

Heat and Gravitation. I. The Action Principle

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ABSTRACT Some features of hydro- and thermodynamics, as applied to atmospheres and to stellar structures, are puzzling: 1. The suggestion, first made by Laplace, that our atmosphere has an adiabatic temperature distribution, is confirmed for the lower layers, but the reason why it should be so is difficult to understand. 2. The standard treatment of relativistic thermodynamics does not allow for a systematic treatment of mixtures, such as the mixture of a perfect gas with radiation. 3. The concept of mass in applications of general relativity to stellar structures is less than completely satisfactory. 4. Arguments in which a concept of energy plays a role, in the context of hydro-thermodynamical systems and gravitation, are not always convincing. It is proposed that a formulation of thermodynamics as an action principle may be a suitable approach to adopt for a new investigation of these matters.

In this first article of a series we formulate the thermodynamics of ideal gases in a constant gravitational field in terms of an action principle. The theory, in its simplest form, does not differ from standard practice, but it lays the foundations for a more systematic approach to the various extensions, such as the incorporation of radiation, the consideration of mixtures and the incorporation of General Relativity. An important conclusion is that a dynamical theory that incorporates the isothermal equilibrium of an ideal gas in a gravitational field does not seem to exist. The true equilibrium state of an ideal gas may in fact be adiabatic, in which case the role of solar radiation is merely to compensate for the loss of energy due to black body radiation into the cosmos. We study the interaction between an ideal gas and the photon gas, and propose a new approach to this problem. An experiment that involves a centrifuge and that has wider implications in view of the equivalence principle is proposed, to determine the influence of gravitation on the equilibrium distribution with a very high degree of precision.

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I. Introduction

The premise of this paper is a conviction that the internal consistency of any physical theory is improved, if not assured, by a formulation that is based on an action principle. The consistency of any set of equations is a delicate matter. One frequently finds oneself in a position of having to make some modification somewhere, to include an additional effect or another degree of freedom, and the need to make corresponding adjustments elsewhere is not always evident. An action principle that treats all dynamical variables on an equal footing and without constraints has the great advantage of being “off shell”, something that has been greatly appreciated in the development of supergravity, for example.

Action principles have been used for some problems in hydro-thermodynamics, but as far as we know the injunction to treat all dynamical variables equally has not been obeyed. Thus, in hydrodynamics the temperature is related to the other variables by what amounts to a constraint. The formalism has the aspect of a partial projection of a complete dynamical theory on the space of solutions. Perhaps this is one of the reasons why thermodynamics is difficult to understand.

This paper is a study of atmospheres consisting of an ideal gas, characterized by the ideal gas law and the expression for the internal energy. We ask to what extent the observed polytropic relations are to be attributed to intrinsic properties of the gas, or to radiation. To answer the question one needs to know what is the natural configuration of an isolated atmosphere, one that is not exposed to radiation. Approaching this problem we are guided by our strong preference for action principles.

The statement that any two thermodynamic systems, each in a state of equilibrium with a well defined temperature, and in thermal equilibrium with each other, must have the same temperature, is a central tenet of thermodynamics. A natural extension is that the temperature, in an extended but closed system in a state of equilibrium, must be uniform, but there does not seem to be universal agreement on whether this is true in the presence of gravitational fields. The question comes up in the investigation of terrestrial or stellar atmospheres, where the gravitational forces create a non-uniform density distribution and it is crucial to this study.

Hydrodynamics is a theory of continuous distributions of matter, described in the simplest case by two fields or distributions: a density field and a velocity field, both defined over \mathbb{R}^3 or a portion thereof. The role of temperature is often disguised, taken to be determined by the density and the pressure. Classical thermodynamics, on the other hand, is the study of states of equilibrium, with uniform density and temperature, and relations between such states. In this context, extremum principles play an important role; see Callen (1960), but the extension of thermodynamics to systems in which the dynamical variables are fields on \mathbb{R}^3 is not immediate and in fact variational principles are seldom invoked in studies of such systems. Investigations that deal with flow of matter or with temperatures that vary in time or space are found under the heading of heat transfer, fluctuations, thermodynamics of irreversible processes and radiation hydrodynamics. See for example Stanyukovich (1960), Castor (2004), Müller (2007).

In this introductory section we study a simple system from the point of view of hydrodynamics, on the basis of a well known action principle. The choice of potential ensures that the pressure is given by the ideal gas law. We stress the role of mass and offer a brief

review of the history of the polytropic atmosphere (Section I.7).

In Section II we extend the action principle to include the temperature as an independent field variable (Section II.5). The potential is refined to give the correct expression for the internal energy; the off shell hamiltonian appears to be new. The energy and the pressure of radiation are incorporated in a natural way (Section II.7). In this theory states of equilibrium (in the presence of a static gravitational field) are polytropic, not isothermal.

When this approach is compared to modern hydro-thermodynamics, restricted to adiabatic processes, we find that the basic equations are nearly identical. The equation of continuity and the hydrodynamical equation are the same, as are the two relations that characterize an ideal gas. The “energy conservation equation” of radiation hydrodynamics is identified with the local conservation law that corresponds to the conservation of the hamiltonian, except that the former allows for an extra solution in which the temperature is constant in time (Section II.6). But the incorporation of additional features is much more straightforward within the variational formulation.

An action principle can also describe an isothermal atmosphere (we are always including a static gravitational field), with a gas that satisfies the ideal gas law, but it has a very different internal energy (Section I.6). The equilibrium state of any ideal gas with a finite adiabatic index is essentially polytropic. Since the density is not uniform, neither is the temperature. This conclusion was accepted after considerable struggle, for it goes against one of the basic tenets of thermodynamics: Clausius’ statement of the second law (Section II.8). We try to reconcile the theory with the second law, but the result has not been satisfactory, so far.

There seems to be a dearth of experimental data to guide us in this situation. We study an ideal gas in a centrifuge and invoke the equivalence principle to relate this situation to atmospheres. Experiments are proposed. (Section II.10).

Some further speculations are in the Appendix.

Summary of the main arguments of this paper

We explain how we reached the conclusion that the equilibrium of an ideal gas in the presence of a gravitational field is polytropic and not isothermal.

The proposed action is a functional of independent fields ρ (density), Φ (velocity potential) and T (temperature). Interaction with a constant gravitational field is included. The potential is adjusted so as to give the correct expression for the pressure and the internal energy of an ideal gas with finite adiabatic index. Variation of ρ gives the standard hydrodynamical equation (Bernoulli’s equation), with $p = \mathcal{R}T\rho$ as appropriate for an ideal gas. Variation of Φ gives the continuity equation. Variation of T yields, in the case that radiation is not taken into account, the polytropic relation $\rho/T^n = \text{constant}$. The theory is in complete agreement with the standard treatment(s), except for a detail that we shall now explain.

Consider

$$\frac{\partial h}{\partial t} + \vec{\nabla} \cdot (h\vec{v} + p\vec{v}) = 0.$$

This is the energy conservation equation, third and last equation that makes up the standard theory, after the equation of continuity and Bernoulli’s equation. On the other hand,

using only the first two equations, continuity and Bernoulli, we derive the identity

$$\frac{\partial h}{\partial t} + \vec{\nabla} \cdot (h\vec{v} + p\vec{v}) = \frac{\partial h}{\partial T} \dot{T}.$$

The variational principle demands that $(\partial h / \partial T) = 0$. Therefore, the only difference between the action principle and standard theory is that the latter allows for a spurious solution of the form $T = \text{constant}$, governed exclusively by the two first equations. This circumstance allows standard theory to postulate the existence of an isothermal equilibrium satisfying the two first equations, independent of the choice of internal energy. The action principle does not offer this freedom; isothermal equilibrium is possible, but only for a “gas” that has internal energy density $u = \mathcal{R}T\rho \log \rho$ and *no thermal excitations*. It is significant that no literature seems to exist, that deals with the propagation of a disturbance (e.g. sound waves or shock waves) in an isothermal gas in a gravitational field. We suggest that this gas may belong to science fiction.

To refute this argument one would need to modify or extend the standard theory of polytropic atmospheres to include a parameter that represents the intensity of radiation and that would allow the effect of radiation to be reduced to zero, to connect to the isothermal gas in the limit of no radiation. In the event that our conclusions should be contradicted by experiment, the need to construct such a theory would become no less urgent.

Applications to astrophysics.

The simplest form of the action principle studied here is the non relativistic approximation to a fully relativistic theory that, with the addition of the Einstein-Hilbert action for the metric, has been applied to the dynamics of certain stars (Fronsdal 2007, 2008). The results are close to those of the traditional approach except for certain academic features that make the present approach more attractive to us, such as the preservation of the non-relativistic equation of continuity. The present study was undertaken as preparation for an attempt to take into account the radiation field. In this paper the radiation correction is introduced into the lagrangian; in a future paper we plan to lift it to a relativistic version and return to the study of stellar dynamics.

I.1. Hydrodynamics

Basic hydrodynamics deals with a density field ρ and a velocity field \vec{v} over \mathbb{R}^3 , subject to two fundamental equations, the equation of continuity,

$$\dot{\rho} + \text{div}(\rho\vec{v}) = 0, \quad \dot{\rho} := \frac{\partial \rho}{\partial t}, \tag{1.1}$$

and the hydrodynamical equation (Bernoulli 1738)

$$-\text{grad } p = \rho \frac{D}{Dt} \vec{v} := \rho(\dot{\vec{v}} + \vec{v} \cdot \text{grad } \vec{v}). \tag{1.2}$$

This involves another field, the scalar field p , interpreted as the local pressure. The theory is incomplete and requires an additional equation relating p to ρ . It is always assumed that this relation is local, giving $p(x)$ in terms of the density (and the temperature) at the same point x , and instantaneous.

I.2. Laminar flow

Since there is enough to do without taking on difficult problems of turbulence, we shall assume, here and throughout, that the velocity field can be represented as the gradient of a scalar field,

$$\vec{v} = -\text{grad } \Phi. \quad (1.3)$$

In this case the hydrodynamical condition is reduced to

$$\text{grad } p = \rho \text{ grad } (\dot{\Phi} - \vec{v}^2/2). \quad (1.4)$$

To complete this system one still needs a relation between the fields p and ρ .

Assume that there is a local functional $V[\rho]$ such that

$$p = \rho V' - V, \quad V' := dV/d\rho. \quad (1.5)$$

In this case $dp = \rho dV'$ and the equation (1.4) becomes, if $\rho \neq 0$,

$$\text{grad } V' = \text{grad } (\dot{\Phi} - \vec{v}^2/2) \quad (1.6)$$

or

$$V' = \dot{\Phi} - \vec{v}^2/2 + \lambda, \quad \lambda \text{ constant.} \quad (1.7)$$

The potential $V[\rho]$ is defined by p modulo a linear term, so that the appearance of an arbitrary constant is natural. It will serve as a Lagrange multiplier.

The introduction of a velocity potential guarantees the existence of a first integral of the motion, a conserved energy functional that will play an important role in the theory.

It will turn out that V , with the inclusion of a term linear in ρ that remains undetermined at this stage, is the internal energy density.

I.3. Variational formulation

Having restricted our scope, to account for laminar flows only, we have reduced the fundamental equations of simple hydrodynamics to the following two equations,

$$\begin{aligned} \partial_\mu J^\mu &= 0, \quad J^t := \rho, \quad \vec{J} := \rho \vec{v}, \\ \partial V/\partial \rho &= \dot{\Phi} - \vec{v}^2/2 + \lambda, \end{aligned} \quad (1.8)$$

together with the defining equations

$$\vec{v} = -\text{grad } \Phi, \quad p := \rho V' - V. \quad (1.9)$$

It is well known that these equations are the Euler-Lagrange equations associated with the action (Fetter and Walecka 1980)

$$A[\rho, \Phi] = \int dt d^3x \mathcal{L}, \quad \mathcal{L} = \rho(\dot{\Phi} - \vec{v}^2/2 + \lambda) - V[\rho]. \quad (1.10)$$

The value of this last circumstance lies in the fact that the variational principle is a better starting point for generalizations, including the incorporation of symmetries, of special relativity, and the inclusion of electromagnetic and gravitational interactions. It also gives us a valid concept of a total energy functional.

I.4. On shell relations

The action (1.10) contains only the fields Φ and ρ . The Euler-Lagrange equations define a complete dynamical framework, but only after specification of the functional $V[\rho]$. The pressure was defined by Eq.(1.9), $p := \rho V' - V$, and one easily verifies that, by virtue of the equations of motion,

$$p = \mathcal{L} \quad (\text{on shell}).$$

This fact has been noted, and has led to the suggestion that the action principle amount to minimization of $\int p$ with respect to variations of p defined by thermodynamics (Taub 1954), (Bardeen 1970), (Schutz 1970). But more is needed, including an off shell action. We note that the lagrangian density is not a thermodynamic function, since it depends on the time derivatives of the variables. After adopting the action (1.10) it remains to relate the choice of the potential V to the thermodynamical properties of the fluid. We shall find that the properties that define an ideal gas lead to a unique expression for V .

It is useful to reflect on the meaning of Eq.(1.5) as well. The pressure does not appear as a variable in the variational principle, it is defined by Eq.(1.2); that is, it is a field the gradient of which can be interpreted as a force acting on the fluid. Eq.(1.5) can be interpreted as an equation of state. In a more general situation, in which the potential V depends on the temperature, we would have

$$\rho \operatorname{grad} \frac{\partial V}{\partial \rho} = \operatorname{grad} \left(\rho \frac{\partial V}{\partial \rho} - V \right) + \rho \frac{\partial V}{\partial T} \operatorname{grad} T.$$

The expression (1.5) for the pressure would be valid “on shell” if the action principle includes variations of the temperature as an independent variable, making $\partial V/\partial T = 0$.

In an equivalent formulation we define a field $e = V/\rho$, then

$$\rho^2 \frac{\partial}{\partial \rho} e = - \frac{\partial e}{\partial \mathcal{V}} \Big|_T = p, \quad \frac{\partial e}{\partial T} \Big|_{\mathcal{V}} = 0. \quad (1.11)$$

This identifies the field e as the specific internal energy density. The second equation is satisfied in the present case because V is independent of T . In the general case it suggests that the potential is an extremal with respect to variations of the temperature.

I.5. The mass

To speak of a definite, isolated physical system we must fix some attributes, and among such defining properties we shall include the mass. We insist on this as it shall turn out to be crucial to the logical coherence of the theory (Fronsdal 2008). The density ρ will be taken to have the interpretation of mass density, and the total mass is the constant of the motion

$$M = \int d^3x \rho.$$

Such integrals, with no limits indicated, are over the domain Σ of definition of ρ and is the total extension of our system in \mathbb{R}^3 .

Since the total mass is a constant of the motion it is natural to fix it in advance and to vary the action subject to the constraint $\int_{\Sigma} d^3x \rho(x) = M$. The parameter λ takes on the role of a Lagrange multiplier and the action takes the form

$$A = \int_{\Sigma} d^3x \left(\rho(\dot{\Phi} - \bar{v}^2/2) - V \right) + \lambda \left(\int_{\Sigma} d^3x \rho - M \right).$$

The conservation of mass has important implications for boundary conditions.

I.6. Equation of state and equation of change

An ideal gas at equilibrium, with constant temperature, obeys the gas law

$$p/\rho = \mathcal{R}T. \quad (1.12)$$

Pressure and density are in cgs units and

$$\mathcal{R} = (1/\mu) \times .8314 \times 10^8 \text{ erg}/K,$$

where μ is the atomic weight. The gas law is assumed to hold, locally at each point of the gas. Effective values of μ are

$$\text{Atomic hydrogen : } \mu = 1, \quad \text{Air : } \mu = 29, \quad \text{Sun : } \mu = 2.$$

Equation (1.12) is the only equation that will be referred to as an ‘equation of state’. Other relations, to be discussed next, are ‘equations of change’, this term taken from Emden’s “Zustandsänderung”, for their meaning is of an entirely different sort. Most important is the polytropic relation

$$p = A\rho^{\gamma'}, \quad A, \gamma' \text{ constant.} \quad (1.13)$$

This relation defines a polytropic path or polytrope in the p, \mathcal{V} diagram ($\mathcal{V} = 1/\rho$). A polytropic atmosphere is one in which, as one moves through the gas, the variables ρ and p change so as to remain always on the same polytrope. Eq.(1.13) is a statement about the system, not about the gas *per se*. The validity of (1.13) for an actual atmosphere cannot be inferred from the early laboratory experiments with near-ideal gases.

The index of the polytrope is the positive number n' defined by

$$\gamma' =: 1 + \frac{1}{n'}.$$

Important special cases are

$$n' = 0, \quad \gamma' = \infty, \quad \rho = \text{constant},$$

$$\gamma' = C_P/C_V, \quad \text{specific entropy} = \text{constant},$$

$$n' = \infty, \quad \gamma' = 1, \quad T = \text{constant}.$$

Numbers γ , n are defined by

$$\gamma := C_P/C_V =: 1 + \frac{1}{n}.$$

The number n is the adiabatic index of the gas. According to statistical mechanics $2n$ is the number of degrees of freedom of each molecule in the gas. That atmospheres tend to be polytropic is an empirical fact.

The case that $\gamma' = \gamma$ is of a special significance. A polytrope with $\gamma' = \gamma$ is a path of constant specific entropy; changes along such polytropes are reversible and adiabatic; these polytropes and no others are adiabats.

Fix the constants A, γ' and consider an associated stationary, polytropic atmosphere. If both (1.12) and (1.13) hold we have

$$p = \text{const. } \rho^{\gamma'}, \quad p = \text{const. } T^{\frac{\gamma'}{\gamma'-1}}, \quad T = \text{const. } p^{1-1/\gamma'}. \quad (1.14)$$

In any displacement along a polytrope from a point with pressure p and temperature T , we shall have $d\rho/\rho = (1/\gamma')dp/p$, so that an increase in pressure leads to an increase in density that is greater for a smaller value of γ' . If a parcel of gas in this atmosphere is pushed down to a region of higher pressure, by a reversible process, then it will adjust to the ambient pressure. If $\gamma > \gamma'$, then it will acquire a density that is lower than the environment; it will then rise back up; this atmosphere is stable. But if $\gamma' > \gamma$ then the parcel will be denser than the environment and it will sink further; this atmosphere is unstable to convection. Thus we have:

A stable, polytropic atmosphere must have $\gamma' < \gamma$, $n' > n$.

Most stable is the isothermal atmosphere, $\gamma' = 1$.

In hydrodynamics, the isothermal atmosphere can be given a lagrangian treatment by taking

$$V = \mathcal{R}T\rho \log \rho. \quad (1.15)$$

We suppose that the gas is confined to the section $z_0 < z < z_0 + h$ of a vertical cylinder with base area \mathcal{A} and expect the density to fall off at higher altitudes. A plausible action density, for a perfect gas at constant temperature T in a constant gravitational field $\phi = gz$, g constant, is

$$\mathcal{L}[\Phi, \rho] = \rho (\dot{\Phi} - \vec{v}^2/2 - gz + \lambda) - \mathcal{R}T\rho \log \rho. \quad (1.16)$$

We may consider this an isolated system with fixed mass and fixed extension.

At equilibrium $\dot{\Phi} = 0, \vec{v} = 0, \dot{\rho} = 0$ and the equation of motion is $V' = \mathcal{R}T(1 + \log \rho) = \lambda - gz$, hence

$$\rho(x, y, z) = e^{-1+\lambda/\mathcal{R}T} e^{-gz/\mathcal{R}T}, \quad M = \mathcal{A} \frac{\mathcal{R}T}{g} e^{-1+\lambda/\mathcal{R}T} (1 - e^{-gh/\mathcal{R}T}) e^{-gz_0/\mathcal{R}T}$$

and after elimination of λ

$$\rho = \frac{gM}{\mathcal{A}\mathcal{R}T} \frac{e^{-g(z-z_0)/\mathcal{R}T}}{1 - e^{-gh/\mathcal{R}T}}, \quad p = \frac{gM}{\mathcal{A}} \frac{e^{-g(z-z_0)/\mathcal{R}T}}{1 - e^{-gh/\mathcal{R}T}}. \quad (1.17)$$

There is no difficulty in taking the limit $h \rightarrow \infty$. The volume becomes infinite but it can be replaced as a variable by the parameter z_0 . This atmosphere is stable.

Later, when it becomes clear that the potential V is the internal energy density, we shall come to suspect that the isothermal atmosphere is not an ideal gas. The isothermal atmosphere is usually abandoned in favor of the polytropic atmosphere.

A polytropic gas can be described by the lagrangian (1.10), with

$$V = \hat{a}\rho^{\gamma'}, \quad \hat{a}, \gamma' \text{ constant.}$$

Variation with respect to ρ gives

$$p = \frac{\hat{a}}{n'}\rho^{\gamma'}, \quad \frac{1}{n'} = \gamma' - 1.$$

The temperature does not appear explicitly but is taken to be determined by the gas law, Eq.(1.12). Among the many applications the following are perhaps the most important. In the case of sound propagation the gas is initially awakened from equilibrium and then left in an isolated, frenzied state of oscillating density and pressure, with the temperature keeping pace in obedience to the gas law (Laplace 1825, Pierce 2008). All three of the relations (1.14) are believed to hold, with $\gamma' = \gamma$. The oscillations are usually too rapid for the heat to disseminate and equalize the temperature, so that the neglect of heat transfer is justified. In applications to atmospheres one often postulates the polytropic equation of change (1.13) and obtains the temperature from the gas law. Understanding the resultant temperature gradient in terms of convection, or as the effect of the heating of the air by solar radiation, or both, is one of the main issues on which we have hoped to gain some understanding.

At mechanical equilibrium $\vec{v} = 0, \dot{\rho} = 0$ and $\lambda - gz = \hat{a}\gamma\rho^{1/n}$, hence

$$\rho = \left(\frac{\lambda - gz}{\hat{a}\gamma}\right)^n.$$

Since the density must be positive one does not fix the volume but assumes that the atmosphere ends at the point $z_1 = \lambda/g$. Then

$$M = \mathcal{A}\left(\frac{g}{\hat{a}\gamma}\right)^n \int_{z_0}^{z_1} (z_1 - z)^n dz = \frac{\mathcal{A}h}{n+1} \left(\frac{gh}{\hat{a}\gamma}\right)^n.$$

This fixes h and thus z_1 and λ . If the atmosphere is an ideal gas then the temperature varies with altitude according to

$$\mathcal{R}T = p/\rho = \frac{\hat{a}}{n}\rho^{1/n} = g\frac{z_1 - z}{n+1} \quad (1.18)$$

Because the lagrangian does not contain T as a dynamical variable it is possible to impose this condition by hand.

One would not apply this theory down to the absolute zero of temperature, but even without going to extremes it seems risky to predict the temperature of the atmosphere without having made any explicit assumptions about the absorption or generation of heat that is required to sustain it. Yet this has been the basis for the phenomenology of stellar structure, as well as the earth's atmosphere, from the beginning (Lane 1870, Ritter 1878).

The success of the polytropic model is notorious, but the theory is incomplete since it does not account for heat flow, nor convection, both of which are needed to complete the picture.

For air, with atomic weight 29, $\mathcal{R} = 2.87 \times 10^6 \text{ergs/gK}$ and $n = 2.5$. At sea level, $g = 980 \text{cm/sec}^2$, the density is $\rho = 1.2 \times 10^{-3} \text{g/cm}^3$, the pressure $p = 1.013 \times 10^6 \text{dyn/cm}^2$. Thus

$$p/\rho = .844 \times 10^9 \text{cm}^2/\text{sec}^2, \quad T = T_0 = 294\text{K}, \quad z_1 = 3.014 \times 10^6 \text{cm} \approx 30\text{km}.$$

and the dry lapse rate at low altitudes is $-T' = 294/z_1 = 9.75 \text{K/km}$. The opacity that is implied by this is mainly due to the presence of CO_2 in the atmosphere. Humidity increases the opacity and decreases the lapse rate by as much as a factor of 2. (A temperature difference of 70 degrees over 12 000 m was observed on a recent flight over Europe.)

The specific internal energy of this model is $np/\rho = n\mathcal{R}T$, as it should be for an ideal gas.

I.7. Historical notes on polytropic atmosphere

Observations of reversible transformations of near-ideal gases, carried out during the 19th century, can be summarized in what is sometimes called the laws of Poisson,

$$\rho \propto T^{n'}, \quad p \propto T^{n'+1}, \quad p \propto \rho^{\gamma'}, \quad \gamma' = 1 + \frac{1}{n'} \text{ constant.}$$

In the original context all the variables are constant and uniform. The exponents as well as the coefficients of proportionality are the same for all states that are related by reversible transformations. Statistical mechanics explained this result and confirmed the experimental value $\gamma' = \gamma = C_P/C_V$. As far as can be ascertained, the presence of terrestrial gravitation and ambient radiation had no effect on these experiments. In a first extrapolation the same relations were taken to hold locally in dynamic situations, as in the case of sound propagation. *The gas is not in thermal equilibrium and the variation of the temperature from point to point, and with time, is obtained from the gas law. This extension of an important thermodynamical relation to the case of a nonuniform system is taken for granted.*

For the atmosphere of the earth it was at first proposed that the temperature would be uniform. However, the existence of a temperature gradient was soon accepted as an incontrovertible experimental fact. The first recorded recognition of this, together with an attempt at explaining the same, may be that of Sidi Carnot, in the paper in which

he created the science of thermodynamics (Carnot 1824). Carnot quotes Laplace: “N’est-ce pas au refroidissement de l’air par la dilatation qu’il faut attribuer le froid des régions supérieures de l’atmosphère? Les raisons données jusqu’ici pour expliquer ce froid sont tout à fait insuffisantes; on dit que l’air des régions élevées, recevant peu de chaleur réfléchi par la terre, et rayonnant lui même vers les espaces célestes, devait perdre de calorique, et que c’était là la cause de son refroidissement; ... ” This may be the first time that the influence of radiation is invoked. The temperature gradient is attributed to the greenhouse effect, and Laplace was an early skeptic, for he continues “...mais cette explication ce trouve détruite si l’on remarque qu’à égale hauteur le froid regne aussi bien et même avec plus d’intensité sur les plaines élevées que sur les sommets des montagnes ou que dans les parties d’atmosphère éloignées du sol.” It is not clear that the two explanations are at odds with each other; Laplace apparently postulates that the atmospheres over lands at different elevations are related by adiabatic transformations, but without explaining why.

By rejecting the role of radiation as the cause of the temperature gradient, Laplace seems to suggest that the same would be observed in an atmosphere subject to gravitation but totally isolated from radiation, neither exposed to the radiation coming from the sun nor radiating outwards. As was strongly emphasized in later phases of this debate, this would contradict the belief that the thermal equilibrium of any isolated system, gravitation and other external forces notwithstanding, is characterized by a uniform temperature.

In 1862 W. Thomson, in the paper “On the convective equilibrium of the temperature in the atmosphere”, defines convective equilibrium with these words “When all parts of a fluid are freely interchanged and not sensibly influenced by radiation and conduction, the temperature is said to be in a state of convective equilibrium.” He then goes on to say that an atmosphere that is in convective equilibrium is a polytrope, and we think that he means an adiabat, because of the words “freely interchanged”, although the value of the polytropic index is taken from experiment and not from statistical mechanics. At first sight the clause “and not sensibly influenced by radiation” would seem to imply that his remarks would apply to an isolated atmosphere, indicating that a temperature gradient would persist in the absence of radiation, but this conclusion would be premature, as we shall see.

In 1870 H.J. Lane made the bold assumption that the laws of Poisson may be satisfied in the Sun. The terrestrial atmosphere (or part of it) had already been found to be well represented by the same relations. Referring to Lane’s paper Thomson, now Lord Kelvin, explains how convective equilibrium comes about (Thomson 1907). He argues that the atmosphere is not, cannot be, at rest, and this time radiation plays an essential role. The upper layers lose heat by radiation and the lower temperature leads to an increase in density. This produces a downward current that mixes with a compensating upward drift of warmer air. This continuing mixing takes place on a time scale that is too short for adjacent currents to exchange a significant amount of heat by conduction or radiation, especially since the variations of temperature are very small. It is evident that Thomson offers his explanation of the temperature gradient to account for its absence in an isolated atmosphere, for he says that, “an ideal atmosphere, perfectly isolated from absorption as well as emission of radiation, will, after enough time has passed, reach a state of uniform temperature, irrespective of the presence of the gravitational field”. Thomson accepts the

mechanism of Laplace and Carnot, as it is at work in the real atmosphere, but he goes further. He believes that the lower temperature aloft is intimately tied to the existence of radiation, implying that it is driven by net outwards radiation. (The effect of solar radiation on the terrestrial atmosphere is not explicitly mentioned.) It is difficult to judge whether or not Thomson is in disagreement with Laplace, but the precision of his statements represents a marked improvement over his predecessors and his earlier work.

The principal developers of the field, Ritter (1878-1883) and Emden (1907), seem to accept the idea of convective equilibrium. It may be pointed out, however, that this mechanism is in no way expressed by the equations that these and other authors use to predict the behaviour of real atmospheres. *The concept of convective equilibrium is introduced to one purpose only: to avoid contradiction with firmly established belief in the isothermal equilibrium of isolated systems. It receives no quantitative theoretical treatment.*

Nor was it accepted by everybody. A famous incidence involves Loschmidt (1876), who believed that an isolated atmosphere, at equilibrium in a gravitational field, would have a temperature gradient. But arguments presented by Maxwell and Boltzmann (1896) led Loschmidt to withdraw his objections, which is hardly surprising given the authority of these two. Nevertheless, it may be pointed out that no attempt was made, to our knowledge, to settle the question experimentally, until recently (Graeff 2008).

An alternative to convective equilibrium was proposed by Schwarzschild (1906) and critically examined by Emden. To understand how it works we turn to Emden's book of 1907, beginning on page 320. Here he invokes a concept that is conspicuously absent from all his calculations on polytropic spheres in the rest of the book: heat flow. He posits that the atmosphere is not completely transparent, and that heat flow is an inevitable consequence of the existence of a temperature gradient. *The most important observation is that heat flow is possible in stationary configurations ($\dot{T} = 0$) provided that the temperature gradient is constant.* We take this to be an implicit reference to the heat equation, the first such reference in the book(!). The heat flux due to conduction and radiation is usually thought to be expressed as

$$\vec{F} = -C\vec{\nabla}T, \quad F^i = -C^{ij}\partial_j T,$$

where the tensor C includes the thermal conductivity as well as the effective coefficient of heat transfer by radiation. The divergence of the flux is the time rate of change of the temperature due to conduction and radiation. In a stationary, terrestrial atmosphere, with no local energy creation, this must vanish. Emden's atmospheres are polytropes, with temperature gradients that are constant. It appears that he takes C to be uniform. That is surprising, and interesting, for it reminds us that the entire edifice implicitly demands that this condition, of a constant heat flow, must be satisfied.

We note that the direction of flow is from hot to cold, outwards. In applications to planetary atmospheres, with no local energy generation, this calls for an explanation, since the ultimate source of energy is above. Here we have to return to the oldest explanation of the existence of a temperature gradient, dismissed by Laplace (*op. cit.*): the greenhouse effect. The atmosphere is highly transparent to the (high frequency) radiation from the Sun but opaque to the thermal radiation to which it is converted by the ground. The atmosphere is thus heated from below!

If the atmosphere is stable in the sense discussed above, when $\gamma' \leq C_P/C_V$, then it is not necessary to assume that any convection takes place. In this case one speaks of (stable) ‘radiative equilibrium’. Convective equilibrium may step in when the stationary atmosphere is unstable, but it is no longer used to explain the existence of a temperature gradient.

A difficulty is present in all accounts of stellar structure up to 1920. The energy observed to be emitted by the Sun, attributed to contraction of the mass and the concomitant release of internal energy, was far too small to account for the age of the sun as indicated by the geological record. The situation changed with the discovery of thermonuclear energy generation. Now there is plenty of energy available. At the same time there arose the realization that convection sometimes plays a very modest role; the concept of convective equilibrium was put aside and with it, Kelvin’s explanation of the temperature gradient. According to Eddington (1926), who is more concerned with stars than with our atmosphere, “convective equilibrium” must be replaced by “radiative equilibrium” in the sense of Schwarzschild. He does not claim that this new concept accounts for the temperature gradient as well as Kelvin’s convective equilibrium does, but in fact the local generation of heat by thermonuclear processes creates an outward flow of heat and explains the existence of a negative temperature gradient.

It is an indication of the incompleteness of this picture that it contains no parameter that can be associated with the strength of radiation and, *a fortiori*, it does not allow us to investigate the result of turning off the radiation.

II. The first law

II.1. Thermodynamic equilibrium

A state of thermodynamical equilibrium of a system that consists of a very large number of identical particles is defined by the values of 3 variables, *a priori* independent, the density D , the pressure P and the temperature T . These are variables taking real values; they apply to the system as a whole. In the case of any particular system there is one relation that holds for all equilibrium states, of the form

$$T = f(D, P).$$

It is written in this form, rather than $F(T, D, P) = 0$, because a unique value of T is needed to define a state of equilibrium between two systems that are in thermal contact with each other: it is necessary and sufficient that they have the same temperature. This statement incorporates the zeroth law.

If we divide our system into subsystems then these will be in thermal equilibrium with each other only if they have the same temperature. This, at least, is inherited wisdom; we shall honor it as long as possible, without being intimidated by it.

The ideal gas at equilibrium is defined by global variables T, D, P , and two relations. The principal one is the gas law

$$P/D = \mathcal{R}T, \quad \mathcal{R} = .8314 \times 10^8 \text{ ergs}/K,$$

where $1/D$ is the volume of a mole of gas. The other may take the form of an expression for the internal energy.

II.2. The ideal gas in statistical mechanics

Here again we consider a gas that consists of identical particles (Boltzmann statistics), each with mass m and subject to no forces. It is assumed that the i th particle has momentum \vec{p}_i and kinetic energy $\vec{p}_i^2/2m$. This is an ideal gas, satisfying the relation $P/D = \mathcal{R}T$ at equilibrium. It is assumed that the number N of particles with energy E is given by the Maxwell distribution

$$N(E) \propto e^{-E/kT}, \quad (2.1)$$

which implies a constant density in configuration space. Now place this gas in a constant gravitational field, with potential $\phi(x, y, z) = gz$, g constant. Since the potential varies extremely slowly on the atomic scale it is plausible that, at equilibrium, each horizontal layer (ϕ constant) is characterized by a constant value of the temperature, density and pressure. Since neighbouring layers are in thermal contact with each other the temperature must (?) be the same throughout,

$$T(z) = T = \text{constant},$$

and in the case of an ideal gas,

$$p(z)/\rho(z) = \mathcal{R}T, \quad z \geq 0. \quad (2.2)$$

The energy of a particle at level z is $\vec{p}^2/2m + mgz$ and (2.1) now implies the following distribution in configuration space,

$$\rho(x, y, z) \propto e^{-mgz/kT}, \quad (2.3)$$

in agreement with (1.17). This supports the expression for the potential, Eq.(1.15), which is strange since that potential is not appropriate for an ideal gas. Both derivations of the distribution rest on the assumption that the temperature is constant throughout the system.

About the influence of gravitation on the temperature distribution there has been some debate, see e.g. Waldram (1985), page 151. It is said that the kinetic energy of each atom in a monatomic gas is $3kT/2$ and that, when the temperature is the same everywhere, this is paradoxical because it does not take account of the potential energy of the atom in the gravitational field. The incident involving Loschmidt, Maxwell and Boltzmann has already been mentioned. See also Feynman (1972). All speculation along these lines falls short of being compelling.

II.3. The first law and the internal energy

Can we extend the action principle to the case that the temperature varies with time? The action must be modified, for the temperature becomes a dynamical field. Is the temperature one of the variables with respect to which the action must be minimized? The usual approach is to lay down the additional equation by fiat (Section I.6); is this completely satisfactory? Would it perhaps be preferable to have it appear as the result of minimizing the action with respect to variations of the temperature field? We think so.

To prepare for the generalization we shall examine some of the main tenets of thermodynamics in the context of the action principle. Assume for the moment that the system is one of uniform density and pressure.

Suppose that the system is in thermal and mechanical isolation except for a force that is applied to the boundary. The system is in an equilibrium state with temperature T . The applied force is needed to hold the gas within the boundary of the domain Σ , then decreased by a very small amount leading to a displacement of the boundary and an increase of the volume by a small amount $d\mathcal{V}$. Assume that this process is reversible. The work done by the applied force is

$$dW = -pd\mathcal{V}. \quad (2.4)$$

The first law states that, if the system is in thermal isolation, then this quantity is the differential of a function $U(T, \mathcal{V})$ that is referred to as the internal energy of the system.

Consider the system that consists of an ideal gas confined to a volume \mathcal{V} and experiencing no external forces, not even gravitation. If the gas expands at constant pressure the work done by the gas is $pd\mathcal{V}$ and the ideal gas law Eq.(1.12) tells us that

$$pd\mathcal{V} = \mathcal{R}T\rho d\mathcal{V} = \mathcal{R}T\frac{M}{\mathcal{V}}d\mathcal{V}. \quad (2.5)$$

The idea of energy conservation suggests a concept of “internal energy”. It is assumed that, under certain circumstances, the work done by the gas is at the expense of an internal energy U so that

$$pd\mathcal{V} + dU = 0,$$

or

$$\mathcal{R}TMd\mathcal{V}/\mathcal{V} + dU = 0.$$

It is an experimental fact (Gay-Lussac 1827, Joule 1850) that the internal energy of an ideal gas is independent of the volume (see below) and the more precise statement that the internal energy density u is proportional to $\mathcal{R}T\rho$ is often included in the definition of the ideal gas (Finkelstein 1969, page 7). Thus

$$u = \hat{c}_V\mathcal{R}T\rho, \quad U = \hat{c}_V\mathcal{R}TM.$$

Statistical mechanics gives $\hat{c}_V = n$, where n is the adiabatic index and takes the value 3/2 for a monatomic gas. Thus $\mathcal{R}TMd\mathcal{V}/\mathcal{V} + dU = \mathcal{R}TMd\mathcal{V}/\mathcal{V} + n\mathcal{R}MdT = 0$, which implies that

$$dT = -\frac{1}{n}\frac{T}{\mathcal{V}}d\mathcal{V}, \quad T \propto \mathcal{V}^{-1/n}.$$

The calculation from (2.4) onward was done with the understanding that $M = \rho\mathcal{V}$ is fixed.

As we see it, the expression for the internal energy in terms of \mathcal{V} and T appears to be deduced from external considerations. At the deepest level the concept of energy derives its importance from the fact that it is conserved with the passage of time, by virtue of the dynamics. The defining equations of hydrodynamics do not admit a first integral, and no unique concept of energy; this is a difficulty that our limitation to laminar flow, and the action principle, will allow us to overcome. In modern versions of thermodynamics, and especially in the thermodynamics of irreversible processes and in radiation thermodynamics, conservation laws are all important, but they are postulated, one by one, not derived from basic axioms as is the case in other branches of physics, and they have a purely formal aspect since they serve only to define various fluxes. See e.g. Stanyukovich (1960), Castor (2004).

II.4. The first law and the hamiltonian

Having adopted an action principle approach we are bound to associate the internal energy with the hamiltonian.

The hamiltonian density is determined by the equations of motion only up to the addition of a constant multiple of the density. When we decide to adopt a particular expression to be used as internal energy over a range of temperatures, we are introducing a new assumption. Any expression for the internal energy, together with the implication that applied forces increase it by an amount determined by the work done, is a statement about a family of systems, indexed by the temperature. This cannot come out of the gas law and implies an independent axiom.

If we adopt the simplest expression for the hamiltonian, that of the isothermal atmosphere, with the lagrangian density of Eq.(1.16),

$$H = \int d^3x (\rho \bar{v}^2 / 2 + V), \quad V = \mathcal{R}T\rho \log \rho,$$

interpreting the potential as the ‘internal energy’, then we shall get, in the static case,

$$pd\mathcal{V} + dH(T, \mathcal{V}) = 0, \quad p = \mathcal{R}TM/\mathcal{V}.$$

with

$$dH = \mathcal{R}M \log(M/\mathcal{V})dT - \mathcal{R}TMd\mathcal{V}/\mathcal{V}.$$

The second term compensates for $pd\mathcal{V}$ and so $dT = 0$, the temperature does not change. This contradicts experimental results for ideal gases. In fact, the hamiltonian density $h = V = \mathcal{R}T\rho \log \rho$ is not the correct expression for the internal energy density of an ideal gas. Or to put it inversely, an isothermal atmosphere cannot be an ideal gas. (We shall have more to say about this later.) Besides, variation of our present, isothermal lagrangian with respect to T does not give a reasonable result, the lagrangian needs to be improved.

II.5. The adiabatic lagrangian

In the absence of gravity the equilibrium configurations all have uniform temperature, density and pressure. The equilibrium configurations described by the lagrangian are related by reversible transformations involving no heat transfer, exactly the configurations examined in the earliest experiments. In the presence of gravity the dynamical variables are fields. The adiabatic atmosphere in equilibrium is one in which the Poisson relations for an ideal gas hold locally, but we shall assume only that the expression for the internal energy density has the same form, namely

$$u = \hat{c}_V \mathcal{R}T\rho. \quad (2.6)$$

Two kinds of additions can be made to the lagrangian (1.16) without spoiling the equations of motion that are essential to hydrodynamics.

Adding a term linear in ρ we consider

$$\mathcal{L}[\Phi, \rho, T] = \rho(\dot{\Phi} - \vec{v}^2/2 - \phi + \lambda) - \mathcal{R}T\rho \log \rho + \rho\mu[T]. \quad (2.7)$$

The continuity equation is unchanged, and the hydrodynamical equation remains

$$\rho \frac{D}{Dt} \vec{v} = -\text{grad } p, \quad p = \mathcal{R}T\rho,$$

where $p = \rho(\partial V/\partial \rho) - V$ is unchanged since the new term in the potential is linear in ρ and since $\partial V/\partial T$ vanishes on shell. (Section I.4.) Variation with respect to T gives

$$\rho\mu'[T] - \mathcal{R}\rho \log \rho = 0. \quad (2.8)$$

On shell or, more precisely, by virtue of this equation, the potential reduces to

$$V[\rho, T] = \mathcal{R}T\rho \log \rho - \rho\mu[T] = \rho(T\mu'[T] - \mu[T])$$

This is the on shell “free”, static hamiltonian density (gravitational potential and kinetic energy omitted) that we expect to identify with the internal energy density (2.6). This determines $\mu[T]$ uniquely,

$$\mu[T] = n\mathcal{R}T \log T. \quad (2.9)$$

The on shell hamiltonian density takes the form

$$h = \rho\vec{v}^2/2 + \rho\phi + u, \quad u = n\mathcal{R}T\rho.$$

Finally, the equation of motion (2.8), with $\mu[T]$ given in (2.9), reduces to

$$\mathcal{R}\rho(n - \log k) = 0, \quad k = \frac{\rho}{T^n}. \quad (2.10)$$

This is just the adiabatic relation $\rho/T^n = \text{constant}$. The equation of motion that is obtained by variation with respect to ρ is

$$\dot{\Phi} - \vec{v}^2/2 - \phi + \lambda + \mu[T] = \mathcal{R}T(1 + \log k). \quad (2.11)$$

Combined with Eq.(10) it reduces, in the static case, to

$$\phi - \lambda + (1 + n)\mathcal{R}T = 0, \quad (2.12)$$

which has the same form as the equation (1.18) studied in Section I.6, expressing the fact that the lapse rate is constant.

We have thus found an action principle, with dynamical variables ρ and T , that reproduces all of the equations that characterize the equilibrium configurations, as well as the standard, hydrodynamical relations of an ideal gas.

II.6. Thermodynamical relations

We have identified the internal energy with the hamiltonian, in the case of configurations with uniform density and temperature. With less justification, we regard the hamiltonian density as a local internal energy density, and this implies local versions of thermodynamical relations.

The (total) hamiltonian includes the kinetic term and the gravitational potential,

$$h = \rho(\bar{v}^2/2 + \phi) + \mathcal{R}T\rho \log k = \rho(\bar{v}^2/2 + \phi) + \rho e. \quad (2.13)$$

The field e thus defined will be identified with the specific, internal energy density, the energy per unit of mass. From now on the volume is $1/\rho$,

$$\mathcal{V} = 1/\rho.$$

The relation $de + pdV = 0$ is expected to hold when the volume and the temperature are varied and the entropy is held fixed, so that

$$\left. \frac{\partial e}{\partial T} \right|_{\mathcal{V}} = 0, \quad \left. \frac{\partial e}{\partial V} \right|_T = -p.$$

Both equations are verified on shell. The first is the new equation of motion,

$$\left. \frac{\partial e}{\partial T} \right|_{\mathcal{V}} = \left. \frac{\partial}{\partial T} \right|_{\mathcal{V}} \mathcal{R}T \log k = \mathcal{R}(\log k - n) = 0. \quad (2.14)$$

The other is

$$\left. \frac{\partial e}{\partial V} \right|_T = -\rho^2 \frac{\partial}{\partial \rho} \mathcal{R}T \log k = -\mathcal{R}T\rho = -p, \quad (2.15)$$

one of the original equations of motion.

At this stage, when the influence of radiation has not yet been included explicitly, we can already compare with the more conventional theories. Our approach agrees with radiation hydrodynamics in so far as the equation of continuity and the hydrodynamical equation are concerned. Taking those equations for granted, the variational principle postulates that $\partial h/\partial T = 0$, while radiation hydrodynamics posits the “energy conservation equation”:

$$\frac{De}{Dt} + p \frac{D(1/\rho)}{Dt} = q, \quad (2.16)$$

where q is a local density of heat generation. Making use of the other equations (continuity and Bernoulli) one transforms this to

$$\frac{\partial}{\partial t}h + \vec{\nabla} \cdot \vec{v}(h + p) = \rho q. \quad (2.17)$$

The specific energy density e and pressure p are not specified, neither is the heat source q .

The only difference between this approach and the one that we recommend is this. Instead of Eq.(2.17) we have the variational equation $\partial h/\partial T = 0$. Now

$$\frac{\partial h}{\partial t} = \frac{\partial h}{\partial \rho} \dot{\rho} + \frac{\partial h}{\partial \vec{v}} \cdot \dot{\vec{v}} + \frac{\partial h}{\partial T} \dot{T}.$$

With the equation of continuity and the hydrodynamical equation we reduce this to

$$\frac{\partial h}{\partial t} + \vec{\nabla} \cdot (h\vec{v} + p\vec{v}) = \frac{\partial h}{\partial T} \dot{T}. \quad (2.18)$$

The variational equation $\partial h/\partial T = 0$ thus implies the energy equation of radiation hydrodynamics, in the case that $q = 0$ and $e = n\mathcal{R}T \log k$, $p = \mathcal{R}T\rho$. The reverse is also true, except that the energy conservation equation has an additional solution: $\dot{T} = 0$.

Radiation hydrodynamics aims at a very general situation with the functions e, p and q being arbitrary. A formulation in terms of a variational principle cannot exist in general, since the pressure is determined by the hamiltonian; the internal energy and the pressure are not unrelated functions to be chosen independently. But applied to the ideal gas both theories are in full agreement. Note that radiation has not yet been taken into account.

II.7. The radiation term

Since the advent of Schwarzschild's paper (1909) the analysis of the effect of radiation has not changed significantly. The conclusion, in the simplest approximation, is that the internal energy is augmented by the Stefan-Boltzmann expression for the energy of black body radiation. Thus

$$\mathcal{L}[\Phi, \rho, T] = \rho(\dot{\Phi} - \vec{v}^2/2 - \phi + \lambda) - \mathcal{R}T\rho \log k + \frac{a}{3}T^4. \quad (2.19)$$

The term $\rho\mu[T]$ in Eq.(2.7) has been included in the potential, with $k = \rho/T^n$. The constant $a = 7.64 \times 10^{-15} \text{ ergs}/K^4$ is the Stefan-Boltzmann constant and the new term is the pressure of the photon gas, or of black body radiation.

The new expression for the internal energy is

$$\rho e = n\mathcal{R}T\rho + aT^4,$$

and the pressure, defined either by the modified hydrodynamical equation or by $dU + pdV = 0$, is

$$p = \mathcal{R}T\rho + \frac{a}{3}T^4,$$

in agreement with the theory of black body radiation and with the principle that the pressure in a mixture of gases is additive.

Variation with respect to T now gives

$$\mathcal{R}\left(n - \log \frac{\rho}{T^n}\right)\rho + \frac{4a}{3}T^3 = 0, \quad (2.20)$$

and in the important case when $n = 3$,

$$\mathcal{R}\left(3 - \log \frac{\rho}{T^3}\right)\frac{\rho}{T^3} + \frac{4a}{3} = 0,$$

which is equivalent to Poisson's law $T^3/\rho = \text{constant}$. This reflects a strong affinity between the polytropic ideal gas with $n = 3$ and radiation, strongly emphasized by Eddington (1926). The value $n = 3$ has a cosmological significance as well, it is characteristic of the changes in ρ, p, T induced by uniform expansion (Ritter, Emden 1907, see Chandrasekhar 1938, page 48). For other values of n , Eq.(2.19) is a mild modification of the polytropic equation of change in the presence of radiation. The standard approach maintains the polytropic relation without change in the presence of radiation.

It deserves to be emphasized that our approach does not allow us to determine a relation between dynamical variables (such as $\rho \propto T^n$ or even $P = \mathcal{R}T\rho$) until the lagrangian has been completed. The modification of the polytropic relation that results from including the energy of the radiation field is not a feature of the traditional theory.

We have thus found an action that, varied with respect to ρ, Φ and T reproduces all of the equations that define the ideal, polytropic gas with polytropic index $n = 3$, radiation included, as well as the standard, hydrodynamical relations. For any value of n , it describes a gas that has its effective polytropic index increased from the 'natural' adiabatic value, approaching the critical value 3 at very high temperatures.

We suggest that using the lagrangian (2.18) is preferable to the usual assumption that $\beta := p_{\text{gas}}/p_{\text{tot}}$ is constant, which is true only when $n = 3$.

II. 8. A puzzle

Let us return to our analysis of the isolated, ideal gas, before we introduced the radiation term in Section II.7. It was based on only two assumptions, the familiar expression for the internal energy of an ideal gas, and the ideal gas law. The adiabatic equation of change, $\rho/T^n = \text{constant}$, was not invoked but derived from those assumptions. In the absence of gravity the result is uncontroversial. In particular, the equations of motion allow for stationary states with uniform density and temperature. The effect of gravity was included by adding the gravitational potential energy to the hamiltonian; which is both standard practice and inevitable. The result of that modification is that, in the presence of the gravitational field there are no longer any stationary solutions with uniform temperature. Instead both density and temperature decrease with elevation. This should be welcome as being in agreement with what is observed in real atmospheres, where the temperature gradient seems to be attributed to the presence of heating in the form of radiation.

The problem is that nothing could justify an application of this theory to phenomena that are significantly influenced by radiation. Indeed it was expected that the effect of radiation was to be taken into account by the inclusion of the black body energy in Section II. 7. Nothing that went into building up the theory suggests that the gas is not in isolation, and indeed the isolated gas is the system to which we had meant to address ourselves. But there is a very strong conviction among physicists that, in an isolated system, the temperature must be uniform, gravitation notwithstanding. We have built a theory that, surprisingly, seems to apply to the irradiated and gravitating atmosphere, but we have not solved the more basic problem that we set out to do, to describe an isolated atmosphere that conforms to established ideas.

We have been led to this paradoxical position by insisting on formulating the theory in terms of a variational principle. Must we conclude that action principles have no place in thermodynamics? We think not!

In fact, the most surprising discovery is that standard theory, for example, radiation hydrodynamics, in the preliminary stage at which radiation has not yet been taken into account, leads to the same paradoxical conclusion; that is, the familiar expressions for internal energy, together with the ideal gas law, lead unambiguously to the polytropic atmosphere.

We have also formulated an action principle for the isothermal atmosphere, in Section I.3. It predicts the same density distribution as does statistical mechanics, but it requires that the internal energy be quite different from that of an ideal gas.

Our conclusion can be summarized very simply. We assume that the system is isolated; this implies that the law

$$dU + pdV = 0$$

holds for changes induced by the passage of time - see (2.16). It is found that it also holds for imaginary changes in which a portion of the gas is compared with a similar portion at a different elevation. Therefore, the vertical profile of the atmosphere is an adiabat and the density gradient implies a temperature gradient.

Why is the prediction of a temperature gradient in an isolated atmosphere so shocking? Imagine a large heat bath located in the region $z > 0$ in \mathbb{R}^3 . A vertical tube, filled with an ideal gas, has its upper end in thermal contact with the bath, otherwise it is isolated. Assume that, at equilibrium, the lower part of the tube has a temperature that is higher than that of the bath. Now extract a small amount of heat from the bottom of the tube; then the restoration of equilibrium demands that heat must flow from the bath to the warmer, lower part of the tube, in violation of one of the statements of the second law, namely:

“Heat cannot pass by itself from a colder to a hotter body” (Clausius 1887).

According to a recent experiment, it can! A recent preprint (Graeff 2008) describes an experiment, carried out over a period of several months, in which a persistent temperature gradient was observed in a column of carefully isolated water.

Concerning the status of this formulation by Clausius of the second law of thermodynamics we quote I. Müller:

“This statement, suggestive though it is, has often been criticised as vague. And indeed, Clausius himself did not feel entirely satisfied with it. Or else he would not have tried to make the sentence more rigorous in a page-long comment, which, however, only succeeds in removing whatever suggestiveness the original statement may have had”. And Müller continues: ”We need not go deeper into this because, after all, in the end there will be an unequivocal *mathematical* statement of the second law”.

We note that Maxwell, in refuting Loschmidt, did not make use of the statement but argued that the arrangement could be turned into a source of energy, a second class perpetuum mobile.

II. 9. The heat equation

As a genuine dynamical variable, the temperature must be governed by an equation for its time derivative, and it must come out of the variational principle. This requires an other variable.

We consider

$$\begin{aligned}\mathcal{L}[\phi, \rho, T] &= \mathcal{L}_1 + \mathcal{L}_2, \\ \mathcal{L}_1 &= \rho(\dot{\Phi} - \vec{v}^2/2 - \phi + \lambda) - \mathcal{R}T\rho \log k + \frac{a}{3}T^4, \\ \mathcal{L}_2 &= \sigma\dot{T} + \vec{v} \cdot (aT\vec{\nabla}\sigma + b\sigma\vec{\nabla}T) + C^{ij}[\rho, T] \partial_i T \partial_j \sigma,\end{aligned}\tag{2.21}$$

where σ is the new variable, canonically conjugate to T . Since the gradient of T is very nearly constant in the polytropic atmosphere, and the heat flow is parallel to it, a reasonable choice for the conductivity tensor is

$$C^{ij} = c \delta^{ij}.$$

with c constant or proportional to $k = \rho/T^n$. It is expected that the parameter c is small and it will be assumed that this term makes a negligible contribution to phenomena with a short time scale.

Variation of the fields Φ, ρ, σ and T gives the equations:

$$\dot{\rho} + \vec{\nabla} \cdot (\rho\vec{v} - aT\vec{\nabla}\sigma - b\sigma\vec{\nabla}T) = 0,\tag{2.22}$$

$$\dot{\Phi} - \vec{v}^2/2 - \phi + \lambda = \mathcal{R}T(1 + \log k)\tag{2.23}$$

$$\dot{T} - \vec{\nabla} \cdot (c\vec{\nabla}T) + (b-a)\vec{v} \cdot \vec{\nabla}T - aT\vec{\nabla} \cdot \vec{v} = 0,\tag{2.24}$$

$$\mathcal{R}\rho(\log k - n) - \frac{4a}{3}T^3 + \dot{\sigma} + \vec{\nabla} \cdot (c\vec{\nabla}\sigma) + (b-a)\vec{v} \cdot \vec{\nabla}\sigma - b\sigma\vec{\nabla} \cdot \vec{v} = 0.\tag{2.25}$$

Eq.(2.22) shows that the mass current is not precisely $\rho\vec{v}$. The second equation is equivalent to

$$\rho \frac{D\vec{v}}{Dt} + \rho\vec{\nabla}\phi + \rho\vec{\nabla}\mathcal{R}T(1 + \log k) = 0$$

On shell, more precisely by virtue of the equation $\partial\mathcal{L}/\partial T = 0$, it can be written as follows,

$$\rho \frac{D\vec{v}}{Dt} + \rho \vec{\nabla}\phi = -\vec{\nabla}p, \quad p := \mathcal{R}T\rho + \frac{a}{3}T^4 + \mathcal{L}_2.$$

The new term, \mathcal{L}_2 , is inevitable.

The parameters a, b are determined as follows. The contribution of convection to the time derivative of the temperature fixes $b - a = 1$. An implication of this is that the value of the field σ also follows the particles of the gas. The value $a = -1/n$ is needed to agree with the theory of sound propagation, always assumed to be adiabatic (Laplace 1925).

The part of (2.25) that is linear in σ must be related to the entropy. This new term is absent in the case of an isolated atmosphere; it is needed if the otherwise isolated gas is subjected to an influx of heat, as by radiation, or if the so far isolated gas is put into thermal contact with empty space. An actual earthly atmospheric equilibrium is a state in which both effects cancel out and the entropy is again zero. The field σ is a stand-in for the electromagnetic fields of radiation, incoming and outgoing (Appendix).

A solution with uniform temperature would require a carefully fine tuned entropy; it is not natural.

II. 10. The centrifuge and the atmosphere

Kelvin justified the polytropic model of the atmosphere in terms of radiation and convection. Eddington discounted the role of convection and relied on a concept of radiative equilibrium. To find out what happens in the case of complete insulation we study the analogous situation in a centerfuge.

Consider an ideal gas. By a series of experiments in which gravity does not play a role, involving reversible changes in temperature and pressure, it is found that, at equilibrium, the laws $p/\rho = \mathcal{R}T$ and $\rho \propto T^n$ are satisfied, constant n fixed. When supplemented by the laws of hydrodynamics, they are found to hold, or at least they are strongly believed to hold, in configurations involving flow, over a limited time span, in the absence of external forces. We have found that these data lead to the statement that ρ/T^n is constant throughout the gas. In addition it is said that, at equilibrium, the temperature must be uniform. Keeping an open mind, let us refer to this last statement as “the axiom”. We are talking about a fixed quantity of gas contained in a vessel, the walls of which present no friction and pass no heat.

Let the walls of the vessel be two vertical, concentric cylinders, and construct a stationary solution of the equations of motion. And why not? We have experimental confirmation of the equations of motion, we applied them to the theory of sound with a degree of confidence that is so high that the prediction of rapid variations in temperature may never have been subjected to verification (?). In terms of cylindrical coordinates, take $v_z = v_r = 0, v_\theta = \omega$, constant. The continuity equation is satisfied with ρ any function of r alone. Then neither T nor p is constant, for the hydrodynamical equations demand that

$$r\omega^2 = cT', \quad c = (n + 1)\mathcal{R} \approx 10^7 \text{ cm/sec}^2 K \quad (\text{for air}).$$

At first sight, this seems to violate the axiom, but perhaps not, for this is not a static configuration. To save the axiom let us suppose that, by conduction, convection or radiation, the temperature will tend towards uniformity. Perhaps after a suitably long time has passed, T has become constant. Let us remember that no heat or any other influence is supposed to go by the walls; then surely energy and angular momentum must both be preserved during the time that the temperature is levelling out. It is reasonable to assume that the final configuration is (macroscopically) stationary and uniform, since the existence of fluctuations would imply that the entropy had not reached its maximum. But a stationary state with non zero density gradient and uniform temperature contradicts the assumptions that we made about the gas, which makes the existence of such a state somewhat problematic.

If we also accept the equivalence principle, then from the point of view of a local observer at rest in the flow there is a centrifugal force field, a density gradient and, by the laws of Poisson, a temperature gradient. The equivalence principle only applies to conditions at one point, and one can question whether the gradient of the temperature or of the density is sufficiently local to be covered by the principle. The entire theory of relativistic thermodynamics has been founded on the belief that it is (Tolman 1934).

If we do accept the equivalence principle (without necessarily embracing the tenets of traditional relativistic thermodynamics), then we shall be lead to expect that a vertical column of an ideal gas, in mechanical equilibrium under the influence of terrestrial gravity, and perfectly isolated, will have a pressure and temperature gradient exactly of the form predicted by Homer Lane. This contradicts what we think is the prevailing opinion of atmospheric scientists, that the temperature gradient owes its existence to the heating associated with solar radiation.

Further measurements in the atmosphere are unlikely to throw light on this, since isolation is difficult. Experiments with a centrifuge may be more realistic. The temperature lapse rate is $r\omega^2 \times 10^{-7} K/cm$. If the acceleration is 1000 g at the outer wall, then the lapse rate will be $.1 K/cm$. The temperature difference between the inner and outer walls will thus be $1 K$ if the distance is $10cm$. In a practical experiment one does not have the gas flow between concentric, stationary cylinders. Instead a tube filled with the gas is oriented radially on a turntable. Friction against the walls is thus eliminated and heat loss is much easier to control. The question of the existence of a temperature gradient is the most urgent. Once this is resolved one way or another the approach to equilibrium is worthy of an investigation.

We have come to doubt that complete equilibrium implies a uniform temperature in all cases. In fact, Tolman (1934, page 314) shows that, according to General Relativity, the temperature of an isolated photon gas in a gravitational field is not quite uniform. The predicted magnitude of this effect is very small, but it shows that there are circumstances in which statistical mechanics is not the absolute truth.

III. Conclusions

III.1. On variational principles

The principal reasons for preferring an action principle formulation of thermodynamics were stressed in the introduction. Here we add some additional comments.

Variational principles have a very high reputation in most branches of physics; they even occupy a central position in classical thermodynamics, see for example the authoritative treatment by Callen (1960). An action is available for the study of laminar flows in hydrodynamics, see e.g. Fetter and Walecka (1960), though it does not seem to have been much used. Without the restriction to laminar flows it remains possible to formulate an action principle (Taub 1954, Bardeen 1970, Schutz 1970), but the proliferation of velocity potentials is confusing and no application is known to us. Recently, variational principles have been invoked in special situations that arise in gravitation.

In this paper we rely on an action principle formulation of the full set of laws that govern an ideal gas, in the presence of gravity and radiation. To keep it simple we have restricted our attention to laminar, hydrodynamical flows.

It was shown that there is an action that incorporates all of the essential properties that characterize an ideal gas, expressed as variational equations. The independent dynamical variables are the density, the velocity potential and the temperature. The idea of varying the action with respect to the temperature is much in the classical tradition. The hamiltonian gives the correct expression for the internal energy and the pressure.

Into this framework the inclusion of a gravitational field is natural. Inevitably, it leads to pressure gradients and thus also temperature gradients. If other considerations, including the heat equation, are put aside, then the theory, as it stands, predicts the persistence of a temperature gradient in an isolated system at equilibrium. The existence of a temperature gradient in an isolated thermodynamical system is anathema to tradition, and further work is required to find the way to avoid it, or to live with it.

The interaction of the ideal gas with electromagnetic fields has been discussed in a provisional manner in the appendix. The transfer of entropy between the two gases is in accord with the usual treatment of each system separately.

III.2. Suggestions

(1) Observation of the diurnal and seasonal variations of the equation of state of the troposphere may lead to a better understanding of the role of radiation in our atmosphere. The centrifuge may be a more practical source of enlightenment. We understand that modern centrifuges are capable of producing accelerations of up to 10^6g . Any positive result for the temperature gradient in an isolated gas would have important theoretical implications.

(2) We suggest the use of the lagrangian (2.7), or its relativistic extension, with T treated as an independent dynamical variable and $n' = n$, in astrophysics. Variation with respect to T yields the adiabatic relations between ρ and T , so long as the pressure of radiation is negligible, but for higher temperatures, when radiation becomes important, the effect is to increase the effective value of n' towards the ultimate limit 3, regardless of the adiabatic index n of the gas. See in this connection the discussion by Cox and Giuli (1968), page 271.

In the case that $n = 3$ there is Eddington's treatment of the mixture of an ideal gas with the photon gas. But most gas spheres have a polytropic index somewhat less than 3 and in this case the ratio $\beta = p_{\text{gas}}/p_{\text{tot}}$ may not be constant throughout the star. The lagrangian (2.19), with n identified with the adiabatic index of the gas, gives all the equations that are used to describe atmospheres, so long as radiation is insignificant. With greater radiative pressure the polytropic index of the atmosphere is affected. It is not quite constant, but nearly so, and it approaches the upper limit 3 when the radiation pressure becomes dominant. Eddington's treatment was indicated because he used Tolman's approach to relativistic thermodynamics, where there is room for only one density and only one pressure. Of course, all kinds of mixtures have been studied, but the equations that govern them do not supplement Tolman's gravitational concepts in a satisfactory manner, in our opinion. Be that as it may, it is patent that the approximation $\beta = \text{constant}$, in the works of Eddington and Chandrasekhar, is a device designed to avoid dealing with two independent gases.

(3) In a subsequent paper we intend to make use of the solid platform that is provided by the action principle to study the stability of atmospheres. It will be argued that expressions for the total energy are not enough to determine stability; what is needed is an expression for the hamiltonian, in terms of the dynamical variables of the theory.

Appendix. Sources and entropy

In Section II.9 we have introduced a new field variable σ in order to include the heat equation in the system of variational equations. There is a close relation between this field and entropy. In this appendix we explain what we have in mind for this field. In the first section we introduce the entropy in the manner of an external source and verify that it fits into the general scheme of thermodynamics. In the next section we consider an example, in which the external source is replaced by a dynamical electromagnetic field. The model presented in Section II.9 should be seen as an "effective" theory, being the simplest dynamical realization of entropy. It seems that this modelling of entropy is consistent with the use of entropy in non-equilibrium thermodynamics, but this suggestion has not been explored till now. The ideas presented here are tentative.

A.1. Generic source

The adiabatic lagrangian describes a single adiabat. To remove this limitation in a formal way, let us add another term to the lagrangian density,

$$\mathcal{L} = \rho(\dot{\Phi} - \vec{v}^2/2 - gz + \lambda) - \mathcal{R}T\rho \log \frac{k}{k_0} + f[T] + \rho TS, \quad (A.1)$$

where S is an external source. The factor ρ in the source term is natural and the factor T is chosen to make S play the role of a local adiabatic parameter. We have introduced the variable k and the parameter k_0 by

$$\rho = kT^n, \quad \rho_0 = k_0 T_0^n, \quad (k_0 \rightarrow 1);$$

Then $k^{-1/n}$ is Emden's "polytropic temperature". It will be recalled that k_0 parameterizes a family of adiabats; in fact, for an isothermal expansion, the variation of $-\mathcal{R} \log k$ is

precisely the change in specific entropy. The introduction of the source S turns $-\mathcal{R} \log k_0$ into a field with the interpretation of entropy. We no longer need the parameter and so, following Lane, we shall use units of density such that $k_0 = 1$.

The internal specific entropy is $\mathcal{R} \log(T^n/\rho)$ and the total specific entropy is

$$S_{tot} = \mathcal{R} \log \frac{T^n}{\rho} + S.$$

With this convention

$$\mathcal{L} = \rho(\dot{\Phi} - \vec{v}^2/2 - \phi + \lambda) + \rho T S_{tot} + \frac{a}{3} T^4, \quad (\text{A.1})$$

where ϕ is the gravitational potential. Variation with respect to T leads to

$$\rho \frac{\partial}{\partial T} (T S_{tot}) + \frac{4a}{3} T^3 = 0. \quad (\text{A.2})$$

As an equation for S_{tot} it has the general solution

$$S_{tot} = -\frac{a}{3\rho} T^3 - \frac{1}{\rho T} V[\rho]. \quad (\text{A.3})$$

Taking this as the definition of the potential (V is the value of $\rho T S_{tot} + \frac{a}{3} T^4$ at the extremum with respect to variation of T) we have

$$\mathcal{L} = \rho(\dot{\Phi} - \vec{v}^2/2 - \phi + \lambda) - V[\rho]. \quad (\text{A.4})$$

The gradient of the equation obtained by variation of ρ is

$$-\rho \frac{D}{Dt} \vec{v} - \rho \text{grad } \phi = \text{grad } p, \quad (\text{A.5})$$

with

$$p = \rho V' - V = (1 - \rho \frac{d}{d\rho})(\rho T S_{tot} + \frac{a}{3} T^4) = -T \rho^2 \frac{\partial S_{tot}}{\partial \rho} + \frac{a}{3} T^4. \quad (\text{A.6})$$

The last equation is justified by the fact that the partial derivative of $\rho T S_{tot} + (a/3) T^4$ with respect to T vanishes on shell, Eq.(A.2).

We shall verify some important relations of thermodynamics, and for this we must take T and ρ to be constant, with $M = \rho \mathcal{V}$, and $\phi = 0$. In this case

$$p = M T \frac{\partial S_{tot}}{\partial \mathcal{V}} + \frac{a}{3} T^4 = \mathcal{R} M T / \mathcal{V} + \frac{a}{3} T^4 + M T \frac{\partial S}{\partial \mathcal{V}}. \quad (\text{A.7})$$

The hamiltonian density is, in the static case, in the absence of gravity,

$$h = -\rho T S_{tot} - \frac{a}{3} T^4, \quad \text{implying that } U = -M T S_{tot} - \frac{a}{3} T^4 \mathcal{V}. \quad (\text{A.8})$$

Variation of h with respect to T gives zero on shell, so this is the same as

$$u = (1 - T\partial_T)h = \rho T^2 \frac{\partial S_{tot}}{\partial T} + aT^4 = n\mathcal{R}\rho T + aT^4 + \rho T^2 \frac{\partial S}{\partial T}.$$

Thus

$$U = MT^2 \frac{\partial S_{tot}}{\partial T} + aT^4 \mathcal{V} = n\mathcal{R}MT + aT^4 \mathcal{V} + MT \frac{\partial S}{\partial T}. \quad (\text{A.9})$$

Using (A.7) and (A.8) one verifies that

$$\frac{\partial U}{\partial \mathcal{V}} = (T\partial_T - 1)p, \quad (\text{A.10})$$

an important consequence of the existence of entropy in general. See Finkelstein (1969) page 26. Also,

$$dU = MT \frac{\partial S_{tot}}{\partial T} dT + Td\left(MT \frac{\partial S_{tot}}{\partial T}\right) + 4aT^3 \mathcal{V} dT + aT^4 d\mathcal{V},$$

$$pd\mathcal{V} = MT \frac{\partial S_{tot}}{\partial \mathcal{V}} d\mathcal{V} + \frac{a}{3} T^4 d\mathcal{V},$$

and the sum is $dU + pd\mathcal{V} = dQ = T(\partial p/\partial T) = 0$, the last on shell.

We used the last expressions in (A.7) and (A.9) because they are familiar, but if we return to (A.8) and the first expression for p in (A.7) we see immediately that $dU + pd\mathcal{V} = 0$.

If instead we consider a change that involves outside forces acting via the source, then $dU + pd\mathcal{V} = MT\delta S$, which confirms the interpretation of S as a contribution to the specific entropy.

A.2. Electromagnetic fields

We write the Maxwell lagrangian as follows,

$$\mathcal{L}_{\text{rad}} = \frac{1}{2\epsilon} \vec{D}^2 - \frac{\mu}{2} \vec{H}^2 + \vec{D} \cdot (\vec{\partial} A_0 - \dot{\vec{A}}) - \vec{H} \cdot \vec{\partial} \wedge \vec{A} + JA, \quad (\text{A.13})$$

and add it to the ideal gas lagrangian

$$\mathcal{L}_{\text{gas}} = \rho(\dot{\Phi} - \vec{v}^2/2 - \phi + \lambda) - \mathcal{R}T\rho \log k + \frac{a}{3} T^4, \quad (\text{A.14})$$

Since the susceptibility of an ideal gas is small, the dielectric constant may be expressed by

$$\epsilon = 1 + \kappa[\rho, T], \quad \text{or} \quad \frac{1}{\epsilon} = 1 - \kappa[\rho, T]. \quad (\text{A.15})$$

Paramagnetic effects will be ignored at present. An interaction between the two systems occurs through the dependence of the susceptibility on ρ . The source S has become

$-(\vec{D}^2/2\rho)(\kappa/T)$. If this quantity has a constant value then it produces a shift in the value of the parameter k .

Two interpretations are possible. The electromagnetic field may represent an external field, produced mainly by the source J , and affecting the gas by way of the coupling implied by the dependence of the dielectric constant on ρ . Alternatively, the field is produced by microscopic fluctuations, quantum vacuum fluctuations as well as effects of the intrinsic dipoles of the molecules of the gas. In this latter case the main effect of radiation is represented by the radiation term $aT^4/3$. Our difficulty is that neither interpretation is complete, and that we do not have a sufficient grasp of the general case when either interpretation is only half right. The following should therefore be regarded as tentative.

Variation of the total action, with lagrangian $\mathcal{L}_{\text{rad}} + \mathcal{L}_{\text{gas}}$, with respect to \vec{A} , \vec{D} , \vec{H} and T gives

$$\dot{\vec{D}} = \vec{\partial} \wedge \vec{H}, \quad (\text{A.14})$$

$$\dot{\vec{A}} = \vec{D}/\epsilon, \quad (\text{A.15})$$

$$\mu\vec{H} = -\vec{\partial} \wedge \vec{A}, \quad (\text{A.16})$$

and

$$\mathcal{R}(n - \log k)\rho - \frac{\vec{D}^2}{2} \frac{\partial \kappa}{\partial T} + \frac{4a}{3}T^3 = 0. \quad (\text{A.17})$$

Taking into account the first 3 equations we find for the static hamiltonian

$$H = \int d^3x \left(\phi\rho + \mathcal{R}\rho T \log k + \frac{\vec{D}^2}{2} + \frac{\mu\vec{H}^2}{2} - \frac{\vec{D}^2}{2} \frac{\kappa}{T} - \frac{a}{3}T^4 \right).$$

With the help of (A.17) it becomes

$$H = \int d^3x \left(\phi\rho + n\mathcal{R}\rho T + \frac{\vec{D}^2}{2} + \frac{\mu\vec{H}^2}{2} + aT^4 \right) - \int d^3x T \frac{\vec{D}^2}{2} \frac{\partial(T\kappa)}{\partial T}. \quad (\text{A.18})$$

The last term, from the point of view of the thermodynamical interpretation of electrostatics, is recognized as the entropy (Panofsky and Phillips 1955). On a suitable choice of the functional κ it merges into the internal energy. For example, if $\kappa = \rho T$ it takes the form ρTS with $S = \vec{D}^2$.

A.3. Using T as a dynamical variable

Let us examine the total lagrangian,

$$\begin{aligned} \mathcal{L} = \mathcal{L}_{\text{rad}} + \mathcal{L}_{\text{gas}} = & \rho(\dot{\Phi} - \vec{v}^2/2 - \phi + \lambda) - \mathcal{R}T\rho \log \frac{\rho}{T^3} \\ & + \frac{\vec{D}^2}{2\epsilon} + \frac{\mu}{2}\vec{H}^2 + \vec{D} \cdot (\vec{\partial}A_0 - \dot{\vec{A}}) - \vec{H} \cdot \vec{\partial} \wedge \vec{A} + JA + \frac{a}{3}T^4. \end{aligned} \quad (\text{A.19})$$

So long as ϵ, μ and J are independent of ρ, T and \vec{v} , the variational equations of motion that are obtained by variation of $\vec{v}, \rho, \vec{A}, \vec{H}$ and \vec{D} are all conventional, at least when $n = 3$ (for all n if radiation is negligible). It would be possible to be content with that and fix T by fiat, as is usual; in the case of the ideal gas without radiation the result is the same. But if ϵ depends on ρ and on T , which is actually the case, then we get into a situation that provides the strongest justification yet for preferring an action principle formulation with T as a dynamical variable. The equations of motion include a contribution from the variation of ϵ with respect to ρ , so that one of the basic hydrodynamical equations is modified. Thus it is clear that the extension of the theory, to include the effect of radiation, is not just a matter of including additional equations for the new degrees of freedom. The presence of the term $\vec{D}^2/2\epsilon[\rho, T]$ certainly introduces the density ρ into Maxwell's equations; that it introduces \vec{D} into the hydrodynamical equations is clear as well. *The over all consistency of the total system of equations can probably be ensured by heeding Onsager's principle of balance, but the action principle makes it automatic.*

Variation of the action with respect to T offers additional advantages. The usual procedure, that amounts to fixing $\rho = kT^n$, k and n constant, gives the same result when radiation is a relatively unimportant companion to the ideal gas, but in the other limiting case, when the gas is very dilute and the material gas becomes an insignificant addition to the photon gas, it is no longer tenable. We need an interpolation between the two extreme cases and this is provided naturally by the postulate that the action is stationary with respect to variations of the temperature field.

In the absence of the ideal gas we have another interesting system, the pure photon gas. The analogy between the photon gas and the ideal gas is often stressed; there is an analogue of the polytropic relation that fixes the temperature in terms of ρ ; the pressure of the photon field is $(a/3)T^4$. Our lagrangian already contains this pressure; we should like to discover a closer connection between it and the electromagnetic field. In the limit when the density of the ideal gas is zero, Eq.(A.17) becomes

$$-\frac{\vec{D}^2}{2} \frac{\partial \kappa}{\partial T} + \frac{4a}{3} T^3 = 0.$$

In the absence of the gas it is reasonable to impose Lorentz invariance, so we include magnetic effects by completing the last to

$$-\frac{F^2}{2} \frac{\partial \kappa}{\partial T} + \frac{4a}{3} T^3 = 0.$$

If we suppose that $\kappa[\rho, T]$, in the limit $\rho = 0$, takes the form αT^2 , then

$$\alpha F^2 = \frac{4a}{3} T^2.$$

The radiation from a gas of Hertzian dipoles can be shown, with the help of the Stefan-Boltzman law and Wien's displacement law, to satisfy a relation of precisely this form. Whether the same relation holds in vacuum is uncertain, but it is suggested by an analysis of the effective Born-Infeld lagrangian calculated on the basis of the scattering of light by light (Euler 1936, Karplus and Neuman 1950). See also McKenna and Platzman (1962), Fronsdal (2007).

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