Optimal FIR and IIR Hilbert Transformer Design Via LS and Minimax Fitting

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Abstract—**The usual way of the implementation of on-line discrete Hilbert transformers is the design of linear phase finite impulse response (FIR) filters. Recently, a method has been published for the design of infinite impulse response (IIR) Hilbert transformers as well.**

The paper introduces a new method for the design of both FIR and IIR Hilbert transformers, based on a parameter estimation method for linear systems. The first approximation is performed in least squares (LS) sense in the complex domain. An iterative extension of the algorithm is also presented. It results in an approximation in minimax (Chebyshev) sense, and is also in the complex domain.

Keywords—**Hilbert transformer, least squares fitting, Chebyshev approximation, minimax criterion, approximation in the complex domain, digital filter design.**

I. Introduction

THE HILBERT transform is a very useful tool in modulation and demodulation. It has applications not only in communications, but also in the measurement of frequency deviations of rotating machines, in the investigation of the impulse response of systems, characterization of acoustical devices, etc. At present, it can be realized by digital means at a reasonable speed; thus design methods for digital Hilbert transformers are of increasing interest [1]–[5].

On-line discrete Hilbert transformers are usually designed in the form of linear phase finite impulse response (FIR) filters [2], [6]. The standard method (the usage of the Remèz algorithm) is well elaborated. There are attempts to design infinite impulse response (IIR) realizations, too [3]. The method in [3] is based on special techniques as transforms of normal or generalized halfband filters.

Recently, Chen and Parks [7] have shown that the performance of FIR filters can be improved by accomplishing the fit in the complex domain, and by allowing small deviations from the linear phase (nearly linear phase filters). This leads to the idea of designing both FIR and IIR Hilbert transformers by fitting the filter to the complex frequency response.

In this paper, a new approach is described. Making use of the maximum likelihood method originally developed for the identification of linear systems from complex frequency-domain input and output data (Estimation of Linear Systems (ELIS), [8], [9]), a design procedure of digital Hilbert transformers is presented. This method provides a least squares approximation with rather limited computational need. After the discussion of the properties of the obtained filters, a technique of fitting in an approximately Chebyshev sense is also presented, making use of the fact that this can be done via a weighted least squares (LS) fitting using appropriate weights.

The method to be described in the paper is not only computationally efficient but allows the selection of 'care'-bands and the use of other frequency band constraints as well. Digital filters other than Hilbert transformers can also be designed by following a similar procedure.

II. Aspects of Digital Hilbert Transformer Design

The design of a (two-sided) digital Hilbert transformer means that we try to approximate the following transfer function:

$$H(f) = \begin{cases} -j & \text{if } 0 < f < f_s/2 \\ j & \text{if } f_s/2 < f < f_s \end{cases} \tag{1}$$

where f_s is the sampling frequency, and $j = \sqrt{-1}$.

Because of the discontinuity at 0 and $f_s/2$, a digital filter can approximate this well only in a given band. Fortunately, in most applications this can be tolerated.

When preparing for the design, we have choices listed as follows.

a) *Filter type.* The filter can be either an FIR or an IIR. The advantage of the FIR filter is that it is always stable (it has no poles), the implementation is straightforward, and with the Remèz algorithm there is an effective algorithm for the design. The IIR filter may often provide the

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same performance with a significantly lower order, however, there is no standard method to assure stability (e.g., the first two examples of [3] are definitely unstable), and the realization is to be made with care (underflows and overflows, etc.).

b) *Measure of goodness of fit.* The approximation of the desired transfer function can be done using one of the minimax (Chebyshev), least squares, least absolute values (etc.) criteria (l_∞, l_2, l_1 norms). Intuitively, the minimax criterion seems to be desirable: using this, the maximum of the error can be kept at a low level. On the other hand, one can argue for the other norms: if the cost function is small enough, the fit will be almost perfect.

Another aspect is the implementation of the minimization procedure. For the minimax fit there is the effective Remez algorithm for linear phase FIR filters, and there is also an approximate method in the complex domain [7]. The LS criterion can often be handled analytically [2], and there are already methods to handle the other criteria [10].

c) *Handling of the "non-Hilbert" bands.* For modulation or demodulation purposes, the Hilbert transform is usually only needed in a certain frequency band. At the same time, there may be other prescriptions in the other frequency intervals: the transfer function should not exceed a certain magnitude.

In some of the applications, the desired Hilbert transformer is not two-sided as defined above, but one-sided (the prescribed frequency shift is provided somewhere in the $(0, f_s/2)$ interval only). This leads to complex signals and complex filter coefficients.

d) *Additional restrictions originating from the design method.* Some of the design methods contain inherent constraints. The Reméz algorithm only allows linear phase, that is, no phase error is allowed. The method presented in [3] only uses all-pass sections, that is, the magnitude function equals exactly one.

These different aspects imply that probably no uniform solution exists for every case. Thus it is reasonable to look for alternative design methods.

III. Estimation of Linear Systems—ELiS

Recently, a method for the identification of linear systems has been published [8], [9]. Starting from the noisy complex input and output amplitudes measured at different frequencies in a linear system, the maximum likelihood estimate of the linear transfer function is determined. The transfer function may be in terms of s or z.

The basic z-domain model is shown in Fig. 1. The input signal X has complex amplitudes X_k at angular frequencies ω_k; the output signal is Y. Both the measured complex input and output amplitudes are corrupted by noise (a_x and a_y, with variances σ_{xk}^2 and σ_{yk}^2 at the corresponding frequencies). The noise is assumed to be independent and Gaussian, which means that the result is nothing more than a weighted *least squares* estimate. The transfer function

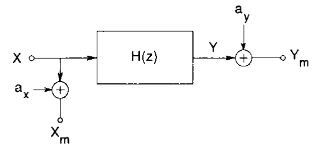

Fig. 1. The z-domain model used in ELiS.

is sought in the following form:

$$H_{\text{est}}(z) = z^{\delta} \cdot \frac{N(z^{-1})}{D(z^{-1})} \tag{2}$$

where $\delta \geq 0$ is a delay (the desired function will be realized in the form $z^{-\delta}H(z)$, that is, we accept this delay in order to obtain a realizable approximation), and $N(z^{-1})$ and $D(z^{-1})$ are the unknown numerator and denominator polynomials of the transfer function. In other words, instead of $H(z)$ we will approximate $z^{-\delta}H(z)$ by $N(z^{-1})/D(z^{-1})$.

The algorithm numerically minimizes the following cost function via the coefficients of the numerator and the denominator of the transfer function of the system:

$$K = \sum_{k=1}^{N} \frac{\left|z^d N(z_k^{-1}) X_{mk} - D(z_k^{-1}) Y_{mk}\right|^2}{\sigma_{xk}^2 \left|N(z_k^{-1})\right|^2 + \sigma_{yk}^2 \left|D(z_k^{-1})\right|^2} \tag{3}$$

where X_{mk} and Y_{mk} are the measured complex input and output amplitudes, respectively, at the frequency points ω_k; σ_{xk}^2 and σ_{yk}^2 are their variances, and

$$z_k = e^{j\omega_k T_s} \tag{4}$$

with T_s being the sampling interval.

If the noise is small, the least squares estimation can be considered in the following way: the transfer function to be determined is $H(f)$ which approximates in LS sense the values

$$Y_{mk}/X_{mk}, \qquad k = 1, \cdots, N. \tag{5}$$

Now, if we impose $X_{mk} \equiv 1$, $\sigma_{xk}^2 \equiv 0$, $\sigma_{yk}^2 \equiv 1$, and Y_{mk} equals our desired transfer function, the minimization procedure will result in a least squares approximation. If $D(z^{-1}) \equiv 1$, the result is an FIR filter, otherwise it is an IIR filter. The delay is to be given in advance. It may be allowed to vary too, but on the one hand the rather complicated surface of the several dimensional cost function will usually lead the minimization procedure to local minima and, on the other hand, fractional delays are often not allowed. Moreover, a poor initial value of δ often results in unstable IIR filters. Thus an appropriate value of δ is to be given in any case, based on experience.

It is easy to see that the "not Hilbert" frequency bands can also be handled. By defining values equal to zero within these bands, and giving large values to the corresponding σ_{yk}^2 values (small weighting of the zero values),

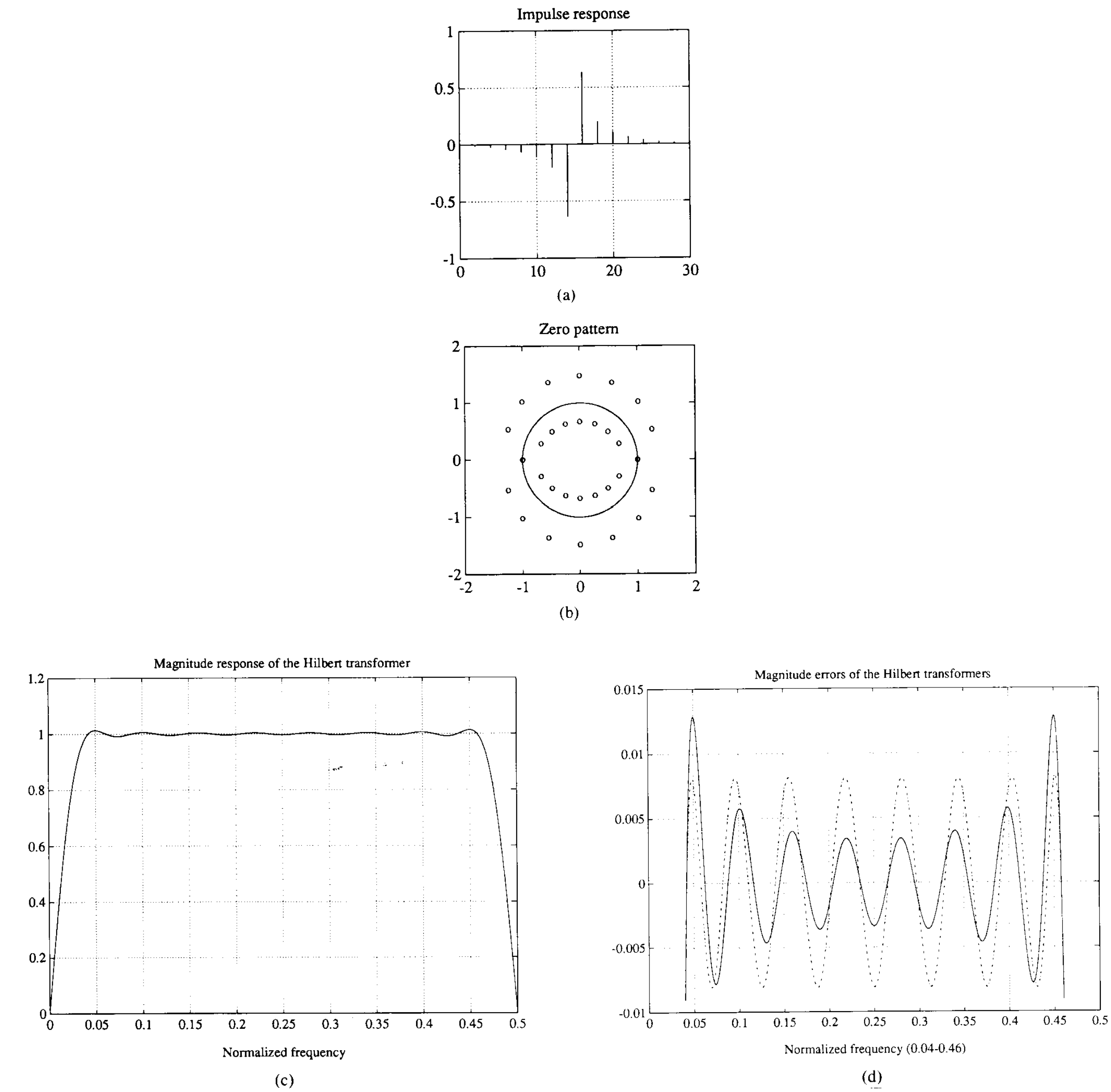

Fig. 2. The designed FIR Hilbert transformer. Order: 30, band: (0.04, 0.46), delay: 15. (a) Impulse response. (b) Zero pattern. (c) Magnitude response. (d) Error of the magnitude in the "care"-band. Solid line: the result obtained by ELiS, dash-dotted line: the result given by the Parks-McClellan algorithm.

the transfer function can be "pulled down" towards zero as much as is desired. This requirement may increase the order necessary to keep the in-band error under the specified level.

IV. Extension to the Minimax Case

The above procedure performs least squares approximation in the complex domain. However, it can be extended to the minimax problem, too. The idea is as follows: the LS approximation results in an error function, which is very small at some frequencies, and larger at others. Since we can weight the fitting by choosing the values σ_{yk}^2 appropriately, it is possible to use the errors of the LS fit for the determination of a new weighting set (dividing the variance values by the squared error of the previous fit), in order to "suppress" the peaks of the error. The error is to be analyzed carefully: since it usually oscillates, the value of the *envelope* can be used for the weighting. In the cases studied by the authors, one to two iteration steps were sufficient to get close to the minimax fit. The theoretical basis for reweighting (although with a different weight updating algorithm) can be found in [11] for the FIR case.

V. Filters Designed using ELiS

The preceding principles have been tested on examples. In this section, some of the results are summarized.

A. *FIR Filter Design, LS Fit*

A FIR Hilbert transformer of order 30 was designed (Example 1 of [2]). The results, presented in Fig. 2, are practically the same as those in [2]. This is no wonder, the eigenfilter approach is nothing more than a least squares method. It can be observed that the error is small in the middle, but increases towards the edges. Here it surpasses the maximum error of the Parks–McClellan algorithm, as it has to, since the latter is the solution of the minimax problem. Thus the remark in [2] that their error is much smaller than that of the Parks–McClellan algorithm is somewhat misleading. The error is indeed generally smaller, but at the edges it is larger than that obtained using the Reméz algorithm.

It is also of interest to notice that the real part of the transfer function of the designed filter is negligible (it is in the order of magnitude of the roundoff errors, shown in Table I), though there was no linear phase constraint. The cause is most probably the fact that since the desired transfer function is purely imaginary, the approximating function has the same property (the impulse response is exactly odd; see [7] for a similar statement). The real part is not negligible any more if the chosen delay value is different from half of the order (15 in this case), but the fitting error is larger so this case is of no practical interest.

B. *FIR Filter Design, Minimax Fit*

Starting from the above results, an iteration step using weighted LS was performed, as shown in Fig. 3. The fit is already very close to the ideal minimax design (Parks–McClellan algorithm).

C. *IIR Filter Design, LS Fit*

A systematic search was necessary in order to find the minimum IIR order and the appropriate delay which provides the same (or even smaller) error as the above FIR filters. For a given order, the value of the delay was found by starting from the value equal to the order and it was decreased gradually until the last value was reached, for which the designed filter was still stable. Then the delay value was chosen for which the error was minimum. When the error was smaller than that desired, the order was decreased. When it was larger, the order was increased and the optimum delay was sought again.

Finally, the order was chosen as 12/12 and the delay

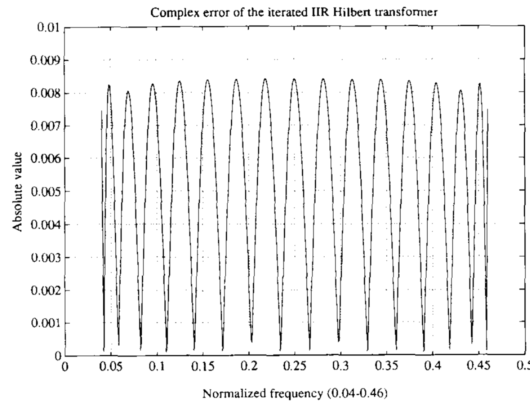

Fig. 3. The result of the first step of the iteration towards the minimax fit.

TABLE I
Coefficients of the FIR Filter, LS Fit, Order 30

b(0) = −0.006057733301139371,	b(30) = +0.006057733301140023
b(1) = +0.000000000000000149,	b(29) = −0.000000000000000262
b(2) = −0.013039367531238520,	b(28) = +0.013039367531238620
b(3) = +0.000000000000000095,	b(27) = −0.000000000000000194
b(4) = −0.023832517737319509,	b(26) = +0.023832517737320359
b(5) = −0.000000000000000062,	b(25) = −0.000000000000000222
b(6) = −0.040114741220746993,	b(24) = +0.040114741220747749
b(7) = −0.000000000000000323,	b(23) = −0.000000000000000634
b(8) = −0.065247413138588742,	b(22) = +0.065247413138590935
b(9) = −0.000000000000000770,	b(21) = −0.000000000000000830
b(10) = −0.107888520426881795,	b(20) = +0.107888520426883905
b(11) = −0.000000000000000464,	b(19) = −0.000000000000000403
b(12) = −0.200092089168674991,	b(18) = +0.200092089168676407
b(13) = −0.000000000000000189,	b(17) = −0.000000000000000439
b(14) = −0.632503847986503276,	b(16) = +0.632503847986504275
b(15) = −0.000000000000000601	

as 11 (the filter became unstable for $\delta < 9$). As shown in Fig. 4, the error is somewhat smaller than that of the above FIR solution. The poles and zeros are easy to implement.

It is remarkable that the amplitude response is practically flat (the magnitude error is less than 1.2 dB (1 mW), that is, the filter is similar to an all-pass filter.

Fig. 4(c)–(d) show the magnitude error, the phase error and the complex error in the "care"-band.

D. *IIR Filter Design, Minimax Fit*

Two steps of the iteration were performed until the error of the fit became practically equiripple, as shown in Fig. 5. Unfortunately, there is no method in the literature which would provide the minimax solution for the IIR case. Thus we can only assume that we obtained at least a suboptimal minimax approximation.

VI. Conclusions

A novel method for the design of digital Hilbert transformers using the least squares and the minimax criterion has been presented. For FIR filters the results are the same

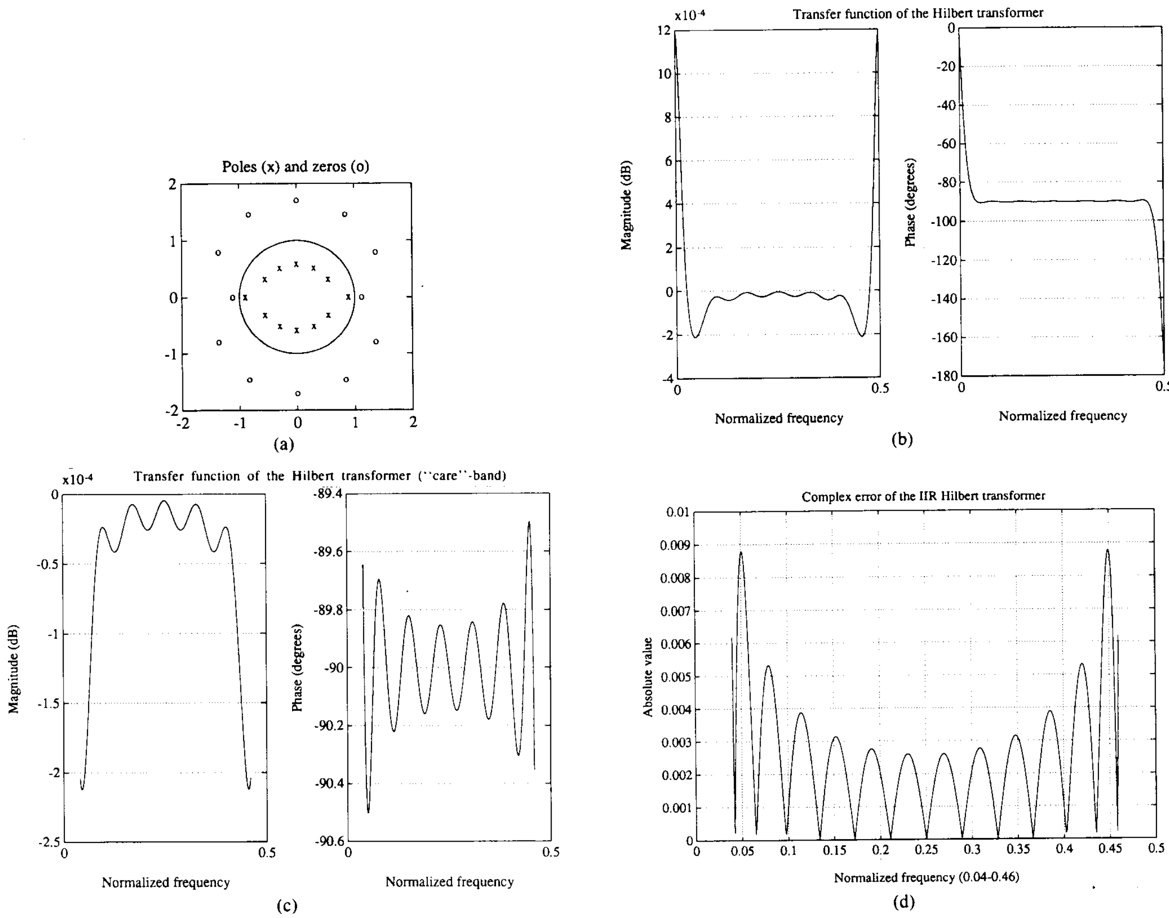

Fig. 4. The designed IIR Hilbert transformer (LS fit). Order: 12/12, delay: 11, band: (0.04, 0.46). (a) Pole-zero pattern:

Zeros (in terms of z)	Poles (in terms of z)
$0.834946 \pm 1.459074j$	± 0.894071
$-0.834946 \pm 1.459074j$	$0.547048 \pm 0.319837j$
$\pm 1.707483j$	$-0.547048 \pm 0.319837j$
$1.362397 \pm 0.796550j$	$.295469 \pm 0.516331j$
$-1.362397 \pm 0.796550j$	$-0.295469 \pm 0.516331j$
± 1.118507	$\pm 0.585696j$

Gain at $z = 0$: -0.00553405. (b) Amplitude and phase response in (0, 0.5). (c) Amplitude and phase response in the 'care'-band (0.04, 0.46). (d) Absolute value of the complex error in the 'care'-band (0.04, 0.46)

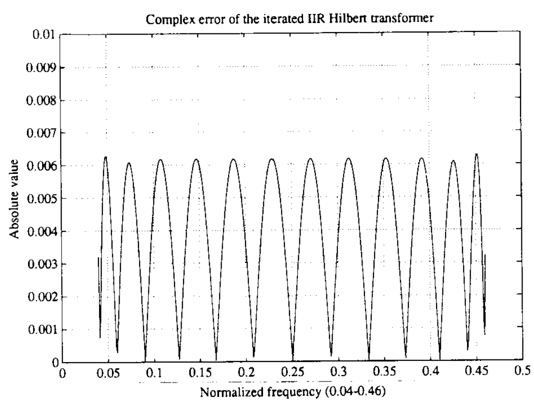

Fig. 5. The result of the second step of the iteration of the IIR filter towards the minimax fit.

as those of the optimal methods known from the literature. For the same task, stable IIR filters were also successfully designed. The complexity of the IIR filters is similar to those of the linear phase FIR filters. The type which is preferred may depend on the implementation.

The procedures proposed here may also be usable for the design of digital filters other than Hilbert transformers since the desired frequency response is given point by point.

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