# Performance Bounds for AF Multi-Hop Relaying over Nakagami Fading

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***Abstract*—This paper presents a new upper bound on the end-to-end signal-to-noise ratio (SNR) of channel-assisted amplify-and-forward (AF) multi-hop relay networks. The harmonic mean of the minimum of the first $P \geq 0$ hop SNRs and the minimum of the remaining hop SNRs forms the new bound. Closed-form expressions are derived for the cumulative distribution function and the moment generating function of this SNR upper bound for independent and non-identically distributed Rayleigh, and independent and identically distributed Nakagami-$m$ fading, where $m$ is an integer. The outage probability and the average symbol error rate bounds are also derived. Our proposed bounds are compared against the existing bounds.**

## I. Introduction

Multi-hop relay networks achieve broader coverage and enhanced throughput due to shorter hops and can also provide network connectivity to locations, where traditional single-hop architectures may not reach [1]–[6]. As well, the battery life of the terminals may be extended due to lower power requirements [4]. Moreover, because the spatial diversity gains of multi-hop relaying enhance the system performance, multi-hop networks have been widely researched [3], [7]–[9].

### A. Prior related research

The exact closed-form analytical results for a number of hops $N \geq 3$ appear to be intractable. Previous analyses of $N$-hop non-regenerative relay networks can be classified into two categories: first one includes various bounds on the end-to-end signal-to-noise ratio (SNR); second one includes approximations. The first category includes [9]–[12]. Reference [10] proposes an upper bound for the end-to-end SNR of multi-hop transmissions; the key idea is to upper bound the harmonic mean by the geometric mean. The moment generating function (MGF), the probability distribution function (PDF), and the cumulative distribution function (CDF) of the upper bound are derived. Closed-form lower bounds on the outage probability and the average bit error rate (BER) of the coherent binary modulation are derived. In [11], the SNR bound of [10] is used to study the performance of multi-hop semi-blind relays over generalized fading channels by using the moments-based approach. In [9], the bound of [11] is further employed for performance analysis of multi-hop relay networks with cooperative diversity. Reference [12] proposes an end-to-end SNR upper bound for a multi-hop channel-assisted amplify-and-forward (CA-AF) relay network by using the minimum SNR of all hops [12, Eq. (11)]. The average BER of several modulation schemes over fading channels is also computed via the moments-based approach. Reference [13] analyze the performance of a multi-hop CA-AF relay network over Weibull fading by using the upper bound of [12].

Examples of the second category are [2], [7], [8], [14], [15]. In [2], the outage probability of a multi-hop CA-AF relay network over Nakagami-$m$ fading is evaluated. The MGF of the reciprocal of the end-to-end SNR is derived in closed-form, and the outage probability is computed via numerical Laplace-transform inversion. A unified framework for the performance analysis of a generic cooperative system with multiple hops and multiple branches is proposed in [7]. The key idea is to relate the MGF of a random variable to the MGF of its reciprocal; this method requires numerical integration in some cases. Reference [14] provides an asymptotic analysis of the error rates of multi-hop multi-branch relay networks. Moreover, the performance of multi-hop AF relays over independent and non-identical Rayleigh fading channels is studied in [8]. In [15], the asymptotic BER of multi-hop AF relaying over Nakagami-$m$ fading is analyzed.

### B. Motivation and our contribution

Although the performance bounds of [9]–[11] are tight for low SNRs, these bounds are significantly loose for high SNR and for more severe fading environments such as Rayleigh fading. As a result, these bounds can not be used to investigate the important system parameters such as the diversity order and coding gains. Moreover, while the outage probability of multi-hop relay networks with CA-AF relays is analyzed [2], no closed-form expression is presented. Furthermore, the performance analysis of the multi-hop multi-branch relay network proposed in [7] is based on a single-integral relation between the MGF of a random variable and the MGF of its reciprocal. Although this approach is accurate, the performance metrics are not in closed-form. Thus, these gaps in the performance analysis of multi-hop relay networks, which are due mainly to intractability of the problem, motivated our development of tight closed-form performance bounds.

In this work, a new upper bound is derived for the end-to-end SNR of $N \geq 2$ hop ideal CA-AF relay network. The key idea is to bound the end-to-end SNR by the harmonic mean of the minimums of the SNR of the first $P$ hops and the next $N-P$ hops, where $0 \leq P \leq N$, and $N$ is the number of hops in the system. Here, $P$ is a free parameter used to

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provide flexibility and generality. For example, if $P = 0$ or $P = N$, we obtain the results of [2]. The CDF and the MGF of the upper bound are derived in closed-form. For the sake of brevity, only the cases of independent and non identically distributed (i.n.i.d.) Rayleigh fading channels, and independent and identically distributed (i.i.d.) Nakagami-$m$ fading, where $m$ is a positive integer, are treated. Closed-form lower bounds for the outage probability and the average symbol error rate are also derived. Numerical results to compare the tightness of the proposed bounds with that of the existing bounds [10], [12] are presented. Monte-Carlo simulation results verify the accuracy of our analytical results. Interestingly, the proposed bounds are asymptotically exact and further details on this fact can be found in [16], which is under review.

The rest of this paper is organized as follows. In Section II, the system model and the channel model are presented. Section III presents the performance analysis. Section IV contains the numerical results. Section VI concludes the paper.

**Notations:** $\mathcal{K}_\nu(z)$ is the *Modified Bessel function of the second kind* of order $\nu$ [17, 9.6]. ${}_2\mathcal{F}_1(\alpha,\beta;\gamma;z)$ is the *Gauss Hypergeometric function* [17, 15.1]. $\mathcal{Q}(z)$ is the *Q-function* [17, 26.2.3]. $\mathcal{E}_\Lambda\{\cdot\}$ denotes the *expected value* over the random variable $\Lambda$. $X \sim \mathcal{G}(\alpha,\beta)$ means $X$ is distributed with *Gamma*$(\alpha,\beta)$ PDF [18, Eq. (2)]. $\mathbb{Z}^+$ is the set of positive integers.

## II. System And Channel Models

We consider a multi-hop relay network with $N$ hops and $N+1$ terminals operating over independent flat fading channels. Out of these terminals, $N-1$ of them act as AF relays between the source (S) and the destination (D) terminals. Only single-antenna terminals are used. We name two commonly used relay types "practical CA-AF" and "ideal CA-AF" [2]. The gain of relay $n$ in the practical CA-AF is $G_n = \sqrt{\frac{\mathcal{P}_n}{\mathcal{P}_n|h_n|^2+N_{0,n}}}$ [2], [19], where $\mathcal{P}_n$ is the average energy per symbol used at the $n$-th relay, $h_n$ is the fading amplitude of the preceding hop and $N_{0,n}$ is the variance of the zero-mean additive white Gaussian noise at the input of the $n$-th receiver. The end-to-end SNR $\gamma_{\text{eq}}$ of a multi-hop practical CA-AF relay network is given by [2] $\gamma_{\text{eq,practical}} = \left[\prod_{n=1}^{N}\left(1+\frac{1}{\gamma_n}\right)-1\right]^{-1}$, where $\gamma_n = \mathcal{P}_n|h_n|^2/N_{0,n}$ is the SNR of the $n$-th hop. The gain of the ideal CA-AF relays, which are capable of inverting the channel of the previous hop (regardless of the fading state of that hop), is given by $G_n = 1/|h_n|$, and the $\gamma_{\text{eq}}$ is given by [2]

$$\gamma_{\text{eq,ideal}} = \left[\sum_{n=1}^{N}\frac{1}{\gamma_n}\right]^{-1}. \tag{1}$$

Note that $\gamma_{\text{eq,ideal}}$ is a tight upper bound for $\gamma_{\text{eq,practical}}$ [2]. We consider a set of ideal CA-AF relays. The performance measures of multi-hop relay networks with such relays serve as benchmarks for systems with various practical relays.

In order to analyze the system performance, statistics for the end-to-end SNR (1) are required. However, the probability distribution of the SNR in (1) is not mathematically tractable, particularly for $N \geq 3$. Thus, in order to develop a more accurate performance analysis framework, we propose a new upper bound for (1). The key idea is to partition the set of $\gamma_l|_{l=1}^{N}$ into two groups. The minimum of $\gamma_l$ of each group is then used to bound (1) as follows:

$$\gamma_{\text{eq,ideal}} \leq \gamma_{\text{eq}}^{\text{ub}} = \left[\frac{1}{\min(\gamma_1,\gamma_2,\ldots,\gamma_P)} + \frac{1}{\min(\gamma_{P+1},\gamma_{P+2},\ldots,\gamma_N)}\right]^{-1}, \tag{2}$$

where $0 \leq P \leq N$. $\gamma_{\text{eq}}^{\text{ub}}$ in (2) is related to the harmonic mean of the minimum of SNR of the first $P$ hops and the minimum of the next $N-P$ hops.

Interestingly, when $P=0$ or $P=N$, (2) reduces to the bound given in [12, Eq. (11)]. We observe that (2) with $N=2$ and $P=1$ reduces to the exact end-to-end SNR for the case of dual-hop systems with ideal CA-AF relays.

## III. Performance Analysis

This section presents a comprehensive performance analysis of multi-hop relay networks by using (2). We first find the distribution of $\gamma_{\text{eq}}^{\text{ub}}$ for i.n.i.d. Rayleigh fading and i.i.d. Nakagami-$m$ fading, $m \in \mathbb{Z}^+$. The outage probability and the average symbol error rate (SER) are derived for both cases.

### A. Statistical characterization of end-to-end SNR

In this section, we obtain closed-form expressions for the CDF and the MGF of the end-to-end SNR $\gamma_{\text{eq}}^{\text{ub}}$. The CDF and the MGF of $\gamma_{\text{eq}}^{\text{ub}}$ in i.n.i.d. Rayleigh fading are given by Theorem 1.

*Theorem 1: Let $\gamma_n \sim \mathcal{G}(1,\bar{\gamma}_n)$, $n = 1,\ldots,N$, be independent hop SNRs. The CDF of $\gamma_{eq}^{ub}$ is then given by*

$$F_{\gamma_{\text{eq}}^{\text{ub}}}(x) = 1 - 2\sqrt{\lambda_1\lambda_2}\,x\exp(-\lambda_0 x)\,\mathcal{K}_1\!\left(2x\sqrt{\lambda_1\lambda_2}\right), \tag{3}$$

*where* $\lambda_1 = \sum_{n=1}^{N-P}\frac{1}{\bar{\gamma}_n}$, $\lambda_2 = \sum_{n=N-P+1}^{N}\frac{1}{\bar{\gamma}_n}$ *and* $\lambda_0 = \lambda_1 + \lambda_2$.

*The MGF of $\gamma_{eq}^{ub}$ is given by*

$$M_{\gamma_{\text{eq}}^{\text{ub}}}(s) = 1 - \frac{64}{3}\lambda_1\lambda_2 s\frac{{}_2\mathcal{F}_1\left(3,\frac{3}{2};\frac{5}{2};\frac{s+\lambda_0-2\sqrt{\lambda_1\lambda_2}}{s+\lambda_0+2\sqrt{\lambda_1\lambda_2}}\right)}{\left(s+\lambda_0+2\sqrt{\lambda_1\lambda_2}\right)^3}. \tag{4}$$

*Proof:* See Appendix. ■

The CDF and the MGF of $\gamma_{\text{eq}}^{\text{ub}}$ in i.i.d. Nakagami-$m$ fading are given by Theorem 2.

*Theorem 2: Let $\gamma_n \sim \mathcal{G}(m,\frac{\bar{\gamma}}{m})$, $n = 1,\ldots,N$, be independent hop SNRs. The CDF of $\gamma_{eq}^{ub}$ is then given by*

$$F_{\gamma_{\text{eq}}^{\text{ub}}}(x) = 1 - \sum_{j=0}^{P(m-1)}\sum_{k=0}^{(N-P)(m-1)} 2\beta_{j,P}\beta_{k,N-P}\left(\frac{m}{\bar{\gamma}}\right)^{j+k}\exp\left(-\frac{mNx}{\bar{\gamma}}\right) \times\left[\frac{m}{\bar{\gamma}}P^{\frac{k+1}{2}}(N-P)^{\frac{1-k}{2}}x^{j+k+1}\,\mathcal{K}_{k+1}\!\left(\frac{2m}{\bar{\gamma}}\sqrt{P(N-P)}x\right)\right.$$

$$+\sum_{l=0}^{j+k-1}\binom{j+k-1}{l}\left(\frac{P}{N-P}\right)^{\frac{l-j+1}{2}}x^{j+k}$$
$$\mathcal{K}_{l-j+1}\!\left(\frac{2m}{\bar{\gamma}}\sqrt{P(N-P)}\,x\right)\!\left(\frac{m(j+k)(N-P)}{\bar{\gamma}(j+k-l)}x-k\right)\Bigg],(5)$$

*where*

$$\beta_{k,N}=\sum_{i=k-m+1}^{k}\frac{\beta_{i,N-1}}{(k-i)!}I_{[0,(N-1)(m-1)]}(i), \qquad (6)$$

$\beta_{0,0}=\beta_{0,N}=1$, $\beta_{k,1}=1/\,k!$, $\beta_{1,N}=N$ *and*
$I_{[a,c]}(b)=\begin{cases}1, & a\le b\le c\\ 0, & \text{otherwise}\end{cases}.$

*The MGF of $\gamma_{eq}^{ub}$ is given by*

$$M_{\gamma_{\text{eq}}^{\text{ub}}}(s)=1-\sum_{j=0}^{P(m-1)}\sum_{k=0}^{(N-P)(m-1)}2\sqrt{\pi}\beta_{j,P}\beta_{k,N-P}\;s$$
$$\times\Bigg[N\left(\frac{m}{\bar{\gamma}}\right)^{2k+j+2}(4P)^{k+1}\frac{\Gamma(2k+j+3)\Gamma(j+1)}{\Gamma(k+j+\frac{5}{2})(s+\beta)^{2k+j+3}}$$
$${}_2\mathcal{F}_1\left(2k+j+3,k+\frac{3}{2};j+k+\frac{3}{2};\frac{s+\alpha}{s+\beta}\right)$$
$$+\sum_{l=0}^{j+k-1}\binom{j+k-1}{l}\left(\frac{m}{\bar{\gamma}}\right)^{l+k+1}(4P)^{l-j+1}\frac{\Gamma(k+l+2)}{\Gamma(j+k+\frac{3}{2})}$$
$$\frac{\Gamma(2j+k-1)}{(s+\beta)^{l+k+2}}\Bigg(\frac{Nm(j+k)(k+l+2)(2j+k-l)}{\bar{\gamma}(j+k-l)(j+k+\frac{3}{2})(s+\beta)}$$
$${}_2\mathcal{F}_1\left(l+k+2,l-j+\frac{3}{2};j+k+\frac{3}{2};\frac{s+\alpha}{s+\beta}\right)$$
$$-k\,{}_2\mathcal{F}_1\left(j+k+2,l-j+\frac{3}{2};j+k+\frac{3}{2};\frac{s+\alpha}{s+\beta}\right)\Bigg)\Bigg],(7)$$

*where* $\alpha=\frac{m}{\bar{\gamma}}\left(N-2\sqrt{P(N-P)}\right)$ *and* $\beta=\frac{m}{\bar{\gamma}}\left(N+2\sqrt{P(N-P)}\right)$.

*Proof:* See Appendix. ■

Since the CDF (5) and the MGF (7) in i.i.d. Nakagami-$m$ fading do not hold for $P=0$ or $P=N$, they are given explicitly by $F_{\gamma_{\text{eq}}^{\text{ub}}}(x)=1-\exp\left(-\frac{mNx}{\bar{\gamma}}\right)\sum_{k=0}^{N(m-1)}\beta_{k,N}\left(\frac{mx}{\bar{\gamma}}\right)^k$ and $M_{\gamma_{\text{eq}}^{\text{ub}}}(s)=1-\sum_{k=0}^{N(m-1)}s\beta_{k,N}\Gamma(k+1)\frac{\bar{\gamma}}{m}\left(\frac{m}{mN+\bar{\gamma}s}\right)^{k+1}$, respectively.

The probability distribution function (PDF) of $\gamma_{\text{eq}}^{\text{ub}}$ can be easily derived by differentiating the relevant CDF expression over $x$ and by using $x\frac{\partial\mathcal{K}_\nu(x)}{\partial x}+\nu\mathcal{K}_\nu(x)+x\mathcal{K}_{\nu-1}(x)=0$ [20, 8.486.12]. However, for the sake of brevity, we do not develop such results.

### B. Outage probability

The outage probability is defined as the probability that the instantaneous end-to-end SNR $\gamma_{\text{eq}}$ falls below a certain target value $\gamma_{th}$. Thus, the lower bounds for the outage probability $P_{out}$ for i.n.i.d. Rayleigh and i.i.d. Nakagami-$m$ fading can immediately be obtained by using the results given in (3) and (5):

$$P_{out}^{lb}=\Pr(0\le\gamma_{\text{eq}}^{\text{ub}}\le\gamma_{th})=F_{\gamma_{\text{eq}}^{\text{ub}}}(\gamma_{th}). \qquad (8)$$

### C. Average error rate

The average SER is one of the most widely used performance metrics of digital communication systems. The average error rates of multi-hop relay networks can be derived by integrating the conditional error probability (CEP) $P_e|\gamma$ over the PDF of the end-to-end SNR $\gamma_{\text{eq}}^{\text{ub}}$. For example, the CEP of coherent binary frequency shift keying (C-BFSK) and $M$-ary pulse amplitude modulation (PAM) can be expressed as $P_e|\gamma=a\mathcal{Q}(\sqrt{b\gamma})$ [21], where $a$ and $b$ are modulation-dependent constants. For example, $(a=1,b=2)$ and $(a=1,b=1)$ represent BPSK and C-BFSK, respectively. Further, the SER of $M$-ary PAM is obtained by using $(a=2(M-1)/M)$ and $(b=6\log_2 M/M^2-1)$. The SER can be simplified to the following integral representation (9) by integrating by parts [22]:

$$\bar{P}_e=\frac{a}{2}-\frac{a}{2}\sqrt{\frac{b}{2\pi}}\int_0^\infty x^{-\frac{1}{2}}\exp\left(-\frac{bx}{2}\right)\bar{F}_{\gamma_{\text{eq}}^{\text{ub}}}(x)\mathrm{d}x, \qquad (9)$$

where $\bar{F}_{\gamma_{\text{eq}}^{\text{ub}}}(x)$ is the complementary cumulative distribution function (CCDF) of $\gamma_{\text{eq}}^{\text{ub}}$ defined by $1-F_{\gamma_{\text{eq}}^{\text{ub}}}(x)$. The average SER for i.n.i.d. Rayleigh fading is given by Theorem 3.

*Theorem 3: Let $\gamma_n\sim\mathcal{G}(1,\bar{\gamma}_n)$, $n=1,\ldots,N$, be independent hop SNRs. The average SER obtained by using $\gamma_{eq}^{ub}$ is given by*

$$\bar{P}_e^{lb}=\frac{a}{2}-3a\pi\sqrt{\frac{b}{2}}\lambda_1\lambda_2\frac{{}_2\mathcal{F}_1\left(\frac{5}{2},\frac{3}{2};2;\frac{\frac{b}{2}+\lambda_0-2\sqrt{\lambda_1\lambda_2}}{\frac{b}{2}+\lambda_0+2\sqrt{\lambda_1\lambda_2}}\right)}{\left(\frac{b}{2}+\lambda_0+2\sqrt{\lambda_1\lambda_2}\right)^{\frac{5}{2}}}. \quad (10)$$

*Proof:* See Appendix. ■

The average SER for i.i.d. Nakagami-$m$ fading is given by Theorem 4.

*Theorem 4: Let $\gamma_n\sim\mathcal{G}(m,\frac{\bar{\gamma}}{m})$, $n=1,\ldots,N$, be independent hop SNRs. The average SER obtained by using $\gamma_{eq}^{ub}$ is then given by*

$$\bar{P}_e^{lb}=\frac{a}{2}-a\sqrt{\frac{b}{2}}\sum_{j=0}^{P(m-1)}\sum_{k=0}^{(N-P)(m-1)}\beta_{j,P}\beta_{k,N-P}$$
$$\times\Bigg[(N-P)(4P)^{l-j+1}\left(\frac{m}{\bar{\gamma}}\right)^{2k+j+2}\frac{\Gamma(2k+j+\frac{5}{2})\Gamma(j+\frac{1}{2})}{\Gamma(j+k+2)(\frac{b}{2}+\beta)^{2k+j+\frac{5}{2}}}$$
$${}_2\mathcal{F}_1\left(2k+j+\frac{5}{2},k+\frac{3}{2};j+k+2;\frac{\frac{b}{2}+\alpha}{\frac{b}{2}+\beta}\right)+\sum_{l=0}^{j+k-1}\binom{j+k-1}{l}$$
$$\left(\frac{m}{\bar{\gamma}}\right)^{l+k+1}(4P)^{l-j+1}\frac{\Gamma(k+l+\frac{3}{2})\Gamma(2j+k-l-\frac{1}{2})}{\Gamma(j+k+1)(\frac{b}{2}+\beta)^{l+k+\frac{3}{2}}}$$
$$\Bigg(\frac{(N-P)m(j+k)(k+l+\frac{3}{2})(2j+k-l-\frac{1}{2})}{\bar{\gamma}(j+k-l)(j+k+1)(s+\beta)}$$
$${}_2\mathcal{F}_1\left(l+k+2,l-j+\frac{3}{2};j+k+\frac{3}{2};\frac{s+\alpha}{s+\beta}\right)$$

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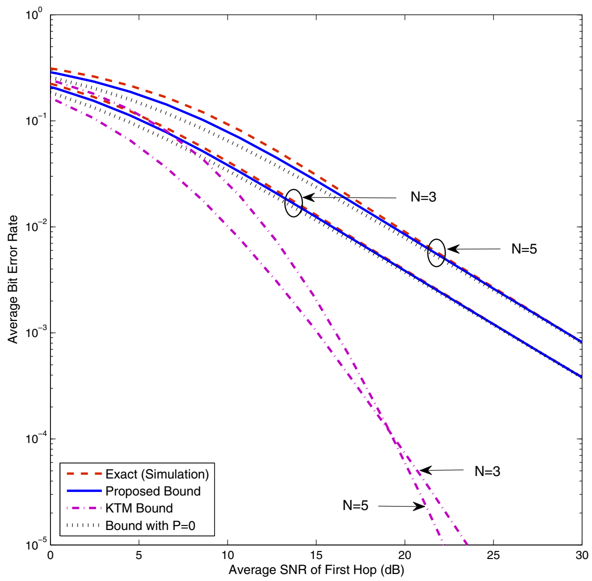


Fig. 1. A comparison of the BPSK average BER bounds of multi-hop relay network with $N = 3$ and $N = 5$ over i.n.i.d. Rayleigh fading channels. Average SNR of first hop is denoted by $\bar{\gamma}_1$. $\bar{\gamma}_2 = 5.3\bar{\gamma}_1$, $\bar{\gamma}_3 = 3.1\bar{\gamma}_1$, $\bar{\gamma}_4 = 1.7\bar{\gamma}_1$ and $\bar{\gamma}_5 = 2.4\bar{\gamma}_1$.

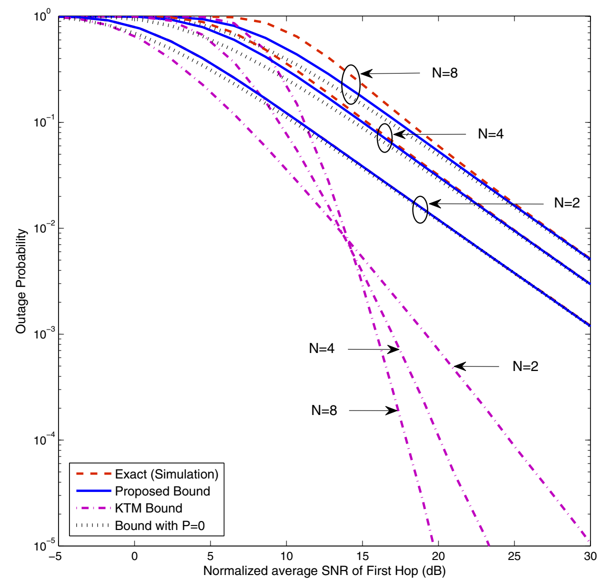


Fig. 2. A comparison of the outage probability bounds of multi-hop relay network with $N = 2$, $N = 4$ and $N = 8$ over i.n.i.d. Rayleigh fading channels. Normalized average SNR of first hop is denoted by $\bar{\gamma}_1/\gamma_{\text{th}}$. $\bar{\gamma}_2 = 5.3\bar{\gamma}_1$, $\bar{\gamma}_3 = 3.1\bar{\gamma}_1$, $\bar{\gamma}_4 = 1.7\bar{\gamma}_1$, $\bar{\gamma}_5 = 7.5\bar{\gamma}_1$, $\bar{\gamma}_6 = 0.5\bar{\gamma}_1$, $\bar{\gamma}_7 = 4.5\bar{\gamma}_1$ and $\bar{\gamma}_8 = 2.4\bar{\gamma}_1$. $P = \left\lceil \frac{N}{2} \right\rceil$.

$$-k_2\mathcal{F}_1\left(j+k+2, l-j+\frac{3}{2}; j+k+\frac{3}{2}; \frac{s+\alpha}{s+\beta}\right)\Bigg)\Bigg]. \quad (11)$$

*Proof:* See Appendix. ■

Since the average BER bound in i.i.d. Nakagani-$m$ fading (11) does not hold for $P = 0$ or $P = N$, it is given explicitly by $\bar{P}_e^{lb} = \frac{a}{2} - \frac{a}{2}\sqrt{\frac{b}{2\pi}}\beta_{k,N}\Gamma\left(k+\frac{1}{2}\right)\left(\frac{m}{\bar{\gamma}}\right)^k\left(\frac{2\bar{\gamma}}{b\bar{\gamma}+2mN}\right)^{k+\frac{1}{2}}$.

## IV. Numerical Results

This section presents our numerical and simulation results in order to investigate the tightness of the proposed performance bounds. They are compared with the existing multi-hop performance bounds [10], [12], [13].

In Fig. 1, the proposed lower bounds for the BPSK average bit error rate (BER) of a multi-hop system operating over i.n.i.d. Rayleigh fading are plotted for three-hop and five-hop cases, respectively. The BPSK average BER bound of [10, Eq. (24)] with $m_i|_{i=1}^{N} = 1$ is plotted for comparison purposes. This bound is named the "KTM" bounds. The proposed bound with $P = 0$, which simplifies to the bound in [12, Eq. (11)] is also plotted. As expected, the proposed bound is tight, particularly in the medium-to-high SNR regime, but becomes looser as the number of hops increases. The KTM bound is quite loose in almost the entire usable SNR regime $\bar{\gamma} > 0$ dB and gets progressively looser as the first hop SNR increases. For example, at a BER of $10^{-3}$, the KTM bound is 10 and 12 dB off, for the three-hop and five-hop cases, respectively. Whereas our proposed bounds are almost exact. The bound with $P = 0$ [12, Eq. (11)] is also tight in the high SNRs. Interestingly, our proposed bound converges to the exact average BER curve at the high SNR regime. The curve corresponding to $N = 2$ is plotted to verify that our proposed bound reduces to the exact outage probability of dual-hop system with ideal CA-AF relays.

In Fig. 2, we plot the outage probability of two-hop, four-hop and eight-hop systems operating over i.n.i.d Rayleigh fading. The proposed outage probability bounds are tight at moderate and high SNR and converge to the exact curves for SNRs above 20 dB for 4-hops and 25 dB for 8-Hops. The bounds deteriorate as $N$ increases. Thus, the proposed outage probability bounds behave a similar way to the BER bounds in comparison to the KTM bound [10, Eq. (16)].

In Fig. 3, the lower bounds for the average BER of a three-hop relay network operating over i.i.d. Makagami-$m$ fading are plotted. In particular, this figure illustrates the effect of the severity of the fading ($m$) parameter. The proposed bounds are tighter than the KTM bounds in the moderate-to-high SNR regime. The KTM bound is tighter than the proposed bounds at low SNRs for less severe fading environments. However, the KTM bound significantly deviates from the exact BER curve in the high SNR regime regardless of the severity of fading or number of hops. Although the bound with $P = 0$ is tighter in the high SNRs for smaller $m$, it is significantly loose in low-to-moderate SNRs for higher $m$.

Fig. 4 shows the tightness of the proposed BER bounds for a multi-hop system operating over i.i.d. Nakagami-$m$ fading as a function of the average SNR of the first hop. This figure illustrates the deterioration of tightness of the bounds as $N$ increases. The higher amount of looseness increases in less severe fading environments.

In Fig. 5, we plot the outage probability of a multi-hop

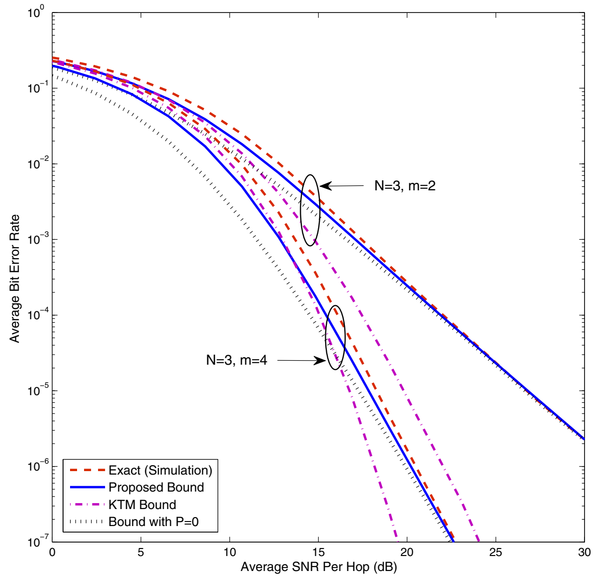


Fig. 3. A comparison of the effect of severity of fading on average BER bounds for a multi-hop relay network in i.i.d. Nakagami-$m$ fading. $N = 3$ and $P = 2$.

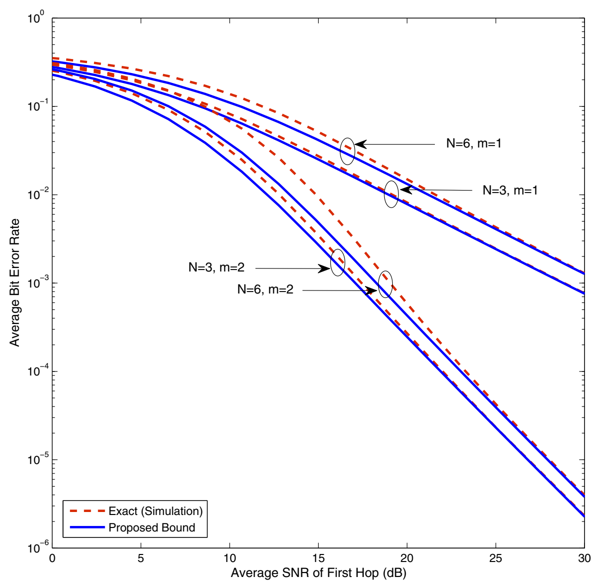


Fig. 4. The effect of number of hops and severity of fading on average BER bounds of a multi-hop relay network in i.i.d. Nakagami-$m$ fading. $P = \left\lceil \frac{N}{2} \right\rceil$.

system over Nakagami-$m$ fading. The curve with $m = 1$ is plotted to show the tightness of our bounds for i.i.d. Rayleigh fading. The proposed bound is tighter at moderate-to-high SNR than the KTM and it is asymptotically exact. The bound looses the tightness as $N$ and $m$ increase. Similar to BER bounds, KTM bound for the outage probability is tighter than our bound for less severe fading conditions (approximately $m > 5$) in the low-to-moderate SNR regime.

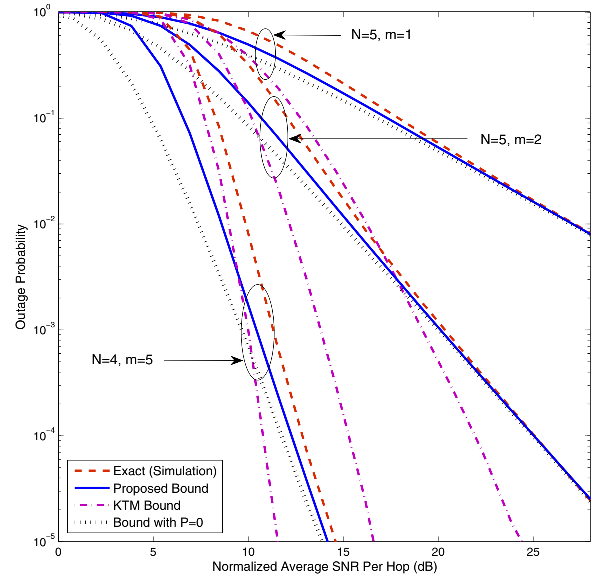


Fig. 5. A comparison of outage probability bounds of multi-hop relay network over i.i.d. Nakagami-$m$ fading channels. $P = \left\lceil \frac{N}{2} \right\rceil$.

## V. Applications

The proposed bounds can readily be applied for performance analysis of multi-hop multi-branch relay networks. The closed-form lower bound expressions for the outage probability and the average SER are useful for solving optimal power allocation problems in multi-hop relay networks. Moreover, our bounds may be useful for not only multi-hop relay networks with single antenna relays, but also for multi-hop relay networks with multiple-input multiple-output terminals. Detailed analyses and numerical examples of these applications can be found in [16].

## VI. Conclusion

This paper proposed a new upper bound for the end-to-end SNR of multi-hop relay networks with ideal CA-AF relays. The closed-form CDF and MGF expressions of upper bounded end-to-end SNR were derived for i.n.i.d. Rayleigh fading and, i.i.d. Nakagami-$m$ fading, $m \in \mathbb{Z}^+$. Lower bounds for the average BER and the outage probability were also derived. The proposed bounds outperform the existing bounds, particularly for severe fading environments and for medium-to-high SNRs. For the sake of brevity, only the case of i.i.d. Nakagami-$m$ fading, $m \in \mathbb{Z}^+$ was considered. Our results may be generalized to i.n.i.d. and non-integer Nakagami-$m$ fading cases.

## VII. Appendix

The sketches of the proofs of the theorems are presented here. Let random variable $\Gamma$ be

$$\Gamma = \frac{\Gamma_1 \Gamma_2}{\Gamma_1 + \Gamma_2}, \tag{12}$$

where $\Gamma = \gamma_{\text{eq}}^{\text{ub}}$, $\Gamma_1 = \min(\gamma_1, \gamma_2 ..., \gamma_{N-P})$ and $\Gamma_2 = \min(\gamma_{N-P+1}, \gamma_{N-P+2} ..., \gamma_N)$. The CDF of $\Gamma$ can be expressed as [22]

$$F_\Gamma(x) = 1 - \int_0^\infty \bar{F}_{\Gamma_1}\left(\frac{(z+x)x}{z}\right) f_{\Gamma_2}(z+x)\mathrm{d}z. \quad (13)$$

*(Proof of Theorem 1):* The CCDF of $\Gamma_1$ and PDF of $\Gamma_2$ for i.n.i.d. Rayleigh fading are given by

$$\bar{F}_{\Gamma_1}(x) = \exp\left(-\sum_{n=1}^{N-P} \frac{x}{\bar{\gamma}_n}\right) \quad \text{and}$$

$$f_{\Gamma_2}(x) = \left(\sum_{n=N-P+1}^{N} \frac{1}{\bar{\gamma}_n}\right) \exp\left(-\sum_{n=N-P+1}^{N} \frac{x}{\bar{\gamma}_n}\right). \quad (14)$$

By substituting (14) into (13) and by using [20, 3.471.9], the desired result (3) can be obtained.

The MGF of $\Gamma$ is given by:

$$M_\Gamma(s) = \mathcal{E}_\Gamma\{\exp(-sx)\} = 1 - \int_0^\infty s[1 - F_\Gamma(x)]\exp(-sx)\mathrm{d}x. \quad (15)$$

The second equality of (15) is obtained by integration by parts and considering the fact that $f_\Gamma(x) = 0, \forall x \leq 0$. Now, the MGF of $\Gamma$ for i.n.i.d. Rayleigh fading (4) can be obtained by substituting (3) into (15) and solving the resulting integral by using [20, 6.621.3]. ■

*(Proof of Theorem 2):* The CCDF of $\Gamma_1$ for i.i.d. Nakagami-$m$ fading with integer $m$ can be obtained by expanding $\left(\frac{\Gamma\left(m, \frac{mx}{\bar{\gamma}}\right)}{\Gamma(m)}\right)^P$ by using [20, 8.352.2] and [23, Eq. (44)] as follows:

$$\bar{F}_{\Gamma_1}(x) = \exp\left(-\frac{mPx}{\bar{\gamma}}\right) \sum_{k=0}^{P(m-1)} \beta_{k,P} \left(\frac{mx}{\bar{\gamma}}\right)^k, \quad (16)$$

where $\beta_{k,P}$ is given by (6). The PDF of $\Gamma_2$ for i.i.d. Nakagami-$m$ fading is given by

$$f_{\Gamma_2}(x) = \frac{m(N-P)}{\bar{\gamma}} \exp\left(-\frac{m(N-P)x}{\bar{\gamma}}\right) \sum_{k=0}^{(m-1)(N-P)} \beta_{k,N-P} \left(\frac{mx}{\bar{\gamma}}\right)^k - \exp\left(-\frac{m(N-P)x}{\bar{\gamma}}\right) \sum_{k=0}^{(m-1)(N-P)} k\beta_{k,N-P} \left(\frac{m}{\bar{\gamma}}\right)^k x^{k-1}. \quad (17)$$

By substituting (16) and (17) into (13) and by using [20, 3.471.9], the desired result (5) can be obtained.

The MGF of $\Gamma$ for i.i.d. Nakagami-$m$ fading (7) can be derived by substituting (5) into (15) and by using [20, 6.621.3]. ■

*(Proof of Theorem 3):* The average SER for i.n.i.d. Rayleigh fading (10) can be derived by substituting (3) into (9) and solving the resulting integral by using [20, 6.621.3]. ■

*(Proof of Theorem 4):* The average SER for i.i.d. Nakagami-$m$ fading (11) can be derived by substituting (5) into (9) and solving the resulting integral by using [20, 6.621.3]. ■

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