J. Evol. Equ.

DOI 10.1007/s00028-013-0215-1

Journal of Evolution Equations

On a time-splitting method for a scalar conservation law with a multiplicative stochastic perturbation and numerical experiments

Caroline Bauzet

Abstract. In this paper, we present a numerical scheme for a first-order hyperbolic equation of nonlinear type perturbed by a multiplicative noise. The problem is set in a bounded domain D of $\mathbb{R}^d$ and with homogeneous Dirichlet boundary condition. Using a time-splitting method, we are able to show the existence of an approximate solution. The result of convergence of such a sequence is based on the work of Bauzet–Vallet–Wittbold (J Funct Anal, 2013), where the authors used the concept of measure-valued solution and Kruzhkov's entropy formulation to show the existence and uniqueness of the stochastic weak entropy solution. Then, we propose numerical experiments by applying this scheme to the stochastic Burgers' equation in the one-dimensional case.

1. Introduction

We are interested in the formal nonlinear scalar conservation law with a stochastic perturbation of type:

$$\mathrm{du} + \operatorname{div} \mathbf{f}(u)\ \mathrm{dt} = h(u)\ \mathrm{dw} \quad \text{in } \Omega\times]0,T[\times D, \tag{1}$$

for an initial condition u_0 and with homogeneous Dirichlet boundary condition.

One assumes that D is a bounded domain of $\mathbb{R}^d$ $(d \geq 1)$ with a Lipschitz boundary if $d \geq 2$, T a positive number, $Q =]0,T[\times D$ and that $W = \{w_t, \mathcal{F}_t; 0 \leq t \leq T\}$ is a 1-D standard adapted Brownian motion defined on the probability space $(\Omega, \mathcal{F}, P)$. We suppose that

H$_1$: $\mathbf{f} : \mathbb{R} \to \mathbb{R}^d$ is a Lipschitz continuous function with $\mathbf{f}(0) = \mathbf{0}$.
H$_2$: $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with $h(0) = 0$.
H$_3$: $u_0 \in L^\infty(D) \cap BV(D)$[1].
H$_4$: There exists $M > 0$ such that supp$h \subset [-M, M]$.

Mathematics Subject Classification (2000): 35L60, 60H15, 35L60

Keywords: Stochastic PDE, First-order hyperbolic equation, Dirichlet problem, Multiplicative stochastic perturbation, Young's measures, Kruzhkov's semi-entropy, Splitting method, Stochastic Burgers' equation, Euler scheme, Godunov scheme.

This research received support from the grant of the GENERAL COUNCIL of the Atlantic Pyrenees.

[1] where $BV(D)$ denotes the set of integrable functions with bounded variation on D.

Let us mention some remarks concerning these assumptions.

REMARK 1. H_1 and H_2 are claimed conditions from the theoretical point of view to ensure the well-posedness in the sense of [3] for an initial condition $u_0 \in L^2(D)$. Then, since D is a bounded domain, assuming H_1 to H_3 yields a result of existence and uniqueness of the entropy solution.

Note that H_1 can be weakened by assuming that $\mathbf{f}$ is a locally Lipschitz continuous function. Indeed, since the solution u is bounded by a constant M_1 depending only on M and $\|u_0\|_\infty$, the result holds by a truncation argument of $\mathbf{f}$ outside $[-M_1, M_1]$. Secondly, since $\text{div}[\mathbf{f}(0)] = 0$, one can assume by convenience that $\mathbf{f}(0) = \mathbf{0}$.

H_2 is a technical condition coming from [3]. Note that, in the present work, $h(0) = 0$ is not necessary to obtain a priori estimates on the approximate solution given by the time-splitting scheme. But as we are interested in proving that such a sequence converges strongly in $L^p(\Omega \times Q)$ for any finite p to the unique stochastic entropy solution of [3], we need to fulfill the assumptions of paper [3].

H_3 and H_4 are specific conditions from the numerical analysis point of view. These are technical assumptions to control the estimates in the forthcoming lemmata, in particular to apply Lemma 3. Note that H_4 is a necessary condition to keep the solution u bounded.

1.1. Former results

Only few papers have been devoted to the study of numerical experiments for stochastic conservation laws. Let us cite the paper of Holden–Risebro [15] where an operator-splitting method is proposed to prove the existence of pathwise weak solutions to the Cauchy problem

$$\mathrm{d}u + f(u)_x \mathrm{d}t = g(u)\mathrm{d}w \quad \text{in } \Omega \times]0, T[\times \mathbb{R}.$$

The operator-splitting approach has also been studied in [4] by Bensoussan–Glowinski–Raşcanu, where the authors are interested in approximating stochastic partial differential equations of parabolic type by some iterative schemes suggested by the Lie–Trotter product formulas. The convergence of the operator-splitting method is based on the continuity of the considered operator, which does not hold in our case.

Concerning the Cauchy problem for a conservation law with multiplicative noise, Feng–Nualart [12] introduced a notion of strong entropy solution in order to prove the uniqueness of the entropy solution. Let us precise that a "strong entropy solution" is a stochastic entropy solution as in the sense of Definition 1 below satisfying an "extra property," which allows to control a particular stochastic integral. We refer the reader to the Definition 2.6 p. 317 of Feng–Nualart [12]. Note that thanks to this additional property, the authors concluded to the uniqueness by comparing a strong entropy solution with a stochastic entropy solution. Using the vanishing viscosity and compensated compactness arguments, the authors established the existence of strong entropy solutions only in the one-dimensional case.

In the recent paper of Chen–Ding–Karlsen [6], the authors proposed a generalization of the work of Feng–Nualart: they considered a multi-dimensional stochastic balance law:

$$\partial_t(t,\mathbf{x}) + \nabla.\mathbf{f}(u(t,\mathbf{x})) = \sigma(u(t,\mathbf{x}))\partial_t W(t), \quad \mathbf{x} \in \mathbb{R}^d, \quad t > 0,$$

with the initial data $u(0,\mathbf{x}) = u_0(\mathbf{x})$. They identified a class of nonlinear balance laws for which uniform spatial BV bound for vanishing viscosity approximations can be achieved. Moreover, they established temporal equicontinuity in L^1 of the approximations, uniformly in the viscosity coefficient. In details, they proved that this stochastic problem is well-posed by using a uniform spatial BV bound. The main tool is a stochastic version of the compensated compactness approach, introduced by Feng and Nualart [12]. They also proved the existence of strong stochastic entropy solutions in $L^p \cap BV$ and developed a "continuous dependence" theory for stochastic entropy solutions in BV, which can be used to derive an error estimate for the vanishing viscosity method. Various further generalizations of their results are discussed.

In the work of Kröker–Rohde [17], the authors are interested in a method of handling the finite volume schemes for the approximate solution of the Cauchy problem for an hyperbolic balance law with random noise. For a class of monotone numerical fluxes, they establish the pathwise convergence of a semi-discrete finite volume solution toward a stochastic entropy solution. The main tool is a stochastic version of the compensated compactness approach. It avoids the use of a maximum principle and total variation estimates.

Using a kinetic formulation, Debussche–Vovelle [8] proved the first complete well-posedness result for multi-dimensional scalar conservation laws set in a d-dimensional torus and driven by a general multiplicative noise. As an extension of this work, in a recent paper, Hofmanová [14] presents a Bhatnagar–Gross–Krook-like approximation to this problem. Using the stochastic characteristics method, the author establishes the existence of an approximate solution and shows its convergence to the kinetic solution of [8].

By the way of the theory of Young's measure-valued solutions, Bauzet–Vallet–Wittbold [2] proved a result of existence and uniqueness of the solution to the multi-dimensional Cauchy problem in $L^2(\Omega \times Q)$. Since the method consists in comparing a weak measure-valued entropy solution to a regular one (the viscous solution in our case) and not to a strong one, the authors could consider very general assumptions on the data.

In Bauzet–Vallet–Wittbold [3], the authors investigated the Dirichlet problem for equation (1) with an initial condition u_0 in $L^2(D)$ and under assumptions H_1 and H_2. They proved a result of existence and uniqueness of the stochastic entropy solution by using the concept of measure-valued solutions and Kruzhkov's semi-entropy formulations. In the present work, we will use their theoretical results.

1.2. Goal of the study

Our aim is to revisit and generalize the time-splitting method introduced by Holden–Risebro [15] for the same scalar conservation law but in a bounded domain of $\mathbb{R}^d$ and prove that the pathwise weak solution they obtained is an entropy weak solution and that the whole sequence of approximation converges. The idea is to complete the work of Bauzet–Vallet–Wittbold [3] by numerical experiments using their theoretical study. For technical reasons, we need to assume the additional hypothesis on the data H_3 and H_4 in order to show the convergence of the method.

The paper is organized as follows. In Sect. 2, we recall for convenience the notion of stochastic entropy (respectively, measure-valued entropy) solution for (1), in particular the way Bauzet–Vallet–Wittbold consider the boundary conditions in [3], and their main result. In Sect. 3, we present a time-splitting method for the stochastic conservation law (1), which allows us to construct an approximate solution. Then, we introduce an entropy formulation satisfied by such a sequence. Using Young's measure compactness arguments, one shows that this approximate solution tends to a measure-valued entropy solution of (1). The study of (1) by Bauzet–Vallet–Wittbold in [3] allows us to conclude that this limit is the unique stochastic entropy solution of (1). Finally, in Sect. 4, we propose a numerical application with the stochastic Burgers' equation in the one-dimensional case. We introduce the scheme used and present simulations of solution obtained for different initial conditions by the free software *Scilab*.

1.3. Notations

Consider $BV(D)$ the set of integrable functions with bounded variation on D endowed with the norm $\|v\|_{BV(D)} = \|v\|_{L^1(D)} + TV_x(v)$, where $TV_x(v)$ denotes the total variation of v on D (see Evans–Gariepy [10] for example).

For a given separable Banach space X, we denote by $N^2_w(0, T, X)$ the space of the predictable X-valued processes. This space is $L^2(]0, T[\times\Omega, X)$ endowed with the product measure $\mathrm{d}t \otimes \mathrm{d}P$ and the predictable σ-field $\mathcal{P}_T$: i.e., the σ-field generated by the sets $\{0\} \times \mathcal{F}_0$ and the rectangles $]s, t] \times A$ for any $A \in \mathcal{F}_s$ (we refer the reader to Da Prato–Zabczyk [7]).

We denote by $\mathcal{E}^+$ the set of nonnegative convex functions η in $C^{2,1}(\mathbb{R})$, approximating the semi-Kruzhkov entropies $x \mapsto x^+$ such that $\eta(x) = 0$ if $x \leq 0$ and that there exists $\delta > 0$ such that $\eta'(x) = 1$ if $x > \delta$.

Then, one denotes by $\mathcal{E}^-$ the set $\{\check{\eta} := \eta(-\cdot),\ \eta \in \mathcal{E}^+\}$; and, for the definition of the entropy inequality, consider the sets

$$\mathbb{A}^+ = \{(k, \varphi, \eta) \in \mathbb{R} \times \mathcal{D}^+(\mathbb{R}^{d+1}) \times \mathcal{E}^+,\ k < 0 \Rightarrow \varphi \in \mathcal{D}^+([0, T] \times D)\},$$
$$\mathbb{A}^- = \{(k, \varphi, \eta),\ (-k, \varphi, \check{\eta}) \in \mathbb{A}^+\},$$
$$\mathbb{A}^+ = \mathbb{A}^+ \cup \mathbb{A}^-,$$

and the flux function $F^\eta(a, b) = \int_b^a \eta'(\sigma - b)\mathbf{f}'(\sigma)\,\mathrm{d}\sigma$, defined for $\eta \in \mathcal{E}^+ \cup \mathcal{E}^-$.

2. Existence and uniqueness result

Let us recall the definitions and the result introduced in the paper of Bauzet–Vallet–Wittbold [3]. Let us mention that these results are obtained under hypotheses H_1 and H_2, for an initial condition $u_0 \in L^2(D)$. For any function u of $\mathcal{N}_w^2(0, T, L^2(D))$, any real k and any regular function η denote P-a.s. in Ω by $\mu_{\eta,k}$, the distribution in $\mathbb{R}^{d+1}$, defined by

$$\mu_{\eta,k}(\varphi) = \int_D \eta(u_0 - k)\varphi(0)\mathrm{d}x + \int_Q \eta(u - k)\partial_t\varphi + F^\eta(u, k)\nabla\varphi\mathrm{d}x\mathrm{d}t \\ + \int_0^T \int_D \eta'(u - k)h(u)\varphi\mathrm{d}x\mathrm{d}w(t) + \frac{1}{2}\int_Q h^2(u)\eta''(u - k)\varphi\mathrm{d}x\mathrm{d}t.$$

DEFINITION 1. (*Entropy solution*) A function u of $\mathcal{N}_w^2(0, T, L^2(D))$ is an entropy solution of the stochastic scalar conservation law (1) with the initial condition $u_0 \in L^2(D)$ if
$u \in L^\infty(0, T, L^2(\Omega, L^2(D)))$ and

$$\forall (k, \varphi, \eta) \in \mathbb{A}, \quad 0 \leq \mu_{\eta,k}(\varphi) \quad P - a.s. \tag{2}$$

REMARK 2. Recall that weak and entropy solutions are not smooth solutions; thus, trace has to be understood in a weak way. We followed the approach of Carrillo [5], which consists in formulating the boundary condition implicitly via global integral entropy inequalities involving the semi-Kruzhkov entropies.

REMARK 3. Let us mention that any entropy solution is, P-a.s., a solution in the sense of distributions in Q to

$$\partial_t\left[u - \int_0^t h(u)\,\mathrm{dw}(s)\right] + \operatorname{div}\mathbf{f}(u) = 0.$$

For technical reasons, as in [3], we also need to consider a generalized notion of entropy solution. In fact, in a first step, we will only prove the existence of a measure-valued solution. Then, thanks to a result of reduction to standard solution, we will be able to deduce the existence of an entropy solution in the sense of Definition 1.

DEFINITION 2. (*Measure-valued entropy solution*) A function $u \in N_w^2(0, T, L^2(D\times]0,1[)) \cap L^\infty\big(]0, T[, L^2(\Omega \times D\times]0,1[)\big)$ is a (Young) measure-valued entropy solution of (1) with the initial condition $u_0 \in L^2(D)$ if

$$\forall (k, \varphi, \eta) \in \mathbb{A}, \quad 0 \leq \int_0^1 \mu_{\eta,k}(\varphi)\,\mathrm{d}\alpha \quad P - a.s. \tag{3}$$

And the main result of [3] is

THEOREM 1. *Assume that* $u_0 \in L^2(D)$. *Under assumptions* $H_1 - H_2$, *there exists a unique measure-valued entropy solution in the sense of Definition 2.*

It is the unique entropy solution in the sense of Definition 1, and it is obtained by viscous approximation for the strong topology in $L^p(\Omega \times Q)$, for any $p < 2$.

If $u_1,\ u_2$ are entropy solutions of (1) corresponding to the initial conditions $u_{1,0}, u_{2,0} \in L^2(D)$, respectively, then, for any t in $[0, T]$

$$E \int_D (u_1(t) - u_2(t))^+ \,\mathrm{dx} \leq \int_D (u_{1,0} - u_{2,0})^+ \,\mathrm{dx}.$$

3. Time-splitting method

3.1. Introduction

Our aim is to prove the convergence of an approximation of Problem (1) under assumptions H_1 to H_4. As proposed by Holden–Risebro in [15], we introduce a method to split the effect of the source term and this technique allows us to construct a sequence to approximate the solution of (1). In few words, this approach is based on considering the equation in two parts, solving first a stochastic differential equation and then using the obtained solution as an initial condition for a scalar hyperbolic conservation law without source term. As an extension of [15], we propose in this paper to generalize their estimates on the approximate sequence to the bounded d-dimensional case, in the idea of Chen–Ding–Karlsen [6] concerning BV estimates. Following the notations introduced in [15], we define here two operators for $s,\ t \in [0, T]$.

Let $R(t, s)$ be the operator which takes a number $\overline{u}$ to the solution u at time t of the stochastic differential equation, $\forall t \in [s, T]$

$$\begin{cases} \mathrm{d}u(t) = h(u)\mathrm{d}w(t) \\ u(t = s) = \overline{u}, \end{cases} \tag{4}$$

i.e $u(t) = R(t, s)\overline{u} = \overline{u} + \int_s^t h(u)\mathrm{d}w$.

And $S(t - s)$ denotes the operator, which takes an initial function $u(x, s)$ at time s to the weak entropy solution u at time t of the conservation law

$$\begin{cases} u_t + \operatorname{div} \mathbf{f}(u) = 0 & \text{in }]0, T[\times D, \\ {}''u = 0'' & \text{on }]0, T[\times \partial D, \\ u(t = s) = u(x, s), & \end{cases} \tag{5}$$

i.e., $u(x, t) = S(t - s)u(x, s)$.

REMARK 4. Let us precise that thanks to the assumptions on the data, both R and S are well defined.

Let us introduce for the sequel of the paper, useful results of such operators.

LEMMA 1. *Consider $s \in [0, T]$. Then, P-a.s. and for all $t \in [s, T]$, $R(t, s)$ will take $[-M, M]$ into itself and be the identity outside this interval, where $M > 0$ is defined in H_4.*

Proof. Consider the process u defined for all $t \in [s, T]$ by $u(t) = R(t, s)u(s)$. Applying the Itô formula to a regular function Ψ independent of the time variable t, vanishing in $[-M, M]$ and increasing outside this interval, one gets, P-a.s.:

$$\Psi(u(t)) = \Psi(u(s)) + \int_s^t \underbrace{\Psi_t(u(\sigma))}_{=0} \mathrm{d}\sigma + \int_s^t \underbrace{\Psi_x(u(\sigma))h(u(\sigma))}_{=0} \mathrm{d}w(\sigma)$$
$$+\frac{1}{2}\int_s^t \underbrace{\Psi_{xx}(u(\sigma))h^2(u(\sigma))}_{=0} \mathrm{d}\sigma, \qquad \forall t \in [s, T].$$

Consider $\omega \in \tilde{\Omega}$, where $\tilde{\Omega}$ is a full measure subset of Ω and $t \in [s, T]$. Thus, if $u(s, \omega) \in [-M, M]$, $\Psi(u(s, \omega)) = 0 = \Psi(u(t, \omega))$ and $u(t, \omega) \in [-M, M]$. Else, $\Psi(u(t, \omega)) = \Psi(u(s, \omega))$, by injectivity of Ψ in $\mathbb{R} - [-M, M]$, $u(t, \omega) = u(s, \omega)$ and $R(t, s) = I_d$. □

LEMMA 2. *Consider $s \in [0, T]$, $v_0 \in L^2(\Omega \times D)$ a $\mathcal{F}_s$-measurable process such that*

$$E[TV_x(v_0)] < \infty.$$

Define the process v for all $t \in [s, T]$ by $v(t) = R(s, t)v_0$. Then, for all $t \in [s, T]$

$$E\|v(t)\|_{BV(D)} \leq E\|v_0\|_{BV(D)}.$$

REMARK 5. Let us mention that using the lower semi-continuity property and the positivity of the total variation TV_x on $L^1(D)$, for all v in $L^1(\Omega \times D)$, $E[TV_x(v)]$ has a sense.

Proof. Consider $s \in [0, T]$ and let $v_0 \in L^2(\Omega \times D)$ be a $\mathcal{F}_s$-measurable process with $E[TV_x(v_0)] < \infty$. Define for all $t \in [s, T]$ $v(t) = R(s, t)v_0$ and consider η_δ a regular approximation of the absolute value function with η''_δ a mollifier sequence satisfying $\mathrm{supp}(\eta''_\delta) \subset [-\delta, \delta]$, $\delta > 0$. Applying Itô's formula with the process v and the function η_δ, one gets by taking the integral over D and the expectation, for every $t \in [s, T]$

$$E\int_D \eta_\delta(v(t))\mathrm{d}x = E\int_D \eta_\delta(v_0)\mathrm{d}x + \frac{1}{2}E\int_D\int_s^t \eta''_\delta(v(t))h^2(v)\mathrm{d}\sigma\mathrm{d}x.$$

Passing to the limit on δ to 0 to get for every $t \in [s, T]$

$$E\|v(t)\|_{L^1(D)} = E\|v_0\|_{L^1(D)}.$$

Let us recall a classical result on approximation of BV functions in the deterministic setting, referring the reader to Evans–Gariepy [10]. For every $w \in BV(D)$, there exists an approximate sequence $(w_\epsilon)_\epsilon \subset \mathcal{C}^\infty(D) \cap BV(D)$ such that w_ϵ converges to w in $L^1(D)$. One is also able to assert the following inequalities, for every $\epsilon > 0$

$$\begin{aligned} \|w_0^\epsilon\|_{L^1(D)} &\leq \|w_0\|_{L^1(D)} + 1, \\ TV_x(w_0^\epsilon) &\leq TV_x(w_0) + 4\epsilon. \end{aligned} \tag{6}$$

For technical reasons in the present proof, one needs to work with Hilbert space, thus using the same notation we consider by a regularization process that w_0^ϵ is also in $W^{1,2}(D)$[2] and satisfies, for every $\epsilon > 0$

$$\|w_0^\epsilon\|_{L^1(D)} \leq \|w_0\|_{L^1(D)} + 1 + \epsilon,$$
$$TV_x(w_0^\epsilon) \leq TV_x(w_0) + 5\epsilon.$$

Notice that in our case, $v_0 \in L^1(\Omega \times D)$ and $E[TV_x(v_0)] < \infty$, thus P-a.s., $v_0 \in L^1(D) \cap BV(D)$. P-a.s., the deterministic regularization process holds and $v_0^\epsilon \to v_0$ P-a.s. in $L^1(D)$. Then, this convergence holds strongly in $L^1(\Omega \times D)$ using (6) and the dominated convergence theorem. By Remark 5, we finally have

$$E[TV_x(v_0^\epsilon)] \leq E[TV_x(v_0)] + 5\epsilon. \tag{7}$$

Now, we need estimate on $\partial_{x_i} v_\epsilon$ in order to obtain BV estimate for v. Let us define for all t in $[s, T]$ $v_\epsilon(t) = R(s,t)v_0^\epsilon$. Applying Itô's formula to the process $d(v_\epsilon - v) = [h(v_\epsilon) - h(v)]\mathrm{d}w$ and the function η_δ, taking the integral over D and the expectation, we obtain for every $t \in [s, T]$

$$\begin{aligned} E\int_D \eta_\delta(v_\epsilon - v)(t)\mathrm{d}x &= E\int_D \eta_\delta(v_0^\epsilon - v_0)\mathrm{d}x \\ &\quad + \frac{1}{2}E\int_D\int_s^t \eta_\delta''(v_\epsilon - v)[h(v_\epsilon) - h(v)]^2 \mathrm{d}\sigma\mathrm{d}x. \end{aligned}$$

Passing to the limit on δ to 0 to get for every $t \in [s, T]$

$$E\|(v_\epsilon - v)(t)\|_{L^1(D)} = E\|v_0^\epsilon - v_0\|_{L^1(D)}.$$

Thus, for every $t \in [s, T]$, $v_\epsilon(t) \to v(t)$ in $L^1(\Omega \times D)$.

As P-a.s. and for all $t \in [0, T]$, $v_\epsilon(t) = v_\epsilon(0) + \int_0^t h(v_\epsilon)\mathrm{d}w$ in $W^{1,2}(D)$, using the linear continuity of the derivation operator $\partial_{x_i} : W^{1,2}(D) \to L^2(D)$ for all $i \in \{1, \dots, d\}$ and the chain-rule derivation formula, we get for all $i \in \{1, \dots, d\}$ $\partial_{x_i} v_\epsilon(0) = \partial_{x_i} v_0^\epsilon$ and:

$$\begin{aligned} \partial_{x_i} v_\epsilon(t) &= \partial_{x_i} v_\epsilon(0) + \partial_{x_i}\int_0^t h(v_\epsilon)\mathrm{d}w \\ &= \partial_{x_i} v_\epsilon(0) + \int_0^t h'(v_\epsilon)\partial_{x_i} v_\epsilon \mathrm{d}w, \text{ in } L^2(D). \end{aligned}$$

Applying Itô's formula with such a process and the function η_δ to get that, after taking the integral over D, the expectation and passing to the limit on δ, for all $t \in [s, T]$

$$E\int_D |\partial_{x_i} v_\epsilon|\mathrm{d}x = E\int_D |\partial_{x_i} v_0^\epsilon|\mathrm{d}x < \infty. \tag{8}$$

[2] $W^{1,2}(D)$ denotes the set of functions u in $L^2(D)$ such that $\partial_{x_i} u \in L^2(D)$, for all $i \in \{1, \dots, d\}$.

Thus, for all $t \in [s, T]$ and P-a.s, $v_\epsilon(t) \in BV(D)$. As $v_\epsilon(t) \to v(t)$ in $L^1(\Omega \times D)$, for a subsequence denoted in the same way, for all $t \in [s, T]$ and P-a.s, $v_\epsilon(t) \to v(t)$ in $L^1(D)$. According to Málek-Nečas–Otto–Rokyta–Růžička [19] p. 36, we thus have for all $t \in [s, T]$ and P-a.s.

$$TV_x(v(t)) \leq \liminf_\epsilon TV_x(v_\epsilon(t)).$$

Using again Remark 5, for all $t \in [s, T]$, $TV_x(v(t))$ is measurable with respect to the probability measure P. Consequently, taking the expectation, using Fatou's Lemma, (8) then (7), one gets that for every $t \in [s, T]$

$$E[TV_x(v(t))] \leq \liminf_\epsilon E[TV_x(v_\epsilon(t))] = \liminf_\epsilon E[TV_x(v_0^\epsilon)] \leq E[TV_x(v_0)],$$

and the result holds. □

From the general theory for scalar conservation law, let us now introduce properties satisfied by the operator S(.).

LEMMA 3. *Let* $u_0 \in L^\infty(D) \cap BV(D)$, $t > 0$, *and* $u(t) = S(t)u_0$. *Then,*
i) For almost every $t > 0$,

$$\|u(t)\|_{L^\infty(D)} \leq \|u_0\|_{L^\infty(D)}.$$

ii) There exists a constant $C > 0$, *such that for all* $t_1, t_2 \in [0, T]$

$$\int_D |u(t_1, x) - u(t_2, x)| dx \leq C \|u_0\|_{BV(D)} |t_1 - t_2|.$$

iii) There exists a constant c depending only on the geometry of the boundary ∂D *of* D, *such that for all* $t \in [0, T]$

$$\|u(t, .)\|_{BV(D)} \leq (1 + ct) \|u_0\|_{BV(D)} e^{K_{\mathbf{f}} t},$$

where $K_{\mathbf{f}}$ *denotes the Lipschitz constant of* $\mathbf{f}$.

Proof. These results are classical ones and the proof would be outside the scope of the present work, we refer the reader to Málek–Nečas–Otto–Rokyta–Růžička [19] but also to Gagneux–Madaune [13] for detailed explanations. These results are obtained by the study of viscous solutions. Let us mention the work of Peyroutet [21], which gives us precisely the expression and also the dependence of the constants introduced in this lemma. □

3.2. Construction of the approximate solution

Let us now explain the construction of the approximate solution as introduced in Holden–Risebro [15]. We consider a positive integer N, denote by $\Delta = \frac{T}{N}$ and split the time interval by denoting $t_n = n\Delta,\ n \in \{0, \ldots, N\}$ each point of the time discretization. For each step of discretization Δ, we consider the function

$$u^{\Delta}(t,x) = \begin{cases} u^n(x) & \text{if } t = t_n \\ R(t,t_n)u^n(x) & \text{if } t \in]t_n, t_{n+1}[, \end{cases}$$

where the sequence $(u^n)_{n\in\mathbb{N}}$ is defined by

$$\begin{cases} u^0(x) = u_0(x) \\ u^{n+1}(x) = S(\Delta)R(t_{n+1}, t_n)u^n(x). \end{cases}$$

For convenience in the sequel, let us introduce some notations.

Notations: $\forall n \in \{0, \ldots, N-1\},\ t \in [0,T]$ and $x \in D$:

- $u_-^{n+1}(x) := R(t_{n+1}, t_n)u^n(x)$.
- $\widetilde{u}(t,x) := S(t-t_n)R(t_{n+1}, t_n)u^n(x) = S(t-t_n)u_-^{n+1}(x)$.

PROPOSITION 1. (A priori estimate) *There exists a constant M_1 independent of n and Δ such that P-a.s and for all $t \in [0,T]$*

$$\|u^{\Delta}(t)\|_{L^\infty(D)} \leq M_1 := max(M, \|u_0\|_{L^\infty(D)}).$$

Proof. Let us mention that the construction of u^{Δ} is done by induction, so the proofs of the associated results also rely on inductive reasoning. Consider $n \in \{0, \ldots, N-1\}$, and $u^{n+1} = S(\Delta)u_-^{n+1}$. Thanks to Lemma 3 i),

$$\|u^{n+1}\|_{L^\infty(D)} \leq \|u_-^{n+1}\|_{L^\infty(D)},\ \ P\text{-a.s.}$$

Moreover, thanks to Lemma 1, P-a.s. and $\forall t \in [t_n, t_{n+1}]$

$$\|R(t,t_n)u^n\|_{L^\infty(D)} \leq max(M, \|u^n\|_{L^\infty(D)})$$

and particularly for $t = t_{n+1}$, one has P-a.s

$$\begin{aligned} \|u_-^{n+1}\|_{L^\infty(D)} &= \|R(t_{n+1}, t_n)u^n\|_{L^\infty(D)} \\ &\leq max(M, \|u^n\|_{L^\infty(D)}) \\ &\leq max(M, \|u^0\|_{L^\infty(D)}) := M_1. \end{aligned}$$

Notice that the construction of u^{Δ} is countable, so P-a.s, for all $t \in [0,T]$ and all possible discretization parameter $N \in \mathbb{N}^*$:

$$\|u^{\Delta}(t,.)\|_{L^\infty(D)} \leq M_1,$$

where M_1 does not depend on Δ and the result holds. □

PROPOSITION 2. (BV(D)-bound) *There exists a constant M_2 such that for every $i \in \{0, \ldots, N\}$:*

$$E\|u^i\|_{BV(D)} \leq M_2\|u_0\|_{BV(D)}.$$

Proof. Consider $i \in \{0, \dots, N-1\}$. As $u^i = S(\Delta)u^i_-$, and $u^i_- = R(t_i, t_{i-1})u^{i-1}$, using Lemma 3 and then Lemma 2, one gets

$$E\|u^i\|_{BV(D)} \leq (1+c\Delta)e^{K_{\mathbf{f}}\Delta}E\|u^i_-\|_{BV(D)} \leq (1+c\Delta)e^{K_{\mathbf{f}}\Delta}E\|u^{i-1}\|_{BV(D)},$$

a reasoning by induction gives us

$$E\|u^i\|_{BV(D)} \leq (1+c\Delta)^i e^{K_{\mathbf{f}}\Delta\times i}E\|u_0\|_{BV(D)}.$$

Elementary calculations lead to $(1+c\Delta)^i \leq e^{c\Delta\times i} \leq e^{cT}$, thus $M_2 := e^{cT}e^{K_{\mathbf{f}}T}$. $\square$

Let us introduce a lemma on the increment of u^Δ, useful for the sequel.

LEMMA 4. *Let $n \in \{1, \dots, N\}$ and consider $t \in [t_n, t_{n+1}[$. Then,*

$$E\int_D |u^\Delta(t_{n+1}, x) - u^\Delta(t, x)|dx \leq CM_2\Delta\|u_0\|_{BV(D)} + \widetilde{C}\sqrt{\Delta},$$

where C is defined in Lemma 3 ii), M_2 in Proposition 2 and $\widetilde{C}$ only depends on h, M_1 and $mes(D)$.

Proof. Let $n \in \{1, \dots, N\}$ and consider $t \in [t_n, t_{n+1}[$. For all $x \in D$,

$$u^\Delta(t,x) = R(t, t_n)u^n(x) = u^n(x) + \int_{t_n}^t h(u^\Delta(\sigma))\mathrm{d}w(\sigma)$$
$$u^\Delta(t_{n+1}, x) = u^{n+1}(x).$$

Thus,

$$E\int_D |u^\Delta(t_{n+1},x) - u^\Delta(t,x)|\mathrm{d}x \leq E\int_D |u^{n+1}(x) - u^n(x)|\mathrm{d}x + E\int_D \left|\int_{t_n}^t h(u^\Delta(\sigma))\mathrm{d}w(\sigma)\right|\mathrm{d}x$$

Using previous results on the sequence $(u_n)_{n\in\mathbb{N}}$ stated in Lemma 3 ii) and Proposition 2, one has

$$\begin{aligned}
&E\int_D |u^{n+1}(x) - u^n(x)|\mathrm{d}x \\
&\leq E\int_D |u^{n+1}(x) - u^{n+1}_-(x)| + |u^{n+1}_-(x) - u^n(x)|\mathrm{d}x \\
&= E\int_D |S(\Delta)u^{n+1}_-(x) - u^{n+1}_-(x)| + |R(t_{n+1}, t_n)u^n(x) - u^n(x)|\mathrm{d}x \\
&\leq EC\Delta\|u^{n+1}_-\|_{BV(D)} + E\int_D \left|\int_{t_n}^{t_{n+1}} h(u^\Delta(s,x))\mathrm{d}w(s)\right|\mathrm{d}x \\
&\leq CM_2\Delta\|u_0\|_{BV(D)} + E\int_D \left|\int_{t_n}^{t_{n+1}} h(u^\Delta(s,x))\mathrm{d}w(s)\right|\mathrm{d}x.
\end{aligned}$$

And it remains to show that

$$E\int_D \Big|\int_{t_n}^{t} h(u^{\Delta}(s,x))\mathrm{d}w(s)\Big|\mathrm{d}x + E\int_D \Big|\int_{t_n}^{t_{n+1}} h(u^{\Delta}(s,x))\mathrm{d}w(s)\Big|\mathrm{d}x \leq \widetilde{C}\sqrt{\Delta}.$$

Notice that $|t_n - t| \leq \Delta$, using Cauchy-Schwarz inequality on $\Omega \times D$ and then Itô isometry, we obtain

$$\begin{aligned} E\int_D \Big|\int_{t_n}^{t} h(u^{\Delta}(s,x))\mathrm{d}w(s)\Big|\mathrm{d}x &\leq \sqrt{mes(D)}\left(E\int_D \Big|\int_{t_n}^{t} h(u^{\Delta}(s,x))\mathrm{d}w(s)\Big|^2\mathrm{d}x\right)^{\frac{1}{2}} \\ &= \sqrt{mes(D)}\left(E\int_D \int_{t_n}^{t} h^2(u^{\Delta}(s,x))\mathrm{d}s\mathrm{d}x\right)^{\frac{1}{2}} \\ &\leq \widetilde{C}'\sqrt{\Delta}, \end{aligned}$$

where $\widetilde{C}'$ only depends on $mes(D)$, M_1 and C_h the Lipschitz constant of h. Similarly, one shows that $E\int_D |\int_{t_n}^{t_{n+1}} h(u^{\Delta}(s,x)\mathrm{d}w(s)|\mathrm{d}x \leq \widetilde{C}'\sqrt{\Delta}$, and so $\widetilde{C} = 2\widetilde{C}'$. □

3.3. Entropy formulation

We follow the idea of Peyroutet [21] for introducing the entropy formulation satisfied by the approximate solution. In order to do this, consider

$$\widetilde{u}(t,x) = S(t-t_n)u_-^{n+1}(x)$$

and write the entropy formulation satisfied by such a solution. In order to be compatible with the Definition 1, as in Bauzet–Vallet–Wittbold [3], we consider boundary conditions in the way Carillo [5] introduced them. Using notations of Sect. 1.3, as a weak entropy solution of a conservation law, $\widetilde{u}$ satisfies the following condition, $\forall(k,\varphi,\eta) \in \mathbb{A}$:

$$\begin{aligned} &\int_D \eta(\widetilde{u}(t_n)-k)\varphi(t_n)\mathrm{d}x - \int_D \eta(\widetilde{u}(t_{n+1})-k)\varphi(t_{n+1})\mathrm{d}x \\ &\quad + \int_D\int_{t_n}^{t_{n+1}} \eta(\widetilde{u}-k)\partial_t\varphi + F^{\eta}(\widetilde{u},k)\nabla\varphi \mathrm{d}t\mathrm{d}x \geq 0. \end{aligned} \tag{9}$$

We would like to approximate this formulation. The idea is to introduce in (9) information coming from the initial condition $\widetilde{u}(t_n)$. We consider $(k,\varphi,\eta) \in \mathbb{A}$ and denote for $s \in [t_n, t_{n+1}]$, $v(s) := R(s,t_n)u^n$ the solution in (t_n, t_{n+1}) of the stochastic differential equation

$$\begin{cases} \mathrm{d}v = h(v)\mathrm{d}w \\ v(t=t_n) = u^n. \end{cases}$$

Applying the Itô formula to the process v and the regular function $\Psi(t,\lambda)=\eta(\lambda-k)$, one gets P-a.s:

$$\eta(v(t_{n+1})-k)=\eta(v(t_n)-k)+\int_{t_n}^{t_{n+1}}\eta'(v(t)-k)h(v(t))\mathrm{d}w(t)$$
$$+\frac{1}{2}\int_{t_n}^{t_{n+1}}\eta''(v(t)-k)h^2(v(t))\mathrm{d}t.$$

Remark that $v(t)=u^{\Delta}(t)$ for all $t\in[t_n,t_{n+1}[$ and $v(t_{n+1})=\widetilde{u}(t_n)$, in this way, P-a.s:

$$\int_D\eta(\widetilde{u}(t_n,x)-k)\varphi(t_n,x)\mathrm{d}x-\int_D\eta(u^{\Delta}(t_n,x)-k)\varphi(t_n,x)\mathrm{d}x$$
$$=\int_D\int_{t_n}^{t_{n+1}}\eta'(u^{\Delta}(t,x)-k)h(u^{\Delta}(t,x))\mathrm{d}w(t)\varphi(t_n,x)\mathrm{d}x$$
$$+\frac{1}{2}\int_D\int_{t_n}^{t_{n+1}}\eta''(u^{\Delta}(t,x)-k)h^2(u^{\Delta}(t,x))\mathrm{d}t\varphi(t_n,x)\mathrm{d}x.$$

Moreover,

$$\int_D\eta(\widetilde{u}(t_{n+1},x)-k)\varphi(t_{n+1},x)\mathrm{d}x=\int_D\eta(u^{\Delta}(t_{n+1},x)-k)\varphi(t_{n+1},x)\mathrm{d}x.$$

Thus, one first gets, for any P-measurable set A

$$E\left(\int_D\eta(u^{\Delta}(t_n,x)-k)\varphi(t_n)\mathrm{d}x1_A-\int_D\eta(u^{\Delta}(t_{n+1},x)-k)\varphi(t_{n+1},x)\mathrm{d}x1_A\right)$$
$$+E\left(\int_D\int_{t_n}^{t_{n+1}}\eta'(u^{\Delta}(t,x)-k)h(u^{\Delta}(t,x))\mathrm{d}w(t)\varphi(t_n,x)\mathrm{d}x1_A\right)$$
$$+\frac{1}{2}E\left(\int_D\int_{t_n}^{t_{n+1}}\eta''(u^{\Delta}(t,x)-k)h^2(u^{\Delta}(t,x))\mathrm{d}t\varphi(t_n,x)\mathrm{d}x1_A\right)$$
$$+E\left(\int_D\int_{t_n}^{t_{n+1}}\eta(\widetilde{u}(t,x)-k)\varphi_t(t,x)+F^{\eta}(\widetilde{u}(t,x),k)\nabla\varphi(t,x)\mathrm{d}t\mathrm{d}x1_A\right)$$
$$\geq 0.$$

We propose to approximate $E(\int_D\int_{t_n}^{t_{n+1}}\eta(\widetilde{u}(t,x)-k)\varphi_t(t,x)\mathrm{d}t\mathrm{d}x1_A)$ by $E(\int_D\int_{t_n}^{t_{n+1}}\eta(u^{n+1}-k)\varphi_t(t,x)\mathrm{d}t\mathrm{d}x1_A)$ making an error only of order Δ^2. Indeed,

$$\left|E\left(\int_D\int_{t_n}^{t_{n+1}}\eta(\widetilde{u}(t)-k)\varphi_t\mathrm{d}t\mathrm{d}x-\int_D\int_{t_n}^{t_{n+1}}\eta(u^{n+1}-k)\varphi_t\mathrm{d}t\mathrm{d}x1_A\right)\right|$$
$$\leq CE\int_D\int_{t_n}^{t_{n+1}}|\eta(\widetilde{u}(t)-k)-\eta(u^{n+1}-k)|.|\varphi_t|\mathrm{d}t\mathrm{d}x$$
$$\leq C\|\varphi_t\|_\infty E\int_D\int_{t_n}^{t_{n+1}}|\widetilde{u}(t)-u^{n+1}|\mathrm{d}t\mathrm{d}x.$$

Using as previously results on the sequence $(u_n)_{n\in\mathbb{N}}$ stated in Lemma 3 ii) and Proposition 2, one has

$$\begin{aligned}
&E\int_{t_n}^{t_{n+1}}\int_D |\widetilde{u}(t)-u^{n+1}|\mathrm{d}x\mathrm{d}t\\
&= E\int_{t_n}^{t_{n+1}} \|S(t-t_n)u_-^{n+1}-S(t_{n+1}-t_n)u_-^{n+1}\|_{L^1(D)}\mathrm{d}t\\
&\le C\int_{t_n}^{t_{n+1}} |t-t_{n+1}|E\|u_-^{n+1}\|_{BV(D)}\mathrm{d}t\\
&\le C\int_{t_n}^{t_{n+1}} |t-t_{n+1}|\|u_0\|_{BV(D)}\mathrm{d}t\\
&\le C\|u_0\|_{BV(D)}\Delta^2.
\end{aligned}$$

In the same way, one shows by using the Lipschitz continuity of $F^\eta(.,k)$ that $E(\int_D\int_{t_n}^{t_{n+1}} F^\eta(u^{n+1}(x),k)\nabla\varphi\mathrm{d}t\mathrm{d}x1_A)$ is an approximation of $E(\int_D\int_{t_n}^{t_{n+1}} F^\eta(\widetilde{u}(t,x),k)\nabla\varphi(t,x)\mathrm{d}t\mathrm{d}x1_A)$ also with an error of order Δ^2.

Finally, we obtain by summing over n

$$\begin{aligned}
&E\left(\sum_{n=0}^{N-1}\int_D\int_{t_n}^{t_{n+1}} \eta(u^\Delta(t_{n+1},x)-k)\varphi_t(t,x)\mathrm{d}t\mathrm{d}x1_A\right)\\
&\quad+E\left(\sum_{n=0}^{N-1}\int_D\int_{t_n}^{t_{n+1}} F^\eta(u^\Delta(t_{n+1},x),k)\nabla\varphi(t,x)\mathrm{d}t\mathrm{d}x1_A\right)\\
&\quad+E\left(\sum_{n=0}^{N-1}\int_D\int_{t_n}^{t_{n+1}} \eta'(u^\Delta(t,x)-k)h(u^\Delta(t,x))\mathrm{d}w(t)\varphi(t_n,x)\mathrm{d}x1_A\right)\\
&\quad+\frac{1}{2}E\left(\sum_{n=0}^{N-1}\int_D\int_{t_n}^{t_{n+1}} \eta''(u^\Delta(t,x)-k)h^2(u^\Delta(t,x))\mathrm{d}t\varphi(t_n,x)\mathrm{d}x1_A\right)\\
&\quad+mes(A)\int_D\eta(u_0(x)-k)\varphi(0,x)\mathrm{d}x-E\left(\int_D\eta(u^\Delta(T,x)-k)\varphi(T,x)\mathrm{d}x1_A\right)\\
&\ge-\epsilon\Delta,
\end{aligned}$$

where $\epsilon\Delta$ tends to 0 when Δ does.

REMARK 6. For technical reasons, we keep the term $\int_D\eta(\widetilde{u}(t_{n+1})-k)\varphi(t_{n+1})\mathrm{d}x$ in the entropy formulation (9), in order to vanish two-by-two terms when we do the summation over n with $\int_D\eta(\widetilde{u}(t_n)-k)\varphi(t_n)\mathrm{d}x$. Then, last term of the sum: $E(\int_D\eta(u^\Delta(T)-k)\varphi(T)\mathrm{d}x1_A)$ is nonnegative and we remove it from the inequality.

3.4. Convergence of the approximate solution

Our aim is to pass to the limit with respect to Δ in the following inequality:

$$
\begin{aligned}
&E\left(\sum_{n=0}^{N-1}\int_D\int_{t_n}^{t_{n+1}}\eta'(u^\Delta(t,x)-k)h(u^\Delta(t,x))\mathrm{d}w(t)\varphi(t_n,x)\mathrm{d}x1_A\right):=I_1^\Delta\\
&+\frac{1}{2}E\left(\sum_{n=0}^{N-1}\int_D\int_{t_n}^{t_{n+1}}\eta''(u^\Delta(t,x)-k)h^2(u^\Delta(t,x))\mathrm{d}t\varphi(t_n,x)\mathrm{d}x1_A\right):=I_2^\Delta\\
&+E\left(\sum_{n=0}^{N-1}\int_D\int_{t_n}^{t_{n+1}}\eta(u^\Delta(t_{n+1},x)-k)\varphi_t(t,x)\mathrm{d}t\mathrm{d}x1_A\right):=I_3^\Delta\\
&+E\left(\sum_{n=0}^{N-1}\int_D\int_{t_n}^{t_{n+1}}F^\eta(u^\Delta(t_{n+1},x),k)\nabla\varphi(t,x)\mathrm{d}t\mathrm{d}x1_A\right):=I_4^\Delta\\
&+mes(A)\int_D\eta(u_0(x)-k)\varphi(0,x)\mathrm{d}x\geq\text{-}\epsilon\Delta,
\end{aligned}
\tag{10}
$$

where A is a P-measurable set. Due to the random variable, even if strong estimates with respect to variables t and x hold, we are not able to use classical results of compactness. The one given by the concept of Young measures is appropriate here and the technique is based on the notion of narrow convergence of Young's measures (or entropy processes), we refer to Eymard–Gallouët–Herbin [11] and Panov [20]. Then, thanks to the uniqueness result of Sect. 2, we will be able to prove that the measure-valued limit is an entropy solution in the sense of Definition 1. Since (u^Δ) is a bounded sequence in $L^\infty(Q\times\Omega)$, the associated Young's measure sequence (u^Δ) converges (up to a subsequence still indexed in the same way) to a Young's measure denoted $\mathbf{u}\in L^\infty(Q\times\Omega\times]0,1[)$. Furthermore, according to Balder [1] but also to Eymard–Gallouët–Herbin [11], for any Carathéodory function Ψ such that $\Psi(.,u^\Delta)$ is uniformly integrable:

$$
E\int_Q\Psi(u^\Delta(t,x))\mathrm{d}t\mathrm{d}x\to E\int_Q\int_0^1\Psi(\mathbf{u}(t,x,\alpha))\mathrm{d}\alpha\mathrm{d}t\mathrm{d}x\text{ when }\Delta\to 0.
$$

Moreover, revisiting the work of Panov [20] on the measurability of $\mathbf{u}$ with respect to all its variables, one shows that as u^Δ is a predictable process with values in $L^2(D)$, $\mathbf{u}$ is in $\mathcal{N}_w^2(0,T,L^2(D\times]0,1[))$. We refer the reader to the "Appendix" of [2] for detailed explanations to obtain this measurability. Let us consider separately the terms of (10) and analyze passage to the limit for each term. In order to make the lecture more fluent, we omit the variable (t,x) when no confusion is possible.

1. $I_1^\Delta\to E\left(\int_D\int_0^T\int_0^1\eta'(\mathbf{u}(\alpha)-k)h(\mathbf{u}(.,\alpha))\mathrm{d}\alpha\varphi\mathrm{d}w(t)\mathrm{d}x1_A\right):=I_1.$

$$\begin{aligned}
&|I_1^\Delta - I_1| \\
&= \Big| E\left(\sum_{n=0}^{N-1}\int_D\int_{t_n}^{t_{n+1}} \eta'(u^\Delta - k)h(u^\Delta)[\varphi(t_n)-\varphi(t)]\mathrm{d}w(t)\mathrm{d}x 1_A\right) \\
&\quad + E\left(\int_D\int_0^T\left[\eta'(u^\Delta - k)h(u^\Delta) - \int_0^1 \eta'(\mathbf{u}(.,\alpha)-k)h(\mathbf{u}(.,\alpha))\mathrm{d}\alpha\right]\varphi(t)\mathrm{d}w(t)\mathrm{d}x 1_A\right)\Big| \\
&:= |I_{1,1}^\Delta + I_{1,2}^\Delta|.
\end{aligned}$$

Using Cauchy-Schwarz inequality on Ω×D and then Itô isometry we obtain

$$\begin{aligned}
|I_{1,1}^\Delta| &= \Big| E\left(\sum_{n=0}^{N-1}\int_D\int_{t_n}^{t_{n+1}} \eta'(u^\Delta - k)h(u^\Delta)[\varphi(t_n)-\varphi(t)]\mathrm{d}w(t)\mathrm{d}x 1_A\right)\Big| \\
&\le C\sum_{n=0}^{N-1}\left[E\int_D\left(\int_{t_n}^{t_{n+1}} \eta'(u^\Delta - k)h(u^\Delta)[\varphi(t_n)-\varphi(t)]\mathrm{d}w(t)\right)^2\mathrm{d}x\right]^{\frac{1}{2}} \\
&= C\sum_{n=0}^{N-1}\left[E\int_D\int_{t_n}^{t_{n+1}} \left[\eta'(u^\Delta - k)h(u^\Delta)[\varphi(t_n)-\varphi(t)]\right]^2\mathrm{d}t\mathrm{d}x\right]^{\frac{1}{2}} \\
&\le C\sum_{n=0}^{N-1}\left[E\int_{t_n}^{t_{n+1}}\int_D [\eta'(u^\Delta - k)h(u^\Delta)]^2\underbrace{(\varphi(t_n)-\varphi(t))^2}_{\le C\Delta^2}\mathrm{d}x\mathrm{d}t\right]^{\frac{1}{2}} \\
&\le C\sum_{n=0}^{N-1}\left[E\int_{t_n}^{t_{n+1}} mes(D)\times\Delta^2\mathrm{d}t\right]^{\frac{1}{2}} \\
&\le C\sum_{n=0}^{N-1}\Delta^{\frac{3}{2}} \\
&= C\sqrt{\Delta}\to 0.
\end{aligned}$$

Let us show that $I_{1,2}^\Delta \to 0$. Denote $v^\Delta = \eta'(u^\Delta - k)h(u^\Delta)\varphi$. Thanks to Proposition 1, v^Δ is bounded in $L^2(Q\times\Omega)$ and there exists $v\in L^2(Q\times\Omega)$ such that $v^\Delta \rightharpoonup v$ in the same space. Moreover, $\Psi : (t,x,\omega,\lambda)\mapsto \eta'(\lambda-k)h(\lambda)\varphi(t,x)$, $(t,x,\omega,\lambda)\in Q\times\Omega\times\mathbb{R}$ is a Carathéodory function and $\Psi(.,u^\Delta)$ is uniformly integrable as it is bounded in $L^2(Q\times\Omega)$. By identification, $v=\int_0^1\Psi(\mathbf{u}(.,\alpha))\mathrm{d}\alpha$. Furthermore, for all $t\in[0,T]$,

$$\begin{aligned}
I_t : L^2(Q\times\Omega) &\to L^2(D\times\Omega) \\
\overline{u} &\mapsto \int_0^t \overline{u}(t,x,\omega)\mathrm{d}w(t)
\end{aligned}$$

is a linear continuous function, and so it is a weakly continuous function from $L^2(Q\times\Omega)$ to $L^2(D\times\Omega)$. Consequently, $I_t(v^\Delta)\to I_t(v)$ in $L^2(D\times\Omega)$. In this manner,

$$E\left(\int_D\int_0^T \eta'(u^\Delta - k)h(u^\Delta)\varphi \mathrm{d}w(t)\mathrm{d}x 1_A\right)$$
$$\to E\left(\int_D\int_0^T\int_0^1 \eta'(\mathbf{u}(.,\alpha)-k)h(\mathbf{u}(.,\alpha))\mathrm{d}\alpha\varphi\mathrm{d}w(t)\mathrm{d}x 1_A\right),$$

and $|I_{1,2}^\Delta| \to 0$.

2. $I_2^\Delta \to \frac{1}{2}E\left(\int_Q\int_0^1 \eta''(\mathbf{u}(.,\alpha)-k)h^2(\mathbf{u}(.,\alpha))\mathrm{d}\alpha\varphi\mathrm{d}t\mathrm{d}x 1_A\right) := I_2.$

$$\begin{aligned}
|I_2^\Delta - I_2| &= \frac{1}{2}\Bigg|E\left(\sum_{n=0}^{N-1}\int_D\int_{t_n}^{t_{n+1}} \eta''(u^\Delta - k)h^2(u^\Delta)[\varphi(t_n)-\varphi(t)]\mathrm{d}t\mathrm{d}x 1_A\right)\\
&\quad + E\int_Q \eta''(u^\Delta - k)h^2(u^\Delta)\varphi(t)\mathrm{d}t\mathrm{d}x 1_A\\
&\quad - E\int_Q\int_0^1 \eta''(\mathbf{u}(.,\alpha)-k)h^2(\mathbf{u}(.,\alpha))\mathrm{d}\alpha\varphi(t)\mathrm{d}t\mathrm{d}x 1_A\Bigg|\\
&:= \frac{1}{2}|I_{2,1}^\Delta + I_{2,2}^\Delta|.\\
|I_{2,1}^\Delta| &\leq E\left(\sum_{n=0}^{N-1}\int_D\int_{t_n}^{t_{n+1}} \left|\eta''(u^\Delta - k)h^2(u^\Delta)[\varphi(t_n)-\varphi(t)]\right|\mathrm{d}t\mathrm{d}x\right)\\
&\leq C\sum_{n=0}^{N-1}\Delta^2\\
&\leq C\Delta \to 0.
\end{aligned}$$

Note that $\Psi(t,x,\omega,\lambda) = \eta''(\lambda - k)h^2(\lambda)\varphi(t,x)1_A$ is a Carathéodory function such that $\Psi(.,u^\Delta)$ is uniformly integrable, thus $I_{2,2}^\Delta \to 0$ and the result holds.

3. $I_3^\Delta \to E\left(\int_Q\int_0^1 \eta(\mathbf{u}(.,\alpha)-k)\mathrm{d}\alpha\varphi_t\mathrm{d}t\mathrm{d}x 1_A\right) := I_3.$

$$\begin{aligned}
|I_3^\Delta - I_3| &\leq \left|E\left(\sum_{n=0}^{N-1}\int_D\int_{t_n}^{t_{n+1}}[\eta(u^\Delta(t_{n+1})-k) - \eta(u^\Delta(t)-k)]\varphi_t\mathrm{d}t\mathrm{d}x 1_A\right)\right|\\
&\quad + \left|E\left(\int_Q[\eta(u^\Delta - k) - \int_0^1 \eta(\mathbf{u}(.,\alpha)-k)\mathrm{d}\alpha]\varphi_t\mathrm{d}t\mathrm{d}x 1_A\right)\right|\\
&:= |I_{3,1}^\Delta| + |I_{3,2}^\Delta|.\\
|I_{3,1}^\Delta| &\leq E\left(\sum_{n=0}^{N-1}\int_D\int_{t_n}^{t_{n+1}}|\eta(u^\Delta(t_{n+1})-k) - \eta(u^\Delta(t)-k)|.|\varphi_t|\mathrm{d}t\mathrm{d}x 1_A\right)\\
&\leq CE\sum_{n=0}^{N-1}\int_D\int_{t_n}^{t_{n+1}}|u^\Delta(t_{n+1}) - u^\Delta(t)|\mathrm{d}t\mathrm{d}x.
\end{aligned}$$

On the other hand, one shows in the proof of Lemma 4 that

$$E\int_D |u^\Delta(t_{n+1}) - u^\Delta(t)|\mathrm{d}x \leq CM_2\Delta\|u_0\|_{BV(D)} + \widetilde{C}\sqrt{\Delta}$$

where C is defined in Lemma 3 ii), M_2 in Proposition 2, $\widetilde{C}$ only depends on h, M_1 and D. Finally, $|I_{3,1}^\Delta| \to 0$. Let us now show that $I_{3,2}^\Delta \to 0$. We consider the Carathéodory function $\Psi(t,x,\omega,\lambda) = \eta(\lambda - k)\varphi_t(t,x)1_A$. As previously, $\Psi(., u^\Delta)$ is uniformly integrable and

$$E\int_Q \eta(u^\Delta - k)\varphi_t 1_A \mathrm{d}t\mathrm{d}x \to E\int_Q\int_0^1 \eta(\mathbf{u}(.,\alpha) - k)\mathrm{d}\alpha\varphi_t 1_A\mathrm{d}t\mathrm{d}x.$$

4. $I_4^\Delta \to E\left(\int_Q\int_0^1 F^\eta(\mathbf{u}(.,\alpha),k)\mathrm{d}\alpha\nabla\varphi\mathrm{d}t\mathrm{d}x1_A\right) := I_4.$

$$\begin{aligned}
&|I_4^\Delta - I_4|\\
&\leq \left|E\left(\sum_{n=0}^{N-1}\int_D\int_{t_n}^{t_{n+1}}\left[F^\eta(u^\Delta(t_{n+1}),k) - F^\eta(u^\Delta(t),k)\right]\nabla\varphi)\mathrm{d}t\mathrm{d}x1_A\right)\right|\\
&\quad+\left|E\left(\int_Q\left[F^\eta(u^\Delta(t),k) - \int_0^1 F^\eta(\mathbf{u}(.,\alpha),k)\mathrm{d}\alpha\right]\nabla\varphi\mathrm{d}t\mathrm{d}x1_A\right)\right|\\
&:= |I_{4,1}^\Delta| + |I_{4,2}^\Delta|.
\end{aligned}$$

As previously, one shows that $|I_{4,1}^\Delta| \to 0$ using the Lipschitz continuity of $F^\eta(.,k)$ for all $k \in \mathbb{R}$. And as $\Psi(t,x,\omega,\lambda) = \int_k^\lambda \eta'(\sigma - k)f'(\sigma)\mathrm{d}\sigma\nabla\varphi 1_A$ is a Carathéodory function with $\Psi(., u^\Delta)$ uniformly integrable, one gets that $I_{4,2}^\Delta \to 0$ by the way of Young measure theory.

Finally, for all $(k,\varphi,\eta) \in \mathbb{A}$ and for any P-measurable set A:

$$\begin{aligned}
&E\left(\int_D\int_0^T\int_0^1 \eta'(\mathbf{u}(\alpha) - k)h(\mathbf{u}(.,\alpha))\mathrm{d}\alpha\varphi\mathrm{d}w(t)\mathrm{d}x1_A\right)\\
&\quad+\frac{1}{2}E\left(\int_Q\int_0^1 \eta''(\mathbf{u}(.,\alpha) - k)h^2(\mathbf{u}(.,\alpha))\mathrm{d}\alpha\varphi\mathrm{d}t\mathrm{d}x1_A\right)\\
&\quad+E\left(\int_Q\int_0^1 [\eta(\mathbf{u}(.,\alpha) - k)\varphi_t + F^\eta(\mathbf{u}(.,\alpha),k)\nabla\varphi]\mathrm{d}\alpha\mathrm{d}t\mathrm{d}x1_A\right)\\
&\quad+mes(A)\int_D \eta(u_0(x) - k)\varphi(0,x)\mathrm{d}x\\
&\geq 0.
\end{aligned}$$

Thus, $\mathbf{u}$ is a measure-valued entropy solution of (1) in the sense of Definition 2. Thanks to the work of Bauzet–Vallet–Wittbold [3] and their main result resumed in Theorem 1, any measure-valued entropy solution in the sense of Definition 2 is unique and is the unique entropy solution in the sense of Definition 1. In this way, our approximate sequence u^{Δ} of the stochastic conservation law (1) converges to the unique weak entropy solution u of such a problem.

REMARK 7. Let us mention that the approximate sequence u^{Δ} converges to u in $L^1(Q \times \Omega)$ thanks to the Young measure theory. Moreover, as u^{Δ} is bounded in $L^{\infty}(Q \times \Omega)$, u^{Δ} converges to u strongly in $L^p(Q \times \Omega)$ for every $1 \leq p < \infty$, using Vitali's theorem.

REMARK 8. (Extension in the $\mathbb{R}^d$-case)
Using the theoretical study of Bauzet–Vallet–Wittbold [2] on the Cauchy problem for Problem (1) setting in $\mathbb{R}^d$ instead of a bounded domain D, one is also able to propose a time-splitting method in the $\mathbb{R}^d$-case to approximate the stochastic weak entropy solution. Indeed, the book of Málek-Nečas–Otto–Rokyta–Růžička [19] gives us necessary tools on scalar conservation laws in unbounded domain as in Lemma 3. Moreover, in order to manage integrals on $\mathbb{R}^d$, it suffices to argue as in Holden–Risebro [15] with compactly supported test functions.

4. Numerical experiments

We propose here an application of this time-splitting method to the stochastic Burgers' equation in the one-dimensional case:

$$\mathrm{d}u + f(u)_x \mathrm{d}x = h(u)\mathrm{d}w \quad \text{in } \Omega \times]0,1[\times]-1,1[,$$

where $f(u) = \frac{u^2}{2}$, $h : \mathbb{R} \to \mathbb{R}$ has a compact support in]0,1[and is defined by

$$h(x) = \begin{cases} 2e^{\frac{1}{|2x-1|^2-1}} & \text{if } 0 < x < 1 \\ 0 & \text{else.} \end{cases} \tag{11}$$

Note that following Remark 1, we are in the framework presented in the previous section. The scheme relies on finite volume method. With an Euler method (let us mention the book of Kloeden–Platen [16] for details), we solve the stochastic differential equation (4). Then, for solving the conservation law (5), we use a Godunov method that seems to be a suitable choice for the Burgers' one-dimensional equation, particularly the way this scheme takes into account the boundary conditions and the behavior of the flux function. We refer the reader to the book of Leveque [18].

We implement simulations with different initial conditions: u_1^0, u_2^0, u_3^0 and u_4^0 defined below:

$$u_1^0(x) = \begin{cases} -\frac{1}{2} & \text{if } x < 0 \\ \frac{1}{2} & \text{else.} \end{cases} \quad u_2^0(x) = \begin{cases} \frac{1}{2} & \text{if } x < 0 \\ -\frac{1}{2} & \text{else.} \end{cases} \quad \begin{aligned} u_3^0(x) &= 1 - \tfrac{2}{\pi}\arctan(x). \\ u_4^0(x) &= -\sin(\pi x). \end{aligned}$$

To illustrate our proposal, we give, for each initial condition, a simulation of the solution in the deterministic case (i.e., when $h = 0$), and for the h given by (11) two sample path simulations. Then, for those two simulations, we propose to highlight the time behavior of the solution at a given point x of the interval $[-1, 1]$. We get the following graphics in the (x, t) plane with $\Delta_x = 0.002$ and $\Delta_t = 0.001$. As expected in the stochastic case, perturbations appear when the solution $u(t, x)$ takes values in the support of h (included in $]0, 1[$). These simulations have been implemented with the free software *Scilab*.

Acknowledgments

The author wishes to thank Guy Vallet for many stimulating discussions and valuable suggestions. Last but not least, many thanks go to the referee for his comments helping to improve the manuscript.

Appendix A: Comments on these numerical experiments

To our knowledge, there are only few papers presenting numerical simulations of solutions of scalar conservation laws with multiplicative stochastic perturbation. Let us mention again the paper of Holden–Risebro [15] where the authors made simulations with data associated with oil reservoirs models. In the paper of Kröker–Rhode [17], the authors are interested in numerical results on the one-dimensional Cauchy problem for a scalar conservation law with random noise. They use combination of finite volume methods and the Euler–Maruyama method. In the section devoted to numerical experiments, they aim to compare solutions of deterministic and stochastic Burgers' equations. In order to have exact expression of the solutions, the particular case of an additive perturbation is considered. Although the average of the stochastic solutions is close to the deterministic solution, it is not equal to it, and further numerical experiments indicate that it does not converge as the number of realizations increases.

REMARK 9. Note that, in the present work, although we are interested in the convergence of the method for a time discretization of the problem, numerical experiments are realized for a full time-space discretized problem. One can use for example the well-known results concerning the approximation of deterministic conservation laws when the initial condition is bounded with bounded variations. In a reverse strategy, Kröker–Rhode investigated in [17] on a space-discretization of the problem and present numerical simulations with a full discretized scheme, and a comparison of the numerical results is not easy.

In the deterministic case, when $h = 0$ (Fig. 1a, e, i, m), one can recognize expected solutions: the propagation in time along the characteristic lines of the initial condition and the Dirichlet boundary condition, with shocks or rarefaction waves.

In the stochastic case, let us first warn the reader that scales are unfortunately not the same for simulations in the deterministic and in the stochastic case.

As expected, perturbations appear when h is active, i.e., when $0 \leq u \leq 1$, and not active else. Note also that since $h(0) = 0$ and $h(1) = 0$, the stochastic perturbation is

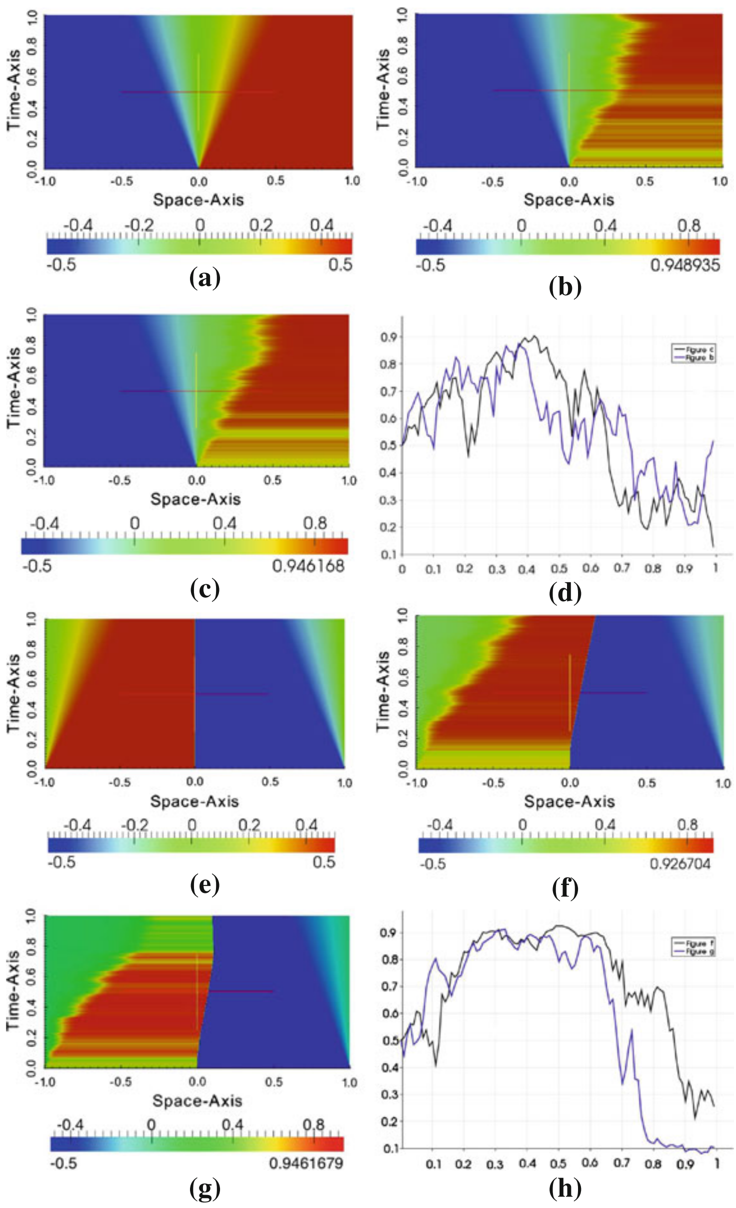

Figure 1. **a** Burgers with u_1^0. **b** Stochastic Burgers with u_1^0. **c** Stochastic Burgers with u_1^0. **d** Pathwise solutions for $x = 0.5$. **e** Burgers with u_2^0. **f** Stochastic Burgers with u_2^0. **g** Stochastic Burgers with u_2^0. **h** Pathwise solutions for $x = 0.5$. **i** Burgers with u_3^0. **j** Stochastic Burgers with u_3^0. **k** Stochastic Burgers with u_3^0. **l** Pathwise solutions for $x = -0.7$. **m** Burgers with u_4^0. **n** Stochastic Burgers with u_4^0. **o** Stochastic Burgers with u_4^0. **p** Pathwise solutions for $x = -0.5$

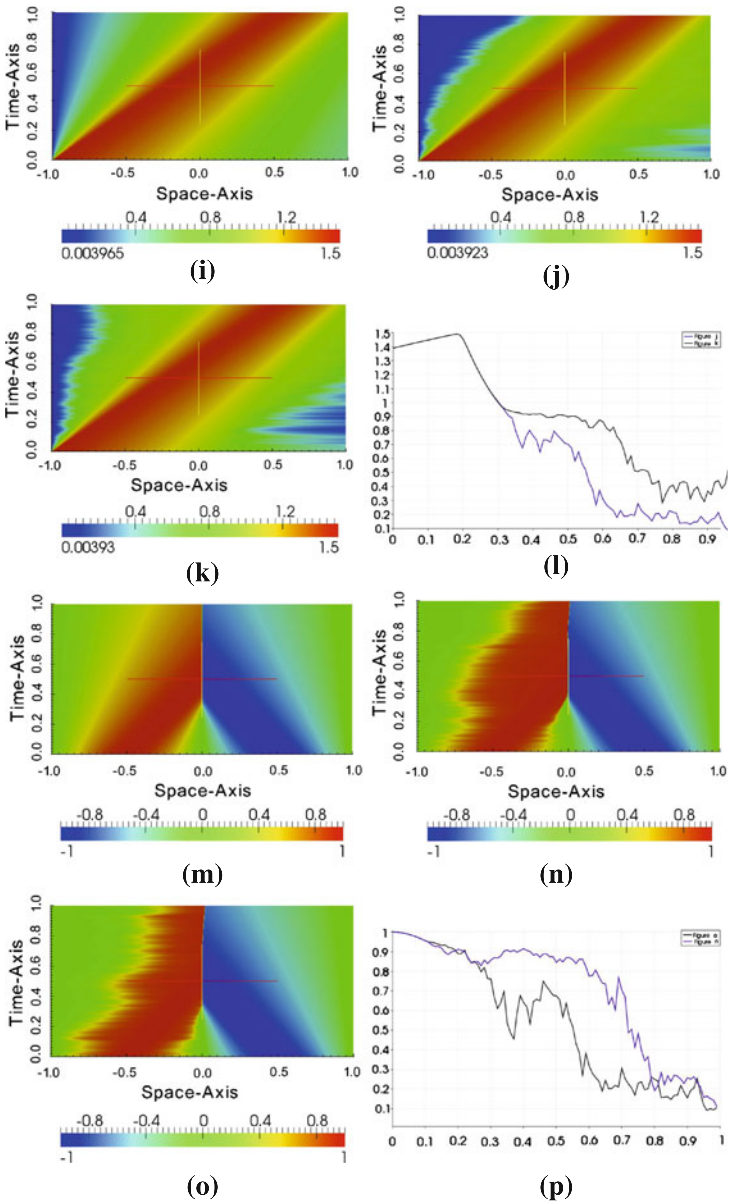

Figure 1. continued

not active if the solution u is equal to 0, or to 1, in a neighborhood. Let us finally remark that since we consider a time noise, horizontal variations appear in the (x, t)-plane, but more classical representations of stochastic perturbation are in the figures denoted "Pathwise solutions."

Visually, we get back illustrations of the stochastic version of characteristics studied by [9].

- **Simulations with** u_1^0. The initial condition generates a rarefaction wave and $h(\mathbb{R}^-) = \{0\}$ yields a division of $(-1, 1)$ in two parts: the one of positive x where $u \geq 0$ and h is active; and the part of negative x where $u \leq 0$ and h is not active.
- **Simulations with** u_2^0. The initial condition generates a shock and h is active only on the part on the left of the shock curve. Then, the perturbation of the positive values of u and the Rankine–Hugoniot condition yields a shock curve that is not a straight line anymore. Notice that, depending on the realization, the monotony of the shock wave is not a priori conserved (see Fig. 1f, g).
- **Simulations with** u_3^0. In this example, the perturbation is mainly active on the extremities of $(-1, 1)$, where $u < 1$. This is illustrated in Fig. 1(l) where the two realizations at $x = -0.7$ are the same for $t \leq 0, 3$, the same than in the deterministic case.

REFERENCES

[1] Balder, E.J.: Lectures on Young measure theory and its applications in economics. Rend. Istit. Mat. Univ. Trieste **31**(suppl. 1), 1–69 (2000). Workshop on Measure Theory and Real Analysis (Italian) (Grado, 1997)

[2] Bauzet, C., Vallet, G., Wittbold, P.: The Cauchy problem for a conservation law with a multiplicative stochastic perturbation. Journal of Hyperbolic Differential Equations **9**(4), 661–709 (2012)

[3] Bauzet, C., Vallet, G., Wittbold, P.: The Dirichlet problem for a conservation law with a multiplicative stochastic perturbation. Journal of Functional Analysis (2013)

[4] Bensoussan, A., Glowinski, R., Raşcanu, A.: Approximation of some stochastic differential equations by the splitting up method. Appl. Math. Optim. **25**(1), 81–106 (1992)

[5] Carrillo, J.: Entropy solutions for nonlinear degenerate problems. Arch. Ration. Mech. Anal. **147**(4), 269–361 (1999)

[6] Chen, G.Q., Ding, Q., Karlsen, K.H.: On nonlinear stochastic balance laws. Arch. Ration. Mech. Anal. **204**(3), 707–743 (2012)

[7] Da Prato, G., Zabczyk, J.: Stochastic equations in infinite dimensions, Encyclopedia of Mathematics and its Applications, vol. 44. Cambridge University Press, Cambridge (1992)

[8] Debussche, A., Vovelle, J.: Scalar conservation laws with stochastic forcing. J. Funct. Anal. **259**(4), 1014–1042 (2010)

[9] E, W., Khanin, K., Mazel, A., Sinai, Y.: Invariant measures for Burgers equation with stochastic forcing. Ann. of Math. (2) **151**(3), 877–960 (2000)

[10] Evans, L.C., Gariepy, R.F.: Measure theory and fine properties of functions. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL (1992)

[11] Eymard, R., Gallouët, T., Herbin, R.: Existence and uniqueness of the entropy solution to a nonlinear hyperbolic equation. Chinese Ann. Math. Ser. B **16**(1), 1–14 (1995). A Chinese summary appears in Chinese Ann. Math. Ser. A **16** (1995), no. 1, 119

[12] Feng, J., Nualart, D.: Stochastic scalar conservation laws. J. Funct. Anal. **255**(2), 313–373 (2008)

[13] Gagneux, G., Madaune-Tort, M.: Analyse mathématique de modèles non linéaires de l'ingénierie pétrolière, Mathématiques & Applications (Berlin) [Mathematics & Applications], vol. 22. Springer, Berlin (1996)

[14] Hofmanová, M.: Bhatnagar-gross-krook approximation to stochastic scalar conservation laws (2013)

[15] Holden, H., Risebro, N.H.: A stochastic approach to conservation laws. In: Third International Conference on Hyperbolic Problems, Vol. I, II (Uppsala, 1990), pp. 575–587. Studentlitteratur, Lund (1991)

[16] Kloeden, P.E., Platen, E.: Numerical solution of stochastic differential equations, Applications of Mathematics (New York), vol. 23. Springer, Berlin (1992)

[17] Kröker, I., Rohde, C.: Finite volume schemes for hyperbolic balance laws with multiplicative noise. Appl. Numer. Math. **62**(4), 441–456 (2012)

[18] LeVeque, R.J.: Finite volume methods for hyperbolic problems. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge (2002)

[19] Málek, J., Nečas, J., Rokyta, M., Ruzicka, M.: Weak and measure-valued solutions to evolutionary PDEs, Applied Mathematics and Mathematical Computation, vol. 13. Chapman & Hall, London (1996)

[20] Panov, E.Y.: On measure-valued solutions of the Cauchy problem for a first-order quasilinear equation. Izv. Ross. Akad. Nauk Ser. Mat. **60**(2), 107–148 (1996)

[21] Peyroutet, F.: Etude d'une méthode de splitting pour des lois de conservation scalaires avec terme de source. Ph.D. thesis, U.P.P.A. (1999)

C. Bauzet
UMR, CNRS 5142,
IPRA BP 1155,
64013 Pau Cedex, France
E-mail: caroline.bauzet@univ-pau.fr