17, 20.
$\mathbb{T}$	$\mathbf{D}_2 \times \mathbb{Z}_2^c$	$\mathbb{Z}_3$	2, 4, 5, 6, 7, 9.	
$\mathbb{T}$	$\mathbf{D}_2$	$\mathbb{Z}_6$		2, 4, 5, 6, 7, 9.
$\mathbf{D}_{2n}$	$\mathbf{D}_n \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$	$n \le \ell < 3n$, $(n \ge 3)$	
$\mathbf{D}_{2n}$	$\mathbf{D}_{2n}^d$	$\mathbb{Z}_2$		$n \le \ell < 3n$, $(n \ge 3)$
$\mathbf{D}_4$	$\mathbf{D}_2 \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$	$2, 4, 5.$	
$\mathbf{D}_4$	$\mathbf{D}_4^d$	$\mathbb{Z}_2$		$2, 4, 5.$

1. We have added the value $\ell = 15$ to the lists of values where the triples $(H, K, \alpha(H)) = (\mathbb{I} \times \mathbb{Z}_2^c, \mathbb{I} \times \mathbb{Z}_2^c, \mathbb{1})$ and $(\mathbb{I} \times \mathbb{Z}_2^c, \mathbb{I}, \mathbb{Z}_2)$ are $\mathbf{C}$-axial subgroups in the plus and minus representations respectively. This is because the (unique) twisted subgroups given by both of these triples are maximal and in the given representations the fixed-point subspace of the twisted subgroups is two-dimensional.

2. In Table 5.1 of [12, Chapter XVIII] the penultimate row states that in the minus representation the unique twisted subgroup given by the triple

$$(H, K, \alpha(H)) = (\mathbf{D}_n \times \mathbb{Z}_2^c, \mathbf{D}_n, \mathbb{Z}_2)$$

is a $\mathbf{C}$-axial subgroup when $\ell/2 < n \leq \ell$. It is true that in the minus representation

$$\dim \mathrm{Fix}_{V_\ell \oplus V_\ell}(\mathbf{D}_n \times \mathbb{Z}_2^c, \mathbf{D}_n, \mathbb{Z}_2) = 2 \quad \text{when} \quad \ell/2 < n \leq \ell \quad \text{and} \quad \ell \text{ is odd,}$$

however it is also true that in the minus representation

$$\begin{aligned} \dim \mathrm{Fix}_{V_\ell \oplus V_\ell}(\mathbf{D}_{2n} \times \mathbb{Z}_2^c, \mathbf{D}_{2n}^d, \mathbb{Z}_2) &= 2 \dim \mathrm{Fix}_{V_\ell}(\mathbf{D}_{2n}^d) \\ &= 2 \left[\frac{\ell + n}{2n} \right] = 2 \text{ when } \ell/3 < n \leq \ell. \end{aligned} \tag{13}$$

Since the twisted subgroup given by the triple $(\mathbf{D}_n \times \mathbb{Z}_2^c, \mathbf{D}_n, \mathbb{Z}_2)$ lies inside the twisted subgroup given by $(\mathbf{D}_{2n} \times \mathbb{Z}_2^c, \mathbf{D}_{2n}^d, \mathbb{Z}_2)$ and they both have two-dimensional fixed-point subspaces when $\ell/2 < n \leq \ell$, Lemma 15.2 of [10] implies that $(\mathbf{D}_n \times \mathbb{Z}_2^c, \mathbf{D}_n, \mathbb{Z}_2)$ cannot be a $\mathbf{C}$-axial subgroup.

Moreover, the unique twisted subgroup given by $(\mathbf{D}_{2n} \times \mathbb{Z}_2^c, \mathbf{D}_{2n}^d, \mathbb{Z}_2)$ is a $\mathbf{C}$-axial subgroup for the values of ℓ given in Table 1 in the minus representation. Although this twisted subgroup is contained in that given by the triples $(\mathbf{D}_{2np} \times \mathbb{Z}_2^c, \mathbf{D}_{2np}^d, \mathbb{Z}_2)$ for any odd value of p, the larger groups have a zero-dimensional fixed-point subspace for all values of ℓ where the twisted subgroup given by $(\mathbf{D}_{2n} \times \mathbb{Z}_2^c, \mathbf{D}_{2n}^d, \mathbb{Z}_2)$ has a two-dimensional fixed-point subspace.

This twisted subgroup does not appear in any previous lists of the $\mathbf{C}$-axial subgroups of $\mathbf{O(3)} \times S^1$ due to the fact that the authors of [10] and [12] falsely assume that the twisted subgroup given by $(\mathbf{D}_{2n} \times \mathbb{Z}_2^c, \mathbf{D}_{2n}^d, \mathbb{Z}_2)$ is contained in the subgroup given by $(\mathbf{O(2)} \times \mathbb{Z}_2^c, \mathbf{O(2)}^-, \mathbb{Z}_2)$. However, this is not true since $\mathbf{D}_{2n}^d$ is not a subgroup of $\mathbf{O(2)}^-$ for any $n \geq 2$.

When $n = 2$, the twisted subgroup, H^α, given by $(\mathbf{D}_4 \times \mathbb{Z}_2^c, \mathbf{D}_4^d, \mathbb{Z}_2)$ has $\dim \mathrm{Fix}(H^\alpha) = 2$ when $\ell = 2, 3, 4$ and 5. However, this subgroup lies inside that given by $(\mathbb{O} \times \mathbb{Z}_2^c, \mathbb{O}^-, \mathbb{Z}_2^c)$ and so, since this larger group has a two-dimensional fixed-point subspace when $\ell = 3$, H^α is only a $\mathbf{C}$-axial subgroup when $\ell = 2, 4$ and 5 by [10, Lemma 15.2].

3. In Table 5.1 of [12, Chapter XVIII] the final row states that in the plus representation the triple $(H, K, \alpha(H)) = (\mathbf{D}_{2n} \times \mathbb{Z}_2^c, \mathbf{D}_n \times \mathbb{Z}_2^c, \mathbb{Z}_2)$ gives a twisted subgroup, H^α, which is a $\mathbf{C}$-axial subgroup when $\ell/2 < n \leq \ell$. [1] We extend this range of values to $\ell/3 < n \leq \ell$ for $n \geq 3$ for the following reason. In the plus representation H^α has a two-dimensional

[1] By comparing Table 14.1 of [10] and Table 5.1 of [12, Chapter XVIII] and the subsequent remarks it is clear that there is a misprint in footnote [2] to Table 5.1 [12, Chapter XVIII] and it should say that the class II subgroup is $\mathbf{D}_{n/2} \times \mathbb{Z}_2^c$ and n is even.

fixed-point subspace when $\ell/3 < n \leq \ell$. In [10], [12] and [13] it is assumed that H^α is contained in the twisted subgroup L^ϕ given by $(\mathbf{D}_{4n} \times \mathbb{Z}_2^c, \mathbf{D}_{2n} \times \mathbb{Z}_2^c, \mathbb{Z}_2)$. This would mean that the range of values of ℓ where H^α gives a **C**-axial subgroup is reduced to $\ell/2 < n \leq \ell$ since when $\ell/3 < n \leq \ell/2$, $\dim \mathrm{Fix}(L^\phi) = 2$. However, this containment relation does not hold. Although $\mathbf{D}_{2n} \times \mathbb{Z}_2^c \subset \mathbf{D}_{4n} \times \mathbb{Z}_2^c$, the homomorphism α does not extend to the homomorphism ϕ. For example, the element $(R_{\pi/n}, \pi)$ where $R_{\pi/n} \in \mathbf{O(3)}$ is a rotation through an angle π/n and $\pi \in S^1$ is the non-identity element in $\mathbb{Z}_2$ is contained in H^α, but not L^ϕ (in fact, the rotation $R_{\pi/n} \in \ker(\phi)$). To specify a particular copy of the group $H = \mathbf{D}_{2n} \times \mathbb{Z}_2^c$ we must specify the two perpendicular axes of rotation. There is only one copy of the group $L = \mathbf{D}_{4n} \times \mathbb{Z}_2^c$ which contains H but then $H = \ker(\phi)$ and hence $H^\alpha \not\subseteq L^\phi$.

The twisted subgroup given by $(\mathbf{D}_{2n} \times \mathbb{Z}_2^c, \mathbf{D}_n \times \mathbb{Z}_2^c, \mathbb{Z}_2)$ is contained in that given by $(\mathbf{D}_{2np} \times \mathbb{Z}_2^c, \mathbf{D}_{np} \times \mathbb{Z}_2^c, \mathbb{Z}_2)$, however the values of ℓ for which each of these twisted subgroups have two-dimensional fixed-point subspaces do not overlap. Hence by [10, Lemma 15.2] $(\mathbf{D}_{2n} \times \mathbb{Z}_2^c, \mathbf{D}_n \times \mathbb{Z}_2^c, \mathbb{Z}_2)$ gives a **C**-axial subgroup in the plus representation for $\ell/3 < n \leq \ell$ when $n \geq 3$.

When $n = 2$, the twisted subgroup, H^α, given by $(\mathbf{D}_4 \times \mathbb{Z}_2^c, \mathbf{D}_2 \times \mathbb{Z}_2^c, \mathbb{Z}_2)$ has $\dim \mathrm{Fix}(H^\alpha) = 2$ when $\ell = 2, 3, 4$ and 5. However, this subgroup lies inside that given by $(\mathbb{O} \times \mathbb{Z}_2^c, \mathbb{T} \times \mathbb{Z}_2^c, \mathbb{Z}_2^c)$ and so, since this larger group has a two-dimensional fixed-point subspace when $\ell = 3$, H^α is only a **C**-axial subgroup when $\ell = 2, 4$ and 5 by [10, Lemma 15.2].

The changes we have made as above to Table 5.1 of [12, Chapter XVIII] result in changes in Table 5.2 of [12, Chapter XVIII], which shows the **C**-axial subgroups for the natural representation of $\mathbf{O(3)} \times S^1$ on $V_\ell \oplus V_\ell$ for $\ell = 1, \ldots, 6$. Table 2 shows the equivalent table after the amendments in Table 1. Recall that branches of solutions to (1) occur in group orbits. Since solutions on the same group orbit have conjugate isotropy subgroups we classify solution branches in terms of their isotropy subgroup. Thus, the number of branches given by the equivariant Hopf theorem stated in Table 2 refers to the number of non-conjugate **C**-axial isotropy subgroups.

3.3 Isotropy subgroups with four-dimensional fixed-point subspaces

Following the method outlined in section 3.1 we can compute that the twisted subgroups of $\mathbf{O(3)} \times S^1$ with four-dimensional fixed-point subspaces. Using the chain criterion as given in Remark 3.1 we can see that such a twisted subgroup H^α is an isotropy subgroup if it is maximal (i.e. not contained in an isotropy subgroup with a two-dimensional fixed-point subspace) or if whenever $H^\alpha \subset L^\phi$, $\dim \mathrm{Fix}(L^\phi) = 0$ or 2. The isotropy subgroups with four dimensional fixed-point subspaces are as listed in Table 3.

Remark 3.2. When the twist type, $\alpha(H)$, is not $\mathbb{Z}_2$ or $\mathbb{1}$ in Table 3 then the triple $(H, K, \alpha(H))$ does not uniquely specify the twisted subgroup. There is more than one possible homomorphism $\alpha : H \to H/K$ and therefore (up to conjugacy) there is more than one twisted subgroup corresponding to each triple $(H, K, \alpha(H))$. In Table 3 these twisted subgroups are specified in terms of the homomorphisms $\psi_j : H \to H/K$. There are three cases to consider:

1. The homomorphisms $\psi_j : \mathbb{Z}_{md} \to \mathbb{Z}_d$ are given by
$$\psi_j(r^k) \;=\; \psi_j(-r^k) \;=\; 2\pi kj/d$$
for $j = 1, 3, \ldots, d-1$ where $\mathbb{Z}_{md} = \langle r = R_{2\pi/md} \rangle$.

Table 2: The **C**-axial subgroups of $\mathbf{O(3)} \times S^1$ for the natural representations on $V_\ell \oplus V_\ell$ for $\ell = 1, \ldots, 6$. Here $H = J \times \mathbb{Z}_2^c$.

ℓ	J	K	$\alpha(H)$	Number of branches given by equivariant Hopf theorem
1	$\mathbf{O(2)}$	$\mathbf{O(2)}^-$	$\mathbb{Z}_2$	2
	$\mathbf{SO(2)}$	$\mathbb{Z}_{2n}^-$	S^1 $[n = 1]$	
2	$\mathbf{O(2)}$	$\mathbf{O(2)} \times \mathbb{Z}_2^c$	$\mathbb{1}$	5
	$\mathbf{SO(2)}$	$\mathbb{Z}_n \times \mathbb{Z}_2^c$	S^1 $[n = 1, 2]$	
	$\mathbb{T}$	$\mathbf{D}_2 \times \mathbb{Z}_2^c$	$\mathbb{Z}_3$	
	$\mathbf{D}_4$	$\mathbf{D}_2 \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$	
3	$\mathbf{O(2)}$	$\mathbf{O(2)}^-$	$\mathbb{Z}_2$	6
	$\mathbf{SO(2)}$	$\mathbb{Z}_{2n}^-$	S^1 $[1 \leq n \leq 3]$	
	$\mathbb{O}$	$\mathbb{O}^-$	$\mathbb{Z}_2$	
	$\mathbf{D}_6$	$\mathbf{D}_6^d$	$\mathbb{Z}_2$	
4	$\mathbf{O(2)}$	$\mathbf{O(2)} \times \mathbb{Z}_2^c$	$\mathbb{1}$	10
	$\mathbf{SO(2)}$	$\mathbb{Z}_n \times \mathbb{Z}_2^c$	S^1 $[1 \leq n \leq 4]$	
	$\mathbb{O}$	$\mathbb{O} \times \mathbb{Z}_2^c$	$\mathbb{1}$	
	$\mathbb{T}$	$\mathbf{D}_2 \times \mathbb{Z}_2^c$	$\mathbb{Z}_3$	
	$\mathbf{D}_8$	$\mathbf{D}_4 \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$	
	$\mathbf{D}_6$	$\mathbf{D}_3 \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$	
	$\mathbf{D}_4$	$\mathbf{D}_2 \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$	
5	$\mathbf{O(2)}$	$\mathbf{O(2)}^-$	$\mathbb{Z}_2$	11
	$\mathbf{SO(2)}$	$\mathbb{Z}_{2n}^-$	S^1 $[1 \leq n \leq 5]$	
	$\mathbb{T}$	$\mathbf{D}_2$	$\mathbb{Z}_6$	
	$\mathbf{D}_{10}$	$\mathbf{D}_{10}^d$	$\mathbb{Z}_2$	
	$\mathbf{D}_8$	$\mathbf{D}_8^d$	$\mathbb{Z}_2$	
	$\mathbf{D}_6$	$\mathbf{D}_6^d$	$\mathbb{Z}_2$	
	$\mathbf{D}_4$	$\mathbf{D}_4^d$	$\mathbb{Z}_2$	
6	$\mathbf{O(2)}$	$\mathbf{O(2)} \times \mathbb{Z}_2^c$	$\mathbb{1}$	15
	$\mathbf{SO(2)}$	$\mathbb{Z}_n \times \mathbb{Z}_2^c$	S^1 $[1 \leq n \leq 6]$	
	$\mathbb{I}$	$\mathbb{I} \times \mathbb{Z}_2^c$	$\mathbb{1}$	
	$\mathbb{O}$	$\mathbb{O} \times \mathbb{Z}_2^c$	$\mathbb{1}$	
	$\mathbb{O}$	$\mathbb{T} \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$	
	$\mathbb{T}$	$\mathbf{D}_2 \times \mathbb{Z}_2^c$	$\mathbb{Z}_3$	
	$\mathbf{D}_{12}$	$\mathbf{D}_6 \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$	
	$\mathbf{D}_{10}$	$\mathbf{D}_5 \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$	
	$\mathbf{D}_8$	$\mathbf{D}_4 \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$	
	$\mathbf{D}_6$	$\mathbf{D}_3 \times \mathbb{Z}_2^c$	$\mathbb{Z}_2$	

2. The homomorphisms $\psi_j : \mathbb{Z}_{(2d-1)m} \to \mathbb{Z}_{2(2d-1)}$ are given by

$$\begin{aligned} \psi_j(r^k) &= 2\pi kj/(2d-1) \\ \psi_j(-r^k) &= 2\pi kj/(2d-1) + \pi. \end{aligned}$$

for $j = 1, 3, 4 \ldots, 2d-2$ where $\mathbb{Z}_{(2d-1)m} = \langle r = R_{2\pi/m(2d-1)} \rangle$.

J	K	Values of ℓ plus representation	Values of ℓ minus representation
$\mathbf{D}_n$	$\mathbf{D}_n \times \mathbb{Z}_2^c$	$\begin{cases} n \le \ell < 2n, & \ell \text{ even} \\ 2n \le \ell < 3n, & \ell \text{ odd} \end{cases}$	–
$\mathbf{D}_n$	$\mathbf{D}_n$	–	$\begin{cases} n \le \ell < 2n, & \ell \text{ even} \\ 2n \le \ell < 3n, & \ell \text{ odd} \end{cases}$
$\mathbf{D}_n$	$\mathbb{Z}_n \times \mathbb{Z}_2^c$	$\begin{cases} 2n \le \ell < 3n, & \ell \text{ even} \\ n \le \ell < 2n, & \ell \text{ odd} \end{cases}$	–
$\mathbf{D}_n$	$\mathbf{D}_n^z$	–	$\begin{cases} 2n \le \ell < 3n, & \ell \text{ even} \\ n \le \ell < 2n, & \ell \text{ odd} \end{cases}$
$\mathbf{D}_{2m}$	$\mathbf{D}_m \times \mathbb{Z}_2^c$	$3m \le \ell < 5m$	–
$\mathbf{D}_{2m}$	$\mathbf{D}_{2m}^d$	–	$3m \le \ell < 5m$
$\mathbb{Z}_{md}$ [*]	$\mathbb{Z}_m \times \mathbb{Z}_2^c$	$\begin{cases} m \le \ell < 3m & d = 2,\ j = 1 \\ \text{[A]} & d \ge 3 \end{cases}$	–
$\mathbb{Z}_{(2d-1)m}$ [*]	$\mathbb{Z}_m$	–	$\begin{cases} 2m \le \ell < 4m & d = 2,\ j = 1 \\ \text{[B]} & d \ge 3 \end{cases}$
$\mathbb{Z}_{2md}$ [*]	$\mathbb{Z}_{2m}^-$	–	$\begin{cases} m \le \ell < 3m & d = 1,\ j = 1 \\ \text{[C]} & d \ge 2 \end{cases}$
$\mathbb{T}$	$\mathbb{T} \times \mathbb{Z}_2^c$	$6, 9, 10, 13, 14, 17$	–
$\mathbb{T}$	$\mathbb{T}$	–	$6, 9, 10, 13, 14, 17$
$\mathbb{T}$	$\mathbf{D}_2 \times \mathbb{Z}_2^c$	$8, 10, 11, 12, 13, 15$	–
$\mathbb{T}$	$\mathbf{D}_2$	–	$8, 10, 11, 12, 13, 15$
$\mathbb{O}$	$\mathbb{O} \times \mathbb{Z}_2^c$	$12, 16, 18, 20–22,$ $25–27, 29, 31, 35$	–
$\mathbb{O}$	$\mathbb{O}$	–	$12, 16, 18, 20–22,$ $25–27, 29, 31, 35$
$\mathbb{O}$	$\mathbb{T} \times \mathbb{Z}_2^c$	$15, 18, 19, 21–26,$ $28, 29, 32$	– –
$\mathbb{O}$	$\mathbb{O}^-$	–	$15, 18, 19, 21–26,$ $28, 29, 32$
$\mathbb{I}$	$\mathbb{I} \times \mathbb{Z}_2^c$	$30, 36, 40, 42, 45, 46,$ $48, 50, 51, 52, 54–58$ $61–65, 67–69, 71, 73,$ $74, 77, 79, 83, 89$	–
$\mathbb{I}$	$\mathbb{I}$	–	$30, 36, 40, 42, 45, 46,$ $48, 50, 51, 52, 54–58$ $61–65, 67–69, 71, 73,$ $74, 77, 79, 83, 89$

Table 3: The values of ℓ for which the twisted subgroups $H^\theta \subset \mathbf{O(3)} \times S^1$ given by the pairs (H, K) have four-dimensional fixed-point subspaces in the representation on $V_\ell \oplus V_\ell$ where H^θ can be an isotropy subgroup. Here $H = J \times \mathbb{Z}_2^c$.
[*] The homomorphisms $\psi_j : H \to H/K$ are given in Remark 3.2
[A]: $\max\{mj, m(d-j)\} \le \ell < \min\{m(d+j), m(2d-j)\}$
[B]: $\max\{mj, m(2d-1-j)\} \le \ell < \min\{m(2d-1+j), m(4d-2-j)\}$
[C]: $\max\{mj, m(2d-j)\} \le \ell < \min\{m(2d+j), m(4d-j)\}$.

3. The homomorphisms $\psi_j : \mathbb{Z}_{2md} \to \mathbb{Z}_{2d}$ are given by

$$\begin{array}{rcl}\psi_j(r^k) & = & \pi kj/d \\ \psi_j(-r^k) & = & \pi kj/d + \pi.\end{array}$$

for $j = 1, 3, 5, 7 \ldots, 2d-1$ where $\mathbb{Z}_{2md} = \langle r = R_{\pi/md} \rangle$.

Remark 3.3. 1. We note that all twisted subgroups of $\mathbf{O(3)} \times S^1$ with four-dimensional fixed-point subspaces in the representation on $V_\ell \oplus V_\ell$ are isotropy subgroups in that representation.

2. The isotropy subgroups $\Sigma \subset \mathbf{O(3)} \times S^1$ given by the pairs of subgroups (H, K)

$$\begin{array}{ccc}(\mathbb{I} \times \mathbb{Z}_2^c, \mathbb{I} \times \mathbb{Z}_2^c), & (\mathbb{I} \times \mathbb{Z}_2^c, \mathbb{I}), & (\mathbb{O} \times \mathbb{Z}_2^c, \mathbb{O} \times \mathbb{Z}_2^c), \\ (\mathbb{O} \times \mathbb{Z}_2^c, \mathbb{O}), & (\mathbb{O} \times \mathbb{Z}_2^c, \mathbb{T} \times \mathbb{Z}_2^c), & (\mathbb{O} \times \mathbb{Z}_2^c, \mathbb{O}^-), \\ (\mathbb{T} \times \mathbb{Z}_2^c, \mathbf{D}_2 \times \mathbb{Z}_2^c), & \text{and} & (\mathbb{T} \times \mathbb{Z}_2^c, \mathbf{D}_2)\end{array}$$

are all maximal isotropy subgroups for representation on $V_\ell \oplus V_\ell$ for the values of ℓ given in Table 3. Hence by Theorem 2.2 (1) is guaranteed to have a branch of periodic solutions with symmetry Σ for these values of ℓ which bifurcates from the Hopf bifurcation at $\lambda = 0$.

In addition, of the isotropy subgroups in Table 3, those given by the pairs (H, K) above are the only isotropy subgroups which can be maximal isotropy subgroups with four-dimensional fixed-point subspaces. All other isotropy subgroups in Table 3 are contained in a **C**-axial subgroup.

3. Suppose that Σ is an isotropy subgroup of $\mathbf{O(3)} \times S^1$ with $\dim \mathrm{Fix}(\Sigma) = 4$ for the representation on $V_\ell \oplus V_\ell$ for some value of ℓ. Suppose that Σ is contained in a **C**-axial isotropy subgroup. It is possible that (1) may admit solutions with Σ symmetry depending on the values of the coefficients in the Taylor expansion of the vector field f.

In sections 4–6 of this paper we consider the case where the representation of $\mathbf{O(3)} \times S^1$ is $V_3 \oplus V_3$. We can see from Tables 2 and 3 that there are 6 branches of periodic solutions given by the equivariant Hopf theorem, all with **C**-axial symmetry.

4 The equivariant vector field and its maximal solutions when $\ell = 3$

Consider the system of ODEs

$$\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = f(\mathbf{z}, \lambda), \tag{14}$$

where $\mathbf{z} \in \mathbf{C}^7$, $\lambda \in \mathbf{R}$ is a bifurcation parameter and $f : \mathbf{C}^7 \times \mathbf{R} \to \mathbf{C}^7$ is a smooth vector field which is equivariant under the action of the group $\mathbf{O(3)}$ on $V_3 \oplus V_3$. In this section we compute explicitly the cubic order truncation of the Birkhoff normal form of the vector field f. This truncation commutes with the action of $\mathbf{O(3)} \times S^1$ for the representation $V_3 \oplus V_3$. We found in section 3 that in this representation the equivariant Hopf theorem guarantees the existence of six branches of periodic solutions within this vector field. In this section we also give details of these solutions.

Let us denote the cubic order truncation of the Birkhoff normal form of $f(\mathbf{z},\lambda)$ by $F_3(\mathbf{z},\lambda)$. The mapping F_3 must satisfy

$$\gamma F_3(\mathbf{z},\lambda) = F_3(\gamma\mathbf{z},\lambda) \qquad \forall\gamma \in \mathbf{O(3)}\times S^1 \tag{15}$$

where

$$\mathbf{z}(t) = (z_{-3},\ z_{-2},\ z_{-1},\ z_0,\ z_1,\ z_2,\ z_3)^{\mathbf{T}} \in \mathbf{C}^7,$$

is the vector of amplitudes of the spherical harmonics of degree 3. It is sufficient to impose that (15) hold for a set of generating elements of $\mathbf{O(3)}\times S^1$. If we choose these elements to be θ', an infinitesimal rotation in the θ direction, ϕ', an infinitesimal rotation in the ϕ direction, the inversion element $-I$ and a time shift ψ then the generators act by multiplication by the matrices

$$M_{\theta'} = \begin{bmatrix} 1 & -\sqrt{\frac{3}{2}}\theta' & 0 & 0 & 0 & 0 & 0 \\ \sqrt{\frac{3}{2}}\theta' & 1 & -\sqrt{\frac{5}{2}}\theta' & 0 & 0 & 0 & 0 \\ 0 & \sqrt{\frac{5}{2}}\theta' & 1 & -\sqrt{3}\theta' & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3}\theta' & 1 & -\sqrt{3}\theta' & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3}\theta' & 1 & -\sqrt{\frac{5}{2}}\theta' & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{5}{2}}\theta' & 1 & -\sqrt{\frac{3}{2}}\theta' \\ 0 & 0 & 0 & 0 & 0 & \sqrt{\frac{3}{2}}\theta' & 1 \end{bmatrix} \tag{16}$$

$$M_{\phi'} = \mathrm{diag}\{\mathrm{e}^{-3\mathrm{i}\phi'}, \mathrm{e}^{-2\mathrm{i}\phi'}, \mathrm{e}^{-\mathrm{i}\phi'}, 1, \mathrm{e}^{\mathrm{i}\phi'}, \mathrm{e}^{2\mathrm{i}\phi'}, \mathrm{e}^{3\mathrm{i}\phi'}\} \tag{17}$$

$$M_{\psi} = \mathrm{e}^{\mathrm{i}\psi} I_7 \tag{18}$$

$$M_{-I} = -I_7 \tag{19}$$

where diag$\{\ldots\}$ indicates a diagonal matrix with elements as listed and I_7 is the 7×7 identity matrix. By imposing that (15) hold for this set of generators we find that the general form of a cubic vector field which commutes with the action of $\mathbf{O(3)}\times S^1$ as above is

$$F_3(\mathbf{z},\lambda) = \mu\mathbf{z} + A\ \mathbf{z}|\mathbf{z}|^2 + B\ \mathbf{P}(\mathbf{z})\hat{\mathbf{z}} + C\ \mathbf{Q}(\mathbf{z}) + D\ \mathbf{R}(\mathbf{z}) \tag{20}$$

where $\mu,\ A,\ B,\ C,\ D$ are smooth complex-valued functions of λ and

$$\begin{aligned} |\mathbf{z}|^2 &= |z_{-3}|^2 + |z_{-2}|^2 + |z_{-1}|^2 + |z_0|^2 + |z_1|^2 + |z_2|^2 + |z_3|^2 \\ \mathbf{P}(\mathbf{z}) &= z_0^2 - 2z_{-1}z_1 + 2z_{-2}z_2 - 2z_{-3}z_3 \\ \hat{\mathbf{z}} &= (-\bar{z}_3, \bar{z}_2, -\bar{z}_1, \bar{z}_0, -\bar{z}_{-1}, \bar{z}_{-2}, -\bar{z}_{-3})^{\mathbf{T}} \\ \mathbf{Q}(\mathbf{z}) &= (\mathbf{Q}_{-3},\ \mathbf{Q}_{-2},\ \mathbf{Q}_{-1},\ \mathbf{Q}_0,\ \mathbf{Q}_1,\ \mathbf{Q}_2,\ \mathbf{Q}_3)^{\mathbf{T}} \\ \mathbf{R}(\mathbf{z}) &= (\mathbf{R}_{-3},\ \mathbf{R}_{-2},\ \mathbf{R}_{-1},\ \mathbf{R}_0,\ \mathbf{R}_1,\ \mathbf{R}_2,\ \mathbf{R}_3)^{\mathbf{T}}. \end{aligned}$$

Here $\mathbf{Q}_m(\tilde{\mathbf{z}}) = \mathbf{Q}_{-m}(\mathbf{z})$, $\mathbf{R}_m(\tilde{\mathbf{z}}) = \mathbf{R}_{-m}(\mathbf{z})$, with $\tilde{\mathbf{z}} = (z_3, z_2, z_1, z_0, z_{-1}, z_{-2}, z_{-3})$ and

$$
\begin{aligned}
\mathbf{Q}_{-3}(\mathbf{z}) &= 5z_{-3}(\, 5|z_{-3}|^2 + 5|z_{-2}|^2 - |z_{-1}|^2 - 4|z_0|^2 - 5|z_1|^2 - 5|z_2|^2 - 8|z_3|^2) \\
&\quad + 5\bar{z}_3 \left(2z_0^2 - 3z_1 z_{-1} + 3z_2 z_{-2}\right) + \sqrt{15}\left(2z_{-1}^2 \bar{z}_1 + 5z_{-2}^2 \bar{z}_{-1}\right) \\
&\quad + 5\sqrt{2}\,(z_0 z_{-1}\bar{z}_2 + z_0 z_{-2}\bar{z}_1 + 3z_{-2}z_{-1}\bar{z}_0) \\
\mathbf{Q}_{-2}(\mathbf{z}) &= 5z_{-2}(\, 5|z_{-3}|^2 + 3|z_{-1}|^2 - 3|z_1|^2 - 8|z_2|^2 - 5|z_3|^2) + 4\sqrt{30}z_{-1}z_0\bar{z}_1 \\
&\quad + 5\bar{z}_2\,(5z_1 z_{-1} + 3z_3 z_{-3}) + 10\sqrt{15}z_{-1}z_{-3}\bar{z}_{-2} + 3\sqrt{30}z_{-1}^2\bar{z}_0 \\
&\quad + 5\sqrt{2}\,(z_1 z_{-3}\bar{z}_0 + z_0 z_1 \bar{z}_3 + 3z_0 z_{-3}\bar{z}_{-1}) \\
\mathbf{Q}_{-1}(\mathbf{z}) &= z_{-1}\left(-5|z_{-3}|^2 + 15|z_{-2}|^2 - 3|z_{-1}|^2 + 12|z_0|^2 - 16|z_1|^2 - 15|z_2|^2 - 25|z_3|^2\right) \\
&\quad + \bar{z}_1\left(24z_0^2 + 25z_2 z_{-2} - 15z_3 z_{-3}\right) + \sqrt{15}\left(4z_1 z_{-3}\bar{z}_{-1} + 2z_1^2\bar{z}_3 + 5z_{-2}^2\bar{z}_{-3}\right) \\
&\quad + 5\sqrt{2}\,(z_2 z_{-3}\bar{z}_0 + z_0 z_2\bar{z}_3 + 3z_0 z_{-3}\bar{z}_{-2}) \\
&\quad + 2\sqrt{30}\,(3z_{-2}z_0\bar{z}_{-1} + 2z_{-2}z_1\bar{z}_0 + 2z_1 z_0\bar{z}_2) \\
\mathbf{Q}_0(\mathbf{z}) &= z_0\left(-20|z_{-3}|^2 + 12|z_{-1}|^2 - 12|z_0|^2 + 12|z_1|^2 - 20|z_3|^2\right) \\
&\quad + 4\bar{z}_0\,(12z_1 z_{-1} + 5z_3 z_{-3}) + 15\sqrt{2}\,(z_1 z_2\bar{z}_3 + z_{-2}z_{-1}\bar{z}_{-3}) \\
&\quad + 5\sqrt{2}\,(z_1 z_{-3}\bar{z}_{-2} + z_2 z_{-3}\bar{z}_{-1} + z_3 z_{-2}\bar{z}_1 + z_3 z_{-1}\bar{z}_2) \\
&\quad + \sqrt{30}\left(4z_{-2}z_1\bar{z}_{-1} + 4z_2 z_{-1}\bar{z}_1 + 3z_1^2\bar{z}_2 + 3z_{-1}^2\bar{z}_{-2}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{R}_{-3}(\mathbf{z}) &= 3z_{-3}(\, 3|z_{-3}|^2 + 3|z_{-2}|^2 + |z_{-1}|^2 - |z_1|^2 - 2|z_2|^2 - 3|z_3|^2) + 3z_{-2}z_2\bar{z}_3 \\
&\quad + 3\sqrt{2}\,(z_0 z_{-2}\bar{z}_1 + z_{-2}z_{-1}\bar{z}_0) + \sqrt{15}\left(z_{-2}^2\bar{z}_{-1} + z_1 z_{-2}\bar{z}_2\right) \\
\mathbf{R}_{-2}(\mathbf{z}) &= z_{-2}\left(9|z_{-3}|^2 + 4|z_{-2}|^2 + 7|z_{-1}|^2 - 2|z_1|^2 - 4|z_2|^2 - 6|z_3|^2\right) + 3z_{-3}z_3\bar{z}_2 \\
&\quad + 3\sqrt{2}\,(z_0 z_{-3}\bar{z}_1 + z_1 z_{-3}\bar{z}_0) + 5z_{-1}z_1\bar{z}_2 + \sqrt{30}\left(z_{-1}^2\bar{z}_0 + z_{-1}z_0\bar{z}_1\right) \\
&\quad + \sqrt{15}\,(z_{-3}z_2\bar{z}_1 + z_2 z_{-1}\bar{z}_3 + 2z_{-1}z_{-3}\bar{z}_{-2}) \\
\mathbf{R}_{-1}(\mathbf{z}) &= z_{-1}\left(3|z_{-3}|^2 + 7|z_{-2}|^2 + |z_{-1}|^2 + 6|z_0|^2 - |z_1|^2 - 2|z_2|^2 - 3|z_3|^2\right) + 6z_0^2\bar{z}_1 \\
&\quad + 3\sqrt{2}\,(z_0 z_{-3}\bar{z}_{-2} + z_0 z_2\bar{z}_3) + \sqrt{30}\,(2z_{-2}z_0\bar{z}_{-1} + z_{-2}z_1\bar{z}_0 + z_0 z_1\bar{z}_2) \\
&\quad + \sqrt{15}\left(z_{-2}^2\bar{z}_{-3} + z_{-2}z_3\bar{z}_2\right) + 5z_{-2}z_2\bar{z}_1 \\
\mathbf{R}_0(\mathbf{z}) &= 6z_0\left(|z_{-1}|^2 + |z_1|^2\right) + 3\sqrt{2}\,(z_{-3}z_1\bar{z}_{-2} + z_1 z_2\bar{z}_3 + z_{-2}z_{-1}\bar{z}_{-3} + z_{-1}z_3\bar{z}_2) \\
&\quad + 12z_1 z_{-1}\bar{z}_0 + \sqrt{30}\left(z_{-2}z_1\bar{z}_{-1} + z_1^2\bar{z}_2 + z_{-1}^2\bar{z}_{-2} + z_{-1}z_2\bar{z}_1\right)
\end{aligned}
$$

Remark 4.1. Antoneli *et al* [1] compute that the number of cubic $\mathbf{O(3)} \times S^1$ equivariant maps for the representation on $V_\ell \oplus V_\ell$ is $\ell + 1$. We have found four cubic equivariant maps for the representation on $V_3 \oplus V_3$ which is in agreement with this result.

As we have seen in section 3, the equivariant Hopf theorem guarantees the existence of six branches of periodic solutions within this vector field. These solutions have the symmetries of the six **C**-axial subgroups listed in Table 4. Using Table 3 we find that the isotropy subgroups with four dimensional fixed-point subspaces for the natural representation on $V_3 \oplus V_3$ are the six given in Table 4. For this specific representation of $\mathbf{O(3)} \times S^1$ on $V_3 \oplus V_3$ we compute also the isotropy subgroups with fixed-point subspaces of dimensions larger than four using the chain criterion. These isotropy subgroups are also listed in Table 4, along with one possible form of their fixed-point subspaces. Figure 1 shows the partial ordering of the conjugacy classes of isotropy subgroups for this representation.

Table 4: The isotropy subgroups Σ of $\mathbf{O(3)} \times S^1$ for the representation on $V_3 \oplus V_3$. Here $H = J \times \mathbb{Z}_2^c$.

Σ	J	K	$\alpha(H)$	$\mathrm{Fix}(\Sigma)$	$\dim \mathrm{Fix}(\Sigma)$	$N(\Sigma)/\Sigma$
$\widetilde{\mathbf{O(2)}}$	$\mathbf{O(2)}$	$\mathbf{O(2)}^-$	$\mathbb{Z}_2$	$\{(0,0,0,w_1,0,0,0)\}$	2	S^1
$\widetilde{\mathbf{SO(2)}}_1$	$\mathbf{SO(2)}$	$\mathbb{Z}_2^-$	S^1	$\{(0,0,w_1,0,0,0,0)\}$	2	S^1
$\widetilde{\mathbf{SO(2)}}_2$	$\mathbf{SO(2)}$	$\mathbb{Z}_4^-$	S^1	$\{(0,w_1,0,0,0,0,0)\}$	2	S^1
$\widetilde{\mathbf{SO(2)}}_3$	$\mathbf{SO(2)}$	$\mathbb{Z}_6^-$	S^1	$\{(w_1,0,0,0,0,0,0)\}$	2	S^1
$\widetilde{\mathbb{O}}$	$\mathbb{O}$	$\mathbb{O}^-$	$\mathbb{Z}_2$	$\{(0,w_1,0,0,0,-w_1,0)\}$	2	S^1
$\widetilde{\mathbf{D}}_6$	$\mathbf{D}_6$	$\mathbf{D}_6^d$	$\mathbb{Z}_2$	$\{(w_1,0,0,0,0,0,-w_1)\}$	2	S^1
$\widetilde{\mathbb{Z}}_6$	$\mathbb{Z}_6$	$\mathbb{Z}_6^-$	$\mathbb{Z}_2$	$\{(w_1,0,0,0,0,0,w_2)\}$	4	$\mathbf{O(2)} \times S^1$
$\widetilde{\mathbb{Z}}_4$	$\mathbb{Z}_4$	$\mathbb{Z}_4^-$	$\mathbb{Z}_2$	$\{(0,w_1,0,0,0,w_2,0)\}$	4	$\mathbf{O(2)} \times S^1$
$\widetilde{\mathbf{D}}_3$	$\mathbf{D}_3$	$\mathbf{D}_3^z$	$\mathbb{Z}_2$	$\{(w_1,0,0,w_2,0,0,-w_1)\}$	4	$\mathbf{D}_2 \times S^1$
$\widetilde{\mathbf{D}}_2$	$\mathbf{D}_2$	$\mathbf{D}_2^z$	$\mathbb{Z}_2$	$\{(0,w_1,0,w_2,0,w_1,0)\}$	4	$\mathbf{D}_2 \times S^1$
$\widetilde{\mathbb{Z}}_3^1$	$\mathbb{Z}_3$	$\mathbb{1}$	$\mathbb{Z}_6$	$\{(0,w_1,0,0,w_2,0,0)\}$	4	$\mathbf{SO(2)} \times S^1$
$\widetilde{\mathbb{Z}}_5$	$\mathbb{Z}_5$	$\mathbb{1}$	$\mathbb{Z}_{10}$	$\{(w_1,0,0,0,0,w_2,0)\}$	4	$\mathbf{SO(2)} \times S^1$
$\widetilde{\mathbb{Z}}_3^2$	$\mathbb{Z}_3$	$\mathbb{Z}_3$	$\mathbb{Z}_2$	$\{(w_1,0,0,w_2,0,0,w_3)\}$	6	$\mathbf{O(2)} \times S^1$
$\widetilde{\mathbb{Z}}_2^1$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\{(0,w_1,0,w_2,0,w_3,0)\}$	6	$\mathbf{O(2)} \times S^1$
$\widetilde{\mathbb{Z}}_2^2$	$\mathbb{Z}_2$	$\mathbb{Z}_2^-$	$\mathbb{Z}_2$	$\{(w_1,0,w_2,0,w_3,0,w_4)\}$	8	$\mathbf{O(2)} \times S^1$
$\widetilde{\mathbb{1}}$	$\mathbb{1}$	$\mathbb{1}$	$\mathbb{Z}_2$	V_3	14	$\mathbf{O(3)} \times S^1$

We use the notation of Golubitsky *et al* [12], whereby $\Sigma = \widetilde{J}$ is an isotropy subgroup with $H = J \times \mathbb{Z}_2^c$ and $\alpha(H)$ a nontrivial subgroup of S^1. A subscript or superscript is added in cases of ambiguity. Notice also that since H is a class II subgroup of $\mathbf{O(3)}$ and K is a class I or III subgroup for all isotropy subgroups in this representation, $\alpha(H) \neq \mathbb{1}$.

In Figure 2 we illustrate the six **C**-axial periodic solution branches. We draw the time-dependent linear combination of spherical harmonics that is invariant under the symmetries of one particular representative of each conjugacy class of **C**-axial subgroups.

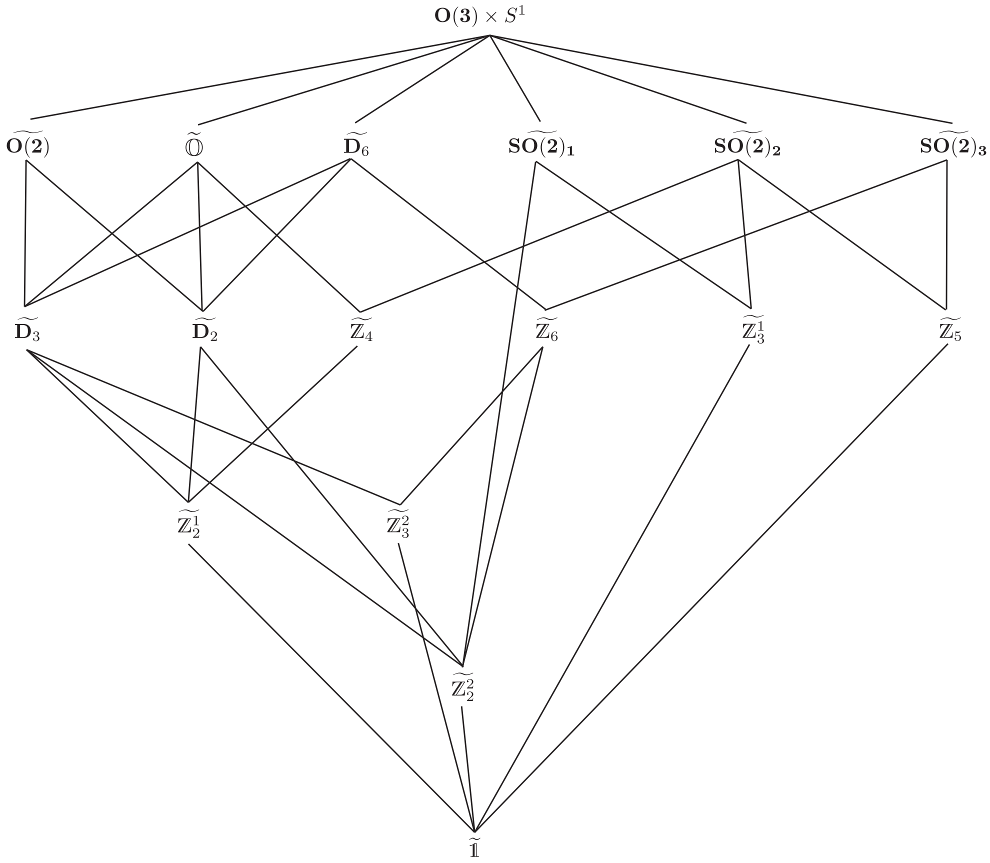
The periodic solutions corresponding to the isotropy subgroups $\widetilde{\mathbf{O(2)}}$, $\widetilde{\mathbb{O}}$ and $\widetilde{\mathbf{D}}_6$ are standing wave solutions and those corresponding to the isotropy subgroups $\widetilde{\mathbf{SO(2)}}_{\mathbf{k}}$, $\mathbf{k} = 1, 2, 3$ are rotating, or travelling, waves, in which the pattern rotates around the axis as indicated in Figure 2.

Notice that in Table 4 the final column lists the group $N(\Sigma)/\Sigma$ for each isotropy subgroup Σ, where

$$N(\Sigma) = \{\gamma \in \mathbf{O(3)} \times S^1 : \gamma \Sigma \gamma^{-1} = \Sigma\}$$

is the normaliser of Σ in $\mathbf{O(3)} \times S^1$ which leaves $\mathrm{Fix}(\Sigma)$ invariant. Since the vector field f is equivariant with respect to the action of $\mathbf{O(3)} \times S^1$, in the restriction to $\mathrm{Fix}(\Sigma)$, f restricts to a $N(\Sigma)/\Sigma$ equivariant equation. Considering the restriction of f to $\mathrm{Fix}(\Sigma)$ enables us to deduce information about the possible existence and bifurcations of other periodic and quasiperiodic solutions with submaximal symmetry, i.e. the symmetries of isotropy subgroups with $\dim \mathrm{Fix}(\Sigma) > 2$. In section 6 we investigate possible submaximal solutions in $\mathrm{Fix}(\Sigma)$ for the isotropy subgroups Σ in Table 4 with $\dim \mathrm{Fix}(\Sigma) = 4$.

Figure 1: The partial ordering of conjugacy classes of isotropy subgroups of $\mathbf{O(3)} \times S^1$ in the representation on $V_3 \oplus V_3$.



5 Stability of maximal solution branches when $\ell = 3$

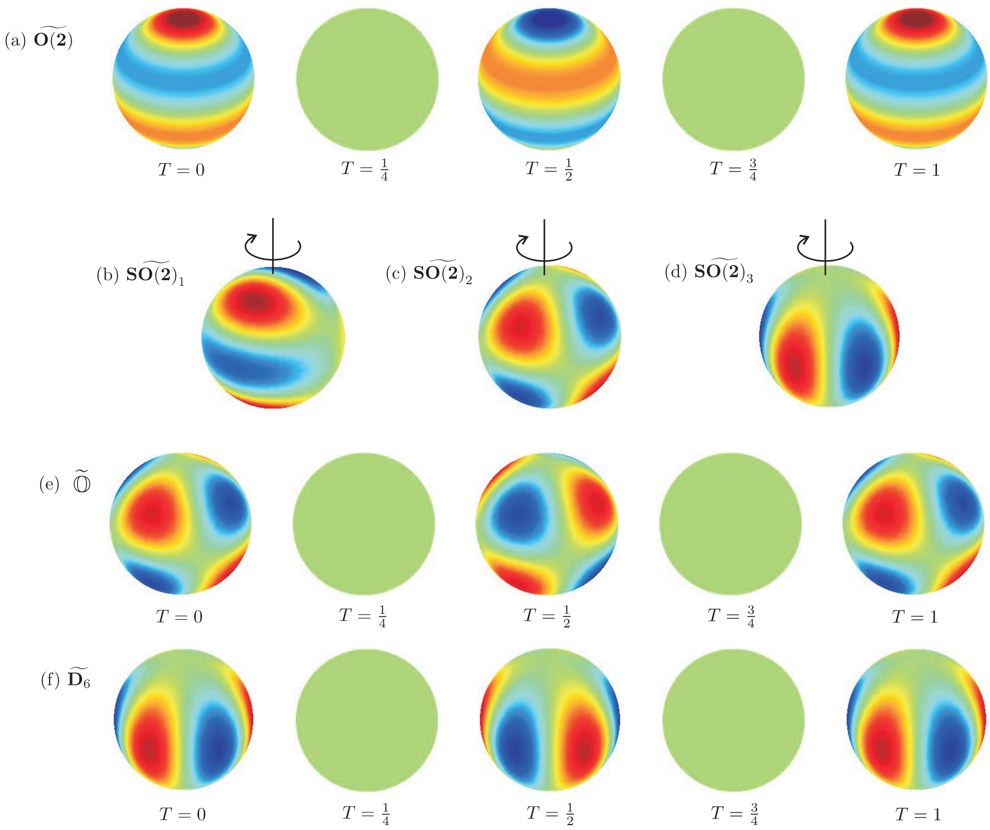
Recall that in section 3 we found that the equivariant Hopf theorem guarantees the existence of six branches of periodic solutions in system (14). In section 4 we computed the Birkhoff normal form of the equivariant vector field f to cubic order. We call this cubic truncation F_3 and it is given by (20). In this section we will use this and the isotypic decomposition of V_3 for the action of each of the $\mathbf{C}$-axial subgroups to determine the branching direction and stability of each of the six periodic solutions.

Recall from section 2 that in order to determine the stability of each of the six branches of $\mathbf{C}$-axial periodic solutions we must compute the eigenvalues of the matrix $(\mathrm{d}g)_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ where g is the reduced equation given by (9) and $(\mathbf{z}_0, \lambda_0, \tau_0)$ is the zero of g corresponding to the periodic solution $\mathbf{z}(t)$ with $\mathbf{C}$-axial symmetry and period $2\pi/(1+\tau_0)$. For our representation of $\mathbf{O(3)} \times S^1$ the matrix J in (9) can be identified with i. We will see that each of the six $\mathbf{C}$-axial subgroups has 3-determined stability and hence the conditions for the solution branches to be stable are given by the real parts of the eigenvalues of $(\mathrm{d}\tilde{g})_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ where

$$\tilde{g}(\mathbf{z}, \lambda, \tau) = F_3(\mathbf{z}, \lambda) - (1 + \tau)\mathrm{i}\mathbf{z} \tag{21}$$

In order to satisfy the conditions of the equivariant Hopf theorem we assume that $\mu(\lambda) \in \mathbf{C}$

Figure 2: The six periodic solution branches with **C**-axial symmetry. (a), (e) and (f) illustrate the evolution of the three standing waves over one period and (b), (c) and (d) illustrate the travelling wave solutions showing the axis and direction of rotation.



in (20) satisfies $\mu(0) = \mathrm{i}$ and $\mathrm{Re}(\mu'(0)) \neq 0$. We will assume that

$$\mathrm{Re}(\mu(\lambda)) = \lambda + \text{ higher order terms in } \lambda,$$

so the trivial solution $\mathbf{z} = 0$ is stable for $\lambda < 0$ and unstable for $\lambda > 0$. This means that for a branch of solutions bifurcating from the trivial solution at $\lambda = 0$ to be stable it must bifurcate supercritically. To determine the dependence of the direction of branching on the coefficients A, B, C and D in F_3, for each periodic solution we compute the *branching equation* by restricting (21) to $\mathrm{Fix}(\Sigma)$ for each of the corresponding **C**-axial subgroups Σ. These branching equations are given in Table 5. Note that we have defined

$$\nu(\lambda) = \mu(\lambda) - (1 + \tau)\mathrm{i}$$

and we denote by subscripts r and i the real and imaginary parts of the coefficients A, B, C and D. Note also that since $\mathrm{Re}(\mu(\lambda)) = \lambda$ to linear order in λ, we have $\mathrm{Re}(\nu(\lambda)) = \lambda$ to linear order also.

We now compute the eigenvalues of the 14×14 matrix $(\mathrm{d}\tilde{g})_{(\mathbf{z}_0, \lambda_0, \tau_0)}$ for each of the **C**-axial solutions. In order to simplify this computation we use the isotypic decomposition of V_3 for the action of the **C**-axial subgroup Σ to block diagonalize the matrix. The isotypic components for the actions of each of the **C**-axial subgroups are given in Table 6. In Table 7 we give the eigenvalues of $(\mathrm{d}\tilde{g})_{(\mathbf{z}_0, \lambda_0, \tau_0)}$ in each of the isotypic components listed in Table 6. Further details on the computations of the isotypic components and eigenvalues can be found in [26].

Table 5: Branching equations for each of the six bifurcating branches of periodic solutions. For the branch to bifurcate supercritically we require that $\lambda > 0$.

Σ	Branching equation	Real part of branching equation
$\widetilde{\mathbf{O(2)}}$	$0 = \nu(\lambda) + (A + B - 12C)\lvert w_1\rvert^2$	$\lambda = -(A_r + B_r - 12C_r)\lvert w_1\rvert^2$
$\widetilde{\mathbf{SO(2)}}_1$	$0 = \nu(\lambda) + (A - 3C + D)\lvert w_1\rvert^2$	$\lambda = -(A_r - 3C_r + D_r)\lvert w_1\rvert^2$
$\widetilde{\mathbf{SO(2)}}_2$	$0 = \nu(\lambda) + (A + 4D)\lvert w_1\rvert^2$	$\lambda = -(A_r + 4D_r)\lvert w_1\rvert^2$
$\widetilde{\mathbf{SO(2)}}_3$	$0 = \nu(\lambda) + (A + 25C + 9D)\lvert w_1\rvert^2$	$\lambda = -(A_r + 25C_r + 9D_r)\lvert w_1\rvert^2$
$\widetilde{\mathbb{O}}$	$0 = \nu(\lambda) + (2A + 2B - 40C)\lvert w_1\rvert^2$	$\lambda = -(2A_r + 2B_r - 40C_r)\lvert w_1\rvert^2$
$\widetilde{\mathbf{D}}_6$	$0 = \nu(\lambda) + (2A + 2B - 15C)\lvert w_1\rvert^2$	$\lambda = -(2A_r + 2B_r - 15C_r)\lvert w_1\rvert^2$

Table 6: Isotypic components for the actions of the **C**-axial subgroups Σ on V_3.

Σ	Isotypic components
$\widetilde{\mathbf{O(2)}}$	$W_0 = \{(0,0,0,u_1,0,0,0)\} = \mathrm{Fix}(\widetilde{\mathbf{O(2)}})$
	$W_1 = \{(0,0,u_1,0,u_2,0,0)\}$
	$W_2 = \{(0,u_1,0,0,0,u_2,0)\}$
	$W_3 = \{(u_1,0,0,0,0,0,u_2)\}$
$\widetilde{\mathbf{SO(2)}}_1$	$W_0 = \{(0,0,u_1,0,0,0,0)\} = \mathrm{Fix}(\widetilde{\mathbf{SO(2)}}_1)$
	$W_1 = \{(0,0,0,0,0,u_1,0)\}$
	$W_2 = \{(0,0,0,0,0,0,u_1)\}$
	$W_3 = \{(0,u_1,0,u_2,0,0,0)\}$
	$W_4 = \{(u_1,0,0,0,u_2,0,0)\}$
$\widetilde{\mathbf{SO(2)}}_2$	$W_0 = \{(0,u_1,0,0,0,0,0)\} = \mathrm{Fix}(\widetilde{\mathbf{SO(2)}}_2)$
	$W_1 = \{(0,0,0,u_1,0,0,0)\}$
	$W_2 = \{(0,0,0,0,u_1,0,0)\}$
	$W_3 = \{(0,0,0,0,0,u_1,0)\}$
	$W_4 = \{(0,0,0,0,0,0,u_1)\}$
	$W_5 = \{(u_1,0,u_2,0,0,0,0)\}$
$\widetilde{\mathbf{SO(2)}}_3$	$W_0 = \{(u_1,0,0,0,0,0,0)\} = \mathrm{Fix}(\widetilde{\mathbf{SO(2)}}_3)$
	$W_1 = \{(0,u_1,0,0,0,0,0)\}$
	$W_2 = \{(0,0,u_1,0,0,0,0)\}$
	$W_3 = \{(0,0,0,u_1,0,0,0)\}$
	$W_4 = \{(0,0,0,0,u_1,0,0)\}$
	$W_5 = \{(0,0,0,0,0,u_1,0)\}$
	$W_6 = \{(0,0,0,0,0,0,u_1)\}$
$\widetilde{\mathbb{O}}$	$W_0 = \{(0,u_1,0,0,0,-u_1,0)\} = \mathrm{Fix}(\widetilde{\mathbb{O}})$
	$W_1 = \{(\sqrt{3}u_1, u_3, \sqrt{5}u_2, 0, -\sqrt{5}u_1, u_3, -\sqrt{3}u_2)\}$
	$W_2 = \{(\sqrt{5}u_1, 0, \sqrt{3}u_2, u_3, \sqrt{3}u_1, 0, \sqrt{5}u_2)\}$
$\widetilde{\mathbf{D}}_6$	$W_0 = \{(u_1,0,0,0,0,0,-u_1)\} = \mathrm{Fix}(\widetilde{\mathbf{D}}_6)$
	$W_1 = \{(0,0,0,u_1,0,0,0)\}$
	$W_2 = \{(0,0,u_1,0,u_2,0,0)\}$
	$W_3 = \{(0,u_1,0,0,0,u_2,0)\}$
	$W_4 = \{(u_1,0,0,0,0,0,u_1)\}$

Table 7: The eigenvalues of $(\mathrm{d}\tilde{g})_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ for each branch of periodic solutions with **C**-axial symmetry Σ by isotypic component.

Σ	Isotypic component	Eigenvalues	Multiplicity
$\widetilde{\mathbf{O(2)}}$	W_0	$(2A_r+2B_r-24C_r)\lvert w_1\rvert^2=-2\lambda$	1
		0	1
	W_1	$(-2B_r+48C_r+12D_r)\lvert w_1\rvert^2$	2
		0	2
	W_2	$\left[-B_r+12C_r+\sqrt{B_r^2+24B_iC_i-144C_i^2}\right]\lvert w_1\rvert^2$	2
		$\left[-B_r+12C_r-\sqrt{B_r^2+24B_iC_i-144C_i^2}\right]\lvert w_1\rvert^2$	2
	W_3	$\left[-B_r-8C_r+\sqrt{(B_r-10C_r)^2-36C_i(B_i-C_i)}\right]\lvert w_1\rvert^2$	2
		$\left[-B_r-8C_r-\sqrt{(B_r-10C_r)^2-36C_i(B_i-C_i)}\right]\lvert w_1\rvert^2$	2
$\widetilde{\mathbf{SO(2)}}_1$	W_0	$(2A_r-6C_r+2D_r)\lvert w_1\rvert^2=-2\lambda$	1
		0	1
	W_1	$\xi=(-12C-3D)\lvert w_1\rvert^2$ and $\overline{\xi}$	1 of each
	W_2	$\xi=(-22C-4D)\lvert w_1\rvert^2$ and $\overline{\xi}$	1 of each
	W_3	$\xi=(33C_r+11D_r+3\mathrm{i}C_i+\mathrm{i}D_i)\lvert w_1\rvert^2$ and $\overline{\xi}$	1 of each
		0	2
	W_4	$\xi_\pm=\left(-C+D+\overline{B}-\frac{13}{2}\overline{C}-\overline{D}\pm\sqrt{\delta}\right)\lvert w_1\rvert^2$ and $\overline{\xi}_\pm$ where $\delta=\left(-C+D+\overline{B}-\frac{13}{2}\overline{C}-\overline{D}\right)^2+(2C-2D)\left(2\overline{B}-13\overline{C}-2\overline{D}\right)+60\lvert C\rvert^2$	1 of each
$\widetilde{\mathbf{SO(2)}}_2$	W_0	$(2A_r+8D_r)\lvert w_1\rvert^2=-2\lambda$	1
		0	1
	W_1	$\xi=-4D\lvert w_1\rvert^2$ and $\overline{\xi}$	1 of each
	W_2	$\xi=(-15C-6D)\lvert w_1\rvert^2$ and $\overline{\xi}$	1 of each
	W_3	$\xi=(2B-40C-8D)\lvert w_1\rvert^2$ and $\overline{\xi}$	1 of each
	W_4	$\xi=(-25C-10D)\lvert w_1\rvert^2$ and $\overline{\xi}$	1 of each
	W_5	$\xi=(40C_r+8D_r+10\mathrm{i}C_i+2\mathrm{i}D_i)\lvert w_1\rvert^2$ and $\overline{\xi}$	1 of each
		0	2
$\widetilde{\mathbf{SO(2)}}_3$	W_0	$(2A_r+50C_r+18D_r)\lvert w_1\rvert^2=-2\lambda$	1
		0	1
	W_1	0	2
	W_2	$\xi=(-30C-6D)\lvert w_1\rvert^2$ and $\overline{\xi}$	1 of each
	W_3	$\xi=(-45C-9D)\lvert w_1\rvert^2$ and $\overline{\xi}$	1 of each
	W_4	$\xi=(-50C-12D)\lvert w_1\rvert^2$ and $\overline{\xi}$	1 of each
	W_5	$\xi=(-50C-15D)\lvert w_1\rvert^2$ and $\overline{\xi}$	1 of each
	W_6	$\xi=(2B-65C-18D)\lvert w_1\rvert^2$ and $\overline{\xi}$	1 of each

Continued on next page

Table 7. – continued from previous page

Σ	Isotypic component	Eigenvalues	Multiplicity
$\widetilde{\mathbb{O}}$	W_0	$(4A_r + 4B_r - 80C_r)\lvert w_1\rvert^2 = -2\lambda$	1
		0	1
	W_1	$(-4B_r + 80C_r + 16D_r)\lvert w_1\rvert^2$	3
		0	3
	W_2	$\left[-2B_r + 40C_r + 2\sqrt{B_r^2 + 40B_iC_i - 400C_i^2}\right]\lvert w_1\rvert^2$	3
		$\left[-2B_r + 40C_r - 2\sqrt{B_r^2 + 40B_iC_i - 400C_i^2}\right]\lvert w_1\rvert^2$	3
$\widetilde{\mathbf{D}_6}$	W_0	$(4A_r + 4B_r - 30C_r)\lvert w_1\rvert^2 = -2\lambda$	1
		0	1
	W_1	$\left[-2B_r - 25C_r + \sqrt{4(B_r - 10C_r)^2 - 45C_i(4B_i + 5C_i)}\right]\lvert w_1\rvert^2$	1
		$\left[-2B_r - 25C_r - \sqrt{4(B_r - 10C_r)^2 - 45C_i(4B_i + 5C_i)}\right]\lvert w_1\rvert^2$	1
	W_2	$\left[-2B_r - 15C_r + \sqrt{(2B_r - 15C_r)^2 - 120B_iC_i}\right]\lvert w_1\rvert^2$	2
		$\left[-2B_r - 15C_r - \sqrt{(2B_r - 15C_r)^2 - 120B_iC_i}\right]\lvert w_1\rvert^2$	2
	W_3	$(-4B_r + 30C_r + 6D_r)\lvert w_1\rvert^2$	2
		0	2
	W_4	$(-4B_r + 130C_r + 36D_r)\lvert w_1\rvert^2$	1
		0	1

5.1 Conditions for stability of the solution branches

We now state a theorem which lists conditions in terms of the coefficients A, B, C and D for each of the individual solution branches to be stable. This theorem is a direct consequence of the results concerning stability of periodic solution branches outlined in section 2.1. The proof can be deduced from the eigenvalues of $(\mathrm{d}\tilde{g})_{(\mathbf{z}_0,\lambda_0,\tau_0)}$ for each of the six branches of periodic solutions as given in Table 7.

Theorem 5.1. *For each* **C***-axial subgroup* Σ *listed in Table 4 let* $\Delta_0, \ldots, \Delta_k$ *be the functions of the coefficients* A*,* B*,* C *and* D *given in Table 8. Then*

1. *For each* Σ *the corresponding branch of periodic solutions is supercritical if* $\Delta_0 < 0$ *and subcritical if* $\Delta_0 > 0$*.*

2. *For each* Σ *the corresponding branch of periodic solutions is stable near* $\lambda = 0$ *if* $\Delta_j < 0$ *for all* j*. If* $\Delta_j > 0$ *for some* $j = 0, \ldots, k$ *then the branch of periodic solutions is unstable.*

5.2 Remarks

From our analysis of the stability of the six branches of periodic solutions with maximal symmetry we make the following observations.

1. For each of the six solution branches the number of eigenvalues which are zero at cubic order for generic values of the coefficients A, B, C and D is equal to the number of eigenvalues which are forced to be zero by symmetry. In other words, each **C**-axial isotropy subgroup

Table 8: Stability conditions for the six branches of periodic solutions. If $\Delta_j < 0$ for all j then the branch of periodic solutions is stable near $\lambda = 0$.
(*) $\delta = \left(-C + D + \overline{B} - \frac{13}{2}\overline{C} - \overline{D}\right)^2 + (2C - 2D)\left(2\overline{B} - 13\overline{C} - 2\overline{D}\right) + 60|C|^2$

Σ	Δ_0	$\Delta_1, \ldots, \Delta_k$
$\widetilde{\mathbf{O(2)}}$	$A_r + B_r - 12C_r$	$-B_r + 24C_r + 6D_r$ $-B_r + 12C_r$ $\mathrm{Re}(B\overline{C}) - 6\|C\|^2$ $-B_r - 8C_r$ $\|C\|^2 - \mathrm{Re}(B\overline{C})$
$\widetilde{\mathbf{SO(2)}}_1$	$A_r - 3C_r + D_r$	$2B_r - 15C_r$ $-4C_r - D_r$ $-11C_r - 2D_r$ $3C_r + D_r$ $\left\|\mathrm{Re}(\sqrt{\delta})\right\| - \left\|B_r - \frac{15}{2}C_r\right\|^{(*)}$
$\widetilde{\mathbf{SO(2)}}_2$	$A_r + 4D_r$	$-D_r$ $-5C_r - 2D_r$ $B_r - 20C_r - 4D_r$ $5C_r + D_r$
$\widetilde{\mathbf{SO(2)}}_3$	$A_r + 25C_r + 9D_r$	$-5C_r - D_r$ $-25C_r - 6D_r$ $2B_r - 65C_r - 18D_r$ $-10C_r - 3D_r$
$\widetilde{\mathbb{O}}$	$A_r + B_r - 20C_r$	$-B_r + 20C_r + 4D_r$ $-B_r + 20C_r$ $\mathrm{Re}(B\overline{C}) - 10\|C\|^2$
$\widetilde{\mathbf{D}}_6$	$2A_r + 2B_r - 15C_r$	$-2B_r - 25C_r$ $-5\|C\|^2 - 4\mathrm{Re}(B\overline{C})$ $-2B_r - 15C_r$ $-\mathrm{Re}(B\overline{C})$ $-2B_r + 15C_r + 3D_r$ $-2B_r + 65C_r + 18D_r$

in this presentation has 3-determined stability. Iooss and Rossi [16] and Haaf *et al* [14] found that this is not the case for the representation on $V_2 \oplus V_2$. They find that in order to determine the stability of two out of the five solution branches, fifth order terms in the equivariant vector field must be considered. In the representation on $V_3 \oplus V_3$ the direction of branching and stability of all six solution branches is fully determined by the functions of the cubic order coefficients A, B, C and D given in Table 8. If any of these functions Δ_j are equal to zero then there is a degeneracy and the stability or direction of branching depends on the coefficients of higher order terms in the equivariant vector field f.

2. For each of the six solution branches it is possible to find a set of values for A, B, C and D such that the solution is stable.

3. It is clear from Table 8 that the pairs of branches of periodic solutions with symmetries

   (a) $\widetilde{\mathbf{SO(2)}}_1$ and $\widetilde{\mathbf{SO(2)}}_2$

(b) $\widetilde{\mathbf{SO(2)}}_2$ and $\widetilde{\mathbf{SO(2)}}_3$

(c) $\widetilde{\mathbf{SO(2)}}_2$ and $\widetilde{\mathbb{O}}$

(d) $\widetilde{\mathbf{SO(2)}}_3$ and $\widetilde{\mathbf{D}_6}$

cannot be simultaneously stable.

4. It is possible for all three standing wave solutions to be simultaneously stable, for example when

$$A_r = -20 \qquad B_r = \frac{5}{2} \qquad B_i = 5 \qquad C_r = -\frac{1}{6} \qquad C_i = \frac{3}{2} \qquad D_r = \frac{3}{5}$$

and A_i and D_i take any values. Note that if the coefficients take these values then all three travelling wave solutions are unstable.

5. It is possible for all six branches of periodic solutions to branch supercritically and be unstable, for example when

$$A_r = -30 \qquad B_r = 50 \qquad B_i = 50 \qquad C_r = 3 \qquad C_i = -150 \qquad D_r = -13$$

and A_i and D_i take any values.

6 Solutions in higher dimensional fixed-point subspaces

In this final section we will investigate possible solutions of (14) in the restriction to $\mathrm{Fix}(\Sigma)$, where Σ is an isotropy subgroup of $\mathbf{O(3)} \times S^1$ in the natural representation on $V_3 \oplus V_3$ with $\dim \mathrm{Fix}(\Sigma) = 4$. These isotropy subgroups are listed in Table 4 along with one possible form of their fixed-point subspaces.

Consider (14) where $f : \mathbf{C}^7 \times \mathbf{R} \to \mathbf{C}^7$ is equivariant with respect to $\mathbf{O(3)} \times S^1$ to all orders i.e. f is the exact Birkhoff normal form, not just a truncated Taylor series. Then recall from section 4 that f restricts to a $N(\Sigma)/\Sigma$ equivariant system on $\mathrm{Fix}(\Sigma)$ for some action of $N(\Sigma)/\Sigma$. For each isotropy subgroup Σ the group $N(\Sigma)/\Sigma$ is given in Table 4. For the isotropy subgroups with $\dim \mathrm{Fix}(\Sigma) = 4$ the action of $N(\Sigma)/\Sigma$ on $\mathrm{Fix}(\Sigma)$ is given in Table 9.

We now consider the restriction of f to $\mathrm{Fix}(\Sigma)$ for each isotropy subgroup Σ given in Table 9. We will look for bifurcations of the maximal solution branches and identify additional periodic and quasiperiodic solutions to (14).

6.1 Solutions in $\mathrm{Fix}(\widetilde{\mathbb{Z}}_6)$ and $\mathrm{Fix}(\widetilde{\mathbb{Z}}_4)$

Calculations show that the restriction of f to $\mathrm{Fix}(\Sigma)$ for $\Sigma = \widetilde{\mathbb{Z}}_6$ or $\widetilde{\mathbb{Z}}_4$ yields equations that have been studied before in the context of a Hopf bifurcation with $\mathbf{O(2)}$ symmetry. Provided that none of the coefficients of the cubic terms in the Birkhoff normal form of this bifurcation vanish (i.e. there are no degeneracies) then there are only two types of solutions which bifurcate from a Hopf bifurcation with $\mathbf{O(2)}$ symmetry: standing waves and rotating or travelling waves.

In $\mathrm{Fix}(\widetilde{\mathbb{Z}}_6)$ the standing waves are solutions with $\widetilde{\mathbf{D}_6}$ symmetry and the rotating waves are solutions with $\widetilde{\mathbf{SO(2)}}_3$ symmetry. In $\mathrm{Fix}(\widetilde{\mathbb{Z}}_4)$ the standing waves are solutions with $\widetilde{\mathbb{O}}$ symmetry

Table 9: The action of $N(\Sigma)/\Sigma$ on $\mathrm{Fix}(\Sigma)$ for isotropy subgroups Σ with $\dim \mathrm{Fix}(\Sigma) = 4$. For each Σ, $\psi \in S^1$ acts as $\psi(w_1, w_2) = \left(e^{i\psi} w_1, e^{i\psi} w_2\right)$.

Σ	$N(\Sigma)/\Sigma$	Action	
$\widetilde{\mathbb{Z}}_6$	$\mathbf{O(2)} \times S^1$	$\phi(w_1, w_2) = \left(e^{-3i\phi} w_1, e^{3i\phi} w_2\right)$	$\phi \in \mathbf{SO(2)}$
		$\kappa(w_1, w_2) = (-w_2, -w_1)$	$\kappa \in \mathbf{O(2)}$
$\widetilde{\mathbb{Z}}_4$	$\mathbf{O(2)} \times S^1$	$\phi(w_1, w_2) = \left(e^{-2i\phi} w_1, e^{2i\phi} w_2\right)$	$\phi \in \mathbf{SO(2)}$
		$\kappa(w_1, w_2) = (-w_2, -w_1)$	$\kappa \in \mathbf{O(2)}$
$\widetilde{\mathbf{D}}_3$	$\mathbf{D}_2 \times S^1$	$R^z_\pi(w_1, w_2) = (-w_1, w_2)$	$R^z_\pi \in \mathbf{D}_2$
		$R^y_\pi(w_1, w_2) = (w_1, w_2)$	$R^y_\pi \in \mathbf{D}_2$
$\widetilde{\mathbf{D}}_2$	$\mathbf{D}_2 \times S^1$	$R^z_\pi(w_1, w_2) = (w_1, w_2)$	$R^z_\pi \in \mathbf{D}_2$
		$R^y_\pi(w_1, w_2) = (-w_1, w_2)$	$R^y_\pi \in \mathbf{D}_2$
$\widetilde{\mathbb{Z}}^1_3$	$\mathbf{SO(2)} \times S^1$	$\phi(w_1, w_2) = \left(e^{-i\phi} w_1, e^{2i\phi} w_2\right)$	$\phi \in \mathbf{SO(2)}$
$\widetilde{\mathbb{Z}}_5$	$\mathbf{SO(2)} \times S^1$	$\phi(w_1, w_2) = \left(e^{-3i\phi} w_1, e^{2i\phi} w_2\right)$	$\phi \in \mathbf{SO(2)}$

and the rotating waves are solutions with $\widetilde{\mathbf{SO(2)}}_2$ symmetry. Since generically these are the only solutions which bifurcate, in the Hopf bifurcation with $\mathbf{O(3)}$ symmetry there are no solutions with $\widetilde{\mathbb{Z}}_6$ or $\widetilde{\mathbb{Z}}_4$ symmetry.

If we allow degeneracies then it is possible for solutions with $\widetilde{\mathbb{Z}}_6$ or $\widetilde{\mathbb{Z}}_4$ symmetry to exist, for example if $2B_r - 65C_r - 18D_r = 0$ then a solution with $\widetilde{\mathbb{Z}}_6$ symmetry can exist and if $B_r - 20C_r - 4D_r = 0$ then a solution with $\widetilde{\mathbb{Z}}_4$ symmetry can exist. For a classification of the possible degeneracies in the $\mathbf{O(2)}$ Hopf bifurcation see [11].

6.2 Solutions in Fix($\widetilde{\mathbf{D}}_3$) and Fix($\widetilde{\mathbf{D}}_2$)

If we compute to cubic order the equations which are equivariant with respect to the actions of $\mathbf{D}_2 \times S^1$ on $\mathrm{Fix}(\widetilde{\mathbf{D}}_3)$ and $\mathrm{Fix}(\widetilde{\mathbf{D}}_2)$ described in Table 9 then we find

$$\dot{w}_1 = \mu_1 w_1 + \alpha_1 w_1 |w_1|^2 + \beta_1 w_1 |w_2|^2 + \gamma_1 w_2^2 \overline{w_1} \tag{22}$$
$$\dot{w}_2 = \mu_2 w_2 + \alpha_2 w_2 |w_2|^2 + \beta_2 w_2 |w_1|^2 + \gamma_2 w_1^2 \overline{w_2}. \tag{23}$$

In the case where $\mu_1 = \mu_2$, these equations also occur in the context of a Hopf bifurcation on a rotating rhombic lattice in the restriction to certain four-dimensional subspaces. See for example [21, 22]. In the restriction of (20) to $\mathrm{Fix}(\widetilde{\mathbf{D}}_3)$ or $\mathrm{Fix}(\widetilde{\mathbf{D}}_2)$ we have $\mu_1 = \mu_2 = \mu$, $\beta_2 = 2\beta_1 = 2\beta$, $\gamma_2 = 2\gamma_1 = 2\gamma$ and in $\mathrm{Fix}(\widetilde{\mathbf{D}}_3)$,

$$\begin{aligned} \alpha_1 &= 2A + 2B - 15C \\ \alpha_2 &= A + B - 12C = \frac{1}{5}(2\alpha_1 + \beta + \gamma) \\ \beta &= A - 20C \\ \gamma &= B - 10C \end{aligned} \tag{24}$$

and in $\text{Fix}(\widetilde{\mathbf{D}_2})$,

$$\begin{aligned}\alpha_1 &= 2A + 2B - 40C \\ \alpha_2 &= A + B - 12C = \frac{1}{10}(3\alpha_1 + 4\beta + 4\gamma) \\ \beta &= A \\ \gamma &= B.\end{aligned} \tag{25}$$

In either case there are three branches of standing wave solutions, with symmetries $\widetilde{\mathbf{O(2)}}$, $\widetilde{\mathbb{O}}$ and $\widetilde{\mathbf{D}_6}$, which bifurcate from the Hopf bifurcation with $\mathbf{O(3)}$ symmetry. Depending on the values of the coefficients α_1, β and γ in $\text{Fix}(\Sigma)$ for $\Sigma = \widetilde{\mathbf{D}_3}$ or $\widetilde{\mathbf{D}_2}$ it is possible to find solutions to (22)–(23) with Σ symmetry.

Here we will consider the equations in $\text{Fix}(\widetilde{\mathbf{D}_2})$, where the values of the coefficients are given by (25). Since the equations in $\text{Fix}(\widetilde{\mathbf{D}_3})$ have the same form, a similar analysis yields similar results.

In $\text{Fix}(\widetilde{\mathbf{D}_2})$ the three standing wave solutions are $w_1 = 0$, with $\widetilde{\mathbf{O(2)}}$ symmetry, $w_2 = 0$, with $\widetilde{\mathbb{O}}$ symmetry and $w_1 = \sqrt{\frac{3}{10}}w_2$, with $\widetilde{\mathbf{D}_6}$ symmetry. This is because alternative forms of $\text{Fix}(\widetilde{\mathbf{D}_6})$ and $\text{Fix}(\widetilde{\mathbb{O}})$ which lie inside $\text{Fix}(\widetilde{\mathbf{D}_2})$ are

$$\text{Fix}(\widetilde{\mathbf{D}_6}) = \left\{\left(0, \sqrt{\frac{3}{10}}w_2, 0, w_2, 0, \sqrt{\frac{3}{10}}w_2, 0\right)\right\} \tag{26}$$

$$\text{Fix}(\widetilde{\mathbb{O}}) = \{(0, w_1, 0, 0, 0, w_1, 0)\}\,. \tag{27}$$

We now consider the points where the stability of the solution branches within $\text{Fix}(\widetilde{\mathbf{D}_2})$ change.

For the periodic solution with $\widetilde{\mathbf{O(2)}}$ symmetry, $\text{Fix}(\widetilde{\mathbf{D}_2})$ is contained in the direct sum of the isotypic components W_0 and W_2. Using the eigenvalues in these components we see that this branch of solutions undergoes a stationary bifurcation if $\text{Re}(B\overline{C}) = 6|C|^2$. It is also possible for this solution branch to undergo a Hopf bifurcation at $-B_r + 12C_r = 0$ if $6|C|^2 - \text{Re}(B\overline{C}) > 0$ there.

A similar analysis for the $\widetilde{\mathbb{O}}$ symmetric branch shows that it undergoes a stationary bifurcation if $\text{Re}(B\overline{C}) = 10|C|^2$. This solution branch can also undergo a Hopf bifurcation at $-B_r + 20C_r = 0$ if $10|C|^2 - \text{Re}(B\overline{C}) > 0$ there.

Finally, the periodic solution with $\widetilde{\mathbf{D}_6}$ undergoes a stationary bifurcation when $\text{Re}(B\overline{C}) = 0$. It also has a zero eigenvalue at $-2B_r - 15C_r = 0$ which represents a Hopf bifurcation if $\text{Re}(B\overline{C}) > 0$ there.

The bifurcations of these solution branches allow for the possibility of periodic and quasiperiodic solutions with $\widetilde{\mathbf{D}_2}$ symmetry. Using the numerical continuation package AUTO, it is possible to demonstrate the existence of these branches of periodic and quasiperiodic solutions with $\widetilde{\mathbf{D}_2}$ symmetry for some particular values of the coefficients A, B and C.

6.2.1 Example

Suppose that when $\lambda = 1$

$$A = -3 + \mathrm{i}, \qquad B = 1 + 3\mathrm{i}, \qquad C = C_r + \frac{3}{40}\mathrm{i}$$

and we vary the value of C_r. Then

$$\alpha = \alpha_1 = 2A + 2B - 40C = \alpha_r + 5\mathrm{i}, \qquad \beta = A = -3 + \mathrm{i}, \qquad \gamma = B = 1 + 3\mathrm{i},$$

where $\alpha_r = -4 - 40C_r$. For these values

1. The $\widetilde{\mathbf{O(2)}}$ symmetric branch of solutions bifurcates supercritically when $\alpha_r < \frac{8}{3}$ and undergoes a stationary bifurcation at $\alpha_r = \frac{1}{3}\left(\sqrt{559} - 22\right)$.
2. The $\widetilde{\mathbb{O}}$ symmetric branch of solutions bifurcates supercritically when $\alpha_r < 0$ and undergoes a stationary bifurcation at $\alpha_r = \sqrt{31} - 6$.
3. The $\widetilde{\mathbf{D}}_6$ symmetric branch of solutions bifurcates supercritically when $\alpha_r < \frac{20}{3}$ and undergoes a stationary bifurcation at $\alpha_r = 5$ and a Hopf bifurcation at $\alpha_r = \frac{4}{3}$.

Using AUTO we find that there is a branch of periodic solutions connecting the $\widetilde{\mathbf{O(2)}}$ and $\widetilde{\mathbf{D}}_6$ symmetric branches and a branch of periodic solutions connecting the $\widetilde{\mathbb{O}}$ and $\widetilde{\mathbf{D}}_6$ symmetric branches. These bifurcate at the stationary bifurcations and have $\widetilde{\mathbf{D}}_2$ symmetry. Neither of these solutions is stable. In addition there is a branch of stable quasiperiodic solutions which bifurcates from the solution with $\widetilde{\mathbf{D}}_6$ symmetry at the Hopf bifurcation. This solution branch also has $\widetilde{\mathbf{D}}_2$ symmetry. These branches of solutions can be seen in Figure 3.

As $\alpha_r \to \alpha_c \approx 2.17806$ the quasiperiodic solution spends an increasing amount of time near the unstable branch of solutions connecting the $\widetilde{\mathbf{O(2)}}$ and $\widetilde{\mathbf{D}}_6$ symmetric branches. At $\alpha_r = \alpha_c$ the system undergoes a global bifurcation to a homoclinic orbit.

6.3 Solutions in Fix($\widetilde{\mathbb{Z}}_3^1$) and Fix($\widetilde{\mathbb{Z}}_5$)

If we compute to cubic order the equations which are equivariant with respect to the actions of $\mathbf{SO(2)} \times S^1$ on Fix($\widetilde{\mathbb{Z}}_3^1$) and Fix($\widetilde{\mathbb{Z}}_5$) described in Table 9 then we find (with some rescaling of w_1)

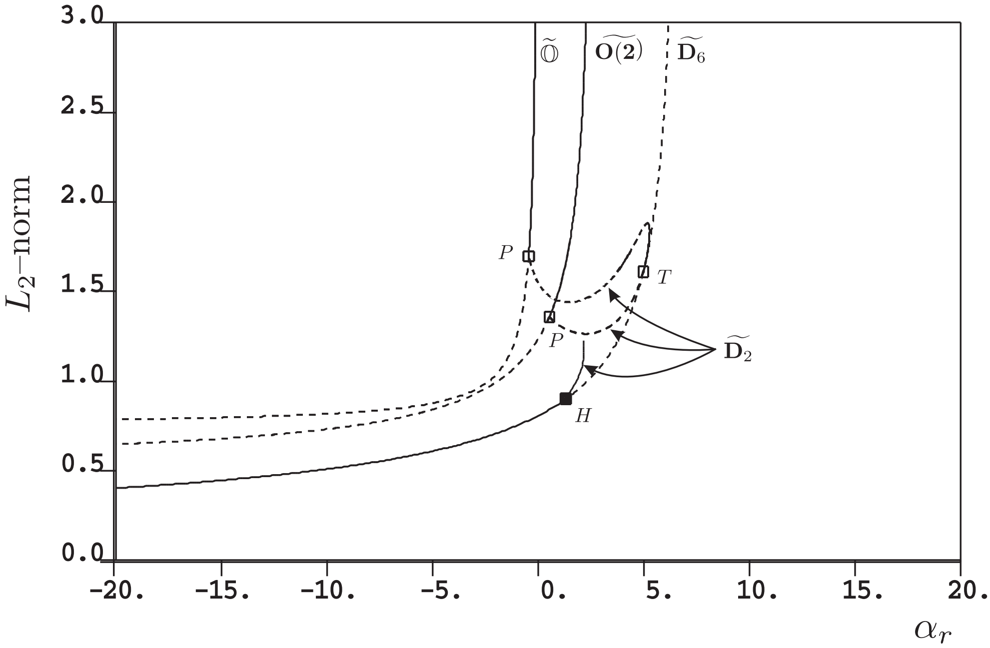
$$\dot{w}_1 = \mu_1 w_1 + \alpha_1 w_1 |w_1|^2 + \beta w_1 |w_2|^2 \tag{28}$$
$$\dot{w}_2 = \mu_2 w_2 + \alpha_2 w_2 |w_2|^2 + \beta w_2 |w_1|^2. \tag{29}$$

These equations describe the interaction of two Hopf bifurcations with $\mathbf{SO(2)}$ symmetry at 1 : 1 resonance, but with different $\mathbf{SO(2)}$ actions. In the case where $\mu_1 = \mu_2$, similarly to (22)–(23), these equations also occur in the context of a Hopf bifurcation on a rotating rhombic lattice in the restriction to other four-dimensional subspaces [21, 22].

In the restriction of (20) to Fix($\widetilde{\mathbb{Z}}_3^1$) or Fix($\widetilde{\mathbb{Z}}_5$) we have $\mu_1 = \mu_2 = \mu$ and in Fix($\widetilde{\mathbb{Z}}_3^1$),

$$\begin{aligned} \alpha_1 &= A + 4D \\ \alpha_2 &= A - 3C + D \\ \beta &= A - 15C - 2D \end{aligned} \tag{30}$$

Figure 3: AUTO generated diagram of the three standing wave solutions with $\widetilde{\mathbf{O(2)}}$, $\widetilde{\mathbb{O}}$ and $\widetilde{\mathbf{D}}_6$ symmetry in $Fix(\widetilde{\mathbf{D}_2})$. The diagram shows the bifurcations of these solution branches and the bifurcating branches of solutions with $\widetilde{\mathbf{D}}_2$ symmetry. P denotes a pitchfork bifurcation, T a transcritical bifurcation and H a Hopf bifurcation. Stable solutions are denoted by solid lines and unstable solutions by dashed lines. The unstable solutions with $\widetilde{\mathbf{D}}_2$ symmetry are periodic and the stable solution with $\widetilde{\mathbf{D}}_2$ symmetry is quasiperiodic.



and in Fix($\widetilde{\mathbb{Z}}_5$),

$$\begin{aligned} \alpha_1 &= A + 25C + 9D \\ \alpha_2 &= A + 4D \\ \beta &= A - 25C - 6D. \end{aligned} \tag{31}$$

In either case there are two 'pure mode' travelling wave solutions (the maximal solution branches) and branches of 'mixed mode' solutions (submaximal solutions) which exist for some values of the coefficients α_1, α_2 and β. The pure mode solutions correspond to $w_1 = 0$ and $w_2 = 0$. These solution branches undergo Hopf bifurcations when $\beta_r = (\alpha_2)_r$ and $\beta_r = (\alpha_1)_r$ respectively. At these bifurcations it is possible for a quasiperiodic branch of mixed mode solutions with submaximal symmetry to be created.

Remark 6.1. In Fix($\widetilde{\mathbb{Z}}_3^1$) the travelling wave solution with $w_1 = 0$ has $\widetilde{\mathbf{SO(2)}}_1$ symmetry and when $w_2 = 0$ the corresponding solution has $\widetilde{\mathbf{SO(2)}}_2$ symmetry. The branch of mixed mode solutions (where it exists) has $\widetilde{\mathbb{Z}}_3^1$ symmetry.

Similarly in Fix($\widetilde{\mathbb{Z}}_5$) the travelling wave solution with $w_1 = 0$ has $\widetilde{\mathbf{SO(2)}}_2$ symmetry and when $w_2 = 0$ the corresponding solution has $\widetilde{\mathbf{SO(2)}}_3$ symmetry. The quasiperiodic branch of mixed mode solutions (where it exists) has $\widetilde{\mathbb{Z}}_5$ symmetry.

It is possible for the quasiperiodic submaximal solutions to be stable within $\mathrm{Fix}(\Sigma)$ for $\Sigma = \widetilde{\mathbb{Z}}_3^1$ or $\widetilde{\mathbb{Z}}_5$. For example, suppose that the pure mode solutions bifurcate supercritically at the Hopf bifurcation with $\mathbf{O(3)}$ symmetry. Then $(\alpha_1)_r < 0$ and $(\alpha_2)_r < 0$. Suppose further that $(\alpha_2)_r < (\alpha_1)_r$. By letting $w_1 = Re^{i\phi}$ and $w_2 = Se^{i\psi}$ and separating the phase and amplitude equations we find that the 'mixed mode' solution is given by

$$R^2 = \frac{\lambda((\alpha_2)_r - \beta_r)}{\beta_r^2 - (\alpha_1)_r(\alpha_2)_r} \qquad\qquad S^2 = \frac{\lambda((\alpha_1)_r - \beta_r)}{\beta_r^2 - (\alpha_1)_r(\alpha_2)_r}$$

and exists when $R^2 > 0$ and $S^2 > 0$. It can be found that when

$$(\alpha_1)_r < \beta_r \qquad \text{and} \qquad \beta_r^2 < (\alpha_1)_r(\alpha_2)_r$$

the quasiperiodic mixed mode solution exists and is stable within $\mathrm{Fix}(\Sigma)$.

7 Conclusions

In this paper we have made changes to the previously published list of isotropy subgroups of $\mathbf{O(3)} \times S^1$. In section 3 we presented an amended table of these isotropy subgroup, Table 1, and justified the changes to Table 5.1 of [12, Chapter XVIII §5] that this represents. The most important changes relate to twisted subgroups H^α where H is a dihedral subgroup of $\mathbf{O(3)}$. We have also computed the isotropy subgroups with four-dimensional fixed-point subspaces in each representation of $\mathbf{O(3)} \times S^1$ on $V_\ell \oplus V_\ell$.

In section 4 a study of the specific example of the Hopf bifurcation where the representation of $\mathbf{O(3)} \times S^1$ is the natural representation on $V_3 \oplus V_3$ revealed six branches of periodic solutions whose existence is guaranteed by the equivariant Hopf Theorem. Of these solution branches three are standing waves and the other three are travelling waves. In contrast to the previously studied case for the representation on $V_2 \oplus V_2$ [14, 16], we found in section 5 that generically the cubic order terms in the Birkhoff normal form of the equivariant differential equations are sufficient to determine fully the stability of each of the six solution branches.

In section 6 we studied the dynamics of the equivariant differential equations (20) in the restriction to each of the invariant subspaces given by $\mathrm{Fix}(\Sigma)$ where Σ is an isotropy subgroup in Table 4 with $\dim \mathrm{Fix}(\Sigma) = 4$. We were able to determine that, depending on the values of the coefficients A, B, C and D in the equivariant differential equations, it is possible for solutions with submaximal symmetry to exist for $\Sigma = \widetilde{\mathbf{D}}_3$, $\widetilde{\mathbf{D}}_2$, $\widetilde{\mathbb{Z}}_3^1$ and $\widetilde{\mathbb{Z}}_5$. We found both periodic and quasiperiodic solutions and we have also shown that it is possible for some of the quasiperiodic solutions to be stable.

The final sections of this paper are concerned with the case where $\ell = 3$. It would be interesting to investigate the dynamics which are possible in general in other representations however, the computations required for the analysis of the stability of solution branches and existence of submaximal solutions use the Taylor expansion of the equivariant vector field. This can only be computed by choosing a particular value of ℓ. Values of ℓ greater than 3 could be considered, but as the number of dimensions increases, so does the number of coefficients in the equivariant vector field. This makes computations analogous to those in the final sections of this paper increasingly impractical with increasing ℓ.

Acknowledgements The author wishes to thank the referees of a previous version of this article whose helpful comments have greatly improved its clarity and readability. The author also wishes to thank Paul Matthews and Stephen Cox for their guidance and support throughout this work. This work formed part of the authors PhD thesis which was funded by the EPSRC.

References

[1] Antoneli F, Dias A P S and Matthews P C 2008 Invariants, equivariants and characters in symmetric bifurcation theory *Proc. Roy. Soc. Edinburgh* A **138** 477–512

[2] Balanov Z, Krawcewicz W and Steinlein H 2006 *Applied Equivariant Degree* AIMS Series on Differential Equations and Dynamical Systems **1**

[3] Busse F H 1975 Patterns of convection in spherical shells *J. Fluid Mech.* **72** 67–85

[4] Busse F H and Riahi N 1982 Patterns of convection in spherical shells, Part 2 *J. Fluid Mech.* **123** 283–301

[5] Byrne H M and Matthews P C 2002 Asymmetric growth of models of avascular solid tumours: exploiting symmetries *IMA J. Math. Appl. Med. Biol.* **19** 1–29

[6] Cross M C and Hohenberg P C 1993 Pattern formation outside of equilibrium *Rev. Mod. Phys.* **65** 851–1112

[7] Dolnik M, Rovinsky A B, Zhabotinsky A M and Epstein I R 1999 Standing waves in a two-dimensional reaction–diffusion model with the short-wave instability *J. Phys. Chem. A* **103** 38–45

[8] Dolnik M, Zhabotinsky A M, Rovinsky A B and Epstein I R 2000 Spatio-temporal patterns in a reaction-diffusion system with wave instability *Chem. Eng. Sci.* **55** 223–31

[9] Fiedler B 1988 *Global bifurcation of periodic solutions with symmetry* Lecture Notes in Math. **1309** (Berlin: Springer)

[10] Golubitsky M and Stewart I 1985 Hopf bifurcation in the presence of symmetry *Arch. Ration. Mech. Anal.* **87** 107-65

[11] Golubitsky M and Roberts M 1987 A classification of degenerate Hopf bifurcations with $\mathbf{O(2)}$ symmetry *J. Diff. Eqns.* **69** 216–64

[12] Golubitsky M, Stewart I and Schaeffer D 1988 *Singularities and Groups in Bifurcation Theory* vol II Appl. Math. Sci. **69** (New York: Springer)

[13] Golubitsky M and Stewart I 1993 An algebraic criterion for symmetric Hopf bifurcation *Proc. R. Soc. London* A **440** 727-32

[14] Haaf H, Roberts M and Stewart I 1993 A Hopf bifurcation with spherical symmetry *Z. Angew. Math. Phys.* **43** 793-826

[15] Ihrig E and Golubitsky M 1984 Pattern selection with $\mathbf{O(3)}$ symmetry *Physica D* **13** 1–13

[16] Iooss G and Rossi M 1989 Hopf bifurcation in the presence of spherical symmetry: Analytical results *J. Math. Anal.* **20** 511–32

[17] Ize J and Vignoli A 2003 *Equivariant Degree Theory* De Gruyter Series in Nonlinear Analysis and Applications **8** (Walter de Gruyter & Co.)

[18] Knightly G H and Sather D 1980 Buckled states of a spherical shell under uniform external pressure *Arch. Ration. Mech. Anal.* **72** 315–80

[19] Knobloch E and Proctor M R E 1981 Nonlinear periodic convection in double–diffusive systems *J. Fluid Mech.* **108** 291–316

[20] Knobloch E 1986 Oscillatory convection in binary mixtures *Phys. Rev.* A **34** 1538–49

[21] Knobloch E and Silber M 1993 Oscillatory convection in a rotating layer *Physica D* **63** 213–32

[22] Knobloch E 1998 Rotating convection: Recent developments *Int. J. Eng. Sci.* **36** 1421–50

[23] Michel L 1980 Symmetry defects and broken symmetry *Rev. Mod. Phys.* **52** 617–51

[24] Montaldi J, Roberts M and Stewart I 1988 Periodic solutions near equilibria of symmetric Hamiltonian systems *Phil. Trans. R. Soc. Lond.* A **325** 237–93

[25] Schubert G, Glatzheimer G A and Travis B 1993 Steady, three-dimensional, internally heated convection *Phys. Fluids* **5** 1928–31

[26] Sigrist R 2010 Bifurcations with spherical symmetry Ph.D. thesis, University of Nottingham

[27] Turing A M 1952 The chemical basis of morphogenesis *Phil. Trans. R. Soc.* B **237** 37–72

[28] Varea C, Aragon J L and Barrio R A 1999 Turing patterns on a sphere *Phys. Rev.* E **80** 4588–92

[29] Zhabotinsky A M, Dolnik M and Epstein I R 1995 Pattern formation arising from wave instability in a simple reaction–diffusion system *J. Chem. Phys.* **103** 10306–14