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Normal form for Hopf bifurcation of partial differential equations on the square

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Abstract. We derive and analyse a normal form governing dynamics of Hopf bifurcations of partial differential evolution equations on a square domain. We assume that the differential operator for the linearized problem decomposes into two one-dimensional self-adjoint operators and a local 'reaction' operator; this gives a basis of i.e. of the form $u(x_1, x_2) = f_1(x_1)f_2(x_2)$. The *normal form reduces to that investigated by* Swift [23] for bifurcation of modes with *odd* parity but is new for modes with *even* parity where the centre eigenspace carries a *reducible* action of $\mathbf{D}_4 \times \mathbf{S}^1$. We consider the Brusselator equations as an example and discover that a separable linearization introduces a degeneracy which causes the three new third order terms in the normal form to be related in an unexpected but simple way.

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1. Introduction

Mathematical models of time-evolving physical systems are often parabolic partial differential equations of the form

$$\frac{\partial u}{\partial t} = Au + g(u) \tag{1.1}$$

where u is in some suitable function space, A is a linear elliptic operator, and g is a (k-times differentiable) nonlinear operator with zero linear part. This equation is assumed to have a quiescent 'trivial' solution with u independent of space and time, say $u = 0$. In order to investigate the stability of this trivial state, we need to know the location of the eigenvalues of A on the complex plane; if they all have negative real part, the trivial state is stable to small perturbations. If the equation *is continuously dependent on a parameter* $\lambda \in \mathbf{R}$ *then* the trivial solution may lose stability if, at say $\lambda = 0$, one or more eigenvalues of A have real part equal to zero. This is a bifurcation of the trivial state, and typically the nonlinear terms g mean that there will be new steady or time periodic solutions with small amplitude at nearby values of $\lambda \approx 0$. *For the case where a complex pair of eigenvalues pass through* the imaginary axis at $\pm i\omega$ with $\omega > 0$, there will typically be branches of time-periodic spatially inhomogeneous solutions near the bifurcation; a *Hopf bifurcation* [18].

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For systems in 'general position' we should expect that there will only be one complex pair of eigenvalues passing through $\pm i\omega$ at $\lambda = 0$ (perturbing the linear part A will typically split any multiple eigenvalues). Thus, we can use the *centre manifold* approach to reduce the time-asymptotic dynamics to be conjugate on an ordinary differential equation on a finite-dimensional submanifold of the original function space, which has the dimension of and is tangent to the centre eigenspace of A at $\lambda = 0$. Thus, Hopf bifurcation typically involves looking at dynamics on a manifold with two real dimensions.

If the system has extra symmetries, the situation is changed; we consider the case of partial differential equations on square domains such that the boundary conditions and equations reflect this symmetry. The theory of Golubitsky *et al* [12, 14] says that if the system has a group of symmetries Γ, then the dimension of the centre eigenspace at Hopf bifurcation must be either double the dimension of a real irreducible representation of Γ or it is the dimension of a complex irreducible representation of Γ. The consequence of this is that for systems with the symmetry $\mathbf{D}_4$ of transformations of a square to itself (the dihedral group with eight elements), the only admissible dimensions of a centre eigenspace are two and four. The two dimensional case is essentially the same as that for Hopf bifurcation without symmetry, but the four dimensional case gives a larger dimensional space, and so there can be many different types of periodic solution; even quasiperiodic solutions and chaos bifurcating from the trivial solutions at Hopf bifurcation. This is discussed in detail by Swift [23].

However, this is not the end of the story. As has been found in many different contexts, notably in the context of steady-state bifurcation, often partial differential equation have centre eigenspaces which are *too big* from the 'generic-with-symmetry' point of view. For example, Crawford has found [5] that firstly, centre eigenspaces at bifurcation can have dimensions that are simply too big and secondly, they can have an admissible dimension, but do not carry an irreducible representation of the symmetry group. The resolution of this problem is to look at genericity within a suitable class of problems; we can then get reducible representations on the centre eigenspace [9]. This was noticed by Shaw [22] in the context of solutions of Schrödinger's equation for a square-well potential. The centre eigenspace may even be of arbitrarily large dimension due to the effects of rotational symmetry in the extension of the problem to $\mathbf{R}^2$ [1, 17, 5].

The particular class of problems we consider here are Hopf bifurcations for systems on square domains whose centre eigenspace is spanned by a basis of *separable* eigenfunctions; roughly speaking, they have the property that they are products of odd or even functions of x_1 and x_2 only. This includes classes of problems that are periodically extendible, or even extendible problems on the plane with Euclidean symmetry. The research of Crawford *et al* [5, 9] and Gomes *et al* [8, 15] has investigated the normal forms of problems where the representation of the group on the centre eigenspace takes into account the presence of hidden symmetries of the problem. They find that at low order, the bifurcation equations have extra symmetry which is only broken to the generic symmetry at an order depending on the mode numbers at bifurcation.

The problems we consider do not typically have hidden translation symmetry. Nevertheless, we find centre eigenspaces that are not irreducible for $\mathbf{D}_4$ symmetry and therefore not generic in the usual sense of equivariant bifurcation theory. The reason for this lies in the special structure of the linear part of the problem. At bifurcation we take explicit account of this structure and obtain a normal form on an eigenspace carrying a reducible representation of the symmetry group. More precisely, bifurcations of modes with odd parity (think of $\sin kx \sin ly$ with $k + l$ odd) give rise to a complex irreducible action of $\mathbf{D}_4 \times \mathbf{S}^1$ on the centre eigenspace (typically isomorphic to $\mathbf{C}^2$) whereas those with even parity ($k+l$ even) give rise to a reducible action, as noted by Shaw [22]. Under the assumption that there are no further linear degeneracies we find a normal form for bifurcation problems where there is a reducible action of $\mathbf{D}_4$ on the centre eigenspace caused by even parity of the bifurcating modes. We concentrate on the problem for Hopf bifurcation, which after reduction to normal form gains an extra $\mathbf{S}^1$ symmetry to arbitrary order [14].

Our work is a generalisation of the normal form of Swift [23] for Hopf bifurcation with the generic action of $\mathbf{D}_4 \times \mathbf{S}^1$ on $\mathbf{C}^2$ which is in turn a normal form derived by Golubitsky and Stewart [13]. We derive a Birkhoff normal form and analyse its generic (codimension one) bifurcation behaviour as a function of the third order coefficients. Although we cannot locate the periodic solutions analytically, we give an algorithm to find the number of branches and their symmetries.

The problems we consider include a large class of physically interesting equations, for example those with translational symmetry of the equation in the x_1 and x_2 directions when extended to the infinite plane, with the domain being a square $[0, \pi]^2$ and Neumann, Dirichlet or Robin boundary conditions imposed on the boundaries. For the case of Euclidean symmetry of the equations, a result of Melbourne [1, 6, 7] gives complex exponential spatial dependence of the eigenfunctions. For such problems, we show that the natural action of $\mathbf{D}_4$ can be reducible on the centre eigenspace, the tangent space to the centre manifold where local bifurcation takes place. For Hopf bifurcation, this action gives structure to the normal form that we derive. The normal form on $\mathbf{C}^2$ has equivariance under an action of $\mathbf{D}_4 \times \mathbf{S}^1$ irreducible for odd parity but reducible for even parity mode numbers.

This normal form has six complex coefficients at third order, reducing to that of Swift (with three complex coefficients) for the case of an irreducible action of $\mathbf{D}_4 \times \mathbf{S}^1$. We analyse the normal form to give the number and symmetry of the bifurcating solutions. The notation follows Swift [23] as far as possible, to allow easy comparison with that paper. We give a set of conditions on the normal form coefficients ensuring generic branching behaviour at Hopf bifurcation.

As an example, we use the normal form analysis to interpret the bifurcation diagram of the Brusselator equations with mixed (Robin) boundary conditions, reduced by a numerical/computer algebraic Liapunov–Schmidt method detailed in [4, 2]. Surprisingly, we find that the three new third order monomials in the reduced bifurcation equation appear only in a very simple combination. We show that this is caused by the fact that there is a basis of separable eigenfunctions for the linearized problem.

2. Reaction–diffusion equations and domain symmetries

Consider the following parameterised semi-linear (e.g. reaction-diffusion) partial differential equation

$$\frac{\partial u}{\partial t} = A(\lambda)u + g(u, \lambda) \tag{2.1a}$$

for $A(\lambda) \in \mathcal{L}(X, Z)$, $g \in C^\infty(X \times \mathbf{R}, Y)$ with continuously embedded Banach spaces $X \hookrightarrow Y \hookrightarrow Z$ and $\lambda \in \mathbf{R}$ as a bifurcation parameter. Moreover, we assume $g(0, \lambda) = 0$ and $Dg(0, \lambda) = 0$ and shall restrict our discussion to the domain being a square $\Omega = [0, \pi]^2$ and X, Y and Z will usually be spaces of functions on Ω taking values in $\mathbf{R}^m$. For reaction–diffusion equations we can decompose the linear operator A into a spatial differential operator and a reaction operator:

$$A = L_d + L_r$$

Moreover, L_d is often self-adjoint and L_r acts on functions $Uf(x)$ with $f \in C(\Omega, \mathbf{R})$ and $U \in \mathbf{R}^m$ simply as a matrix operation on U.

Definition 1. *A self-adjoint linear operator L_d is separable if it can be written as the sum of two operators:*

$$L_d = L_1 + L_2$$

such that the L_1 and L_2 have identity action in the x_2, resp. x_1 directions (and are therefore self-adjoint as one-dimensional operators).

Because the operator L_d commutes with the symmetry $\mu(x_1, x_2) \mapsto (x_2, x_1)$ of the square, we have $L_1\mu = \mu L_2$ while the self-adjointness means that the eigenfunctions of L_1 form a basis for $C([0, \pi], \mathbf{R}^m)$. If this eigenbasis is given by

$$\{f_l(x_1), \lambda_l\}$$

with λ_l real, then L_d has an eigenbasis for Z given by

$$\{f_k(x_1)f_l(x_2), \lambda_k + \lambda_l\}.$$

Note also that the square symmetry and the $\mathbf{Z}_2$ symmetry $u(x) \mapsto -u(x)$ of the linear operator means that the $f_k(x_i)$ will be either odd or even.

Definition 2. *A continuous function $u : \Omega = [0, \pi]^2 \to \mathbf{R}^m$ is separable if it can be written in the form*

$$u(x_1, x_2) = Uf(x_1)g(x_2) \quad \textit{for all} \quad x := (x_1, x_2) \in \Omega$$

for a constant vector $U \in \mathbf{R}^m$ and two continuous function f, $g : [0, \pi] \to \mathbf{R}$ with $f(\pi - x) = \pm f(x)$ and $g(\pi - x) = \pm g(x)$.

We remark that the square domain does not force symmetries of the form $f(\pi - x) = -f(x)$ for the full nonlinear equations. However, this is the case for solutions of the linearized equations at a trivial solution.

Definition 3. *Consider $f : [0, \pi] \to \mathbf{R}$. If there is a $p \in \{0, 1\}$ such that*

$$f(\pi - x_1) = (-1)^p f(x_1),$$

we say that f has parity p. For a separable function $u(x) = f(x_1)g(x_2)$ we define its parity to be the sum of the parities of f and g. Note that the parity is defined modulo two and so we refer to parity as being even or odd.

Linear parts of problems (2.1a) with homogeneous boundary conditions are typically separable. Consider X_Ω, Y_Ω and Z_Ω to be spaces of functions $u : \Omega \to \mathbf{R}^m$ with boundary conditions on X of the form

$$au_i(x) + b\frac{\partial u_i}{\partial n}(x) = 0 \quad \text{for all} \quad x \in \partial\Omega, \quad i \in \{1, \ldots, d\}. \tag{2.1b}$$

Here $\frac{\partial}{\partial n}$ is the outward normal derivative on the boundary. The constants $a \geqslant 0$ and $b \in \mathbf{R}$ satisfy $|a| + |b| > 0$ and are so-called Robin (or Cauchy) boundary conditions on $\partial\Omega$. To ensure the problem is well-posed, it is necessary that $d = m$ for a second order differential operator A. The cases $a = 0$ and $b = 0$ correspond to Neumann and Dirichlet boundary conditions respectively. It is possible to generalise the boundary conditions to having a and

b dependent on i, but this requires restriction of the bifurcating modes and introduces more complexity without illuminating the discussion here.

For steady state or Hopf bifurcation at $\lambda = 0$ we assume that $A(0)$ has an eigenvalue with zero real part and an eigenspace of finite dimension. For steady state bifurcation there are eigenvalues (which may be multiple) of $A(\lambda)$ acrossing the imaginary axis at $\lambda = 0$ through zero only. For Hopf bifurcation, we assume there are eigenvalues of $A(\lambda)$ which across the imaginary axis when $\lambda = 0$ at $\pm i\omega$ with non-zero rates. These eigenvalues must be isolated. Assuming A to be sectorial (see Henry [16]), extra hypotheses on g can ensure local existence of time-dependent solutions. Another approach is that of Vanderbauwhede and Iooss [24] who work with Banach spaces of functions $\mathbf{R}^+ \to X, Y, Z$ exponentially bounded in time and prove existence and smoothness of a centre manifold under weaker assumptions than Henry by not considering the dynamics away from the centre manifold.

Instead of taking one or other of these approaches, we are concerned with the structure of the normal form on the generic centre eigenspaces. Because the centre manifold can be chosen to be invariant under any symmetry acting on the centre eigenspace, we examine the action there for generic nonlinear terms. For our example, we do not perform a centre manifold reduction but instead use the Liapunov–Schmidt method to gain information about the branching of periodic solutions. Thus, although we are motivated by problems of the form (2), we only need discuss the functional analytic setting of the linear problem.

2.1. *Euclidean symmetries*

For many physically interesting problems, especially reaction–diffusion problems, the governing equation can be extended from Ω to the whole plane and becomes equivariant under the Euclidean group on the plane, i.e. the group $\mathcal{E}(2)$ of transformations on $\mathbf{R}^2$ generated by

$$\rho_\psi \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := \begin{pmatrix} \cos\psi & -\sin\psi \\ \sin\psi & \cos\psi \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\mu \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := \begin{pmatrix} x_2 \\ x_1 \end{pmatrix}$$

$$\tau_{(y_1,y_2)} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} := \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \end{pmatrix}.$$

Here the ρ are rotations about the origin, μ is reflection in the line $x_1 = x_2$ and the τ are translations. This induces an action of $\mathbf{O}(2)$ on the tangent space (also $\mathbf{R}^2$) by the ρ and μ acting as above and the τ acting trivially. For $\gamma \in \mathcal{E}(2)$ we denote the former action by $\alpha(\gamma)$ and the latter by $\tilde{\alpha}(\gamma)$. As noted by Crawford [6, 7] we can write any element in $\gamma \in \mathcal{E}(2)$ as

$$\gamma = \tau_a \mu^k \rho_\psi.$$

These actions of $\mathcal{E}(2)$ on $\mathbf{R}^2$ induce a natural action on functions $u \in X_{\mathbf{R}^2}$ defined on the plane

$$\gamma \cdot u(x) := u(\alpha(\gamma^{-1})x).$$

The action on the operator $F : X \to Z$ defining the partial differential equation $\frac{\partial u}{\partial t} = F(u, \lambda)$ is given by

$$\gamma \cdot F(u(x), \lambda) := \tilde{\alpha}(\gamma) F(\gamma \cdot u(x), \lambda)$$

for all $\gamma \in \mathcal{E}(2)$. Turning back to the original equation (2), we shall usually be interested in $u \in X_\Omega \cap \{u|_\Omega \ : \ u \in X_{\mathbf{R}^2}\}$.

The action of $\mathbf{D}_4$ on the domain $\Omega = [0, \pi]^2$ is generated by

$$\rho : (x_1, x_2) \to (x_2, \pi - x_1)$$
$$\mu : (x_1, x_2) \to (x_2, x_1).$$

and we note that these group elements can be trivially extended to act on the plane $\mathbf{R}^2$ and can be seen as a subgroup of $\mathcal{E}(2)$.

A result of Crawford [6, 7] states that the centre eigenspace for linear problems with $\mathcal{E}(2)$ symmetry is spanned by exponentials of the form

$$u(x) = U e^{ik\cdot x} + c.c.$$

for $|k| = \kappa$, the critical wavenumber at bifurcation and U a constant vector. Here $c.c.$ represents the complex conjugate of the first term. Generically there is a unique critical wavenumber for one parameter problems. Imposing Robin boundary conditions (2.1b) leads to a basis for the centre eigenspace consisting of eigenfunctions

$$\sum_k a_k U_k \cos(l_{1,k}x_1 + \phi_{1,k}) \cos(l_{2,k}x_2 + \phi_{2,k}) \tag{2.2}$$

with $U_k \in \mathbf{R}^m$ constant and $\|U_k\| = 1$, $a_k \in \mathbf{R}$ and $l_{1,k}^2 + l_{2,k}^2 = \kappa^2$. The constants $l_{i,k}$ link the mode numbers of the Neumann and Dirichlet problems whereas $\phi_{i,k}$ are phase shifts to ensure the Robin boundary conditions (2.1b) are satisfied, more precisely, $l_{i,k}$ satisfy

$$2a\,b\,l_{i,k}\cos(l_{i,k}\pi) + (a^2 - b^2 l_{i,k}^2)\sin(l_{i,k}\pi) = 0$$

and $\phi_{i,k}$ are solutions of

$$\cos(\phi_{i,k}) = \frac{bl_{i,k}}{a^2 + b^2 l_{i,k}^2} \qquad \sin(\phi_{i,k}) = -\frac{a}{a^2 + b^2 l_{i,k}^2} \qquad i = 1, 2$$

in $[0, 2\pi)$, where a and b are given constants in the boundary conditions (2.1b). Both these sets of constants are calculated numerically (see Ashwin and Mei [2] for a discussion of this). An important property is that parity of eigenfunctions is defined for Robin boundary conditions and stays constant on varying a and b.

For corank two steady state bifurcations, the centre eigenspace can be identified with $(a_1, a_2) \in \mathbf{R}^2$, and the action of $\mathbf{D}_4$ on the basis (2.2) is given by

$$\rho(\cos(l_{1,k}x_1 + \phi_{1,k})\cos(l_{2,k}x_2 + \phi_{2,k})) = (-1)^{p_1}\cos(l_{2,k}x_1 + \phi_{2,k})\cos(l_{1,k}x_2 + \phi_{1,k})$$
$$\rho(\cos(l_{2,k}x_1 + \phi_{1,k})\cos(l_{1,k}x_2 + \phi_{2,k})) = (-1)^{p_2}\cos(l_{1,k}x_1 + \phi_{1,k})\cos(l_{2,k}x_2 + \phi_{2,k})$$
$$\mu f(x_1, x_2) = f(x_2, x_1) \quad \text{for all} \quad f \in C(\Omega),$$

where $p_i \in \{0, 1\}, i = 1, 2$. If $p_1 + p_2$ is even then the bifurcating mode has even parity, otherwise it has odd parity. Moreover, these modes have a separable form.

Separability of bases of eigenspaces holds for such equations which have the subgroup of symmetries $\mathcal{E}(1) \times \mathcal{E}(1) \times_s \mathbf{Z}_2 \prec \mathcal{E}(2)$ in the extension to the plane (for example, a subgroup of the former group is the $\mathbf{D}_4 \times_s \mathbf{T}^2$ considered by Crawford [5]; '$\times_s$' denotes the semidirect product). Imposing periodic boundary conditions in both directions gives the symmetry group $\mathbf{T}^2 \prec \mathcal{E}(1) \times \mathcal{E}(1)$.

3. The action of $\mathbf{D}_4$ on the centre eigenspace

3.1. *Steady-state bifurcation*

For the case of corank two and odd parity, the action of $\mathbf{D}_4$ induces an irreducible action on $\mathbf{R}^2$, otherwise it is reducible; it action is isomorphic to either $\mathbf{Z}_2$ or $\mathbf{Z}_2 \times \mathbf{Z}_2$. As discussed by

Ashwin and Mei [2], Robin boundary conditions generically remove any number theoretic degeneracies (i.e. bifurcations with corank higher than two), but they preserve the corank two degeneracies. It is interesting then to note that the work of Crawford *et al* [9] also shows that one cannot remove such degeneracies by considering Robin boundary conditions. They do succeed in removing them by perturbing the boundaries away from straight edges whilst retaining the square symmetry. We suggest that the degeneracy would also be removed by making the boundary conditions (2.1*b*) non-homogeneous (for example, taking $a(x)$ and $b(x)$ in (2.1*b*)).

3.2. Hopf bifurcation

For Hopf bifurcation the linearization $A(0)$ has an even dimensional centre eigenspace corresponding to eigenvalues $\pm i\omega$, and generically these eigenspaces are spanned by functions of the form

$$\{U_k \cos(l_{1,k}x_1 + \phi_{1,k}) \cos(l_{2,k}x_2 + \phi_{2,k}) e^{i\omega t} + c.c. \ : \ l_{1,k}^2 + l_{2,k}^2 = \kappa^2\}$$

with $U_k \in \mathbf{C}^m$ constant up to multiples in $\mathbf{C}$. We assume that for λ near zero, $A(\lambda)$ has eigenvalues

$$\lambda \pm i\omega$$

which is satisfied after appropriately rescaling of the bifurcation parameter if this complex pair of eigenvalues goes through the imaginary axis with a non-zero rate.

By [2] we can generically exclude 'number-theoretic' degeneracies and assume there is a Hopf bifurcation at $\lambda = 0$ with critical wavenumber κ and centre eigenspace is of dimension 2 or 4. For this case we denote $p = l_{1,1}$, $q = l_{2,1}$, $\phi_1 = \phi_{1,1}$ and $\phi_2 = \phi_{2,1}$. We note that the case $p = q$ gives a normal form on $\mathbf{C}$ which is that for standard Hopf bifurcation without symmetry, and so concentrate on the case $p \neq q$. We define the spatial dependence of the eigenvectors to be

$$\begin{aligned} S_1(x) &= \cos(px_1 + \phi_1)\cos(qx_2 + \phi_2) \\ S_2(x) &= \cos(qx_1 + \phi_2)\cos(px_2 + \phi_1). \end{aligned}$$

The exponential of the adjoint $A(0)^*$ induces an action of $\mathbf{S}^1$ on the centre eigenspace [10] given by $e^{sA(0)^*}$ for $s \in [0, 2\pi\omega)$ and so there is a natural complex structure on the eigenspace by $\mathcal{I} = e^{\omega A(0)^*/2}$. For the generic case, choosing $U \in \mathbf{R}^m$ such that

$$US_k(x)e^{i\omega t} + c.c. \quad k = 1, 2$$

is in the centre eigenspace, we define $V = \mathcal{I}U$ and coordinates $(z_+, z_-) \in \mathbf{C}^2$ for the centre eigenspace by

$$\begin{aligned} u(x,t) := \ & z_+(S_1(x)U + iS_2(x)V)e^{i\omega t} \\ & + z_-(S_2(x)U + iS_1(x)V)e^{i\omega t} + c.c. \end{aligned} \tag{3.1}$$

We remark that this differs from the coordinates $(\tilde{z}_+, \tilde{z}_-)$ in our previous work [4] by a transformation $(z_+, z_-) = (-i\tilde{z}_+, \tilde{z}_-)$. The induced action of $\mathbf{D}_4$ on $\mathbf{C}^2$ is given by

$$\begin{aligned} \rho &: (z_+, z_-) \to (\pm iz_+, \mp iz_-) \\ \mu &: (z_+, z_-) \to (z_-, z_+) \end{aligned}$$

for odd parity modes and

$$\begin{aligned} \rho &: (z_+, z_-) \to \pm(z_-, z_+) \\ \mu &: (z_+, z_-) \to (z_-, z_+). \end{aligned}$$

for even parity. We shall fix on the case where $\rho(z_+, z_-) = (\mathrm{i}z_+, -\mathrm{i}z_-)$ for odd parity and $\rho(z_+, z_-) = (z_-, z_+)$ for even parity. The case $\rho(z_+, z_-) = -(z_-, z_+)$ leads to the same normal form, just with a slightly different interpretation of the solution branches.

The natural $\mathbf{S}^1$ action on the centre eigenspace becomes a symmetry of the Birkhoff normal form to all orders, although there will typically be error terms not in Birkhoff normal form which break this symmetry. As in Golubitsky *et al* [14] we shall take the approach of examining the bifurcation by using the Birkhoff normal form of the vector field on the centre manifold and so including this extra $\mathbf{S}^1$ symmetry. The bifurcation behaviour in the full system is then the normal form bifurcation behaviour perturbed by a generic $\mathbf{S}^1$ symmetry breaking perturbation. Such perturbations will cause hyperbolic structures (for example, hyperbolic periodic orbits) in the dynamics to persist. As before, the action of $\mathbf{D}_4 \times \mathbf{S}^1$ is reducible for even parity and irreducible for odd parity.

This $\mathbf{S}^1$ symmetry acts via temporal phase shift as indicated before. The induced action on $\mathbf{C}^2$ is

$$T_\psi : (z_+, z_-) \to (\mathrm{e}^{\mathrm{i}\psi} z_+, \mathrm{e}^{\mathrm{i}\psi} z_-)$$

for Hopf bifurcations with both odd and even parity mode. Note that the action of $\mathbf{D}_4 \times \mathbf{S}^1$ on $\mathbf{C}^2$ is the same as Swift's for the case of odd parity and gives an action of $\mathbf{Z}_2 \times \mathbf{S}^1$ for even parity. Upon performing a centre manifold reduction we get that the dynamics on the centre manifold is determined by an ordinary differential equation on $\mathbf{C}^2$:

$$\begin{aligned} \dot{z}_+ &= (\lambda + \mathrm{i}\omega) z_+ + f_+(z_+, z_-) \\ \dot{z}_- &= (\lambda + \mathrm{i}\omega) z_- + f_-(z_+, z_-) \end{aligned}$$

where $f_\pm(z_+, z_-)$ are k times differentiable functions with zero linear part, and equivariant under the appropriate action of $\mathbf{D}_4 \times \mathbf{S}^1$ [24]. For a smooth operator equation, we can choose k arbitrarily large.

4. The normal form

The ring of invariants for both actions of $\mathbf{D}_4 \times \mathbf{S}^1$ on $\mathbf{C}^2$ are generated by

$$|z_+|^2 + |z_-|^2, \quad |z_+|^2 |z_-|^2$$

for odd parity, and the extra generators

$$z_+\bar{z}_- + z_-\bar{z}_+, \quad |z_+|^2 z_+\bar{z}_- + |z_-|^2 z_-\bar{z}_+$$

for even parity. The equivariants are the module over this ring generated by

$$\begin{pmatrix} z_+ \\ z_- \end{pmatrix}, \begin{pmatrix} |z_+|^2 z_+ \\ |z_-|^2 z_- \end{pmatrix}, \begin{pmatrix} \bar{z}_+ z_-^2 \\ \bar{z}_- z_+^2 \end{pmatrix}$$

for odd parity and three additional generators

$$\begin{pmatrix} z_- \\ z_+ \end{pmatrix}, \begin{pmatrix} |z_-|^2 z_- \\ |z_+|^2 z_+ \end{pmatrix}, \begin{pmatrix} \bar{z}_- z_+^2 \\ \bar{z}_+ z_-^2 \end{pmatrix}$$

for even parity. This is proved by defining

$$z^t = \begin{cases} z^t & \text{for } t \geqslant 0 \\ \bar{z}^{-t} & \text{for } t < 0 \end{cases}$$

and considering the action of the groups on sums of homogeneous monomials of the form $z_+^l z_-^m |z_+|^{2p} |z_-|^{2q}$. For the invariants, the $\mathbf{S}^1$ action means that p, q are arbitrary and $l + m = 0$. A minimal set of generators is then given as above. For the equivariants, the μ action swaps the components, so considering monomials in the first component gives again p, q arbitrary and $l+m=1$. It is a routine calculation then to show the equivariants are as above.

Thus, C^k (for $k>4$) vector fields commuting with the action of $\mathbf{D}_4 \times \mathbf{S}^1$ can be written in the form

$$\begin{pmatrix} \dot{z}_+ \\ \dot{z}_- \end{pmatrix} = A_1 \begin{pmatrix} z_+ \\ z_- \end{pmatrix} + A_2 \begin{pmatrix} |z_+|^2 z_+ \\ |z_-|^2 z_- \end{pmatrix} + A_3 \begin{pmatrix} \overline{z}_+ z_-^2 \\ \overline{z}_- z_+^2 \end{pmatrix}$$
$$+ A_4 \begin{pmatrix} z_- \\ z_+ \end{pmatrix} + A_5 \begin{pmatrix} |z_-|^2 z_- \\ |z_+|^2 z_+ \end{pmatrix} + A_6 \begin{pmatrix} \overline{z}_- z_+^2 \\ \overline{z}_+ z_-^2 \end{pmatrix}.$$

with $A_1 \in C^{k-1}$ and $A_i \in C^{k-3}$ ($i=2,\ldots,6$) complex valued functions of the invariants ($A_4 = A_5 = A_6 = 0$ for odd parity), in particular, $A_4(0)=0$. Truncating at cubic order gives

$$\begin{aligned} \dot{z}_+ &= (\lambda + \mathrm{i}\omega) z_+ + [A(|z_+|^2 + |z_-|^2) + B|z_+|^2] z_+ + C\overline{z}_+ z_-^2 \\ &\quad + [X_1(|z_+|^2 + |z_-|^2) + X_2 |z_-|^2] z_- + X_3 \overline{z}_- z_+^2, \\ \dot{z}_- &= (\lambda + \mathrm{i}\omega) z_- + [A(|z_+|^2 + |z_-|^2) + B|z_-|^2] z_- + C\overline{z}_- z_+^2 \\ &\quad + [X_1(|z_+|^2 + |z_-|^2) + X_2 |z_+|^2] z_+ + X_3 \overline{z}_+ z_-^2 \end{aligned} \tag{4.1}$$

with six complex coefficients A, B, C, X_1, X_2 and X_3. For odd parity, $X_1 = X_2 = X_3 = 0$ and the equations reduce to Swift's [23, equation (25)].

5. Analysis of the normal form

Table 1 shows the isotropy lattices for the two different actions of $\mathbf{D}_4 \times \mathbf{S}^1$ on $\mathbf{C}^2$; as usual, the isotropy subgroups are classified up to conjugacy. Note that there are three maximal isotropy subgroups for the odd parity (irreducible) case, but only one maximal isotropy subgroup for the even parity case. In both cases, the kernel of the action is $\tilde{\mathbf{Z}}_2(\rho^2)$ corresponding to equivariance under temporal phase shift of half a period and spatial rotation by half a period. This equivariance will be lost in the original equations by the introduction of terms that are not in the centre eigenspace at bifurcation.

Table 2 gives the size of and some representative elements in the fixed point spaces of the isotropy subgroups displayed in table 1, as well as the names assigned by Swift. By the Equivariant Hopf Lemma [13], there are generically branches at bifurcation with isotropy Σ if $\dim_{\mathbf{R}} \mathrm{fix}(\Sigma) = 2$ (This implies that Σ is maximal for the actions discussed). For odd parity, we get the three maximal isotropy types of Golubitsky and Stewart while for even parity we get only one. This is a noticeable difference in the branching behaviour of bifurcation; we only get branches with $\tilde{\mathbf{Z}}_2(\rho^2) \times \mathbf{Z}_2(\mu)$ symmetry for even parity. However, for even parity we may get up to four distinct group orbits of periodic solutions with *submaximal* symmetry, as opposed to only one group orbit of submaximal solutions for odd parity.

Using the following transformation of (z_+, z_-) to (u, v, w, ψ):

$$\begin{aligned} u &= |z_+|^2 - |z_-|^2, \\ v &= 2\mathrm{Im}(z_+\overline{z}_-) = \frac{z_+\overline{z}_- - z_-\overline{z}_+}{\mathrm{i}} \\ w &= 2\mathrm{Re}(z_+\overline{z}_-) = z_+\overline{z}_- + z_-\overline{z}_+ \\ \mathrm{e}^{\mathrm{i}\psi} &= \frac{z_+\overline{z}_-}{|z_+\overline{z}_-|} \end{aligned}$$

Table 1. Isotropy lattices for the actions of $\mathbf{D}_4 \times \mathbf{S}^1$ on $\mathbf{C}^2$.

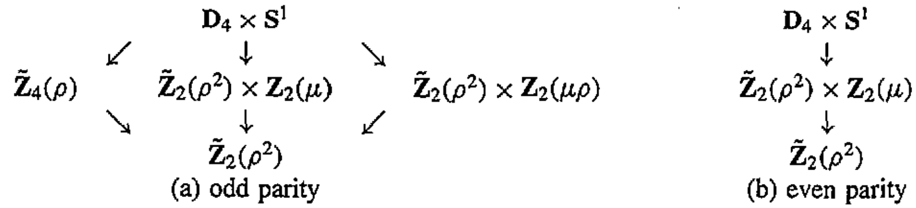

Table 2. Representative fixed point spaces for the actions of $\mathbf{D}_4 \times \mathbf{S}^1$ on $\mathbf{C}^2$.
(a) odd parity,

Isotropy	Fix	$\dim_{\mathbf{C}}$ Fix	Name ofsolutions
$\mathbf{D}_4 \times \mathbf{S}^1$	$(0, 0)$	0	homogeneous equilibrium
$\tilde{\mathbf{Z}}_4(\rho)$	$(z, 0)$	1	rotating wave
$\tilde{\mathbf{Z}}_2(\rho^2) \times \mathbf{Z}_2(\mu)$	(z, z)	1	edge oscillation
$\tilde{\mathbf{Z}}_2(\rho^2) \times \mathbf{Z}_2(\mu\rho)$	(z, iz)	1	vertex oscillation
$\tilde{\mathbf{Z}}_2(\rho^2)$	(z, w)	2	submaximal solution

(b) even parity

Isotropy	Fix	$\dim_{\mathbf{C}}$ Fix
$\mathbf{D}_4 \times \mathbf{S}^1$	$(0, 0)$	0
$\tilde{\mathbf{Z}}_2(\rho^2) \times \mathbf{Z}_2(\mu)$	(z, z)	1
$\tilde{\mathbf{Z}}_2(\rho^2)$	(z, w)	2

it is possible to write (4.1) as

$$\begin{aligned}\dot{z}_+ &= (\lambda + \mathrm{i}\omega)z_+ + \alpha_u u z_+ + \mathrm{i}\alpha_v v z_- + \alpha_w w z_- + \beta_u u z_- + \mathrm{i}\beta_v v z_+ + \beta_w w z_+ \\ \dot{z}_- &= (\lambda + \mathrm{i}\omega)z_- - \alpha_u u z_- - \mathrm{i}\alpha_v v z_+ + \alpha_w w z_+ - \beta_u u z_+ - \mathrm{i}\beta_v v z_- + \beta_w w z_-.\end{aligned} \tag{5.1}$$

At this point, we depart from Swift [23] in which u and w are interchanged. The reason for this will become clear; it is to ensure that the coordinate singularity is at a point of maximal isotropy. The parameters α_u, α_v, α_w, β_u, β_v, β_w are complex numbers and $A_r, A_i \ldots, X_{3r}, X_{3i}$ are real and imaginary parts of A, B, C and X_1, X_2, X_3 respectively. They are related by

$$\begin{aligned}A = A_r + iA_i &= -\alpha_u + \alpha_v + \alpha_w \\ B = B_r + iB_i &= 2\alpha_u - \alpha_v - \alpha_w \\ C = C_r + iC_i &= -\alpha_v + \alpha_w \\ X_1 = X_{1r} + iX_{1i} &= \beta_u - \beta_v + \beta_w \\ X_2 = X_{2r} + iX_{2i} &= -2\beta_u + \beta_v - \beta_w \\ X_3 = X_{3r} + iX_{3i} &= \beta_v + \beta_w.\end{aligned}$$

We denote the real and imaginary parts of the α's and β's by

$$\begin{aligned}\alpha_u &= R_u + \mathrm{i}I_u \quad \alpha_v = R_v + \mathrm{i}I_v \quad \alpha_w = R_w + \mathrm{i}I_w \\ \beta_u &= S_u + \mathrm{i}J_u \quad \beta_v = S_v + \mathrm{i}J_v \quad \beta_w = S_w + \mathrm{i}J_w\end{aligned}$$

and define r by

$$r^2 = u^2 + v^2 + w^2.$$

We rewrite (5.1) in the form of a modified 'Euler equation':

$$\begin{aligned}
\frac{1}{2}\dot{u} &= u(R_u r + \lambda) + (I_w - I_v)vw - J_v vr + (S_u + S_w)uw \\
\frac{1}{2}\dot{v} &= v(R_v r + \lambda) + (I_u - I_w)uw + J_u ur + (S_v + S_w)vw \qquad (5.2)\\
\frac{1}{2}\dot{w} &= w(R_w r + \lambda) + (I_v - I_u)uv + S_w w^2 - S_u u^2 - S_v v^2 \\
\dot{\psi} &= \omega + O(r) \qquad (5.3)
\end{aligned}$$

(when all parameters are zero except for I_u, I_v and I_w, this is Euler's equation for the motion of a rigid body). The ψ equation uncouples from those for u, v and w (this is a consequence of the $\mathbf{S}^1$ symmetry of the normal form) and so *periodic solutions* in (z_+, z_-)–space correspond to *fixed points* in (u, v, w)–space.

Because (5.2) is homogeneous except for the λ term, we can further reduce it to an *associated spherical system*, still following Swift [23]. To do this, we change the coordinate from (u, v, w) into (r, θ, ϕ) via the transformation

$$\begin{aligned}
u + \mathrm{i}v &= r \sin\theta \mathrm{e}^{\mathrm{i}\phi} \\
w &= r\cos\theta.
\end{aligned}$$

Note that for $\sin\theta = 0$, ϕ is not defined; this coordinate singularity is the 'pole' of the sphere. This leads to the following system for r, θ and ϕ

$$\begin{aligned}
\dot{r} = \frac{r}{2}\Big[4\lambda + r\Big(&4A_r + 3B_r - C_r + (B_r + C_r)\sin^2\theta\cos 2\phi - (X_{2i} + X_{3i})\sin^2\theta\sin 2\phi \\
&+(3C_r - B_r)\cos^2\theta + (4X_{1r} + 2X_{2r} + 2X_{3r})\cos\theta\Big)\Big] \qquad (5.4)
\end{aligned}$$

$$\begin{aligned}
\dot{\theta} = \frac{r\sin\theta}{2}\Big\{&\cos\theta\Big[(-X_{2i} - X_{3i})\sin\phi\cos\phi + (B_r + C_r)\cos^2\phi - 2C_r\Big] \\
&+(B_i + C_i)\sin\phi\cos\phi - (X_{2r} + X_{3r})\cos^2\phi - 2X_{1r} - X_{2r} + X_{3r}\Big\} \qquad (5.5)
\end{aligned}$$

$$\begin{aligned}
\dot{\phi} = \frac{r}{2}\Big\{&\Big[(X_{2r} + X_{3r})\cos\theta - (B_r + C_r)\Big]\sin 2\phi \\
&+\Big[(B_i + C_i)\cos\theta - (X_{2i} + X_{3i})\Big]\cos 2\phi \qquad (5.6)\\
&+(B_i - 3C_i)\cos\theta - 4X_{1i} - 3X_{2i} + X_{3i}\Big\}.
\end{aligned}$$

This system is defined for $r \geqslant 0$, $\theta \in [0, 2\pi)$ and $\phi \in [0, \pi)$. The action of the reflection and rotation is given by

$$\begin{aligned}
\mu &: (r, \theta, \phi) \to (r, 2\pi - \theta, \phi) \\
\rho &: (r, \theta, \phi) \to (r, \pi - \theta, \pi - \phi) \quad \text{(odd parity only)}.
\end{aligned}$$

As noted by Swift, the equations for θ and ϕ can be made independent of $r > 0$ by rescaling time, and then we can find periodic solutions of the full problem (steady state solutions for (r, θ, ϕ)) in two stages:

1. Solve the *associated steady state spherical system* $(\dot{\theta}, \dot{\phi}) = (0, 0)$ for $r > 0$.
2. Solve the radial equation $\dot{r} = 0$ at zeroes of the associated spherical system.

For A_r, B_r and C_r asymptotically close to zero compared to the other coefficients and $X_k \equiv 0$, this system describes the motion of a rigid body. Van Gils and Silber [11] have also investigated 2-tori for this system with $X_i \equiv 0$ using other techniques. In addition to those found by Swift, we believe there may be new types of heteroclinic orbit or quasiperiodic behaviour near bifurcation but we have not pursued this idea. The equation $\dot{\theta} = 0$ is always satisfied if

$$\sin\theta = 0. \tag{5.7}$$

This corresponds to there always being a branch with maximal symmetry $\mathbf{Z}_2$ that bifurcates, (this corresponds to $u = v = 0$ and $w = r$. Our reason for choosing u, v, w differently from Swift is to ensure that this maximal solution is at the pole of the coordinates). Alternatively, $\dot{\theta} = 0$ has a root at

$$\cos\theta = -\frac{(B_i + C_i)\sin\phi\cos\phi - (X_{2r} + X_{3r})\cos^2\phi - 2X_{1r} - X_{2r} + X_{3r}}{(-X_{2i} - X_{3i})\sin\phi\cos\phi + (B_r + C_r)\cos^2\phi - 2C_r} \tag{5.8}$$

corresponding to periodic solutions with trivial symmetry; i.e. submaximal symmetry solutions. Note that this is different from Swift's equation for $\cos\theta$ due to our interchanging the definitions of u and w. If we were to keep to Swift's coordinates, problems begin at this point as there is no easy solution giving $\cos\theta$ when terms $X_k \neq 0$ are included. Substituting (5.8) into $\dot{\phi} = 0$ we get the following equation for ϕ

$$\alpha\cos 4\phi + \beta\sin 4\phi + \gamma\cos 2\phi + \delta\sin 2\phi + \epsilon = 0 \tag{5.9}$$

where the constants $\alpha, \ldots, \epsilon$ are given by

$$\alpha = \tfrac{1}{4}\mathrm{Im}[(\bar{B} + \bar{C})(X_2 + X_3)] \tag{5.10}$$

$$\beta = \tfrac{1}{8}[|B|^2 + |C|^2 - |X_2|^2 - |X_3|^2 + 2\mathrm{Re}(\bar{B}C - \bar{X}_2 X_3)] \tag{5.11}$$

$$\gamma = \mathrm{Im}[(\bar{B} + \bar{C})X_1 + C\bar{X}_3 + \bar{B}X_2] \tag{5.12}$$

$$\delta = \tfrac{1}{4}[|B|^2 + |X_3|^2 - 3(|C|^2 + |X_2|^2) - 2\mathrm{Re}(B\bar{C} + 2X_1(\bar{X}_2 + \bar{X}_3) + X_2\bar{X}_3)] \tag{5.13}$$

$$\epsilon = \tfrac{1}{4}\mathrm{Im}[(\bar{B} - 3\bar{C})(4X_1 + 3X_2 - X_3)]. \tag{5.14}$$

For odd parity ($X_k = 0$), we have $\alpha = \gamma = \epsilon = 0$ and so either $\sin 2\phi = 0$ (corresponding to maximal branches) or

$$\cos 2\phi = -\frac{\delta}{2\beta}.$$

Equation (5.9) for ϕ can be written as a fourth order polynomial of $\exp(i2\phi)$ and as such there exists a closed form solution to this equation. There may be zero, two or four nondegenerate (isolated) solutions ϕ in the interval $[0, \pi)$ which we write as $\phi_1 < \phi_2$ (and $\phi_1 < \phi_2 < \phi_3 < \phi_4$ if they exist).

Having solved this equation, we solve (5.8) with $\phi = \phi_i$ which will have zero or two (symmetrically placed) solutions according as the right-hand side has modulus greater than or less than unity.

Given a pair (θ_{ij}, ϕ_i) that solves these equations, the equation $\dot{r} = 0$ (5.4) implies there will be a branch of solutions in the direction $\mathrm{sign}(l_i)$

$$l_i := -4\Big[4A_r + 3B_r - C_r + \Big((B_r + C_r)\cos 2\phi_i - (X_{2i} + X_{3i})\sin 2\phi_i\Big)\sin^2\theta_{ij} + (3C_r - B_r)\cos^2\theta_{ij} + (4X_{1r} + 2X_{2r} + 2X_{3r})\cos\theta_{ij}\Big]^{-1} \tag{5.15}$$

and

$$r = l_i\lambda.$$

Since the solutions indexed by j are symmetrically related, l_i is independent of j.

5.1. Procedure to find branching information

We summarise the calculations already performed to find the number and direction of branching of periodic solutions at a Hopf bifurcation point

• Find the complex normal form coefficients (A, B, C, X_1, X_2, X_3) using centre manifold or Liapunov–Schmidt reduction.

• Use these to calculate $\alpha, \ldots, \epsilon$ via equations (5.10)–(5.14).

• Determine the solutions (including multiplicities) ϕ_i of equation (5.9). We exclude the case that

(I) the solutions ϕ_i are not isolated.

• Determine the solutions θ_{ij} of equation (5.8), i.e. such that $|\cos\theta_{ij}| \leqslant 1$. We exclude the case

(II) $|\cos\theta_{ij}| = 1$.

• Compute l_i using equation (5.15) and observe that the bifurcating branch corresponding to submaximal solutions with (θ_{ij}, ϕ_i) is subcritical for $l_i > 0$ and supercritical for $l_i < 0$. The value of r is then given to lowest order by $r = l_i\lambda$. We exclude the case

(III) $l_i = 0$.

This procedure will work for normal form coefficients which do not satisfy any of the degeneracies marked (I)–(III). It is clear that there is an open everywhere dense set of parameters for which none of (I)–(III) satisfied, and so we have a procedure that will work for a generic set of third order coefficients.

6. Application to Brusselator equations

As an example of a normal form derived from a partial differential equation on the square, we briefly present some Liapunov–Schmidt reduced bifurcation equations for the Brusselator on a square. Because an analytical reduction is not possible for the boundary conditions assumed, a hybrid numerical–analytical reduction is used [4, 2].

We consider the following equations with diffusive coupling:

$$\begin{aligned}\frac{\partial u_1}{\partial t} &= a - (b + \lambda + 1)u_1 + u_1^2 u_2 + \nabla^2 u_1 \\ \frac{\partial u_2}{\partial t} &= (b + \lambda)u_1 - u_1^2 u_2 + d\nabla^2 u_2\end{aligned} \tag{6.1}$$

on the square domain $\Omega := [0, \pi]^2$ with Robin boundary conditions

$$\mu(u_i - u_i^s) + (1 - \mu)\frac{\partial u_i}{\partial n} = 0 \quad \text{on } \partial\Omega. \tag{6.2}$$

(u^s is the homogeneous state $(u_1, u_2) = (a, (b + \lambda)/a)$ and n is the outward normal derivative). Setting $(u_1, u_2) = (a, (b + \lambda)/a) + (\tilde{u}_1, \tilde{u}_2)$ and rescaling time $t \leftarrow (\omega_0 + \tau)t$ we get the following operator equation for periodic solutions near the trivial state after dropping the ˜s:

$$\Phi(u, \lambda, \tau) := \left[Lu - \omega_0\frac{\partial u}{\partial t}\right] + R(u, \lambda, \tau) = 0 \tag{6.3}$$

where $u := (u_1, u_2)^T$,

$$L := \begin{pmatrix} \Delta + b - 1 & a^2 \\ -b & d\Delta - a^2 \end{pmatrix} \tag{6.4}$$

and

$$R(u, \lambda, \tau) := \begin{pmatrix} 1 \\ -1 \end{pmatrix} \left(\frac{b+\lambda}{a} u_1^2 + 2au_1u_2 + u_1^2 u_2 + \lambda u_1 \right) - \tau \frac{\partial u}{\partial t}. \tag{6.5}$$

The operator L maps $X := (C_0^{2,\alpha}(\Omega))^2$ into $Y := (C^{0,\alpha}(\Omega))^2$ (functions with Hölder condition, exponent $0 < \alpha < 1$ and suitable boundary conditions). Let

$$C_{2\pi} := \{u(\cdot) \in C(\mathbf{R}, Y) \ : \ u(s + 2\pi) = u(s)\}$$

and

$$C_{2\pi}^1 := \{u(\cdot), \frac{\partial u}{\partial s}(\cdot) \in C(\mathbf{R}, X) \ : \ u(s + 2\pi) = u(s)\}.$$

The operator Φ maps $C_{2\pi}^1$ into $C_{2\pi}$ and

$$\Phi(0, \lambda, \tau) = 0 \quad \text{for all} \ \ \lambda, \ \tau \in [0, \infty).$$

Fixing the parameters a, b and d at a Hopf bifurcation point, we study the periodic solution branches of (6.1) bifurcating from the trivial solution curve upon varying λ, i.e. steady solutions of the operator equation $\Phi = 0$.

6.1. Linear stability analysis

As discussed in [2], the eigenvalue problem is solvable for $\kappa(\mu) = \kappa_x(\mu)^2 + \kappa_y(\mu)^2$ with eigenfunctions

$$\begin{aligned} \phi_1 &:= [\mu \sin(\kappa_x(\mu)x) + (1-\mu)\kappa_x(\mu)\cos(\kappa_x(\mu)x)] \\ &\quad \cdot [\mu \sin(\kappa_y(\mu)y) + (1-\mu)\kappa_y(\mu)\cos(\kappa_y(\mu)y)] \\ \phi_2 &:= [\mu \sin(\kappa_y(\mu)x) + (1-\mu)\kappa_y(\mu)\cos(\kappa_y(\mu)x)] \\ &\quad \cdot [\mu \sin(\kappa_x(\mu)y) + (1-\mu)\kappa_x(\mu)\cos(\kappa_x(\mu)y)] \end{aligned}$$

($\phi_1 = \phi_2$ if and only if $\kappa_x = \kappa_y$, implying the multiplicity is one), and κ_x, κ_y each satisfy

$$2\mu(1-\mu)\kappa\cos(\kappa\pi) + [\mu^2 - (1-\mu)^2\kappa^2]\sin(\kappa\pi) = 0. \tag{6.6}$$

Since the Laplacian operator in $C_0^{2,\alpha}(\Omega)$ is self-adjoint and elliptic, the set of its eigenfunctions make up a basis for $C_0^{2,\alpha}(\Omega)$. Consequently, the operator L leaves the spaces spanned by eigenfunctions of the Laplacian invariant. This means that there is Hopf bifurcation at $\lambda = 0$ if for an eigenvalue $\kappa(\mu)$ of $-\Delta$ the parameters a_0, b_0, d_0 satisfy

$$b_0 - 1 - a_0^2 - (1 + d_0)\kappa(\mu) = 0 \tag{6.7a}$$

$$\omega_0^2 := (1 + \kappa(\mu))(a_0^2 + d_0\kappa(\mu)) - b_0 d_0 \kappa(\mu) > 0. \tag{6.7b}$$

For further details of the linear analysis, we refer the interested reader to [2]; we merely pause to mention that if the eigenvalue $\kappa(\mu)$ has multiplicity 2, the null space $\mathrm{Ker}(\partial\Phi)$ is 4 dimensional and belongs to the case considered in the first part of the paper.

We use the method described in [4] to numerically approximate the complex quantities A, B, C, X_1, X_2 and X_3 at a Hopf bifurcation point and then the procedure summarised in section 5.1 to find branching behaviour at the bifurcation. The branches found were checked to be solutions of the normal form equation (4.1).

We define

$$P = (P_{ij})_{i,j=1,2} := \frac{1}{\omega_0}\begin{pmatrix} b_0 - 1 - \kappa(\mu) & a_0^2 \\ -b_0 & -d_0\kappa(\mu) - a_0^2 \end{pmatrix}$$

$$\psi_1 := (e_1 \cos t + Pe_1 \sin t)\phi_1 \quad \psi_2 := (-Pe_1 \cos t + e_1 \sin t)\phi_1$$

$$\psi_3 := (e_1 \cos t + Pe_1 \sin t)\phi_2 \quad \psi_4 := (-Pe_1 \cos t + e_1 \sin t)\phi_2$$

where $e_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $e_2 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ was used to compute the Liapunov–Schmidt reduced equations by writing elements in the null space as

$$\psi = \sum_{i=1}^{4} \alpha_i \psi_i. \tag{6.8}$$

The inner product

$$\langle u,\ v\rangle = \frac{2}{\pi^3}\int_0^{2\pi}\int_\Omega (u,\, v)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t \tag{6.9}$$

in $C_{2\pi}$ was used to define the adjoint problem. Via this, we obtain the dual basis $\{\psi_i^*\}$ and define the projection

$$Qw := w - \sum_{i=1}^{4}\langle\psi_i^*, w\rangle\psi_i \quad w \in C_{2\pi}.$$

This is a projection from $C^1_{2\pi}$ into Range$(\partial\Phi) \cap C^1_{2\pi}$.

Writing the operator Φ as a power series in $\eta = (\psi, \tau, \lambda)$ about the trivial solution, that is $\Phi(\eta) = D\Phi_0(\eta) + R_2(\eta, \eta) + R_3(\eta, \eta, \eta)$ from [4] we note that the Liapunov–Schmidt reduced equation at third order is given by

$$(I - Q)(R_2(\eta, w_2(\eta)) + R_3(\eta, \eta, \eta)) = 0$$

with $w_2(\eta)$ *being the solution in* Ker$\partial\Phi^\perp$ *of*

$$\partial_u\Phi_0 w_2(\eta) = -QR_2(\eta, \eta).$$

We refer the reader to [2, 4] for details.

7. Results

Tables 3 and 4 show the third order coefficients obtained as a function of μ, the homotopy parameter in boundary condition for $d_0 = 0.05$ and $d_0 = 0.01$ respectively. We show results for the even parity branches going from the $(1, 3)$ and $(2, 4)$ Neumann modes to the $(2, 4)$ and $(3, 5)$ Dirichlet modes (tables 3 and 4 respectively). It can be checked that there are no mode interactions which might complicate the bifurcation scenario.

It can be seen from the results that very surprisingly, $X_1 = -2X_2 = 2X_3$ in all of these cases. Numerical experiments on other mode bifurcations imply that this is always the case for these equations. This means that instead of the three additional third order equivariants, there is a degeneracy that forces them to always appear in the combination

$$\begin{pmatrix} |z_-|^2 z_- + 2|z_+|^2 z_- + \bar{z}_- z_+^2 \\ |z_+|^2 z_+ + 2|z_-|^2 z_+ + \bar{z}_+ z_-^2 \end{pmatrix}.$$

Dr Gabriela Gomes has pointed out that this is the gradient of the invariant function

$$|z_+|^2|z_-|^2(\bar{z}_+ z_- + \bar{z}_- z_+).$$

In section 8 we discuss the source of this degeneracy.

For the case of Neumann boundary conditions, there is a hidden symmetry arising from the fact that solutions can be smoothly extended to doubly periodic solutions on the torus obtained by identifying opposite edges of the square. By looking at the Neumann boundary conditions as a symmetry restriction of the problem with periodic boundary conditions, it becomes generic by symmetry arguments that the centre eigenspace of even parity Hopf bifurcation can be two (complex) dimensional. This explains why $X_1 = X_2 = X_3 = 0$ in the results for $\mu = 0$.

Consequences of the presented values of A, B, C and $X_1 = -2X_2 = 2X_3$ for the branching are described in tables 5 and 6.

Table 3. This table shows the calculated values of the third order complex coefficients that determine the branching behaviour in the λ direction of periodic solutions near Hopf bifurcation. The homotopy parameter μ varies from Neumann boundary conditions ($\mu = 0$) to Dirichlet boundary conditions ($\mu = 1$) and $d_0 = 0.05$, $a_0 = 1$ for the computations presented here. The results were calculated using a 40 by 40 grid for the spatial discretisation for the branch connecting the (1, 3) Neumann mode to the (2, 4) Dirichlet mode. Note how the coefficients $X_1 = -2X_2 = 2X_3$ are zero at the Neumann limit (where there is a hidden symmetry of translation) and become non-zero for $\mu \neq 0$.

μ	A	B	C	$X_1 = -2X_2 = 2X_3$
0	−2.28506 − 0.02634i	1.40773 − 0.06248i	0.14184 + 0.00928i	0.00000 + 0.00000i
1/9	−2.28660 − 0.02773i	1.43026 − 0.08191i	0.12962 + 0.01898i	−0.00920 − 0.00981i
2/9	−2.28642 − 0.02545i	1.45898 − 0.10496i	0.11620 + 0.03116i	−0.02063 − 0.02246i
3/9	−2.28496 − 0.01901i	1.49673 − 0.13175i	0.10126 + 0.04623i	−0.03452 − 0.03861i
4/9	−2.28403 − 0.00853i	1.54805 − 0.16133i	0.08396 + 0.06409i	−0.05059 − 0.05872i
5/9	−2.28877 + 0.00382i	1.62067 − 0.19015i	0.06244 + 0.08307i	−0.06727 − 0.08265i
6/9	−2.31316 + 0.01066i	1.72865 − 0.20966i	0.03353 + 0.09821i	−0.08063 − 0.10914i
7/9	−2.39389 − 0.00803i	1.90197 − 0.20602i	−0.00358 + 0.10190i	−0.08415 − 0.13767i
8/9	−2.61188 − 0.10182i	2.21470 − 0.17007i	−0.03309 + 0.10064i	−0.06981 − 0.17877i
1	−2.97648 − 0.38048i	2.75807 − 0.10249i	−0.02528 + 0.13237i	−0.01403 − 0.25490i

Table 4. As in table 3 except these results are calculated for the branch connecting the (2, 4) Neumann mode to the (3, 5) Dirichlet mode and $a_0 = 1$, $d_0 = 0.01$.

μ	A	B	C	$X_1 = -2X_2 = 2X_3$
0	-4.69443+ 0.96305i	2.95487-0.74399i	-0.13504-0.33405i	0.00000+ 0.00000i
1/9	-4.73780+ 0.99776i	2.99837-0.79350i	-0.14463-0.31677i	-0.00100-0.00081i
2/9	-4.79001+ 1.04133i	3.05536-0.85072i	-0.15651-0.28904i	-0.00511-0.00390i
3/9	-4.85233+ 1.09567i	3.13034-0.91711i	-0.17197-0.24639i	-0.01487-0.01064i
4/9	-4.92590+ 1.16299i	3.22953-0.99450i	-0.19315-0.18334i	-0.03446-0.02279i
5/9	-5.01349+ 1.24552i	3.36253-1.08484i	-0.22375-0.09518i	-0.07005-0.04195i
6/9	-5.12985+ 1.34472i	3.54877-1.18923i	-0.27006+ 0.01636i	-0.12838-0.06760i
7/9	-5.34685+ 1.46328i	3.84318-1.30675i	-0.34310+ 0.13068i	-0.21002-0.09442i
8/9	-5.94761+ 1.63667i	4.42698-1.45787i	-0.45809+ 0.20979i	-0.30365-0.12271i
1	-7.49637+ 2.00104i	5.66954-1.75650i	-0.61536+ 0.28048i	-0.41589-0.19121i

Table 5. This table shows the possible solutions ϕ, θ and l along the homotopy path corresponding to the values in table 3. The signs of l determine the bifurcation direction on changing λ. We denote no solution by 'n.s.'. On this branch, there are two group orbits of submaximal solutions up until $\mu = 5/9$; the solutions ϕ_3 and ϕ_4 of the ϕ-equation are annihilated. From $\mu = 7/9$ there are again two group orbits of solutions.

μ	ϕ_1 ϕ_3	θ_{11} θ_{31}	θ_{12} θ_{32}	l_1 l_3	ϕ_2 ϕ_4	θ_{21} θ_{41}	θ_{22} θ_{42}	l_2 l_4
0	0	1.570796	4.712389	1.139826	1.129473	n.s.	n.s.	n.s.
	1.570796	1.570796	4.712389	1.139826	2.0121197	n.s.	n.s.	n.s.
1/9	0.006723964	1.577545	4.705641	1.167746	1.151941	n.s.	n.s.	n.s.
	1.565340	1.536620	4.746565	0.6110883	1.988384	n.s.	n.s.	n.s.
2/9	0.01551817	1.585313	4.697872	1.208514	1.177226	n.s.	n.s.	n.s.
	1.559913	1.485305	4.797880	0.6188707	1.959732	n.s.	n.s.	n.s.
3/9	0.02674702	1.593917	4.689269	1.268420	1.205644	n.s.	n.s.	n.s.
	1.556158	1.405486	4.877699	0.6289309	1.923841	n.s.	n.s.	n.s.
4/9	0.04042879	1.602729	4.680456	1.357882	1.237870	n.s.	n.s.	n.s.
	1.557934	1.272077	5.011109	0.6412422	1.876156	n.s.	n.s.	n.s.
5/9	0.05562084	1.610297	4.672888	1.494196	1.276148	n.s.	n.s.	n.s.
	1.575267	0.9971433	5.286042	0.6545510	1.805353	n.s.	n.s.	n.s.
6/9	0.06969786	1.614028	4.669158	1.703950	1.326229	n.s.	n.s.	n.s.
	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.
7/9	0.07934294	1.610901	4.672285	2.015995	1.389855	2.849157	3.434028	0.7346098
	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.
8/9	0.08591220	1.599423	4.683762	2.474485	1.434446	2.174610	4.108575	0.6918723
	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.
1	0.09436033	1.576897	4.706289	4.338267	1.456928	1.774929	4.508256	0.6385345
	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.

8. Discussion

In summary, we have shown that the analysis of Swift [23] for $\mathbf{D}_4$ Hopf bifurcation can be extended to cover generic Hopf bifurcation problems of PDEs on the square where the spatial parts of the eigenfunctions are separable. For Hopf bifurcations with odd parity (or with even parity and Neumann boundary conditions), the normal form of Swift applies, but for even parity there are extra third order terms in the normal form for the bifurcation; these are important for determining the branching behaviour.

For the example presented, the Brusselator equations on a square with Robin boundary conditions, we have found that there is an extra degeneracy of the new third order terms. We now show that this arises because the linearized problem has a basis of eigenfunctions that are separable. This is in turn implied by the fact that the linearized problem has separable spatial part.

It turns out that a different coordinate system enables one to investigate this degeneracy more easily. We define

$$z_x = \frac{z_+ + z_-}{2}$$
$$z_y = \frac{z_+ - z_-}{2}$$

and we note that elements in the centre eigenspace (3.1) can be written

$$u = z_x \Psi_x + z_y \Psi_y + \text{c.c.}$$

Table 6. This table is generated with the values in table 4. There are up to 4 solutions for ϕ-equation, and for each of these there are up to two solutions of θ along the homotopy path starting from (2,4) mode of Neumann problem to (3,5) mode of Dirichlet problem. The signs of l determine the direction of branching. Note that in this case, the solutions with maximal isotropy type for $\mu = 0$ all have unique continuations to $\mu = 1$; there are no secondary bifurcations on this branch.

μ	ϕ_1	θ_{11}	θ_{12}	l_1	ϕ_2	θ_{21}	θ_{22}	l_2
	ϕ_3	θ_{31}	θ_{32}	l_3	ϕ_4	θ_{41}	θ_{42}	l_4
0	0	1.570796	4.712389	0.5748576	1.570796	1.570796	4.712389	0.5748576
	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.
1/9	0.0003176902	1.571003	4.712183	0.5748991	1.574077	1.586851	4.696334	0.3158247
	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.
2/9	0.001548596	1.571839	4.711346	0.5764834	1.577760	1.612483	4.670702	0.3140639
	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.
3/9	0.004275930	1.573794	4.709391	0.5807234	1.579042	1.641955	4.641230	0.3124132
	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.
4/9	0.009336105	1.577652	4.705533	0.5895024	1.575964	1.675929	4.607257	0.3112329
	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.
5/9	0.01769188	1.584513	4.698672	0.6057593	1.566185	1.715654	4.567532	0.3110555
	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.
6/9	0.02964825	1.595335	4.687851	0.6327034	1.548260	1.760115	4.523070	0.3120088
	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.
7/9	0.04271166	1.609049	4.674136	0.6658259	1.526578	1.800929	4.482256	0.3105186
	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.
8/9	0.05160103	1.619921	4.663264	0.6592501	1.513119	1.822905	4.460281	0.2897166
	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.
1	0.05906714	1.623301	4.659884	0.5487133	1.504274	1.827378	4.455808	0.2337781
	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.	n.s.

where

$$\Psi_x = (S_1 + S_2)(U + \mathrm{i}V)\mathrm{e}^{\mathrm{i}\omega t}/2$$
$$\Psi_y = (S_1 - S_2)(U - \mathrm{i}V)\mathrm{e}^{\mathrm{i}\omega t}/2.$$

with $S_1 = f_k(x_1)f_l(x_2)$ and $S_2 = f_l(x_1)f_k(x_2)$ as before. In this case, equation (4.1) transforms to give

$$\begin{aligned}\dot{z}_x &= (\lambda + \mathrm{i}\omega)z_x + f_x(z_x, z_y)\\ &:= (\lambda + \mathrm{i}\omega)z_x + (2A + B + C)|z_x|^2 z_x + 2(A + B - C)|z_y|^2 z_x + (B + C)\bar{z}_x z_y^2\\ &\quad +(2X_1 + X_2 + X_3)|z_x|^2 z_x + 2(X_1 + X_2 - X_3)|z_y|^2 z_x + (X_2 + X_3)\bar{z}_x z_y^2\\ \dot{z}_y &= (\lambda + \mathrm{i}\omega)z_y + f_y(z_x, z_y)\\ &:= (\lambda + \mathrm{i}\omega)z_y + (2A + B + C)|z_y|^2 z_y + 2(A + B - C)|z_x|^2 z_y + (B + C)\bar{z}_y z_x^2\\ &\quad -(2X_1 + X_2 + X_3)|z_y|^2 z_y - 2(X_1 + X_2 - X_3)|z_x|^2 z_y - (X_2 + X_3)\bar{z}_y z_x^2.\end{aligned} \tag{8.1}$$

In these coordinates, the degeneracy $X_1 = -2X_2 = 2X_3$ means that the only term that breaks the symmetry of the odd parity representation is of the form

$$\begin{pmatrix} |z_x|^2 z_x \\ -|z_y|^2 z_y \end{pmatrix}.$$

We can see that this is the case by considering the spatial dependence in the inner product giving these coefficients. Noting that complex conjugation does not affect the spatial dependence of the eigenfunctions, we use $(|z_x|^2 z_x)_y$ to signify the coefficient of $|z_x|^2 z_x$ in f_y. The Liapunov–Schmidt reduction gives rise to the following formulae defining C_i, $i = 1, \cdots, 3$

$$C_1 := (|z_x|^2 z_x)_x - (|z_y|^2 z_y)_y = K_1 \int_\Omega 4\Big((S_1^2 + S_2^2) P_1 S_1 S_2 + S_1 S_2 P_1 (S_1^2 + S_2^2)\Big) \mathrm{d}x\mathrm{d}y$$

$$C_2 := (|z_y|^2 z_x)_x - (|z_x|^2 z_y)_y = K_2 \int_\Omega \Big(4 S_1 S_2 P_2 (S_1^2 + S_2^2) - (S_1^2 + S_2^2) P_2 S_1 S_2\Big) \, \mathrm{d}x\mathrm{d}y$$

$$C_3 := (z_y^2 \bar{z}_x)_x - (z_x^2 \bar{z}_y)_y = K_3 \int_\Omega \Big(4 S_1 S_2 P_3 (S_1^2 + S_2^2) - (S_1^2 + S_2^2) P_3 S_1 S_2\Big) \, \mathrm{d}x\mathrm{d}y.$$

where P_i are linear operators that are of the form $c_1 I + c_2 A(0)^{-1}$ with $c_1, c_2 \in \mathbf{R}$ and $A(0)$ in (2.1a) (and therefore the P_i are separable). In section 6.1 the P_i are the spatial parts of the linear operators used to calculate the coefficients of $w_2(\eta)$. The constants K_i come from the reaction terms and the temporal parts of the eigenfunctions. Similarly as in section 2 we rewrite the linear operator into $P_2 = L_d + L_r$ with $L_d := L_1 + L_2$ representing the self-adjoint spatial operator and L_r the reaction operator, in particular, L_i is self-adjoint, giving an eigenbasis $\{f_i(x_1), \lambda_i\}$, and L_r behaves like a matrix operator, moreover

$$\langle u_1,\ L_r u_2 \rangle_2 = U_1 M_r U_2 \int_\Omega f_1(x) f_2(x) \, \mathrm{d}x_1 \, \mathrm{d}x_2$$

for all $u_k(x) = U_k f_k(x) \in C(\Omega, \mathbf{R}^m)$, $f_k \in C(\Omega, \mathbf{R})$. By $\langle \cdot \rangle_k$ we denote the L^2-product in $C([0, \pi]^k, \mathbf{R}^m)$. Based on this and the self-adjointness of L_d, we derive

$$\begin{aligned}
C_2 &= K_2[\langle 4 S_1 S_2,\ P_2(S_1^2 + S_2^2)\rangle_2 - \langle (S_1^2 + S_2^2),\ P_2 S_1 S_2 \rangle_2] \\
&= 2K_2[\langle S_1 S_2,\ L_d S_1^2 \rangle_2 - \langle S_1^2,\ L_d S_1 S_2 \rangle_2] \\
&= K_2\Big\{\Big[\langle f_k(x_1) f_l(x_1), L_1 f_k^2(x_1)\rangle_1 - \langle f_k(x_1)^2, L_1 f_k(x_1) f_l(x_1)\rangle_1\Big]\langle f_k(x_2) f_l(x_2), f_l^2(x_2)\rangle_1 \\
&\quad + \langle f_k(x_1) f_l(x_1), f_k^2(x_1)\rangle_1 \Big[\langle f_k(x_2) f_l(x_2),\ L_2 f_l^2(x_2)\rangle_1 - \langle f_l(x_2)^2, L_2 f_k(x_2) f_l(x_2)\rangle_1\Big]\Big\} \\
&= 0 \quad (\text{ due to the self-adjointness of } L_i,\ i = 1, 2).
\end{aligned}$$

Similarly C_3 can be shown to be zero and these imply that $2X_1 + X_2 - X_3 = 0$ and $X_2 + X_3 = 0$, giving

$$X_1 = -2X_2 = 2X_3.$$

We note that the above argument cannot be adapted to get $C_1 = 0$ and this is supported by the numerical results we have presented.

The linear degeneracy discussed in section 3.2 will always occur for branching from a trivial solution of a reaction–diffusion problems with Laplacian spatial coupling and homogeneous boundary conditions on the square. We have found that there is also a nonlinear degeneracy in this case. Fortunately, the normal form is still determined by third order terms even with this degeneracy and so this does not mean that we need to consider any higher order terms.

We are not aware of specific examples where the linear degeneracy occurs but the nonlinear degeneracy does not. In order to get this, it seems that we need to consider problems were there is no separable basis for the spatial part of the eigenfunctions but the eigenfunctions for the centre eigenspace are separable. This seems possible although exceptional.

We conjecture that degeneracies in nonlinear terms will also appear at orders higher than third but have not attempted to find them.

Finally, we remark that the analysis here may be of use when examining mode interactions of Hopf bifurcations with $\mathbf{Z}_2$ symmetry; in this case, the normal form (8.1) will naturally arise, but with an extra parameter at the linear level.

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