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PRESERVATION OF PROPERTIES IN DISCRETE-TIME SYSTEMS UNDER SUBSTITUTIONS

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ABSTRACT

For discrete-time systems, using substitutions of the variable z by $\mathbb{M}_1$ functions, we give results on preservation of stability, stabilization, positive real (PR), strictly positive real (SPR), bounded real (BR), and strictly bounded real (SBR) functions, along with the $\mathscr{H}_\infty$-norm. These results can be interpreted in the sense of robust control or uniform systems. Based on these properties, we present results about preservation of the SPR Lemma and absolute stability for discrete-time descriptor systems using linear matrix inequalities (LMIs) and substitutions.

Key Words: Discrete-time systems, SPR and PR functions, substitutions, $\mathbb{M}_1$ functions, preservation of properties.

I. INTRODUCTION

The study of substitutions in the frequency domain for discrete-time systems is not new. For instance, in [1], a spectral transformation for digital filters is used and it is proven that these spectral transformations (all-pass functions) transform a given low-pass-digital-filter into a pulse-transfer function that has the same type of amplitude characteristics and that belongs to certain classes of digital filters.

In [2], a theory is developed for spectral transformation of two-dimensional digital filters. It is proven that these transformations take the form of stable two-dimensional all-pass functions and that the result of the spectral transformation is stable, provided that the original transfer function is stable. This result is used to obtain new designs of two-dimensional digital filters from a previous one.

In [3], a technique is presented to design low-pass circularly symmetric 2D Infinite Impulse Response (IIR) digital filters from a 1D digital filter transfer function via the use of substitutions of the variable z in the 1D digital filter transfer function by a 2D system. In this case, the Bounded-Input Bounded-Output (BIBO) stability property is preserved if the transformation is free of poles in the unit disk.

In continuous-time systems, a justification for studying substitutions comes from the so-called uniform systems, *i.e.* linear time-invariant systems consisting of identical components and amplifiers described in terms of a proper transfer function $W(s)=\frac{N(f(s))}{D(f(s))}$, where $N(s)$ and $D(s)$ are real polynomials and $f(s)$ is a proper transfer function (Fig. 1 illustrates a uniform system).

If $f(s)$ is the correct kind of strictly positive real function, one could study the system $N(s)/D(s)$ in place of the more complex and higher order $W(s)$. Concerning the study of uniform systems, a general criterion for robust stability was established in [4]. By applying such a criterion, one attains a generalization of the celebrated Kharitonov's theorem [5], as well as some robust stability criteria for $\mathscr{H}_\infty$ -uncertainty. As far as robust stability of polynomial families is concerned, some Kharitonov's like results are given in [6]

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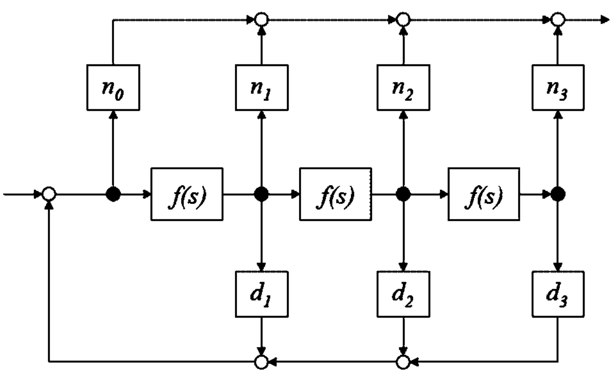

Fig. 1. An example of a uniform system, in which $f(s)$ is a single-input single-output transfer function and n_i (for $i = 0, \ldots, 3$) and d_j (for $j = 1, 2, 3$) are gains. The transfer function of the system is $W(s) = \sum_{i=0}^{3} n_i f^i / 1 + \sum_{j=0}^{3} d_j f^j = N(f(s))/D(f(s))$.

(for a particular class of polynomials), when interpreting substitutions as nonlinearly correlated perturbations on the coefficients. With respect to the robustness-oriented properties preservation in a rational transfer function (modified by strictly positive real (SPR) substitutions), some results linked to $\mathscr{H}_\infty$-robustness are given in [7] for linear time invariant (LTI) single-input single-output (SISO) systems, while the corresponding multiple input multiple output (MIMO) case is presented in [8]. [9] provides the characterization of families of algebraic Riccati equations associated with SISO systems bounded in $\mathscr{H}_\infty$-norm terms (the positive real (PR) substitutions acting on the bounded SISO systems). All these results have the common factor of corresponding to scalar PR substitutions performed in scalar rational functions or/and matrix rational functions.

In this paper, several results reported in [8, 10] for the continuous-time case are extended for SISO LTI discrete-time systems by using M_1 functions (see Definition 4). It should be stressed that spectral transformations have been of particular importance in the design of digital filters. In fact, $\mathbb{M}_1$ functions can be seen as similar to spectral transformation that is bounded in the unitary circle. In this paper, it will be shown that $\mathbb{M}_1$ functions preserve stability, positive real (PR) functions, strictly positive real (SPR) functions, bounded real (BR) functions, strictly bounded real (SBR) functions, and the $\mathscr{H}_\infty$-norm for discrete-time systems when the substitution of the variable z by a $\mathbb{M}_1$ function is used. Additionally, $\mathbb{M}_1$ functions are closed under composition of functions. Also, spectral transformations preserve stability of digital filters (linear discrete-time systems) where spectral transformations must be all-pass transfer functions which transform the unitary complex circle internally [1]. However, since the $\mathbb{M}_1$ functions do not transform the unitary complex circle internally, the $\mathbb{M}_1$ functions are different from the spectral transformations.

The previous preservation results are applied to obtain results on preservation of the SPR Lemma and absolute stability for discrete-time descriptor systems, when the substitution of variable z by an $\mathbb{M}_1$ function is used. These results can be interpreted in the sense of robustness when considering nonlinear perturbations on the plant parameters induced by the substitution of the $\mathbb{M}_1$ function in the sense of [6, 9].

II. PRELIMINARIES

This section presents the notation and definitions which will be used throughout the paper. Let $\mathbb{R}$ and $\mathbb{C}$ be fields of real and complex numbers, respectively, and $\mathbb{C}^+$ the open right-half complex plane; define the following sets:

$$T = \{ z \in \mathbb{C} : |z| > 1 \}$$
$$\partial T = \{ z \in \mathbb{C} : |z| = 1 \}$$
$$\bar{T} = T \cup \partial T$$
$$\bar{T}_e = \bar{T} \cup \{\infty\}$$

In this paper, we will study the cases of SISO and LTI discrete-time systems.

Definition 1. The rational real function $G(z) = \frac{N(z)}{D(z)}$ with $N(z)$ and $D(z)$ polynomials in $z \in \mathbb{C}$ is Schur stable if $D(z)$ has all its roots inside the unitary circle, *i.e.*, the roots of $D(z)$ are in $|z| < 1$.

Definition 2. ([*11–13*]) A rational real function $G(z)$ is a strictly positive real (SPR) function if

(2.1) $G(z)$ is analytic in $\bar{T}$, *i.e.*, $G(z)$ is Schur stable.
(2.2) $\mathrm{Re}[G(z)] > 0$ for $z \in \bar{T}$.

In [13] it was shown that if $G(z)$ is SPR then $G(\infty) > 0$. Therefore, Definition 2 is equivalent to the following: A real rational function $G(z)$ is SPR if $G(z)$ is analytic in $\bar{T}_e$ and $\mathrm{Re}[G(z)] > 0$ for $z \in \bar{T}_e$, *i.e.* $G(\bar{T}_e) \subseteq \mathrm{C}^+$.

Definition 3. ([*11–13*]) A rational real function $G(z)$ is a positive real (PR) function if

(3.1) $G(z)$ is analytic in $\bar{T}$, *i.e.*, $G(z)$ is Schur stable.
(3.2) $\mathrm{Re}[G(z)] \geq 0$ for $z \in \bar{T}$.

Definition 4. A rational real function $G(z)$ of zero relative-degree is called a $\mathbb{M}_1$ function if

(4.1) $G(z)$ is analytic in $\bar{T}$, *i.e.* Schur stable.
(4.2) $\frac{1}{G(z)}$ is analytic in $\bar{T}$, *i.e.* minimum phase.

(4.3) $|G(e^{j\theta})| > 1$ for all $\theta \in [0, 2\pi]$, *i.e.* $G(\partial T) \subseteq T$

(4.4) $|G(\infty)| > 1$.

Notice that if $G(z)$ is an $\mathbb{M}_1$ function, then $-G(z)$ is also an $\mathbb{M}_1$ function. Moreover, by (4.4) $G(z)$ has zero relative degree. On the other hand, $\mathbb{M}_1$ functions can be interpreted as a generalization of bilinear transformations, *i.e.*, real rational transformations.

Definition 5. ([*14, 13*]) A rational real function $G(z)$ is a strictly bounded real (SBR) function if

(5.1) $G(z)$ is analytic in $\bar{T}$.

(5.2) $\|G(z)\|_\infty < 1$, with $\|G(z)\|_\infty := \sup_{\theta\in[0,2\pi]} |G(e^{j\theta})|$

In the case where condition 5.1 is satisfied and $\| G(z) \|_\infty \leq 1$, the function $G(z)$ is called a bounded real (BR) function.

The following lemma is proven in [15].

Lemma 6. Let $G(z)$ be analytic in the closed bounded domain $\bar{T}$ and assume that $G(z) \neq 0$ anywhere in $\bar{T}$. Then, the minimum of $|G(z)|$ is reached on the boundary of $\bar{T}$, unless $G(z)$ is a constant.

III. RESULTS ON PRESERVATION OF PROPERTIES UNDER SUBSTITUTIONS

In this section, the main technical results of the paper are presented. These results will be used to prove the results of the following applications section.

Theorem 7. If $H(z)$ is a rational stable function and $G(z)$ is a $\mathbb{M}_1$ function, then $H(G(z))$ is a rational stable function and $G(\bar{T}_e) \subseteq T$.

Proof. First, it is proven that $\mathbb{M}_1$ functions are analytic in $\bar{T}_e$. From Definition 4, we have:

$$\begin{aligned} G(\bar{T}_e) &= G(T \cup \partial T \cup \{\infty\}) \\ &= G(T) \cup G(\partial T) \cup G(\infty) \\ &\subseteq G(T) \cup G(\partial T) \cup \{c\} \end{aligned}$$

where $G(\infty) = c$ and $|c| > 1$, $c \in T$; therefore, $G(z)$ is analytic in $\bar{T}_e$. Since $G(z)$ and $\frac{1}{G(z)}$ are analytic in $\bar{T}$, then $G(z) \neq 0$ and $\min_{\theta\in[0,2\pi]} |G(e^{j\theta})| > 1$. From Lemma 6, we also know that $|G(z)| > 1$ for each $z \in \bar{T}_e$. Therefore, $G(T) \subseteq T$ and $G(\bar{T}_e) \subseteq T$. Remember that the composition of analytic functions is an analytic function. The function $H(z)$ is an analytic function in $\bar{T}_e$ and as $H(G(\bar{T}_e)) \subseteq H(T)$ in consequence $H(G(z))$ is an analytic function in $\bar{T}_e$. □

Theorem 7 establishes that $\mathbb{M}_1$ functions preserve stability of SISO LTI discrete-time systems when substituted for the variable z; the same property is valid for SPR0 functions in the continuous case [10].

Theorem 8. The composition of $\mathbb{M}_1$ functions is an $\mathbb{M}_1$ function.

Proof. It is enough to show that Definition 4 holds for the function $G_2(G_1(z))$ where $G_1(z)$ and $G_2(z)$ are $\mathbb{M}_1$ functions. From Theorem 7, it is known that if $G_1(\bar{T}_e) \subseteq T$; $G_1(z)$ and $G_2(z)$ are analytic in $\bar{T}_e$, then $G_2(G_1(z))$ is analytic in T. Since $G_1(\bar{T}_e) \subseteq T$, $G_1(z)$ is analytic in $\bar{T}_e$ and $\frac{1}{G_2(z)}$ is analytic in T, $\frac{1}{G_2(G_1(z))}$ is analytic in T. As $G_1(\partial T) \subseteq T$ and $G_2(\partial T) \subseteq T$, $G_2(G_1(\partial T)) \subseteq T$. As $|G_1(\infty)| \in T$ and $G_2(T) \subseteq T$, $|G_2(G_1(\infty))| \in T$. □

Theorem 8 establishes that $\mathbb{M}_1$ functions are closed under the composition of functions. As a matter of fact, these results are similar to Theorem 1 and 2 in [10] for continuous-time systems.

Theorem 9. (*SPR and PR properties*) If $H(z)$ is a SPR (PR) function and $G(z)$ is an $\mathbb{M}_1$ function, then $H(G(z))$ is a SPR (PR) function.

Proof. We need to show that the function $H(G(z))$ is analytic in $\bar{T}_e$ and $H(G(\bar{T}_e)) \subseteq \mathbb{C}^+$. By Theorem 7, we know that when $G(\bar{T}_e) \subseteq T$, $G(z)$ and $H(z)$ are analytic in $\bar{T}_e$, then $H(G(z))$ is analytic in $\bar{T}_e$. Since $G(\bar{T}_e) \subseteq T$ and $\mathrm{Re}[H(z)] > 0$ for each $z \in \bar{T}_e$ then $H(G(\bar{T}_e)) \subseteq \mathbb{C}^+$. The proof for PR functions is similar. □

Theorem 7, Theorem 8, and Theorem 9 establish that the substitution of the variable z by $\mathbb{M}_1$ functions preserves stability, SPR and PR properties and that the class of $\mathbb{M}_1$ functions is closed under composition. Notice that substitution of a SPR function by the z variable does not preserve stability for discrete-systems as is the case for continuous systems [8, 10].

Proposition 10. Suppose that $H(z)$ is Schur stable, proper and $H(z) \neq 1$ for $|z| = 1$.

(10.1) If $H(z)$ is a SBR function and $G(z)$ is a $\mathbb{M}_1$ function, $H(G(z))$ is a SBR function.

(10.2) If $H(z)$ is a BR function and $G(z)$ is a $\mathbb{M}_1$ function, then $H(G(z))$ is a BR function.

(10.3) If $\|H(z)\|_\infty \leq \gamma_H$, then $\|H(G(z))\|_\infty \leq \gamma_H$ for each $\mathbb{M}_1$ function $G(z)$.

Proof.

(10.1) Define $H_\gamma(z) \equiv \lambda_H^{-1} H(z)$ where $\gamma_H = \|H(z)\|_\infty$. By Lemma 1 in [13], if $H_\gamma(z) \neq 1$,

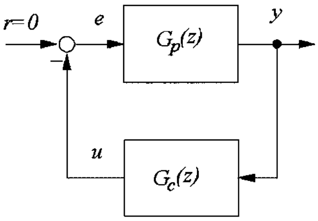

Fig. 2. Closed loop system.

then $\bar{H}_\gamma(z) = \frac{1+H_\gamma(z)}{1-H_\gamma(z)}$ is SPR if and only if $H_\gamma(z)$ is SBR. Now, by Theorem 9, $\bar{H}_\gamma(G(z)) = \frac{1+H_\gamma(G(z))}{1-H_\gamma(G(z))}$ is SPR for each $\mathbb{M}_1$ function $G(z)$. Then, by Lemma 1 in [13], $\bar{H}_\gamma(G(z))$ is SBR for each $\mathbb{M}_1$ function $G(z)$.

(10.2) The proof is similar to the proof of the above item.

(10.3) Using Definition 5 and Item (1) of this theorem, we have that $\bar{H}_\gamma(G(z))$ is analytic in $|z| \geq 1$, and $\|H_\gamma(G(z))\|_\infty \leq 1$ for each $\mathbb{M}_1$ function $G(z)$. Now, as $\|H_\gamma(G(z))\|_\infty = \gamma_H^{-1}\|H(G(z))\|_\infty \leq 1$, $\|H(G(z))\|_\infty \leq \gamma_H$ for each $\mathbb{M}_1$ function $G(z)$. □

Proposition 10 establishes that when substitutions of the variable z by $\mathbb{M}_1$ functions are used, it preserves BR and SBR functions and the $\mathscr{H}_\infty$-norm. This result is similar to Proposition 9 in [8] but uses $\mathbb{M}_1$ functions instead of SPR functions.

Corollary 11. Consider the closed-loop system shown in Fig. 2. If the output feedback $u(z) = G_c(z)y(z)$ stabilizes the system $G_p(z) = \frac{N_p(z)}{D_p(z)}$, *i.e.*, the closed-loop system

$$M_c(z) = \frac{G_c(z)G_p(z)}{1 + G_c(z)G_p(z)} \tag{1}$$

is Schur stable. Then, the output feedback $u(z) = G_c(G(z))y(z)$ stabilizes system

$$G_p(G(z)) = \frac{N_p(G(z))}{D_p(G(z))} \tag{2}$$

for each $\mathbb{M}_1$ function $G(z)$. Moreover, if the constant output feedback $u(z) = Ky(z)$ stabilizes system $G_p(z) = \frac{N_p(z)}{D_p(z)}$, the same constant output feedback stabilizes the plant $M_K(G(z))$ for each $\mathbb{M}_1$ function $G(z)$ where $M_k(z) = \frac{KG_p(z)}{1+KG_p(z)}$.

Proof. The stability of $M_c(G(z))$ is a direct consequence of the substitution $z \to G(z)$ with $G(z)$, an $\mathbb{M}_1$ function that preserves sums, products, quotients, constants and Theorem 7. The stability of $M_K(G(z))$ by a constant output feedback $u(z) = Ky(z)$ is a consequence of Theorem 7 and the previous paragraph. □

This result establishes that the substitution $z \to G(z)$ with $G(z)$ a $\mathbb{M}_1$ function preserves stabilization *i.e.* the new controller stabilizes the new plant after the substitution, and proportional controllers robustly stabilize systems under nonlinearly correlated perturbations induced by substitutions with $\mathbb{M}_1$ functions as long as the parameters of the $\mathbb{M}_1$ functions vary continuously; this interpretation is also possible for all the previous results. If we vary the order of the $\mathbb{M}_1$ function $G(z)$, then we obtain a result of simultaneous stabilization for the systems $M_K(G_1(z)), \ldots, M_K(G_n(z))$, using the proportional controller K where the order of the systems $G_1(z), \ldots, G_n(z)$ are different [7, 8, 10].

IV. RESULTS FOR DISCRETE-TIME DESCRIPTOR SYSTEMS

In this section, we extend the result for discrete-time descriptor systems about the SPR Lemma [13, 16] and absolute stability for these systems with a memoryless time-varying nonlinearity in the feedback path [13, 17].

We consider the following discrete-time descriptor system:

$$\begin{aligned} Ex(k+1) &= Ax(k) + Bw(k) \\ y(k) &= Cx(k) + Dw(k) \end{aligned} \tag{3}$$

where $x(k) \in \mathbb{R}^n$ is the descriptor variable, $w(k) \in \mathbb{R}$ is the exogenous input, and $y(k) \in \mathbb{R}$ is the measured output. The matrix $E(k) \in \mathbb{R}^{n\times n}$ has $\text{rank}(E) = r \leq n$. The other matrices are assumed to have appropriate sizes. The pair (E, A) is employed when only the behavior of the descriptor variable in an unforced system (3) is concerned.

Definition 12. ([*14, 13*]) For the descriptor system given by (3)

(12.1) A pair (E, A) is called regular if $\det(zE - A)$ is not identically zero.

(12.2) A pair (E, A) is called impulse-free if $(zE - A)^{-1}$ is proper.

(12.3) A pair (E, A) is called admissible if it is regular, impulse-free and stable (Schur stable).

For a regular descriptor system (3), the transfer function from w to y is well-defined by

$$T_{yw}(z) = C(zE - A)^{-1}B + D$$

and the existence and uniqueness of the solution to (3) are guaranteed for any specified initial condition. It is proven in [14] that the pair (E, A) is regular if and only if there exist two (non-unique) real nonsingular matrices M and N such that:

$$MEN = \begin{bmatrix} I_r & 0 \\ 0 & J \end{bmatrix}, \quad MAN = \begin{bmatrix} A_r & 0 \\ 0 & I_{n-r} \end{bmatrix}$$

where $J \in \mathbb{R}^{(n-r)\times(n-r)}$ is nilpotent and $A \in \mathbb{R}^{r\times r}$ determines exactly the r finite eigenvalues of the matrix pencil $zE - A$. By partitioning MB and CN as follows:

$$MB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad CN = [C_1 \;\; C_1]$$

we have

$$\begin{aligned} T_{yw}(z) &= C(zE - A)^{-1}B + D \\ &= C_1(zI_r - A_r)^{-1}B_1 \\ &\quad + C_2(zJ - I_{n-r})^{-1}B_2 + D \end{aligned}$$

By Definition 12, the regular pair is impulsive-free if and only if $J = 0$. Therefore, the partitions of MEN and MAN are

$$MEN = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}, \quad MAN = \begin{bmatrix} A_r & 0 \\ 0 & I_{n-r} \end{bmatrix}$$

and transfer function $T_{yw}(z)$ is simplified to

$$T_{yw}(z) = C_1(zI_r - A_r)^{-1}B_1 - C_2B_2 + D$$

This indicates the equivalence, in the sense of transfer matrix representation, of the n-dimensional regular and impulse-free descriptor system (3) denoted by (E, A, B, C, D), and the r-dimensional state-space system $(I_r, A_r, B_1, C_1, D - C_2B_2)$ associated with the transfer function $T_{yw}(z)$. It is clear that the transfer function $T_{yw}(z)$ is Schur stable if and only if the matrix A_r is stable.

At this point, we consider the substitution of the variable z by a $\mathbb{M}_1$ function $G(z)$ in the transfer function $T_{yw}(z)$, as follows:

$$\begin{aligned} T_{yw}(G(z)) &= C_1(G(z)I_r - A_r)^{-1}B_1 - C_2B_2 + D \\ &= C_3(zI_{rm} - A_3)^{-1}B_3 + D_3 \end{aligned}$$

As a consequence, we have linked the transfer function $T_{yw}(G(z))$ to an rm-dimensional state-space system $(I_{rm}, A_3, B_3, C_3, D_3)$ where the order of $G(z)$ is m. $T_{yw}(G(z))$, though, is also the transfer function of some descriptor system $(\hat{E}, \hat{A})$ with matrices $\hat{B}, \hat{C}, D$, *i.e.*, $T_{yw}(G(z)) = \hat{C}(z\hat{E} - A)^{-1}B + D$.

Proposition 13. If there exists a matrix $P = P^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ satisfying the following linear matrix inequalities (LMI)

$$\begin{bmatrix} A^{\mathrm{T}}PA - E^{\mathrm{T}}PE + C^{\mathrm{T}}C & A^{\mathrm{T}}PB + C^{\mathrm{T}}D \\ B^{\mathrm{T}}PA + D^{\mathrm{T}}C & -\gamma^2 I + B^{\mathrm{T}}PB + D^{\mathrm{T}}D \end{bmatrix} < 0$$

and $E^{\mathrm{T}}PE \geq 0$, then $(\hat{E}, \hat{A})$ is admissible and $\|T_{yw}(G(z))\|_\infty < \gamma$ for each $\mathbb{M}_1$ function $G(z)$.

Proof. By Theorem 1 in [13] (*cf.* [16]), the conditions of this proposition imply that the pair (E, A) is admissible and $\|T_{yw}(z)\|_\infty < \gamma$. Now, we prove first that the pair $(\hat{E}, \hat{A})$ is admissible. Using Definition 12, the pair (E, A) is regular, thus, $\det(G(z)E - A)$ is not identical to zero for each function G(z) $\in \mathbb{M}_1$. Also, the pair (E, A) is impulse-free, then $(G(z)E - A)^{-1}$ is proper for each function G(z) $\in \mathbb{M}_1$ because each $\mathbb{M}_1$ function has zero relative degree by Definition 4. If the pair (E, A) is stable, then $T_{yw}(z)$ is a rational stable function, thus by Theorem 7, $T_{yw}(G(z))$ is a rational stable function, therefore, the pair $(\hat{E}, \hat{A})$ is stable and admissible. Now, by Proposition 10, if $\|T_{yw}(z)\|_\infty < \gamma$, then $\|T_{yw}(G(z))\|_\infty < \gamma$ for each function G(z) $\in \mathbb{M}_1$. □

Proposition 14. If there exists a matrix $P = P^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ satisfying the following LMIs:

$$\begin{bmatrix} A^{\mathrm{T}}PA - E^{\mathrm{T}}PE & A^{\mathrm{T}}PB + C^{\mathrm{T}} \\ B^{\mathrm{T}}PA + C & -D - D^{\mathrm{T}} + B^{\mathrm{T}}PB \end{bmatrix} < 0$$

and $E^{\mathrm{T}}PE \geq 0$, then $(\hat{E}, \hat{A})$ is admissible and $T_{yw}(G(z))$ is SPR for each $\mathbb{M}_1$ function $G(z)$.

Proof. The proof is similar to the previous one, but Theorem 9 is used instead of Proposition 10. In consequence, if $T_{yw}(z)$ is SPR, then $T_{yw}(G(z))$ is SPR for each function G(z) $\in \mathbb{M}_1$. □

This result is a generalization of Theorems 1 and 3 in [13]. Our results guarantee that the admissibility, $\mathscr{H}_\infty$-norm, and SPR property of the pair (E, A) (rational stable function $T_{yw}(z)$) are preserved when considering nonlinear perturbations on the plant parameters resulting from the substitution of the variable z by a $\mathbb{M}_1$ function $G(z)$. Notice that the LMIs are the same as in the original results.

The following provides an extension of a result of absolute stability for discrete-time descriptor systems. Consider the feedback connection of the discrete-time descriptor system (3) and the memoryless time-varying nonlinearity in the feedback path:

$$w(k) = -\Phi(k, y(k)) \tag{4}$$

where the memoryless time-varying nonlinearity $\Phi : \mathbb{Z}^+ \times \mathbb{R}^m \to \mathbb{R}^m$ is restricted to the first and third quadrants, *i.e.*, the following sector condition is satisfied for all nonnegative integer k

$$\Theta(k, y(k))y(k) \geq 0, \text{ for all } y(k) \in \mathbb{R}^m \tag{5}$$

and $\Phi(k, 0) = 0$.

The following necessary definition of global uniformly asymptotic stability is recalled.

Definition 15. ([*17*]) Consider the well-defined feedback connection of the discrete-time descriptor system (3) and the nonlinear control (4). The equilibrium point $x = 0$ is

(15.1) Uniformly stable if, for each ε and any initial time $k_0 \geq 0$, there is $\delta(\varepsilon)$, independent of k_0, such that $\|x(k_0)\| < \delta$ implies $\|x(k)\| < \varepsilon$ for all $k \geq k_0$;

(15.2) Globally uniformly asymptotically stable if it is uniformly stable and, for each pair of positive numbers ε and σ, there is a nonnegative constant that may depend on σ and ε, $K(\varepsilon, \sigma) \geq 0$ such that $\|x(k)\| < \varepsilon$ for all $k \geq k_0 + K(\varepsilon, \sigma)$ whenever $\|x(k_0)\| < \sigma$

Corollary 16. If (E, A) is admissible and $T_{yw}(z)$ is SPR, then the equilibrium point of the feedback system of the discrete-time descriptor system

$$\begin{aligned} \hat{E}x(k+1) &= \hat{A}x(k) + \hat{B}w(k) \\ y(k) &= \hat{C}x(k) + \hat{D}w(k) \end{aligned} \tag{6}$$

with a memoryless time-varying nonlinearity in the feedback path (4) that satisfies the sector condition (5), is globally uniformly asymptotically stable for each $\mathbb{M}_1$ function $G(z)$, where $T_{yw}(G(z))$ is the transfer function of some descriptor system $(\hat{E}, \hat{A}, B, C, D)$, *i.e.*, $T_{yw}(G(z)) = \hat{C}(z\hat{E} - A)^{-1}B + D$

Proof. By Theorem 9, if $T_{yw}(z)$ is SPR, then $T_{yw}(G(z))$ is SPR for each $\mathbb{M}_1$ function $G(z)$. As shown in Proposition 14, if (E, A) is admissible, then $(\hat{E}, \hat{A})$ is admissible for each $\mathbb{M}_1$ function $G(z)$. Now, by Theorem 4 in [13], if $(\hat{E}, \hat{A})$ is admissible and $T_{yw}(G(z))$ is SPR, then the equilibrium point of the feedback system of (6) and the control (4) is globally uniformly asymptotically stable for each $\mathbb{M}_1$ function $G(z)$. □

This result is a generalization of Theorem 4 in [13]. Our result guarantees that this closed-loop system is globally uniformly asymptotically stable for each $\mathbb{M}_1$ function $G(z)$. Thus, the new closed-loop system is robustly globally uniformly asymptotically stable when considering nonlinear perturbations on the plant parameters induced by the substitution of the $\mathbb{M}_1$ function $G(z)$, and for any memoryless time-varying nonlinearity Φ satisfying the sector condition (5).

Example 17. Consider the discrete-time descriptor system (3) with system matrices (Example 1 in [13]):

$$E = \begin{bmatrix} 4 & 0 & 0 & 4 \\ -8 & 4 & 0 & -4 \\ 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 4 \end{bmatrix}$$

$$A = \begin{bmatrix} -6 & 2 & 0 & -4 \\ 7 & -4 & 0 & 3 \\ 0 & 0 & 1 & 0 \\ -6 & 2 & 0 & -3 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad C = [-1\ 0\ 0\ 0], \quad D = 3$$

The corresponding transfer function

$$T_{yw}(z) = 2\frac{16z^2 + 23z + 9}{8z^2 + 12z + 5}$$

is an SPR function. Now, consider the following family of $\mathbb{M}_1$ functions (bilinear transformations):

$$\begin{aligned} &\Delta(a, b, c, d) \\ &\quad \triangleq \left\{ G(z) = N\frac{az + b}{cz + d} : -\frac{b}{a}, -\frac{d}{c} \right. \\ &\qquad \left. \in (-1, 1),\ N\frac{a}{c} > 1,\ N\left|\frac{a-b}{c+d}\right| > 1 \right\} \end{aligned}$$

Then, by Theorem 9, the family of transfer functions

$$\Omega(a, b, c, d) \triangleq \{T_y w(G(z)) : G(z) \in \Delta(a, b, c, d)\}$$

can be interpreted like the robustness of the property SPR of the transfer function $T_{yw}(z)$ under the nonlinear perturbation on the parameters of the system $T_{yw}(z)$ induced by the substitution of the $\mathbb{M}_1$ functions $G(z)$

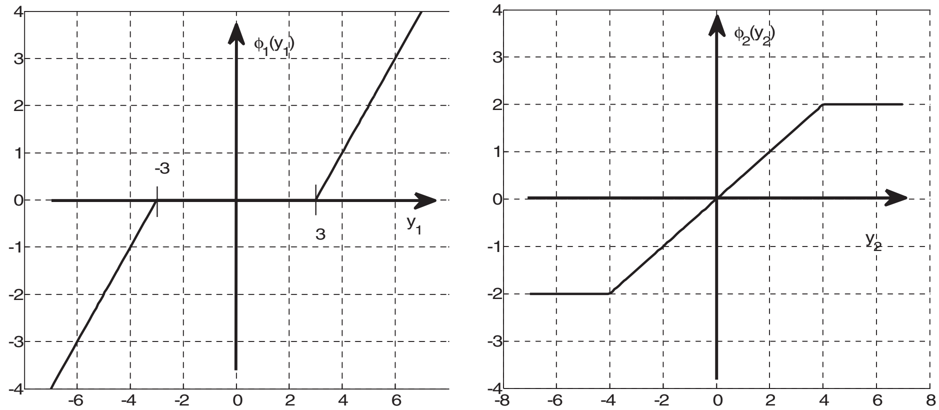

Fig. 3. Memoryless nonlinearities.

taken from the set $\Delta(a, b, c, d)$. In other words, a transfer function

$$T_{yw}(G(z)) = \frac{z^2(46Nac + 18c^2 + 32N^2a^2) + z(36cd + 46Nad + 46Nbc + 64N^2ab) + 46Nbd + 18d^2 + 32N^2b^2}{z^2(12Nac + 5c^2 + 8N^2a^2) + z(10cd + 12Nad + 12Nbc + 16N^2ab) + 12Nbd + 5d^2 + 8N^2b^2}$$

is an SPR function, for all reals a, b, c, d such that the following conditions are met:

1. $-\frac{b}{a}, -\frac{d}{c} \in (-1, 1)$
2. $N\frac{a}{c} > 1$
3. $N\left|\frac{a-b}{c+d}\right| > 1$

i.e. the transfer functions $N\frac{az+b}{cz+d}$ are $\mathbb{M}_1$ functions.

Now, consider two forms of memoryless nonlinearity in the feedback path:

(i) The decoupled time-invariant nonlinearity, $\Phi(y(k)) \stackrel{\Delta}{=} [\phi_1(y_1(k)) \;\; \phi_2(y_2(k))]^{\mathrm{T}}$, where ϕ_1(? is a dead-zone nonlinearity and ϕ_2(? is a saturation nonlinearity, as shown in Fig. 3.

(ii) The couple time varying nonlinearities $\bar{\Phi}(y(k)) \stackrel{\Delta}{=} [\bar{\phi}_1(y_1(k)) \;\; \bar{\phi}_2(k, y_2(k))]^{\mathrm{T}}$ where $\bar{\phi}_1(y(k)) = y_1(k)y_2^2(k)$, $\bar{\phi}_2(k, y(k)) = 2ky_2(k)$.

Since (E, A) is admissible and $T_{yw}(z)$ is SPR, then, by Corollary 16, the equilibrium point of the feedback system of the discrete-time descriptor system (3) with each one of the memoryless nonlinearities in the feedback path i) and ii) is globally uniformly asymptotically stable for each $\mathbb{M}_1$ function $G(z)$, in particular for all $G(z) \in \Delta(a, b, c, d)$, where $T_{yw}(G(z))$ is the transfer function of some descriptor system $(\hat{E}, \hat{A})$ and the new matrices $(\hat{E}, \hat{A}, \hat{B}, \hat{C}, D)$ are functions of the parameters (a, b, c, d). In that sense, we have the robust global uniformly asymptotic stability property of the equilibrium point of the feedback system of the discrete-time descriptor system (3), with each of the memoryless nonlinearities (i) and (ii) in the feedback path under nonlinear perturbation on the parameters of the system $T_{yw}(z)$, induced by the substitution of the $\mathbb{M}_1$ functions $G(z)$, taken from the set $\Delta(a, b, c, d)$, and more general for all $\mathbb{M}_1$ functions $G(z)$.

The class of parametric perturbations that can be modeled by the substitution of $\mathbb{M}_1$ functions are nonlinear polynomial relations among the parameters of the transfer function and the coefficients of the $\mathbb{M}_1$ function, which is induced by the composition.

V. CONCLUSION

New results on the preservation of stability $\mathscr{H}_\infty$-norm, PR, SPR, BR, and SBR properties for discrete-time systems under substitutions by $\mathbb{M}_1$ functions were proven. These results are different from the continuous case because the substitution is via $\mathbb{M}_1$ functions instead of SPR functions. Nevertheless, the performance is similar to the continuous case [7–9], and [10]. Also, notice that $\mathbb{M}_1$ functions are not spectral transformations since $\mathbb{M}_1$ functions do not transform the unitary complex circle internally [1]. Based on the previous results, conditions for the preservation of the SPR Lemma and absolute stability from discrete-time descriptor systems, when the substitution of the variable z by a $\mathbb{M}_1$ function is used, are presented. The LMIs involved in Propositions 13 and 14 are the same LMIs in

Theorems 1 and 3 in [13], respectively; also, the conditions in Corollary 16 are the same as Theorem 4 in [13]. Consequentially, Theorems 1, 3, and 4, presented in [13], are robust when considering nonlinear perturbations on the system's parameters induced by the substitution of $\mathbb{M}_1$ functions in the sense of [6]. In particular, Corollary 16 is a result of robust global uniformly asymptotic stability of the equilibrium point of the feedback connection for descriptor systems. These results can be easily tested by using currently available software packages for solving problems in LMIs. Further work on the relationship between the $\mathbb{M}_1$ functions and the SPR0 functions currently is under development.

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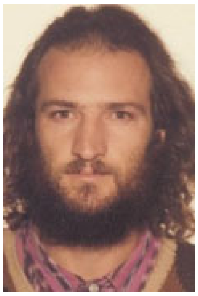

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